Cross-Sectional Dynamics Under Network Structure: Theory & Macroeconomic Applications

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Motivation

- Common in economics: cross-section of units/agents, linked by network ties
- Theory and empirics: network amplifies unit-level shocks, implies comovement of cross-sectional variables
- How does network-induced comovement play out over time?
- **Literature:** Two restrictive cases:
  - **innovations transmit contemporaneously**
    e.g. Acemoglu et al. 2012, Elliott et al. 2014

    $\rightarrow$ static model, links of all order play out simultaneously
  - **innovations transmit one link per period**
    e.g. Long & Plosser 1983, Golub & Jackson 2010

    $\rightarrow$ tenable in theory, less so in empirics

Contribution

- Econometric framework that can speak to dynamics implied by networks
  - VAR parameterized s.t. innovations transmit cross-sectionally via bilateral links
  - Can accommodate general patterns on how innovations travel through network over time
- Applicable in two distinct lines of empirical work with cross-sectional time series
  - estimate dynamic network (peer) effects, with network given or estimated (+ shrink to observed links)
  - dimensionality-reduction technique for modeling (c.s.) time series

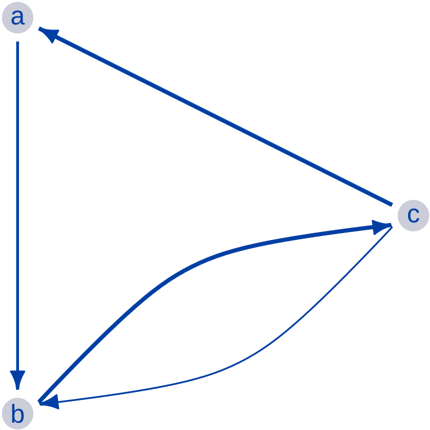
$\rightarrow$ Two applications

Related Literature: Model

Networks in econometrics

- Spatial Autoregressive Models:
  - identify network effects in static framework
    Manski 1993, Lee 2007, Bramouillé et al. 2009, de Paula et al. 2020, ...
    → I look at dynamic, contagion-like network effects
  - some work on lagged/dynamic network effects
    Knight et al. 2016, Zhu et al. 2017, Yang & Lee 2019, ...
    → I relate TS properties to network and timing of network effects, generalize mapping, & show how to conduct inference on both
- Networks in time series (TS) econometrics:
  - represent TS model output as network
    Diebold & Yilmaz 2009, 2014, Barigozzi & Brownlees 2018, ...
    → I use network to obtain a TS model
  - restrict TS models using networks
    Pesaran et al. 2004, Chudik & Pesaran 2011, Caporin et al. 2022, ...
    → I focus on simpler/clearer case & assume tranmission via links → analytical results

Bilateral Connections in Networks details



$$A = \begin{bmatrix} 0 & 0 & .8 \\ .7 & 0 & .6 \\ 0 & .8 & 0 \end{bmatrix}$$

shows direct links

$$A^2 = \begin{bmatrix} 0 & .64 & 0 \\ 0 & .48 & .56 \\ .56 & 0 & .48 \end{bmatrix}$$

shows 2nd order connections

...

Lagged Innovation Transmission via Bilateral Links

VAR(1):

$$y_t = Ay_{t-1} + u_t \ ,$$

$$\to y_{it} = \sum_{j=1}^{n} a_{ij} y_{j,t-1} + u_{it}$$

- Interpret A as network: innovations travel one link per period

→ Granger Causality at horizon $h = 1, 2, ...$ given by hth order network connections:

illustration

$$\frac{\partial y_{i,t+h}}{\partial y_{j,t}} = (A^h)_{ij} \ .$$

- Used in theory:
  - Long & Plosser (1983): sectoral output under one period delay in I-O conversion
  - Golub & Jackson (2010): study of societal opinion formation through friendship ties

Lagged Innovation Transmission via Bilateral Links

NVAR(p, 1): (particular version of NAR(p) in Zhu et al. 2017)

$$\tilde{y}_\tau = \alpha_1 A \tilde{y}_{\tau-1} + ... + \alpha_p A \tilde{y}_{\tau-p} + \tilde{u}_\tau \ , \quad \alpha = (\alpha_1, ..., \alpha_p)' \in \mathbb{R}^p \ .$$

- Assuming $\alpha_l \neq 0 \; \forall \; l$, $\tilde{y}_j$ Granger-causes $\tilde{y}_i$ at horizon h iff there exists a connection from i to j of at least one order $k \in \{\underline{k}, \underline{k}+1, ..., h\}$, where $\underline{k} = ceil(h/p)$.

$$\rightarrow \frac{\partial \tilde{y}_{i,\tau+h}}{\partial \tilde{u}_{j,\tau}} = c^h_{\underline{k}}(\alpha) \left[A^{\underline{k}}\right]_{ij} + ... + c^h_h(\alpha) \left[A^h\right]_{ij} \ .$$

- i.e. $\tilde{y}_\tau$ driven by lagged network effects, with transmission spread out over p periods
- α shows time profile of transmission

Lagged Innovation Transmission via Bilateral Links

$$\tilde{y}_\tau = \alpha_1 A \tilde{y}_{\tau-1} + ... + \alpha_p A \tilde{y}_{\tau-p} + \tilde{u}_\tau \ , \quad \alpha = (\alpha_1, ..., \alpha_p) \in \mathbb{R}^p \ .$$

If $\tilde{y}_\tau$ observed every $q > 1$ periods, then $\{y_t\}_{t=1}^T = \{\tilde{y}_{tq}\}_{t=1}^T$

- for GC at horizon h, links of order $k \in \{\underline{k}, \underline{k}+1, ..., hq\}$ matter, $\underline{k} = ceil(hq/p)$
- network-induced correlation in observed innovations u_t
- holds for $q \in \mathbb{Q}_{++}$, and also for flow variables under $q \in \mathbb{N}$

$\rightarrow$ " **NVAR**(p, q) " stationarity relation to contemp. transmision VARMA approx.

Inference: $\alpha \mid A$, in NVAR$(p, 1)$

$$y_t = \alpha_1 A y_{t-1} + ... + \alpha_p A y_{t-p} + u_t = X_t(A)\alpha + u_t$$

- LS estimator for α:

$$\hat{\alpha}|A = \left[\sum_{t=1}^{T} X_t' \Sigma^{-1} X_t\right]^{-1} \left[\sum_{t=1}^{T} X_t' \Sigma^{-1} y_t\right] , \quad X_t = [A y_{t-1}, ..., A y_{t-p}] .$$

- OLS ($\Sigma = I$): consistent and asymp. Normal for n, T & $(n, T) \to \infty$ conditions

Inference: $\alpha \mid A$, NVAR(p,q), $q>1$

$$\tilde{y}_\tau = \alpha_1 A \tilde{y}_{\tau-1} + \ldots + \alpha_p A \tilde{y}_{\tau-p} + \tilde{u}_\tau = X_\tau(A)\alpha + \tilde{u}_\tau \ , \quad \tau = 1 : T_\tau \ ,$$

$$y_{\tau/q} = \tilde{y}_\tau \quad \text{if } \tau/q \in \mathbb{N} \ ,$$

- Data augmentation. But: point ID not guaranteed; e.g. for $q=2, p=1$, can identify α_1 up to sign: $y_t = \alpha_1^2 A^2 y_{t-1} + \eta_t$
- Akin to AR(p) observed every $q>1$ periods (Palm & Nijman 1984)
- Shrink towards lower-dimensional function; e.g. $\alpha_l \sim N\left(\mu_l, \lambda_\alpha^{-1}\right)$, $\mu_l = (1, l, l^2)\beta_\alpha$
- Gives full-sample posterior $\alpha_l | \tilde{Y}_{1:T_\tau} \sim N\left(\bar{\alpha}, \bar{V}_\alpha\right)$ with

$$\bar{V}_\alpha = \left[\sum_{\tau=1}^{T_\tau} \tilde{X}_\tau' \tilde{\Sigma}^{-1} \tilde{X}_\tau + \lambda_\alpha I_p\right]^{-1} \ , \quad \bar{\alpha} = \bar{V}_\alpha \left[\sum_{\tau=1}^{T_\tau} \tilde{X}_\tau' \tilde{\Sigma}^{-1} \tilde{y}_\tau + \lambda_\alpha I_p \mu\right] \ .$$

- Uniform hyperpriors for β_α and λ_α: shrink towards MLE/OLS $\hat{\beta}_\alpha$, optimizing predictive ability (Giannone, Lenza & Primiceri 2015)

Inference: (α, A)

$$\tilde{y}_\tau = \alpha_1 A \tilde{y}_{\tau-1} + ... + \alpha_p A \tilde{y}_{\tau-p} + \tilde{u}_\tau = A z_\tau(\alpha) + \tilde{u}_\tau \ , \quad \tau = 1 : T_\tau \ ,$$

$$y_{\tau/q} = \tilde{y}_\tau \quad \text{if } \tau/q \in \mathbb{N} \ ,$$

NVAR$(p, 1)$:

- Ridge-prior $a_{ij} \sim N(b_{ij}, \lambda_a^{-1})$ gives posterior $A|(\alpha, \Sigma) \sim MN\left(\bar{A}', \Sigma, \bar{U}_A\right)$ with

$$\bar{U}_A = \left[Z'Z + \lambda_a \Sigma\right]^{-1} \ , \quad \bar{A} = \bar{U}_A \left[Z'Y + \lambda_a B' \Sigma\right] \ .$$

- Can shrink to actual links: set $b_{ij} = w_{ij}^{b\prime} \beta_b$
- Iterate on posteriors (or modes) of $A|\alpha$ and $\alpha|A$, normalizing $||\alpha||_1 = 1$ (e.g.)
- $(\hat{\alpha}, \hat{A})_{OLS}$ consistent and asymp. Normal for $T \to \infty$

NVAR(p, q): add data augmentation step (Carter-Kohn Gibbs sampler / EM algo)

Application 1: Motivation

Macro literature on production networks:

- assuming contemporaneous input-output-conversion, shows:
  Horvath 2000, Acemoglu et al. 2012, 2016, Bouakez et al. 2014, ...
  - supply chain network amplifies sectoral shocks
  - strength of effect on aggregates depends on sector's position in network
- exception: one period-lagged I-O-conversion → NVAR(1,1)
  Long & Plosser (1983), Foerster et al. (2011), Carvalho & Reischer (2021)
  - generates endogenous BCs (persistence in aggregates)
  - model-persistence matches empirics,
    calibrated model gives improved forecasts of agg. IP relative to statistical models

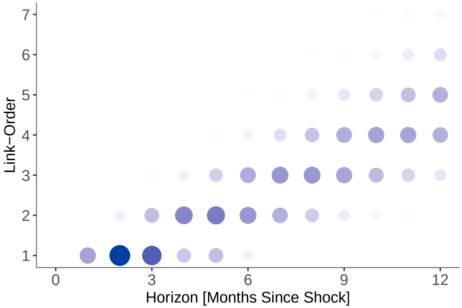
This application: setup theory data

- How does amplification materialize over time?
- Does network-position shape timing of effect?
- Estimate roles of exogenous shock persistence vs. lagged IO conversion Foerster et al. (2011)

Results: Impulse Responses & Their Composition

more results

Relevance of Link-Orders Across Horizons



Input-Output Links to Utilities Sector

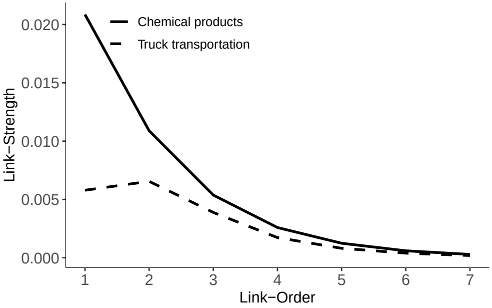
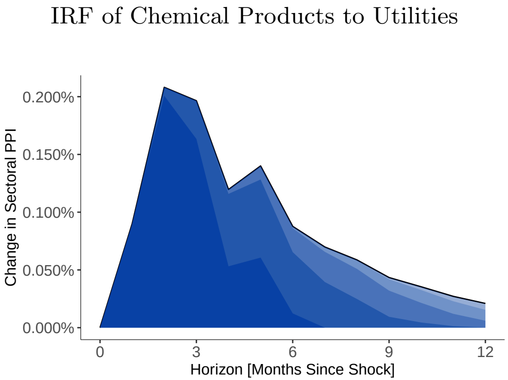


Figure: Transmission of Price Shocks via Supply-Chain Links (1)

Recall: $$\frac{\partial y_{i,t+h}}{\partial u_{j,t}} = c^h_{\underline{k}}(\alpha)\left(A^{\underline{k}}\right)_{ij} + \ldots + c^h_h(\alpha)\left(A^h\right)_{ij} , \quad \underline{k} = ceil(h/p) .$$

Results: Impulse Responses & Their Composition



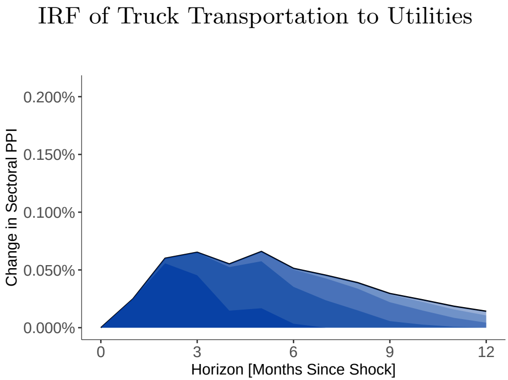


Figure: Transmission of Price Shocks via Supply-Chain Links (2)

Recall: $$\frac{\partial y_{i,t+h}}{\partial u_{j,t}} = c^h_{\underline{k}}(\alpha)\left(A^{\underline{k}}\right)_{ij} + \ldots + c^h_h(\alpha)\left(A^h\right)_{ij} \;, \quad \underline{k} = ceil(h/p) \;.$$

Application 2: Motivation

How to model industrial production dynamics across 44 countries?

- Even for this moderate size of cross-section, unrestricted VAR not feasible
- NVAR(p, q): well-performing, simple-to-estimate and interpretable alternative details

$\rightarrow$ Estimate (α, A), A sparse !

- Assumption: a few bilateral links drive dynamics of whole cross-section

Relation to Alternative Dimensionality-Reduction Techniques

- Combines insights from factor models / RR regression (Velu et al. 1986, Stock & Watson 2002, ...) and sparse / shrinkage methods (Tibshirani 1996, ...)

$$\text{Recall NVAR}(p,1): \quad y_t = A\,[y_{t-1}, ..., y_{t-p}]\,\alpha + u_t \;.$$

- Equivalence betw. factor model & NVAR$(p,1)$, with # factors = rank(A): details
  - $y_t \sim \text{NVAR}(p,1) \quad \Rightarrow \quad y_t \sim \text{FM}$
  - $y_t \sim \text{FM} + f_t \sim \text{NVAR}(p,1) \quad \Rightarrow \quad y_t \sim \text{NVAR}(p,1)$, for n large
- Expect: Network-VAR preferred when dynamics driven by many micro links rather than few influential units (see Boivin & Ng, 2006)
- Rationalize sparse factors as locally important units in network

Results: Forecasting setup more results

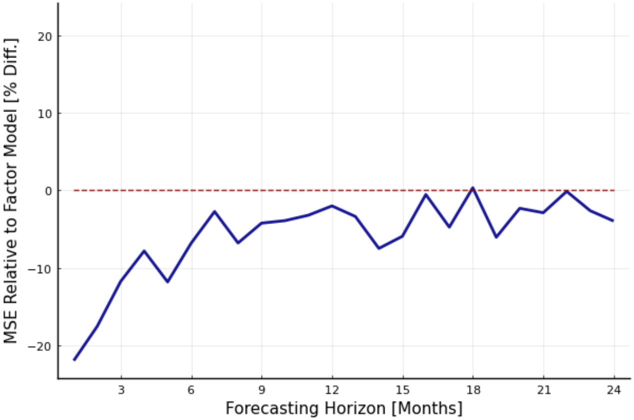


Figure: Out-Of-Sample Forecasting Performance: NVAR(4, 1) vs. Factor Model

Notes: Plot depicts percentage difference between out-of-sample Mean Squared Errors generated by NVAR(4, 1) to those generated by Principal Components Factor Model.

Conclusion

- I propose econometric framework for cross-sectional time series exploiting network structure
- I apply it to estimate how supply shocks propagate through US supply chain network and affect dynamics of sectoral prices
- I apply it to forecast cross-country IP dynamics, assuming & estimating network

Bilateral Connections in Networks back

- Network: $n \times n$ adjacency matrix A with elements a_{ij}
- Directed and weighted: $a_{ij} \in [0,1]$ shows strength of (direct) link from i to j
- Walk: product of direct links a_{ij} that lead from i to j over some intermediary units

$$\text{e.g.} \quad a_{i,k_1} a_{k_1,k_2} a_{k_2,j} \quad : \quad \text{walk from } i \text{ to } j \text{ of length } 3$$

- $(A^K)_{ij}$: sum of all walks from i to j of length K ("Kth order connection from i to j")

$$\text{e.g.} \quad A = \begin{bmatrix} 0 & 0 & .8 \\ .7 & 0 & .6 \\ 0 & .8 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & .64 & 0 \\ 0 & .48 & .56 \\ .56 & 0 & .48 \end{bmatrix}, \quad A^3 = \begin{bmatrix} .448 & 0 & .384 \\ .336 & .448 & .288 \\ 0 & .384 & .448 \end{bmatrix}.$$

Lagged Innovation Transmission via Bilateral Links back

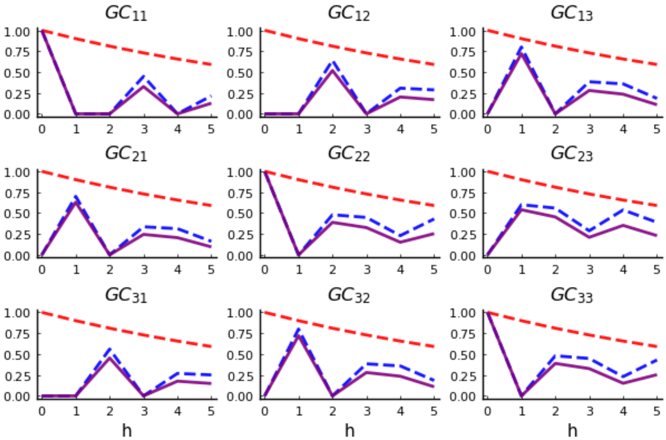


Figure: Example Generalized Impulse Responses For NVAR$(1, 1)$

Notes: Panel (i, j) shows $(A^h)_{ij}$ in blue, α^h in red and GC^h_{ij}, their product, in purple.

Time Aggregation of Lagged Transmission Patterns back

- Let $\tilde{y}_\tau = \alpha_1 A\tilde{y}_{\tau-1} + \alpha_2 A\tilde{y}_{\tau-2} + \alpha_3 A\tilde{y}_{\tau-3} + \tilde{u}_\tau$, and $\{y_t\}_{t=1}^T = \{\tilde{y}_{2t}\}_{t=1}^T$.
- We get

$$\tilde{y}_\tau = \left[\alpha_2 A + \alpha_1^2 A^2\right]\tilde{y}_{\tau-2} + \left[(\alpha_1\alpha_2 + 2\alpha_1\alpha_3)A^2\right]\tilde{y}_{\tau-4} \\ + \tilde{u}_\tau + \alpha_1 A\tilde{u}_{\tau-1} + (\alpha_3 A + \alpha_1\alpha_2 A^2)\tilde{u}_{\tau-3} + \text{terms in } \tilde{y}_{\tau-6}, \tilde{y}_{\tau-7} \ .$$

$$\rightarrow y_t \approx \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \Theta_0 u_t + \Theta_1 u_{t-1} \ ,$$

$$\text{with } \Phi_1 = \alpha_2 A + \alpha_1^2 A^2 \ , \quad \Phi_2 = (\alpha_1\alpha_2 + 2\alpha_1\alpha_3)A^2$$

$$u_t = [\tilde{u}_\tau', \tilde{u}_{\tau-1}']' \ , \quad u_{t-1} = [\tilde{u}_{\tau-2}', \tilde{u}_{\tau-3}']', \ ,$$

$$\Theta_0 = [I_n, \alpha_1 A] \ , \quad \Theta_1 = \left[0_n, \alpha_3 A + \alpha_1\alpha_2 A^2\right] \ .$$

Contemporaneous Innovation Transmission via Bilateral Links back

- Under contemporaneous network interactions,

$$x = Ax + \varepsilon = (A + A^2 + A^3 + ...)\varepsilon \ .$$

→ Acemoglu et al. (2012): network A amplifies granular shocks ε_j, implies cross-sectional comovement in $\{x_i\}_{i=1}^n$

- Result: for NVAR$(p, 1)$, $y_t = \alpha_1 A y_{t-1} + ... + \alpha_p A y_{t-p} + u_t$, we have that

$$lim_{h\to\infty} \sum_{j=0}^{h} \frac{\partial y_{t+h}}{\partial u_{t+j}} = \frac{\partial x}{\partial \varepsilon} = (I - A)^{-1} \ , \qquad (\text{for} \sum_{l=1}^{p} \alpha_l = 1)$$

→ Taking stance on timing of network effects, y_t can speak to (transition) dynamics

Stationarity of NVAR$(p, 1)$ back

Let $\tilde{y}_\tau$ follow an NVAR$(p, 1)$

$$\tilde{y}_\tau = \alpha_1 A \tilde{y}_{\tau-1} + ... + \alpha_p A \tilde{y}_{\tau-p} + \tilde{u}_\tau \ ,$$

where $\tilde{u}_\tau \sim WN$, and assume $\alpha_l \neq 0$ for at least one l. Let $a = \sum_{l=1}^{p} |\alpha_l|$.

1a $\tilde{y}_\tau$ is WS if for all Eigenvalues λ_i of A it holds that $|\lambda_i| < 1/a$.

1b If in addition $\alpha_1, ..., \alpha_p \geq 0$, this condition is both necessary and sufficient.

2 $\tilde{y}_\tau$ is WS iff the univariate AR(p)

$$\check{x}_\tau = \lambda_i \alpha_1 \check{x}_{\tau-1} + ... + \lambda_i \alpha_p \check{x}_{\tau-p} + \check{v}_\tau$$

is WS for all Eigenvalues λ_i of A.

Asymptotics: $\hat{\alpha}_{OLS}|A$ in NVAR$(p,1)$ back

$T \longrightarrow \infty$

- Model correct: $y_t = X_t \alpha + u_t$
- $\mathbb{E}_{t-1}[u_t] = 0 \,, \quad \mathbb{E}_{t-1}[u_t u_t'] = \Sigma$
- y_t ergodic and strictly stationary

$n \longrightarrow \infty$

- Model correct: $y_{it} = x_{it}' \alpha + u_{it}$
- $\mathbb{E}_{t-1}[u_t] = 0 \,, \quad \mathbb{E}_{t-1}[u_{it} u_{is}] = \sigma^2$ if $t = s$ and zero otherwise
- A_n converges to some limit s.t.
  - $\frac{1}{n} \sum_{i=1}^{n} \left(A_{n,i\cdot} y_{t-l}\right)' \left(A_{n,i\cdot} y_{t-k}\right) \longrightarrow \mathbb{E}\left[\left(A_{i\cdot} y_{t-l}\right)' \left(A_{i\cdot} y_{t-k}\right)\right]$
  - $\frac{1}{n} \sum_{i=1}^{n} \left(A_{n,i\cdot} y_{t-l}\right)' u_{it} \longrightarrow \mathbb{E}\left[\left(A_{i\cdot} y_{t-l}\right)' u_{it}\right]$
  - $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(A_{n,i\cdot} y_{t-l}\right)' u_{it} \Rightarrow N\left(\mathbb{E}\left[\left(A_{i\cdot} y_{t-l}\right)' u_{it}\right], \mathbb{V}\left[\left(A_{i\cdot} y_{t-l}\right)' u_{it}\right]\right)$

Estimation/Setup back

- Generalized version of LP: firms use inputs produced in last p periods
  $\rightarrow$ at some model-frequency, sectoral prices $\sim$ NVAR$(p, 1)$:

$$\tilde{y}_\tau \approx \alpha_1 A \tilde{y}_{\tau-1} + \ldots + \alpha_p A \tilde{y}_{\tau-p} + \tilde{u}_\tau \ ,$$

  with $\alpha_l \geq 0 \ \forall \ l$ and $\sum_{l=1}^{p} \alpha_l = 1$ and $A =$ I-O-matrix theory

- Observation freq. potentially $\neq$ network interaction freq.: $\{y_t\}_{t=1}^T = \{\tilde{y}_{qt}\}_{t=1}^T$
  $\rightarrow$ I consider $q \in \{1/3,\ 1/2,\ 1,\ 2,\ 4\}$,
  i.e. quarterly, bi-monthly, monthly, bi-weekly and weekly network interactions

- 51 sectors, Jan 2005 - Aug 2022, annual I-O-matrix from 2010 data

- For now, let $\tilde{u}_{i\tau} \overset{iid}{\sim} N(0, \sigma_i^2)$, get $(\hat{\alpha}, \hat{\sigma})_{MLE}$ for different (p, q) & select model via IC

- Work in progress: $\tilde{u}_{i\tau} = \lambda_i f_\tau + \varepsilon_{i\tau}$, $\quad f_\tau, \varepsilon_{i\tau} \sim$ AR(2)
  $\rightarrow$ Determine roles of exogenous shock persistence vs. lagged I-O-conversion

Theory back

Assume n sectors, rep. firm produces variety i by using labor and inputs $j = 1 : n$:

$$y_{i\tau} = z_{i\tau} l_{i\tau}^{b_i} \prod_{j=1}^{n} x_{ij\tau}^{a_{ij}} , \quad b_i > 0 , \ a_{ij} \geq 0 , \quad b_i + \sum_{j=1}^{n} a_{ij} = 1 .$$

- If $x_{ij\tau}$ is variety j bought at τ: $p_\tau = A p_\tau + \varepsilon_\tau$, $\varepsilon_\tau = -log(z_\tau)$ (e.g. Acemoglu et al., 2012)
- If $x_{ij\tau}$ is variety j bought at $\tau - 1$: $p_\tau = A p_{\tau-1} + \varepsilon_\tau$ (Long & Plosser 1983, Carvalho & Reischer 2021)

→ If $x_{ij\tau}$ is CES-aggregate of variety j bought at $\{\tau - p, ..., \tau - 1\}$:

$p_\tau \approx \alpha_1 A p_{\tau-1} + ... + \alpha_p A p_{\tau-p} + \varepsilon_\tau$, for some $\alpha_l \geq 0$, $l = 1 : p$, and $\sum_{l=1}^{p} \alpha_l = 1$

Data back

Input-Output Matrix from Bureau of Economic Analysis (BEA)

- 64 mostly 3- and 4-digit sectors (due to PPI availability)
- I take data for 2010
- Following Acemoglu et al. (2016), links defined as $a_{ij} \equiv \frac{sales_{j \to i}}{sales_i}$ (valid for general p as $\beta \to 1$)

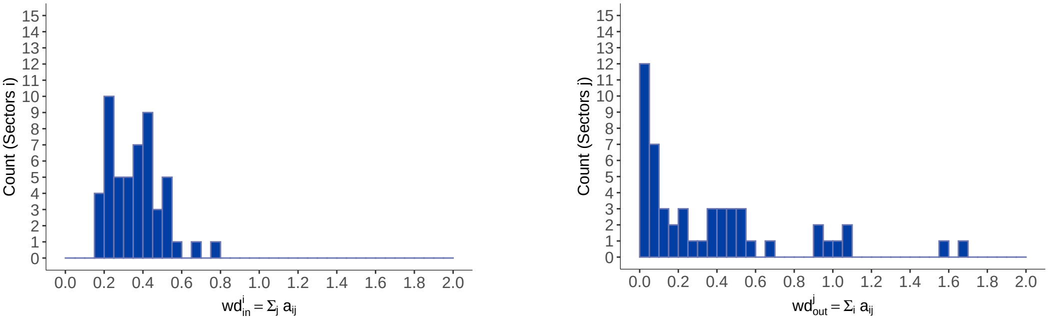
Monthly sector-level PPI data from Bureau of Labor Statistics (BLS)

- 51 BEA-sectors, January 2005 - August 2022
- I take logs and subtract sector-specific linear time trend and seasonality (since the assumed process is stationary)

Data: Input-Output Network

- Density: 16.88 %
- Average shortest path: 2.41, longest shortest path: 7

(a) Weighted In-Degrees (b) Weighted Out-Degrees



Notes: Left panel plots weighted in-degrees (column-wise sums of A), shows sectors' differing reliance on intermediate inputs. Right panel plots weighted out-degrees (row-wise sums of A), shows sectors' differing importance as suppliers to other sectors.

Data: Input-Output Network

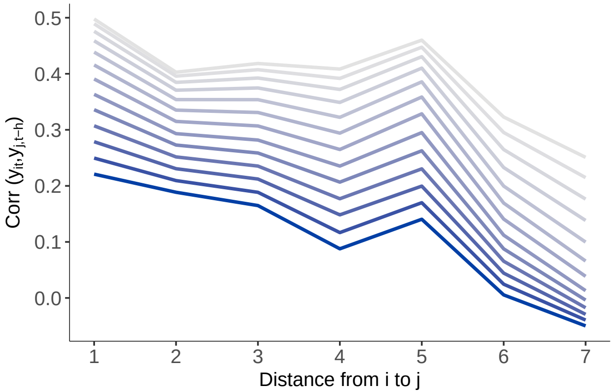


Figure: Network Distance And The Correlation of Sectoral Inflation

Notes: Figure plots average correlation of sectoral prices for different distances between them. Lightest blue line refers to contemporaneous correlations. Darker lines show average correlation of sector i with lagged values of sector j as function of distance from i to j. Lags from 1 to 12 months. Series are de-trended and de-seasonalized log PPIs.

Data: PPI back

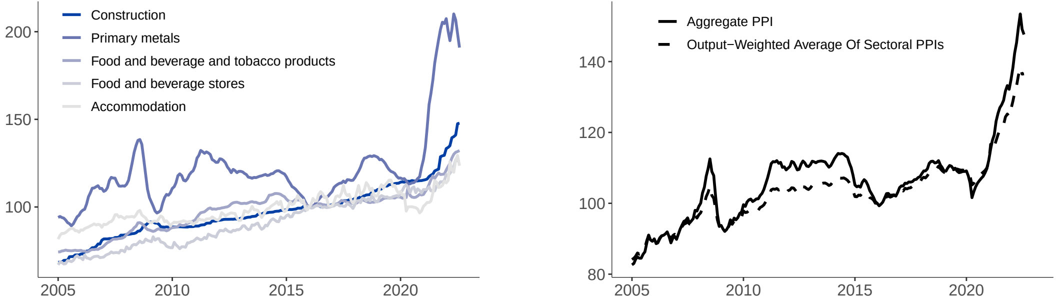


Figure: Aggregate & Sectoral PPIs

Notes: Left panel shows raw PPI series for few selected sectors. Right panel compares aggregate PPI (FRED Database) and output-weighted average of PPIs of studied sectors.

Estimation Results: Model Selection back

Table: Model Selection: Log MDD

		p					
		$1q$	$2q$	$3q$	$4q$	$5q$	$6q$
q	1/3			19079			19044
	1/2		19384		18768		18690
	1	20153	20056	19675	19879	18899	20218
	2	17546	19570	19248	20142	18662	19636
	4	18517	19808	19754	19655	18904	19301

Notes: Table shows log Marginal Data Density (MDD) across model specifications. Values for q (from top to bottom) refer to quarterly, bi-monthly, monthly, bi-weekly and weekly network interactions, while $p = mq$ implies last m months matter for dynamics.

Estimation Results back

Table: Estimation Results: α

	MLE	Mean	Low	High
α_1	0.1550	0.1557	0.1370	0.1745
α_2	0.3460	0.3382	0.3168	0.3605
α_3	0.2816	0.2865	0.2644	0.3129
α_4	0.0915	0.0991	0.0785	0.1174
α_5	0.1045	0.0975	0.0837	0.1135

Notes: First column shows Maximum Likelihood or Maximum A-Posteriori (MAP) Estimator, second refers to posterior mean, and Low and High report the bounds of the 95% Bayesian HPD credible sets.

Application 2: Motivation back

NVAR(p, q): sparse, flexible and interpretable dimensionality-reduction

$$\tilde{y}_\tau = \sum_{l=1}^{p} \alpha_l A \tilde{y}_{\tau-l} + \tilde{u}_\tau \ , \quad \{y_t\}_{t=1}^T = \{\tilde{y}_{tq}\}_{t=1}^T \ .$$

- Sparsity:
  - $y_{i\tau} = x'_{i\tau}\alpha + u_{i\tau}$ with $X_\tau = A\big[\tilde{y}_{\tau-1}, ..., \tilde{y}_{\tau-p}\big]_{(n \times p)}$

  $\rightarrow$ reduce n^2 parameters in VAR to $n^2 + p - 1$ parameters in NVAR
  - A can be sparse: higher-order network effects through A^2, A^3, ...
- Flexibility:
  - estimated network + general time dimension of network effects
  - like functional approximation using A as basis (recall: $y_t \overset{approx.}{\sim}$ restricted VARMA)
- Interpretability:
  - dynamics driven by innovation transmission along bilateral links
  - estimate network & whole set of spillover and spillback effects

Relation to Factor Model back

NVAR $\rightarrow$ FM

- $y_t = A[\alpha_1 y_{t-1} + \alpha_2 y_{t-2}] + u_t$ with A of rank $r \in 1:n$
- Write $A = B_{n\times r} C_{r\times n}$

$\rightarrow$ $y_t = \Lambda f_t + u_t$, with $\Lambda = B$ and $f_{kt} = \alpha_1 C_{k\cdot} y_{t-1} + \alpha_2 C_{k\cdot} y_{t-2}$ for $k = 1:r$

- (not unique: $A = BC = BQQ^{-1}C = \tilde{B}\tilde{C}$ for any $r \times r$ full-rank matrix Q)

Relation to Factor Model back

FM → NVAR

- $y_t = \Lambda f_t + \xi_t \ , \quad f_t = \Phi_1 f_{t-1} + \Phi_2 f_{t-2} + \eta_t$, with $f_t \in \mathbb{R}^r$
- Take r distinct vectors of weights $w^k = (w_1^k, ..., w_n^k)$, $k = 1 : r$, and consider $\sum_{i=1}^n w_i^k y_{it} = \sum_{i=1}^n w_i^k \Lambda_{i\cdot} f_t + \sum_{i=1}^n w_i^k \xi_{it}$
- If n large enough, $\bar{\xi}_t^k \equiv \sum_{i=1}^n w_i^k \xi_{it} \sim O_p(n^{-1/2})$ is negligible $\rightarrow Wy_t = W\Lambda f_t$

$$\begin{aligned} y_t &= \Lambda\left(\Phi_1 f_{t-1} + \Phi_2 f_{t-2} + \eta_t\right) + \xi_t \\ &= \Lambda\Phi_1 (W\Lambda)^{-1} W y_{t-1} + \Lambda\Phi_2 (W\Lambda)^{-1} W y_{t-2} + u_t \ , \end{aligned}$$

- If $\Phi_l = \phi_l \Phi$ for $l = 1, 2$ (i.e. $f_t \sim$ NVAR(2,1)), then

$$y_t = \Lambda\Phi(W\Lambda)^{-1} W[\phi_1 y_{t-1} + \phi_2 y_{t-2}] + u_t$$

- Let $A = \Lambda\Phi(W\Lambda)^{-1}W$, $\alpha_l = \phi_l$

Data & Forecasting Setup back

Data:

- Use IMF & OECD data on monthly IP series
- Compute growth rate relative to same month previous year, subtract mean
- January 2001 - January 2020, 44 countries

Forecasting Exercise:

- Use sample end dates from December 2017 to December 2019
- Consider forecasts of up to 24 months ahead (COVID-19 excluded)
- For $p = 1:6$, compare
  - NVAR$(p, 1)$ + Lasso-shrinking of a_{ij} to zero, select λ based on BIC (Zou, Hastie & Tibshirani 2007) details
  - PC-FM: select # of factors based on Bai & Ng (2002), fit VAR(p) for factors

Estimation back

$$y_t = \sum_{l=1}^{p} \alpha_l A y_{t-l} + u_t \ , \quad \alpha \equiv (\alpha_1, ..., \alpha_p) \in \mathbb{R}^p \ , \quad a_{ij} \in [0,1] \ ,$$

- To identify (α, A), normalize $||\alpha||_1 = 1$ and change domain of a_{ij} to $\mathbb{R}_+$
- Consider OLS with Lasso penalty (λ) on a_{ij}
- Get $(\hat{\alpha}, \hat{A})$ by iterating on

$$\hat{\alpha}|A = \left[\sum_{t=1}^{T} X_t' X_t\right]^{-1} \left[\sum_{t=1}^{T} X_t' y_t\right] \ ,$$

$$\hat{a}_{ij}|(\alpha, A_{i,-j}) = max\{0, \check{a}_{ij}\} \ , \quad \check{a}_{ij} = \frac{\sum_{t=1}^{T} (y_{it} - A_{i,-j} z_{-j,t}) z_{jt} - \lambda}{\sum_{t=1}^{T} z_{jt}^2} \ .$$

Results: Estimated Network back

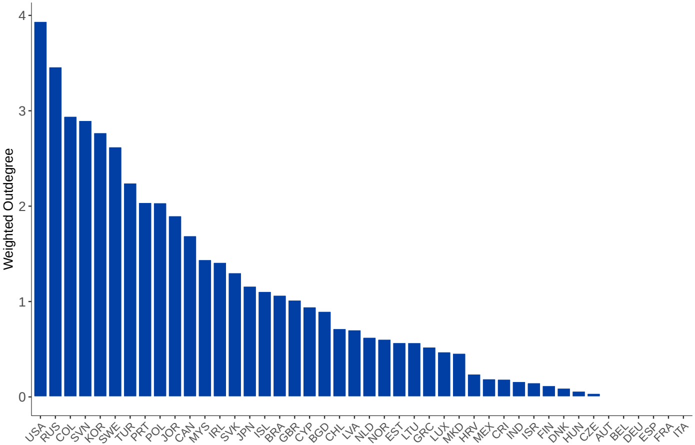


Figure: Weighted Outdegrees In The Estimated Network

Notes: Plot shows weighted outdegrees in estimated network as relevant for cross-country monthly IP dynamics.

Results: Impulse Responses & Their Composition back

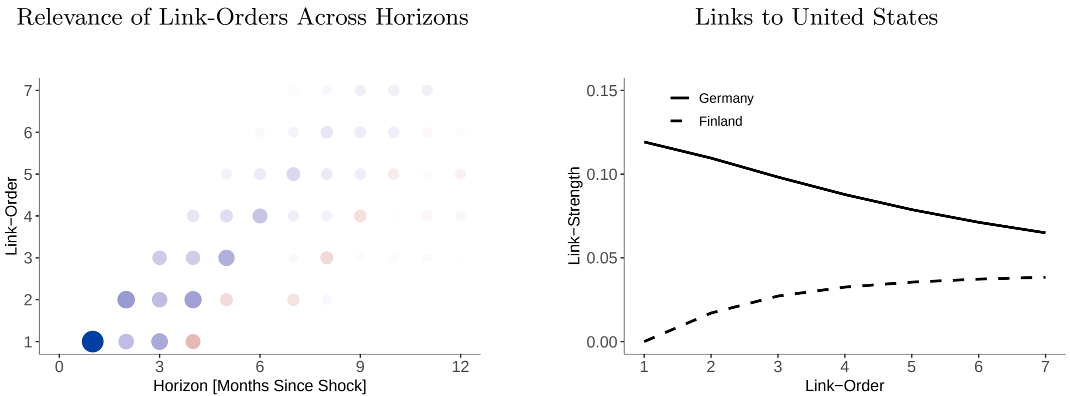


Figure: Network-Induced Transmission of Inudstrial Production Innovations (1)

Notes: Left panel shows importance of different connection-orders for transmission as function of time elapsed since shock took place. Right panel shows connections of different order from Germany and Finland to United States.

Results: Impulse Responses & Their Composition

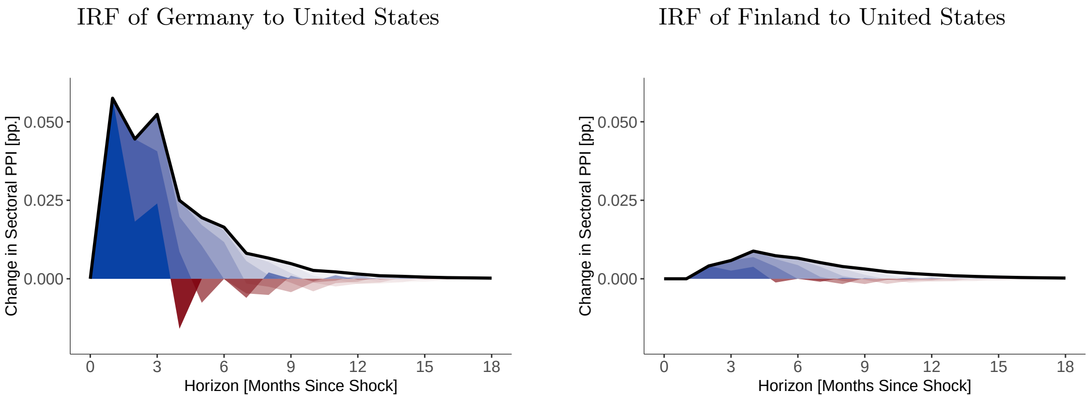


Figure: Network-Induced Transmission of Inudstrial Production Innovations (2)

Notes: The two panels show the Impulse-Response Functions (IRFs) of German and Finnish IP growth, respectively, to a one standard deviation increase in US IP growth.