Intervention analysis, causality and generalized impulse responses in VAR models: theory and inference

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Outline

1. Introduction
2. GIRs and intervention analysis
3. Impulse response decomposition and mediation analysis
4. Inference
5. MC simulation
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Previous work - Impulse responses and Granger Causality

- Causality in VARMA models: non-causality restrictions nonlinear possibly non-regular (nonstandard asymptotic theory)
  1. Boudjellaba, H., Dufour, J.-M., and Roy, R. (1992) Testing Causality Between Two Vectors in Multivariate ARMA Models, *Journal of the American Statistical Association* 87, 1992, 1082-1090.
  2. Boudjellaba, H., Dufour, J.-M., and Roy, R. (1994) Simplified Conditions for Non-Causality Between Two Vectors in Multivariate ARMA Models, *Journal of Econometrics* 63, 271-287.
- Impulse responses do not represent Granger causality
  1. Dufour, J.-M., and Tessier, D. (1993) On the Relationship between Impulse Response Analysis, Innovation Accounting and Granger Causality, *Economics Letters* 42, 1993, 327-333.

Previous work - Granger Causality and VAR at horizon h

- Causality at multiple horizons and (generalized) impulse responses, indirect effects (mediation), causality measures at multiple horizons: possibly non-regular restrictions (nonstandard asymptotic theory), VAR at horizon h as a solution
  1. Dufour, J.-M., and Renault, E. (1998) Short-Run and Long-Run Causality in Time Series: Theory, *Econometrica* 66, 1099-1125.
  2. Dufour, J.-M., Pelletier, D., and Renault, E. (2006) Short run and long run causality in time series: inference, *Journal of Econometrics*, 132, 2, 337-362. [in Pelletier (2004, Ph.D. thesis, U. Montréal)]
- Causality measures at multiple horizons
  1. Dufour, J.-M., and Taamouti, A. (2010) Short and long run causality measures: theory and inference, *Journal of Econometrics*, 154, 1, 42-58.
  2. Dufour, J.-M., Garcia, R., and Taamouti, A. (2012) Measuring high-frequency causality between returns, realized volatility and implied volatility, *Journal of Financial Econometrics*, 1, 124-163.
  3. Zhang, H. J., Dufour, J.-M., and Galbraith, J. (2016) Exchange rates and commodity prices: measuring causality at multiple horizons, *Journal of Empirical Finance*, 36, 100-120.

IR and Granger causality

- Impulse response and Granger causality
  - Similarity
    1. Based on a VAR structure, typically
    2. 'Casual philosophy': if A occurs then B must occur
  - Disparity
    1. Impulse response focuses on a 'pulse' shock; Granger causality depends on the information set generated by variables.
    2. Granger causality is inclined to predication; impulse response interprets treatment effect.
- Any possibility to unify two concepts?
- What impulse response can (can't) do?
  - It gauges the average treatment effect between experimental group (shock$=1$) and the control group (shock$=0$)
  - 'Total effect' of shock of interest on target variable
  - Is the causal mechanism well understood by impulse response itself?
  - Total effect $=$ Direct effect $+$ Indirect effect
- The concept of Granger causality and Generalized impulse response could contribute to causal mechanism.

Granger causality and Generalized impulse responses

- **Generalized impulse response (GIR) coefficients** [Dufour and Renault 1998 Econometrica; Dufour, Pelletier and Renault (2004; 2006, JE)]. Linear projection:

$$P[\boldsymbol{y}_{t+h}|\mathcal{F}_t] = \Phi_1^{(h)}\boldsymbol{y}_t + \Phi_2^{(h)}\boldsymbol{y}_{t-1} + \cdots + \Phi_p^{(h)}\boldsymbol{y}_{t-p}$$

  We showed that the first coefficient $\Phi_1^{(h)}$ is the usual impulse response of order h (Sims). We called this projection an "h-autoregression"].
  This shows that estimation and inference on impulse response coefficients can be made by linear regression methods [later called "local projections"].
  The other coefficients $\Phi_2^{(h)}, \Phi_3^{(h)}, \cdots$, carry important information:

$$\Phi_k^{(h)} = \Phi_{k+1}^{(h-1)} + \Phi_1^{(h-1)}\Phi_k.$$

- Noncausality from variable $j \to i$ at horizon h

$$\mathcal{H}_0 : \Phi_{ij,k}^{(h)} = 0, \quad \forall k = 1, 2, \cdots$$

- Indirect effect: for multivariate models, Granger noncausality (noncausality at horizon one) is not equivalent to noncausality at horizon h [Dufour and Renault (1998)].

An illustrative example: IR vs GIR

VAR(6)

$$
\begin{aligned}
y_t = & \begin{bmatrix} 0.6 & 0 & 0.2 \\ 0.2 & 0.6 & 0 \\ -0.2 & 0.4 & 0.7 \end{bmatrix} y_{t-1} + \begin{bmatrix} -0.4 & -0.08 & 0.36 \\ 0 & -0.2 & 0.1 \\ 0.1 & 0 & -0.5 \end{bmatrix} y_{t-2} \\
& + \begin{bmatrix} 0.1 & -0.2 & 0 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0 & -0.2 \end{bmatrix} y_{t-3} + \begin{bmatrix} 0.3 & -0.1 & 0.19 \\ 0 & 0.2 & 0 \\ 0 & 0.05 & 0.15 \end{bmatrix} y_{t-4} \\
& + \begin{bmatrix} 0 & -0.04 & -0.1 \\ 0 & 0.08 & 0.03 \\ 0 & 0 & -0.02 \end{bmatrix} y_{t-5} + \begin{bmatrix} -0.1 & 0.01 & 0.03 \\ -0.08 & 0.03 & 0.06 \\ 0 & 0 & 0 \end{bmatrix} y_{t-6} + u_t .
\end{aligned} \tag{1}
$$

An illustrative example: IR vs GIR

Table

Dynamic causality table: IR vs GIR

horizon h	$\phi_{12,1}^{(h)}$	$\phi_{12,2}^{(h)}$	$\phi_{12,3}^{(h)}$	$\phi_{12,4}^{(h)}$	$\phi_{12,5}^{(h)}$	$\phi_{12,6}^{(h)}$
1	0	-0.080	-0.200	-0.100	-0.040	0.010
2	0	-0.248	-0.220	-0.090	-0.014	0.006
3	0	-0.214	-0.074	0.025	0.009	-0.001
4	0	-0.051	0.083	0.065	0.011	-0.003
5	0	0.068	0.027	-0.002	-0.011	0.002
6	-0.062	-0.013	-0.101	-0.076	-0.018	0.005
7	-0.124	-0.104	-0.128	-0.059	-0.008	0.000
8	-0.074	-0.087	-0.044	0.001	-0.002	-0.006
9	0.044	-0.012	0.029	0.027	-0.003	-0.004
10	0.054	0.016	0.026	0.006	-0.003	0.002

Limitations

- Limitation of impulse responses
  - Zero impulse responses do NOT imply no causality, but only these indirect effects perfectly offset these others.
  - Impulse responses (reduced-form or structural form/orthogonalized) can easily be misleading, in the sense of understanding causal mechanism.
  - Absence of a measure of the dynamic contribution of variables over the causal transmission to an interested IR/Total effect.
  - Understanding the causal mechanism could contribute to policy decision/counterfactual analysis.
- What Granger causality and Generalized impulse responses could contribute
  - Granger-type causality (or noncausality) properties at different horizons correspond to having zero rows in the causality table.
  - The elements in a dynamic causality table represent direct, indirect (mediation) and total effects.
  - Indirect effects are not well captured by usual IRs.
  - A complete causal analysis requires one to look at complete dynamic causality tables.

IR vs GIR and causal mediation analysis in macroeconomics

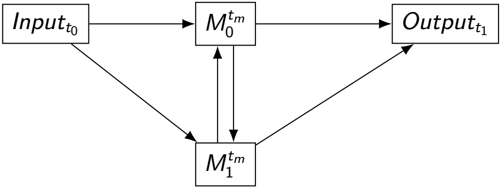
- Dynamic causal analysis in macro econ, e.g., Bernanke and Gertler (1995)

$$\text{Monetary policy} \xrightarrow{\text{horizon } h} \text{Movement in real output}, \quad \text{for } h \in \mathbb{N}$$

- The textbook interpretation overlooks certain mediators, e.g., credit channel

$$\text{Monetary policy} \to \text{Cost of capital} \to \text{Durable goods} \to \text{Real output}.$$

- We formalize dynamic causality through mediation analysis and use GIRs to assess the contribution of each channel to the total effect/IR.



where $\{M^{t_m}\} = \{M_0^{t_m}, M_1^{t_m}\}$ indicates mediators up to time t_m.

Relevant literature

- IR and GIR
  - The first GIR = reduced-form impulse response [Dufour and Renault (1998), Plagborg-Møller and Wolf (2021)]
- Counterfactual analysis in macroeconomics
  - Set hypothetical policy shock(s) to enforce the counterfactual rule ex post [Bernanke et al. (1997), Sims and Zha (2006), Kilian and Lewis (2011), Brunnermeier et al. (2021)]
  - Critics: shall allow counterfactual rule ex ante in private-sector expectations [Lucas Jr (1976) McKay and Wolf (2023)]
- Economic interpretation of GIRs
  - The first GIR = reduced-form impulse response [Dufour and Renault (1998), Plagborg-Møller and Wolf (2021)]
  - The interpretation of higher order GIRs remains unclear.
- Estimation and inference
  - Recursive VAR: nonstandard Wald test statistics [Lütkepohl and Burda (1997), Dufour, Renault and Zinde-Walsh (2023)]
  - Multiple-horizon VAR: serial correlated residuals [Dufour, Pelletier, and Renault (2006), Montiel Olea and Plagborg-Møller (2021), Xu (2023)]
    - Reliability of HAC estimators and the corresponding CI shall be questioned.
    - Losing efficiency when horizon h grows: increasing variance of the residual.

Contributions

- Conceptual:
  - Causal mediation analysis
    - Sims' impulse responses are shown to reflect total effects and Granger-type causality elucidates variables' contribution.
    - We bridge the impulse response decomposition with mediation analysis for macro dynamics.
    - We propose index to reveal the weighted contribution of causal channels/variables over the life of shock transmission.
  - Theory and interpretation of GIRs
    - We recast standard and generalized impulse responses in a multivariate intervention model which extends the bivariate framework of Box and Tiao (1975).
    - The GIRs are shown to be able to represent different types of dynamic interventions, short-lived and long-lived.
    - The GIR coefficients are described to measure direct and indirect (mediation) effects in a dynamic framework.
- Statistical: Inference of GIRs - 2SLS IV-based model
  - IV-based multi-horizon VAR model (IV=VAR residuals)
    - Simple and robust inference/Obviate HAC inference
    - More efficient than the standard LS estimates for persistent data and long horizon.
    - Long horizon, $h \propto T$.

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GIRs recast in a multivariate intervention model

- The intervention model [Box and Tiao (1975)] involves adding date dummy variables to an ARMA model.
- Denote a indicator function as

$$p_t(t_0) = 1 \text{ if } t = t_0, \quad p_t(t_0) = 0 \text{ if } t \neq t_0. \tag{2}$$

- Define the one-time pulse (the intervention) at time t as

$$I_{1,t}(\omega, t_0) = \omega p_t(t_0), \tag{3}$$

where $\omega := y_t - \tilde{y}_t$.

- Construct GIR polynomials,

$$\Phi_i(L) = I + \sum_{h=1}^{\infty} \Phi_i^{(h)} L^{h^*} \tag{4}$$

for all $i \in \mathbb{Z}^+$, $h^* = h + i - 1$.

- The first GIR(standard IR), $\Phi_1^{(h)}$, measures the dynamic causal effect to an one-time shock $I_{1,t}$ – "*response of y_{t+h} of a shock on y_t with no control in the future*".

$$y_t^{(1)}(I_{1,t}) = y_t^{(0)} + \Phi_1(L) I_{1,t} \tag{5}$$

GIRs recast in a multi-variate intervention model

- The second GIR, $\Phi_2^{(h)}$, is the dynamic causal effect of a "compound" shock,

$$I_{2,t} = I_{1,t} - \underbrace{\Psi_1 I_{1,t-1}}_{\text{hypo shock to impose } y_{t+1} \perp I_{1,t}} \tag{6}$$

The causal effect is defined as

$$\begin{aligned} y_t^{(1)}(I_{2,t}) &= y_t^{(0)} + \Phi_1(L) I_{2,t} \\ &= y_t^{(0)} + \underbrace{\Phi_2(L)}_{\text{2nd GIRs}} I_{1,t} \end{aligned} \tag{7}$$

- It implies shutting down the endogenous variable response in subsequent periods following a shock [Bernanke, Gertler, Watson, Sims, and Friedman (1997), Sims and Zha (2006), and Kilian and Lewis (2011)]
- Economic interpretation: Campbell et al. (2012) call "Delphic":

"... the policymaker (has) potentially superior information about future economic fundamentals" (Antolin-Diaz et al. (2021))

- e.g., Monetary policy evaluation from the perspective of central bank

GIRs recast in a multi-variate intervention model

- The GIR, $\Phi^{(h)}_{n+1}$, is the dynamic causal effect of a "compound" shock

$$I_{n,t} = I_{1,t} - (\Psi_1 I_{n-1,t} + \Psi_2 I_{n-2,t} + \cdots + \Psi_{n-1} I_{1,t-n+1}). \tag{8}$$

It entails the response of y_{t+h} to a shock on y_t while holding the subsequent observations up to y_{t+n} constant, that is $y_{t+1:t+n} \perp I_{1,t}$

- The causal effect is defined as

$$\begin{aligned} y_t^{(1)}(I_{n,t}) &= y_t^{(0)} + \Phi_1(L) I_{n,t} \\ &= y_t^{(0)} + \underbrace{\Phi_{n+1}(L)}_{(n+1)\text{-th GIRs}} I_{1,t} \end{aligned} \tag{9}$$

- Implications and applications of GIRs
  - Provide flexible experimental intervention analysis
  - Evaluate the causal effect from certain variable to target variable at given horizon
  - For interested causal effects of a 'structural shock', if variables' performance in the presence of the impulse can be computed, it is feasible to evaluate the 'contribution' of variables in the causality.
  - $\Rightarrow$ Impulse response decomposition

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Interpretation of direct and indirect effect

- Recall: a standard approach to compute structural IRs,

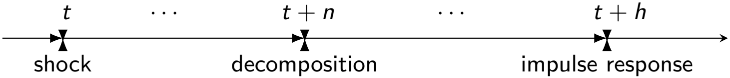
$$\theta_h = \underbrace{\Phi_1^{(h)}\theta_0}_{\text{effect from } y_t}, \text{ for all } h > 0. \tag{10}$$

  We name it as decomposing IR at the time of intervention. 'θ_0' is contemporaneous causal effect and $\Phi_1^{(h)}$ maps the variation of variables to h-period ahead.

- Decompose IR at one period after the intervention

$$\theta_h = \underbrace{\Phi_1^{(h-1)}\theta_1}_{\text{effect from } y_{t+1}} + \underbrace{\Phi_2^{(h-1)}\theta_0}_{\text{effect from } y_t}, \text{ for all } h > 1. \tag{11}$$

- Decompose IR at n-period after the intervention



$$\underbrace{\Phi_{n+1}^{(h-n)}\theta_0}_{\text{effect from } y_{t+n}} + \cdots + \underbrace{\Phi_{n-1}^{(h-n)}\theta_{n-1}}_{\text{effect from } y_{t+1}} + \underbrace{\Phi_1^{(h-n)}\theta_n}_{\text{effect from } y_t} = \underbrace{\theta_h}_{\text{total effect}}. \tag{12}$$

IR decomposition

Causal mediation analysis - one time intervention
Direct and indirect effect on target variable Y_{t+h} interpreted through multiple-horizon VAR

Decomposition	direct effect	indirect effect				total effect
	Y_t	Y_{t+1}	Y_{t+2}	$\cdots$	Y_{t+h-1}	Y_{t+h}
t	$\Phi_1^{(h)}\theta_0$	0	0	0	0	θ_h
$t+1$	$\Phi_2^{(h-1)}\theta_0$	$\Phi_1^{(h-1)}\theta_1$	0	0	0	θ_h
$t+2$	$\Phi_3^{(h-2)}\theta_0$	$\Phi_2^{(h-2)}\theta_1$	$\Phi_1^{(h-2)}\theta_2$	0	0	θ_h
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t+h-1$	$\Phi_h^{(1)}\theta_0$	$\Phi_{h-1}^{(1)}\theta_1$	$\Phi_{h-2}^{(1)}\theta_2$	$\cdots$	$\Phi_1^{(1)}\theta_{h-1}$	θ_h

- IR decomposition reveals a fact, that is an IR is realized only through the endogenous responses of mediators/variables.
- The impulse response of variable j, $\theta_{j,h}$, at decomposition time n, consists of the contribution from K variables,

$$\theta_{j,h} = \Sigma_{i=1}^{K} c_{i\to j}^{(h,n)}, \text{ for all } 0 \le n < h, h > 0. \tag{13}$$

where $c_{i\to j}^{(h,n)}$ denotes the contribution of variable i at time n to the response at h, $c_{i\to j}^{(h,n)} = \Sigma_{m=0}^{n}\Phi_{ji,m+1}^{(h-n)}\theta_{i,n-m}$, in which $\theta_{i,n-m}$ indicates the response of variable i at horizon $n-m$ in the presence of intervention.

Mediation analysis

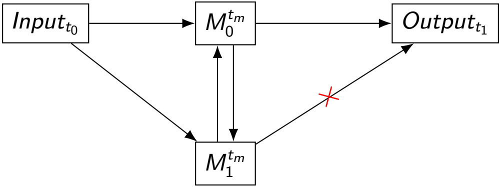
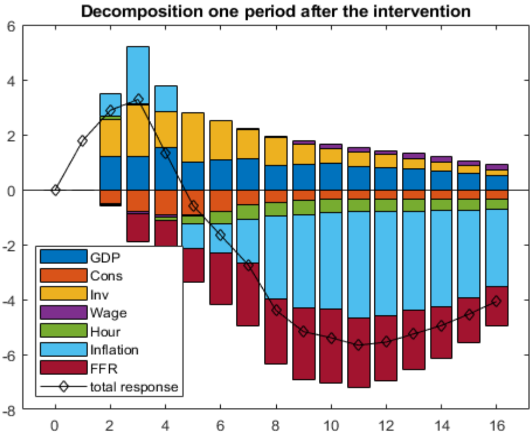


Figure: The channel from mediator to output $M_1^{t_m} \rightarrow \text{Output}_{t_1}$ is cutoff.

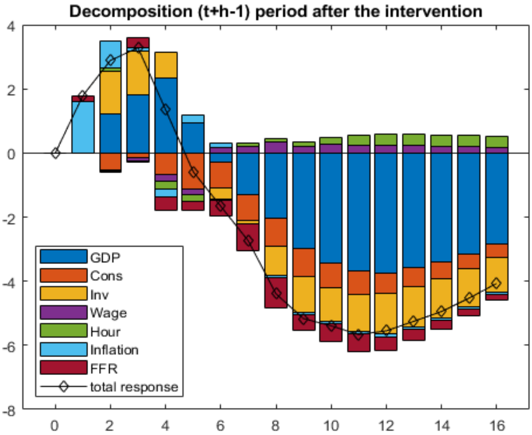
1. **The Focal Causal Effect**: Originates from a one-time shock intervention at time t_0 and extends to the output variable observed at time $t_1 = t_0 + h$.
2. **The Evaluation Time**: A specific causal channel's measurement occurs at time $t_m = t_0 + n$, $t_0 \leq t_m \leq t_1$. This designated moment is named the "time of evaluation."
3. **The Contribution Assessment**: To ascertain the contribution of a causal channel to the output at the time of evaluation, a counterfactual analysis is employed. This analysis considers the output variable's response to the movements of variables up to time t_m, excluding the specific variable representing the channel of interest.

Illustrative example - inflation $\xrightarrow{h}$ GDP



The data set comprises crucial economic indicators, including Gross Domestic Product, Consumption, Investment, Real Wage, Working Hours, Inflation, and the Federal Funds Rate, from 1976 Q3 to 2019 Q4. Estimation model: VAR(4) with Cholesky decomposition.

Illustrative example - inflation $\xrightarrow{h}$ GDP



Weighted contribution

- Two issues
  - First issue: The contribution $c_{i\to j}^{(h,n)}$ could be either positive or negative
  - Desired outcome: It would be more desirable to have contributions that are proportional $[0, 1]$.
  - Second issue: Each figure displays the contribution of variables to a sequence of impulse responses at a single time of decomposition.
  - Desired outcome: It would be more beneficial to have a figure with the x-axis representing the time of decomposition and displaying the proportional contributions of variables to all future impulse responses up to an upper bound horizon, for example, $H = 16$.
- Solutions
  - Solution to the first issue: proportional contribution by absolute value

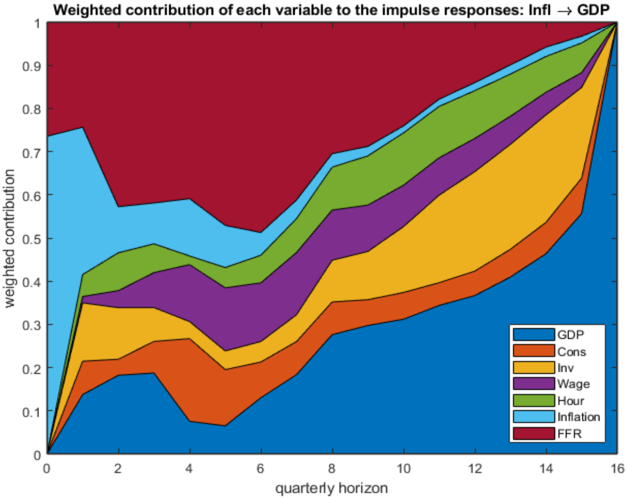
$$\omega_i(h, n) := |c_{i\to j}^{(h,n)}| / \sum_{l=1}^{k} |c_{l\to j}^{(h,n)}| \tag{14}$$

  - Solution to the second issue: average the weighted contribution to all future impulse response up to horizon H

$$\overline{\omega}_i(H, n) := \frac{1}{H-n} \sum_{h=n+1}^{H} \omega_i(h, n) \tag{15}$$

Illustrative example - inflation $\xrightarrow{h}$ GDP

Inflation $\rightarrow$ FFR $\rightarrow$ Wage&Cons $\rightarrow$ Inv $\rightarrow$ Output



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2SLS model

- Note: For estimation and inference, the method proposed in Dufour, Pelletier and Renault (2006) is sufficiently general to estimate and test GIRs through linear methods. Joint tests are easy. We propose here a simpler method based on two-stage approach.
- Multiple horizon autoregression with order p, (wolg, 1st equ)

$$y_{1,t+h} = \beta_h' x_t + e_{t,h} \tag{16}$$

where β_h includes p-lag coefficients in the first row of the equations, $\beta_h = (\Phi_{1\bullet,1}^{(h)}, \Phi_{1\bullet,2}^{(h)}, \cdots, \Phi_{1\bullet,p}^{(h)})'$; x_t contains p-lag regressors, $x_t := (y_t', \cdots, y_{t-p+1}')'$; and $e_{t,h}$ is the residual.
- Standard LS induces HAC inference, due to serial correlation of $x_t e_{t,h}$
- We apply the IV-based estimation, the IV is

$$z_t := (u_t', u_{t-1}', \cdots, u_{t-p+1}')'. \tag{17}$$

- The IV equation:

$$y_{1,t+h} = \beta_h' x_t + e_{t,h}, \tag{18}$$
$$x_t = \Psi(p) z_t + v_t \tag{19}$$

where $\Psi(p)$ is a upper triangular matrix, $\Psi(p) = [\Psi_{ij}(p)]_{1\le i,j\le p}$, $\Psi_{ij}(p) = \Psi_{j-i}$ if $j \ge i$ and zero otherwise, v_t is a $pK \times 1$ vector of residuals, $v_t = (v_{1,t}', v_{2,t}', \cdots, v_{p,t}')'$, and $v_{i,t} = y_{t-i+1} - \sum_{l=0}^{p-i} \Psi_l u_{t-l}$.

IV identification

- Replacing the IV in the original equation yields

$$y_{1,t+h} = \beta_h' \Psi(p) z_t + (e_{t,h} + \beta_h' v_t). \quad (20)$$

- The full rankness of $\mathbb{E}[x_t z_t'] = \Psi(p)(I_p \otimes \Sigma_u)$ implies 'relevant' IV.
- The exogeneity condition holds:

$$\mathbb{E}[z_t e_{t,h}] = 0. \quad (21)$$

- The parameter is identified as

$$\beta_h = \mathbb{E}[z_t x_t']^{-1} \mathbb{E}[z_t y_{1,t+h}] \quad (22)$$

- It can also be viewed as the result of moment condition:

$$\mathbb{E}[(y_{1,t+h} - \beta_h' x_t) z_t] = 0 \quad (23)$$

- Why are we using 2SLS?
  - Obviate HAC inference
  - More efficient than the standard LS estimates for persistent data and long horizon.
  - Long horizon, $h \propto T$.

2SLS

- VAR residuals as instruments:

$$\hat{z}_t = (\hat{u}_t', \hat{u}_{t-1}', \cdots, \hat{u}_{t-p+1}')', \tag{24}$$

- It induces 2SLS estimates (only consider i-th equation)

$$\hat{\beta}_h^{IV} = (\sum_{t=p}^{T-h} \hat{z}_t x_t')^{-1} (\sum_{t=p}^{T-h} \hat{z}_t y_{1,t+h}), \tag{25}$$

- Note if I(1), $1/T \sum z_t x_t'$ conv in law. Lag-augmentation.

$$\hat{\beta}_h^{\text{LA}(\delta)\text{-IV}} = H_\delta' (\sum_{t=p+\delta}^{T-h} \hat{z}_{t,\delta} x_{t,\delta}')^{-1} (\sum_{t=p+\delta}^{T-h} \hat{z}_{t,\delta} y_{1,t+h}) \tag{26}$$

For $\delta = 1, 2$, $\hat{z}_{t,\delta} = (\hat{z}_t', y_{t-p}', y_{t-p-\delta+1}')'$, $x_{t,\delta} = (x_t', y_{t-p}', y_{t-p-\delta+1}')'$, H_δ is the selection matrix.

- Denote $s_t := z_t e_{t,h}$,

$$\bar{T}^{-1/2} \sum_{t=p}^{T-h} s_t \xrightarrow{d} N(0, \Omega_s) \tag{27}$$

where $\bar{T} = T - h - p + 1$, $\Omega_s = \sum_{k=-h+1}^{h-1} \mathbb{E}[s_t s_{t+k}']$.

LRV

- **Contribution:** The LRV of the 2SLS regression score is shown to equal to the variance of a re-ordered series. Therefore, it help obviate HAC standard error.
- The reordered regression score

$$s_t^* = (e_{t,h}, e_{t+1,h}, \cdots, e_{t+p-1,h})' \otimes u_t. \tag{28}$$

where $e_{t+i,h}$ contains only future shocks to u_t.

- With a slightly stronger assumption, for instance, mean independence, it turns out

$$\begin{aligned} \mathbb{E}[s_t^* s_\tau^{*'}] &= \mathbb{E}[\mathbb{E}[s_t^* s_\tau^{*'} \mid u_{t+1}, u_{t+2}, \cdots]] \text{ (LIE)} \\ &= \mathbb{E}[(e_{t,h}, e_{t+1,h}, \cdots, e_{t+p-1,h})' \otimes \underbrace{\mathbb{E}[u_t | u_{t+1}, u_{t+2}, \cdots]}_{=0} s_\tau^{*'}] \\ &= 0. \end{aligned} \tag{29}$$

- It induces

$$\text{LRV}(s_t^*) = \text{Var}(s_t^*) = \Omega_s = \text{LRV}(s_t). \tag{30}$$

Clearly, Ω_s can be estimated by sample variance.

VAR parameter space

Definition

VAR Parameter Space: Given positive constant $c = (c_0, c_1)'$, $\epsilon \in (0,1)$, and integer $\delta = \{0,1,2\}$. let $\mathcal{B}(\delta, c, \epsilon)$ denote the space of potentially infinite number of autoregressive coefficients $(\Phi_1, \Phi_2, \cdots, \Phi_p)$ such that the lag polynomial $\Phi(L)$ admits the factorization

$$\Phi(L) = B(L)U_1(L)U_2(L)\Pi \tag{31}$$

where $\lambda_{min}(\Pi) > c_0$, $U_i(L) = \mathbf{1}_{(\delta \geq i)}(I - P_i L)$, $P_i = \text{diag}[\rho_{m,i}]$, $\rho_{m,i} \in [-1, 1]$, for $1 \leq m \leq K$ and $i = 1, 2$. The polynomial $B(L) = \sum_{i=0}^{p-\delta} B_i L^i$ satisfies $B_0 = \Pi^{-1}$ and $\|\mathbf{B}^h\|_2 < c_1(1-\epsilon)^h$, for all $h \in \mathbb{Z}^+$, where $\mathbf{B}$ is the companion matrix containing coefficients in $B(L)$.

Assumptions

Assumption

The process u_t is mean-independent; $\mathbb{E}[u_t|u_\tau] = 0$ a.s. for all $t \neq \tau$.

Assumption

Assume

1. $\mathbb{E}[\|u_t\|^8] < \infty$, *and* $\lambda_{min}(\mathbb{E}[u_t u_t'|\mathcal{F}_{t-1}]) > c > 0$ *for some constant* c, *almost surely.*
2. *The process* $u_t \otimes u_t$ *has absolutely summable cumulants up to order four.*

Assumption

For any $\delta = \{0, 1, 2\}$, $c = (c_0, c_1)'$, $c_0, c_1 > 0$, *and* $\epsilon \in (0, 1)$,

$$\lim_{N\to\infty} \lim_{T\to\infty} \inf_{\Phi\in\mathcal{B}(\delta,c,\epsilon)} \mathrm{P}_\Phi \left(\lambda_{min} \left(T^{-1} G_{\delta,T} \sum_{t=1}^{T} x_{t,2} x_{t,2}' G_{\delta,T}' \right) \geq 1/N \right) = 1 \quad (32)$$

Note $G_{\delta,T}$ is the product of probability scaling matrix and Dickey-Fuller type matrix to transform potential unit root process into stationary ones. Check our paper for the explicit form of matrix $G_{\delta,T}$.

Uniform inference

Proposition

Suppose the above Assumptions hold. Let $w \in \mathbb{R}^{pK} \backslash \{0\}$, $x \in \mathbb{R}$, *and* $Z \sim N(0,1)$,

(i) *Let* $\alpha \in (0,1)$. *Then*

$$\lim_{T\to\infty} \sup_{\Phi\in\mathcal{B}(0,c,\epsilon)} \sup_{0<h\leq\alpha T} \left| \mathrm{P}_{\Phi}\left(\bar{T}^{1/2} \frac{w'(\hat{\beta}_h^{IV} - \beta_h)}{(w'\hat{\Omega}_{\beta,0}w)^{1/2}} < x \right) - \mathrm{P}(Z < x) \right| \xrightarrow{p} 0.$$

(ii) *Let* $\{\bar{h}_T\}$ *of positive integers satisfied* $\frac{\bar{h}}{T} \max(1, \frac{\bar{h}}{w'\Omega_\beta w}) \xrightarrow{p} 0$. *Then*

$$\lim_{T\to\infty} \sup_{\Phi\in\mathcal{B}(1,c,\epsilon)} \sup_{0<h\leq\bar{h}_T} \left| \mathrm{P}_{\Phi}\left(\bar{T}^{1/2} \frac{w'(\hat{\beta}_h^{LA(1)\text{-}IV} - \beta_h)}{(w'\hat{\Omega}_{\beta,1}w)^{1/2}} < x \right) - \mathrm{P}(Z < x) \right| \xrightarrow{p} 0.$$

(iii) *Let* $\{\bar{h}_T\}$ *of positive integers satisfied* $\frac{\bar{h}^3}{T} \max(1, \frac{\bar{h}}{w'\Omega_\beta w}) \xrightarrow{p} 0$. *Then*

$$\lim_{T\to\infty} \sup_{\Phi\in\mathcal{B}(2,c,\epsilon)} \sup_{0<h\leq\bar{h}_T} \left| \mathrm{P}_{\Phi}\left(\bar{T}^{1/2} \frac{w'(\hat{\beta}_h^{LA(2)\text{-}IV} - \beta_h)}{(w'\hat{\Omega}_{\beta,2}w)^{1/2}} < x \right) - \mathrm{P}(Z < x) \right| \xrightarrow{p} 0.$$

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DGP

- Stationary process

$$y_t = \begin{bmatrix} 0.4 & 0.1 \\ -0.1 & 0.4 \end{bmatrix} y_{t-1} + \begin{bmatrix} 0.25 & 0.13 \\ 0.17 & 0.22 \end{bmatrix} y_{t-2} + u_t. \quad (33)$$

  The eigenvalues are 0.84 and 0.56; -0.44 and -0.16.

- I(1) process

$$y_t = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.4 \end{bmatrix} y_{t-1} + \begin{bmatrix} 0.27 & 0.16 \\ 0.23 & 0.24 \end{bmatrix} y_{t-2} + u_t. \quad (34)$$

  The eigenvalues are 1 and 0.4; -0.44 and -0.16.

- I(2) process

$$y_t = \begin{bmatrix} 1.3 & 0.7 \\ 0.7 & 1.3 \end{bmatrix} y_{t-1} + \begin{bmatrix} -0.54 & -0.46 \\ -0.46 & -0.54 \end{bmatrix} y_{t-2} + u_t. \quad (35)$$

  The eigenvalues are 1 and 0.4; 1 and 0.2.

- Innovations

$$u_t \overset{i.i.d.}{\sim} N\left(\mathbf{0}, \begin{bmatrix} 1 & 0.8 \\ 0.8 & 2 \end{bmatrix}\right) \quad (36)$$

Generalized impulse response coefficients (white noise)

$\phi_{12,1}^{(h)}$ and $\phi_{12,2}^{(h)}$ measure the causality from $y_{2,t} \rightarrow y_{1,t+h}$

Monte Carlo simulations: white noise process

	$\phi_{12,1}^{(h)}$						$\phi_{12,2}^{(h)}$					
h	1	6	12	24	36	60	1	6	12	24	36	60
value	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
coverage ratio of nomial 95% confidence interval												
hvar	0.949	0.941	0.934	0.909	0.877	0.844	0.957	0.940	0.942	0.914	0.909	0.844
la-hvar	0.949	0.944	0.945	0.943	0.934	0.939	0.951	0.948	0.953	0.946	0.944	0.947
la-hvar$_b$	0.960	0.951	0.949	0.950	0.949	0.951	0.957	0.951	0.961	0.956	0.957	0.956
var	0.952	1.000	1.000	1.000	1.000	1.000	0.947	1.000	1.000	1.000	1.000	1.000
2-stage	0.949	0.945	0.932	0.909	0.875	0.740	0.957	0.939	0.936	0.908	0.893	0.736
2-stage$_b$	0.953	0.952	0.953	0.949	0.938	0.950	0.962	0.953	0.946	0.948	0.958	0.938
average width of nomial 95% confidence interval												
hvar	0.109	0.109	0.108	0.106	0.103	0.101	0.109	0.109	0.107	0.107	0.106	0.102
la-hvar	0.109	0.109	0.111	0.114	0.117	0.125	0.108	0.110	0.111	0.114	0.118	0.125
la-hvar$_b$	0.113	0.114	0.116	0.119	0.124	0.134	0.112	0.114	0.116	0.119	0.124	0.134
var	0.110	0.002	0.000	0.000	0.000	0.000	0.110	0.000	0.000	0.000	0.000	0.000
2-stage	0.109	0.108	0.110	0.113	0.116	0.123	0.109	0.109	0.110	0.114	0.117	0.124
2-stage$_b$	0.112	0.113	0.119	0.129	0.146	0.213	0.111	0.114	0.119	0.130	0.146	0.214

Figure: Note - The number of replication is 1000. For each replication, the number of bootstrap simulations is 500. The "value" row shows the true value of the parameters. The "b" subscripts indicates bootstrap.
Model: (1) hvar: multiple horizon VAR without lag-augmentation, (2) la-hvar: lag-augmented multiple horizon VAR, (3) var: iterated VAR-based method, (4) 2-stage: Two-stage estimation method. The subscript, "b", indicates bootstrap. The asymptotic variance of hvar is computed by "Neweywest" function in "Sandwich" package from R.

Generalized impulse response coefficients (stationary)

Monte Carlo simulations: stationary process

	$\phi_{12,1}^{(h)}$						$\phi_{12,2}^{(h)}$					
h	1	6	12	24	36	60	1	6	12	24	36	60
value	0.100	0.161	0.062	0.008	0.001	0.000	0.130	0.061	0.023	0.003	0.000	0.000
coverage ratio of nomial 95% confidence interval												
hvar	0.950	0.925	0.930	0.900	0.876	0.828	0.948	0.920	0.930	0.898	0.895	0.862
la-hvar	0.948	0.931	0.944	0.937	0.935	0.936	0.948	0.933	0.942	0.939	0.956	0.941
la-hvar$_b$	0.954	0.945	0.961	0.954	0.959	0.966	0.955	0.942	0.955	0.943	0.961	0.960
var	0.952	0.938	0.900	0.839	0.781	0.699	0.945	0.927	0.880	0.814	0.758	0.692
2-stage	0.950	0.926	0.923	0.909	0.881	0.743	0.948	0.933	0.936	0.917	0.901	0.741
2-stage$_b$	0.950	0.952	0.949	0.950	0.937	0.946	0.955	0.948	0.946	0.950	0.962	0.947
average width of nomial 95% confidence interval												
hvar	0.108	0.161	0.174	0.173	0.171	0.161	0.107	0.159	0.169	0.169	0.166	0.160
la-hvar	0.109	0.161	0.174	0.180	0.185	0.195	0.114	0.135	0.142	0.146	0.150	0.159
la-hvar$_b$	0.113	0.171	0.186	0.193	0.200	0.216	0.118	0.141	0.149	0.154	0.160	0.172
var	0.109	0.101	0.059	0.018	0.005	0.000	0.108	0.101	0.059	0.018	0.005	0.000
2-stage	0.108	0.131	0.119	0.113	0.116	0.123	0.107	0.114	0.111	0.112	0.116	0.122
2-stage$_b$	0.111	0.149	0.130	0.130	0.145	0.214	0.109	0.122	0.120	0.128	0.144	0.212

Generalized impulse response coefficients (I(1))

Monte Carlo simulations: I(1) process

	$\phi_{12,1}^{(h)}$						$\phi_{12,2}^{(h)}$					
h	1	6	12	24	36	60	1	6	12	24	36	60
value	0.200	0.359	0.359	0.359	0.359	0.359	0.160	0.143	0.144	0.144	0.144	0.144
coverage ratio of nomial 95% confidence interval												
hvar	0.947	0.918	0.912	0.887	0.844	0.764	0.950	0.924	0.920	0.900	0.873	0.803
la-hvar	0.943	0.923	0.934	0.934	0.912	0.890	0.953	0.929	0.939	0.927	0.934	0.925
la-hvar$_b$	0.949	0.944	0.959	0.962	0.946	0.921	0.955	0.933	0.958	0.950	0.954	0.949
var	0.952	0.936	0.902	0.862	0.839	0.799	0.940	0.932	0.909	0.885	0.843	0.784
2-stage	0.947	0.913	0.892	0.817	0.800	0.732	0.950	0.921	0.911	0.885	0.880	0.769
2-stage$_b$	0.949	0.932	0.930	0.908	0.903	0.882	0.953	0.946	0.932	0.929	0.951	0.927
average width of nomial 95% confidence interval												
hvar	0.108	0.207	0.288	0.379	0.437	0.503	0.111	0.211	0.290	0.384	0.443	0.505
la-hvar	0.108	0.208	0.290	0.404	0.490	0.624	0.114	0.182	0.244	0.333	0.401	0.508
la-hvar$_b$	0.112	0.225	0.319	0.458	0.569	0.754	0.118	0.192	0.260	0.359	0.437	0.568
var	0.109	0.143	0.150	0.172	0.199	0.258	0.112	0.089	0.091	0.095	0.102	0.120
2-stage	0.108	0.156	0.162	0.176	0.196	0.241	0.111	0.133	0.135	0.139	0.146	0.162
2-stage$_b$	0.111	0.181	0.194	0.229	0.275	0.413	0.114	0.146	0.152	0.165	0.185	0.265

Generalized impulse response coefficients (I(2))

Monte Carlo simulations: I(2) process

	$\phi^{(h)}_{12,1}$						$\phi^{(h)}_{12,2}$					
h	1	6	12	24	36	60	1	6	12	24	36	60
value	0.700	3.496	6.500	12.500	18.500	30.500	-0.460	-2.999	-6.000	-12.000	-18.000	-30.000
coverage ratio of nomial 95% confidence interval												
hvar	0.948	0.914	0.877	0.827	0.767	0.670	0.944	0.917	0.888	0.821	0.758	0.639
la-hvar	0.945	0.913	0.897	0.884	0.863	0.802	0.951	0.919	0.908	0.856	0.827	0.762
la-hvar$_b$	0.956	0.935	0.935	0.947	0.952	0.937	0.950	0.941	0.948	0.933	0.920	0.893
var	0.942	0.936	0.909	0.858	0.804	0.740	0.945	0.926	0.888	0.830	0.797	0.751
2-stage	0.948	0.862	0.828	0.771	0.722	0.658	0.944	0.844	0.802	0.747	0.722	0.672
2-stage$_b$	0.951	0.932	0.917	0.920	0.926	0.939	0.942	0.936	0.926	0.951	0.939	0.946
average width of nomial 95% confidence interval												
hvar	0.107	1.059	2.673	6.599	10.944	19.876	0.109	1.098	2.852	7.268	12.290	22.313
la-hvar	0.108	1.086	2.876	7.700	13.634	27.546	0.160	1.137	2.780	7.083	12.327	24.673
la-hvar$_b$	0.112	1.191	3.286	9.424	17.954	42.709	0.165	1.201	3.074	8.506	16.193	39.120
var	0.108	1.034	2.340	5.173	8.414	16.106	0.109	0.726	1.595	3.951	7.018	14.728
2-stage	0.107	0.833	1.888	4.177	6.794	13.028	0.109	0.592	1.291	3.200	5.688	11.965
2-stage$_b$	0.110	1.061	2.493	6.195	11.458	27.542	0.111	0.760	1.745	4.882	9.830	25.779

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Conclusion on conceptual part

- Theoretical implication of GIRs
    - Reinterpret GIRs from the perspective of intervention analysis
    - GIRs carry the meaning of treatment effect with certain endogenous responses being shut down
    - Impulse responses, as a dynamic causal effect, can be decomposed at various time periods using GIRs
    - Links with mediation analysis in time series dynamic model
- These studies cast light on the understanding of dynamic causal effect and empirical policy evaluation research.
- Research on causality channel for dynamic model

Conclusion on inference part

- For researchers are favor of linear projection method but are bothered with HAC estimator and non-stationarity, our 2SLS method could be a simple solution.
- All the researchers need to do is to reorder to the regression score and compute the variance estimates. Then HAC standard error estimator can be obviated.
- Uniform results on parameter space and projection horizon.
- Our statistical inference results have huge benefit on a variety applications of the multiple horizon VAR, e.g., causality test, causality measure, forecasting and etc.

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