Testing spatial correlation for spatial models with heterogeneous coefficients when both n and T are large

Shi Ryoung Chang* Robert de Jong†

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Abstract

The widely used approach to testing spatial correlation is to formulate a hypothesis on a homogenous spatial coefficient in spatial models. This paper proposes a novel test for spatial correlation in spatial panel data models with heterogeneous spatial autoregressive coefficients. In small reciprocal interactions, the proposed test asymptotically follows a standard normal distribution when both n and T tend to infinity jointly. The power under local alternatives is investigated. We show that the traditional test may lose power when spatial effects are heterogeneous in nature. Monte Carlo simulations demonstrate that our proposed test has better power compared to the traditional one in these types of networks. We provide an empirical example to illustrate that the proposed and traditional tests can draw different conclusions on spatial correlation.

JEL classifications: C12, C33

Keywords: Spatial panels, LM test, Dispersion test, Near-epoch dependence, Quasi-maximum likelihood

*Correspondence to: Department of Economics, The Ohio State University, Email: chang.1846@osu.edu

†Department of Economics, The Ohio State University, Email: dejong.36@osu.edu.

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1 Introduction

A natural first step in the spatial economic analysis is a test for spatial correlation. The standard econometric approach is to formulate a hypothesis as a restriction on the spatial coefficient in spatial models. For cross-sectional data, the most popular procedure is the Moran I test, which dates back to Moran (1950) and is further advanced by Cliff and Ord (1973). Burridge (1980) explores the Lagrange multiplier (LM) interpretation of the Moran I test. Kelejian and Prucha (2001) derive the asymptotic distribution of Moran I type test statistics by introducing the central limit theorem (CLT) for linear-quadratic forms.

Over the last decades, the spatial econometrics literature has extended the models from cross-section data to spatial panels. Along with these advances in the estimation of various spatial models, numerous contributions to hypothesis testing have been made. Among the classical approaches, the LM tests are popular in spatial settings because they only require restricted estimates and can be computationally simpler. One seminal contribution of spatial panels is Yu et al. (2008). Consider the following spatial dynamic panel data model:

$$Y_{nt} = \lambda_0 W_n Y_{nt} + \gamma_0 Y_{nt-1} + \rho_0 W_n Y_{nt-1} + X_{nt}\beta_0 + \mathbf{c}_{n0} + V_{nt}, \quad t = 1, ..., T \tag{1}$$

where $Y_{nt} = (y_{1t}, ..., y_{nt})'$ is an $n \times 1$ vector of a dependent variable for all units in period t, W_n is an $n \times n$ spatial weights matrix, X_{nt} is an $n \times k_x$ matrix of nonstochastic regressors, $\mathbf{c}_{n0}$ is an $n \times 1$ vector of individual fixed effects, and $V_{nt} = (\varepsilon_{1t}, ..., \varepsilon_{nt})'$ is an $n \times 1$ vector of disturbance terms. The LM tests for the hypotheses of $\lambda_0 = 0$, $\gamma_0 = 0$, and/or $\rho_0 = 0$ can be found in Bera et al. (2019)[1].

However, in almost all contributions, the hypotheses are formulated on the scalar spatial autoregressive or autocorrelation coefficient[2]. Consider a special case of (1), the pure spatial

[1]For recent surveys of LM tests in the spatial literature, see Baltagi et al. (2003, 2007), Debarsy and Ertur (2010), Yang (2010), Born and Breitung (2011), Qu and Lee (2012), Baltagi and Yang (2013a, 2013b), Robinson and Rossi (2014), Yang (2015), Cheng and Lee (2017), among others.

[2]In conventional spatial models, spatial spillover or network effects are assumed to be homogeneous across economic units.

autoregressive (SAR) panel data model:

$$
\begin{aligned}
Y_{nt} &= \lambda_0 W_n Y_{nt} + \mathbf{c}_{n0} + V_{nt} \\
&= \begin{pmatrix} \lambda_0 & 0 & 0 & \dots & 0 \\ 0 & \lambda_0 & 0 & \dots & 0 \\ 0 & 0 & \lambda_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_0 \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & w_{13} & \dots & w_{1n} \\ w_{21} & w_{22} & w_{23} & \dots & w_{2n} \\ w_{31} & w_{32} & w_{33} & \dots & w_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & w_{n3} & \dots & w_{nn} \end{pmatrix} \begin{pmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \\ \vdots \\ y_{nt} \end{pmatrix} + \mathbf{c}_{n0} + V_{nt}, \quad t = 1, ..., T
\end{aligned} \tag{2}
$$

where λ_0 is the spatial autoregressive (lag) parameter. To test $H_0 : \lambda_0 = 0$ against $H_1 : \lambda_0 \neq 0$ for (2), one may use a standard normal test, $M = \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} \tilde{Y}_{nt}' W_n \tilde{Y}_{nt}}{\sqrt{tr(W_n' W_n + W_n^2)}}$ when both n and T are large, derived in Appendix D. We refer to this underlying statistic of the traditional approach as the M test.

Recently, some interest has been in the heterogeneous version of the standard SAR models (LeSage and Chih, 2016; LeSage et al., 2017; Geniaux and Martinetti, 2018; Aquaro et al., 2021). LeSage and Chih (2016) point out that allowing for heterogeneous coefficients holds a natural appeal when contrasted with conventional spatial models[3]. Aquaro et al. (2021) discuss the estimation and inference of the spatial panel data models with fully heterogeneous coefficients in the sense that the assumption of a homogeneous spatial coefficient is likely to be restrictive when the time dimension T is large. Consider the heterogeneous version of (2):

$$
\begin{aligned}
Y_{nt} &= \Psi_0 W_n Y_{nt} + \mathbf{c}_{n0} + V_{nt} \\
&= \begin{pmatrix} \delta_{10} & 0 & 0 & \dots & 0 \\ 0 & \delta_{20} & 0 & \dots & 0 \\ 0 & 0 & \delta_{30} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \delta_{n0} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & w_{13} & \dots & w_{1n} \\ w_{21} & w_{22} & w_{23} & \dots & w_{2n} \\ w_{31} & w_{32} & w_{33} & \dots & w_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & w_{n3} & \dots & w_{nn} \end{pmatrix} \begin{pmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \\ \vdots \\ y_{nt} \end{pmatrix} + \mathbf{c}_{n0} + V_{nt}, \quad t = 1, ..., T
\end{aligned} \tag{3}
$$

where $\Psi_0 = diag(\delta_{10}, ..., \delta_{n0})$.

[3]LeSage et al. (2017) apply the heterogeneous coefficients spatial panel data model to explore retail fuel pricing. Geniaux and Martinetti (2018) consider the spatial model with spatially varying coefficients due to the misspecification of explanatory variables or the unknown structure of the spatial weights matrix.

In many empirical applications, we have data, but we do not know the true model. If one believes that the spatial spillover or network effects are heterogeneous, the widely used M test is not applicable[4]. Before going into estimation and inference for panel data, one might be interested in testing whether the spatial correlation exists or not in this heterogeneous setting of (3). Our hypothesis of interest is $H_0 : \delta_{i0} = 0$ for all $i = 1, ..., n$ against $H_1 : \delta_{i0} \neq 0$ for a non-zero fraction of units. Furthermore, some econometrics literature discusses that the power of the Cliff-Ord type tests can be very low or vanish under certain circumstances (Krämer, 2005; Martellosio, 2010, 2012; Preinerstorfer and Pötscher, 2017; Preinerstorfer 2023)[5]. The analysis in this paper will also provide a new perspective on how the power of the traditional tests may be low or vanish.

In the case of fixed n and large T, one may test the hypothesis formulated in (3) following a recent discussion. Elhorst et al. (2021) show a spatial econometric model can be viewed as a special case of a GVAR model and propose the likelihood ratio (LR) test to choose the homogeneous coefficient (SAR) in (2) against the heterogeneous coefficients (GVAR) in (3). However, this test procedure is theoretically only valid for fixed n, which implies that n should be notably smaller than T in practice. In particular, spatial empirical applications typically focus on large n cases; this existing test procedure may not apply to many microeconomic questions because n is often large in micro-datasets. This small n issue can be more severe when the number of observations over time periods is limited.

To the best of our knowledge, there are few formal tests of spatial correlation for spatial panel data models with heterogeneous spatial lag coefficients, especially when both n and T are large. If n is growing, theoretical challenges arise because the dimension of a standard test statistic increases[6]. Outside of the spatial econometrics literature, Pesaran and Yamagata

[4]Section 5.1 discusses the M test may lose power when the spatial effects are heterogeneous in nature and the sample size is small. Monte Carlo results in Section 5.2 are shown to be in line with these key findings. Section 6 offers an empirical application to illustrate our discussions.

[5]The similar discussion for the Durbin-Watson test in time series regression can be found in the early contribution of Krämer (1985).

[6]For example, an $n \times 1$ vector of the score function for the LM test and its corresponding $n \times n$ variance matrix will have an infinite length.

(2008) propose the test of slope homogeneity for panel data models with strictly exogenous regressors when n could be larger relative to T. We will follow their approach to propose a test statistic because they formulate a hypothesis on fully heterogeneous slope coefficients and derive the asymptotic results of the test when both n and T are large. However, more considerations are required for our setting due to the presence of endogenous regressors Y_{nt} in (3). Therefore, the hypothesis test about a set of n restrictions in (3), when n is large in addition to T, is essential in theoretical and empirical perspectives.

In this sense, this paper aims to fill these gaps by proposing the test statistic for spatial correlation in the pure SAR panel data models with heterogeneous coefficients when both n and T are large. We begin by constructing an LM test statistic for large T asymptotics[7]. The most important reason for deriving the LM test in our case is that such models with heterogeneous spatial lag coefficients in (3) raise intractable difficulties at the level of identification and estimation (Elhorst, 2014). To avoid the issue of identification of (3), we use the LM principle. We then propose a standardized version of the LM test, denoted by S, following Pesaran and Yamagata (2008). The proposed tests are not based on the assumption that the error terms are normally distributed. This quasi-maximum likelihood framework yields robust tests for error distributions.

This paper is organized as follows. In Section 2, we introduce the model specification and its likelihood function. Section 3 derives the LM test for large T asymptotics. Using Le Cam's theory, we analyze the power of the LM test under local alternatives. In Section 4, we propose the S test and derive the limiting distributions of S under the null hypothesis and local alternatives when both n and T tend to infinity jointly. Section 5 discusses the power properties and finite sample properties of the proposed S test compared to the traditional M test. Section 6 presents an empirical example to illustrate the usefulness of our proposed S test. Some basic lemmas are provided in Appendix A. All proofs are given in Appendix C. The asymptotic results of the M test are shown in Appendix D.

[7]If n is fixed, there is no dimensionality issue on the test statistic. This LM test can be used for empirical cases when T is notably larger than n.

2 The model and concentrated likelihood function

2.1 The heterogeneous SAR panel data model

Define s_i as an $n \times 1$ vector of zeros, except for one at the ith element for $i = 1, ..., n$[8]. Using $s_i s_i'$ for all i, we rewrite (3) as

$$\begin{aligned} Y_{nt} &= \Psi_0 W_n Y_{nt} + \mathbf{c}_{n0} + V_{nt} \\ &= \big(\sum_{i=1}^{n} \delta_{i0} s_i s_i'\big) W_n Y_{nt} + \mathbf{c}_{n0} + V_{nt}, \qquad t = 1, ..., T \end{aligned} \tag{4}$$

where $Y_{nt} = (y_{1t}, ..., y_{nt})'$ and $V_{nt} = (\varepsilon_{1t}, ..., \varepsilon_{nt})'$ are $n \times 1$ vectors, W_n is an $n \times n$ spatial weights matrix, and $\mathbf{c}_{n0}$ is an $n \times 1$ vector of individual fixed effects[9].

Define $S_n(\psi) = I_n - \Psi W_n$ where $\Psi = \sum_{i=1}^{n} \delta_i s_i s_i'$ for any $\psi = (\delta_1, ..., \delta_n)'$. At the true parameter, $S_n(\psi_0) = I_n - \Psi_0 W_n$. Then, presuming $S_n(\psi_0)$ is invertible, (4) can be written as $Y_{nt} = (I_n - \Psi_0 W_n)^{-1}(\mathbf{c}_{n0} + V_{nt}) = S_n(\psi_0)^{-1}(\mathbf{c}_{n0} + V_{nt})$.

2.2 The concentrated likelihood function

Denote $\theta = (\psi', \sigma^2)'$ and $\zeta = (\psi', \mathbf{c}_n')'$ where $\psi = (\delta_1, ..., \delta_n)'$. At the true value, $\theta_0 = (\psi_0', \sigma_0^2)'$ and $\zeta_0 = (\psi_0', \mathbf{c}_{n0}')'$ where $\psi_0 = (\delta_{10}, ..., \delta_{n0})'$. The likelihood function of (4) is

$$lnL_{nT}(\theta, \mathbf{c}_n) = -\frac{nT}{2} ln(2\pi) - \frac{nT}{2} ln(\sigma^2) + T ln|S_n(\psi)| - \frac{1}{2\sigma^2} \sum_{t=1}^{T} V_{nt}(\zeta)' V_{nt}(\zeta) \tag{5}$$

where $V_{nt}(\zeta) = (I_n - \Psi W_n) Y_{nt} - \mathbf{c}_n = S_n(\psi) Y_{nt} - \mathbf{c}_n$. Thus, $V_{nt} = V_{nt}(\zeta_0)$.

For analytical purposes, it is convenient to concentrate $\mathbf{c}_n$ out in (5). We define $\tilde{Y}_{nt} = Y_{nt} - \bar{Y}_{nT}$ for $t = 1, ..., T$, where $\bar{Y}_{nT} = \frac{1}{T} \sum_{t=1}^{T} Y_{nt}$. Similarly, $\tilde{V}_{nt} = V_{nt} - \bar{V}_{nT}$. Using the first order condition that $\frac{\partial lnL_{nT}(\theta, \mathbf{c}_n)}{\partial \mathbf{c}_n} = \frac{1}{\sigma^2} \sum_{t=1}^{T} V_{nt}(\zeta)$ from (5), the concentrated likelihood

[8] Then, $s_i s_i'$ is an $n \times n$ matrix of zeros, except for one at the (i,i)th element.

[9] For the elements of V_{nt}, we assume that ε_{it} is i.i.d. across i and t with zero mean and variance σ_0^2.

function is

$$lnL_{nT}(\theta) = -\frac{nT}{2}ln(2\pi) - \frac{nT}{2}ln(\sigma^2) + Tln|S_n(\psi)| - \frac{1}{2\sigma^2}\sum_{t=1}^{T}\tilde{V}_{nt}(\psi)'\tilde{V}_{nt}(\psi) \quad (6)$$

where $\tilde{V}_{nt}(\psi) = (I_n - \Psi W_n)\tilde{Y}_{nt} = S_n(\psi)\tilde{Y}_{nt}$.

Define $G_n(\psi) = W_n(I_n - \Psi W_n)^{-1} = W_n S_n(\psi)^{-1}$ for any $\psi = (\delta_1, ..., \delta_n)'$. From (6), the first and second order derivatives of the concentrated likelihood function can be derived: see Appendix B for their expressions[10]:

3 Test statistic for large T asymptotics

In this section, we derive an LM test statistic, asymptotically chi-square distributed with n degrees of freedom for large T asymptotics. To analyze the asymptotic properties of the LM test, we need the following assumptions:

Assumption 1. *The spatial weights matrix W_n is time-invariant and its diagonal elements satisfy $w_{ii} = 0$ for $i = 1, ..., n$.*

Assumption 2. *The disturbances ε_{it}, $i = 1, ..., n$ and $t = 1, ..., T$, are i.i.d. across i and t with zero mean, finite variance $\sigma_0^2 > 0$, and $E|\varepsilon_{it}|^{4+\eta} < \infty$ for some $\eta > 0$.*

Assumption 3. *$S_n(\psi)$ is invertible for all ψ in a small neighborhood around zero.*

Assumptions 1-2 are the standard regularity conditions used in the spatial econometrics literature[11]. Assumption 3 is needed to show the asymptotic power under local alternatives.

3.1 LM test

Consider the first order derivative with respect to $\psi = (\delta_1, ..., \delta_n)'$ in (B.2). Under H_0 ($\Psi = 0_{n\times n}$), $\tilde{V}_{nt}(\psi) = (I_n - 0_{n\times n}W_n)\tilde{Y}_{nt} = \tilde{Y}_{nt}$ and $G_n(\psi) = W_n(I_n - 0_{n\times n}W_n)^{-1} = W_n$ such

[10] Detailed derivation steps are available in the supplementary material.
[11] See Lee (2004), Yu et al. (2008), and Yu and Lee (2010), among others.

that $\psi = (0, ..., 0)'$. Therefore, the LM test statistic is based on the $n \times 1$ vector of

$$\frac{\partial lnL_{nT}(0,...,0,\sigma^2)}{\partial \psi} = \begin{pmatrix} \frac{\partial lnL_{nT}(0,...,0,\sigma^2)}{\partial \delta_1} \\ \frac{\partial lnL_{nT}(0,...,0,\sigma^2)}{\partial \delta_2} \\ \vdots \\ \frac{\partial lnL_{nT}(0,...,0,\sigma^2)}{\partial \delta_n} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma^2}\sum_{t=1}^{T}\left(\tilde{Y}'_{nt}s_1s'_1W_n\tilde{Y}_{nt} - \sigma^2 s'_1W_ns_1\right) \\ \frac{1}{\sigma^2}\sum_{t=1}^{T}\left(\tilde{Y}'_{nt}s_2s'_2W_n\tilde{Y}_{nt} - \sigma^2 s'_2W_ns_2\right) \\ \vdots \\ \frac{1}{\sigma^2}\sum_{t=1}^{T}\left(\tilde{Y}'_{nt}s_ns'_nW_n\tilde{Y}_{nt} - \sigma^2 s'_nW_ns_n\right) \end{pmatrix} \quad (7)$$

where s_i is an $n \times 1$ vector of zeros, except for one at the ith element. Define $g_{nT}(\sigma^2) \equiv \frac{\partial lnL_{nT}(0,...,0,\sigma^2)}{\partial \psi}$. Also, denote $g_{nT,i}(\sigma^2)$ as the ith element of $g_{nT}(\sigma^2)$.

Let $\tilde{\sigma}^2$ be the restricted QML estimator with the restriction $\psi = (0, ..., 0)'$ imposed, so $\tilde{\sigma}^2 = max_\sigma lnL^c_{nT}(\sigma^2)$ where

$$lnL^c_{nT}(\sigma^2) = -\frac{nT}{2}ln(2\pi) - \frac{nT}{2}ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^{T}\tilde{Y}'_{nt}\tilde{Y}_{nt} \quad (8)$$

From (8), we derive $\tilde{\sigma}^2 = \frac{1}{nT}\sum_{t=1}^{T}\tilde{Y}'_{nt}\tilde{Y}_{nt}$.

Proposition 1. *Under H_0 and Assumption 2, as $T \to \infty$,*
$\tilde{\sigma}^2 \xrightarrow{p} \sigma_0^2$ and hence $\frac{\sigma_0^2}{\tilde{\sigma}^2} \xrightarrow{p} 1$.

We now investigate the asymptotic distribution of $g_{nT}(\tilde{\sigma}^2)$ under H_0. From (7) evaluated at $\tilde{\sigma}^2$ under H_0 ($\tilde{Y}_{nt} = \tilde{V}_{nt}$) and Assumption 1 ($s'_iW_ns_i = w_{ii} = 0$ for all i), we have

$$g_{nT}(\tilde{\sigma}^2) = \begin{pmatrix} \frac{1}{\tilde{\sigma}^2}\sum_{t=1}^{T}\tilde{V}'_{nt}s_1s'_1W_n\tilde{V}_{nt} \\ \frac{1}{\tilde{\sigma}^2}\sum_{t=1}^{T}\tilde{V}'_{nt}s_2s'_2W_n\tilde{V}_{nt} \\ \vdots \\ \frac{1}{\tilde{\sigma}^2}\sum_{t=1}^{T}\tilde{V}'_{nt}s_ns'_nW_n\tilde{V}_{nt} \end{pmatrix} = \frac{\sigma_0^2}{\tilde{\sigma}^2}\begin{pmatrix} \frac{1}{\sigma_0^2}\sum_{t=1}^{T}\tilde{V}'_{nt}s_1s'_1W_n\tilde{V}_{nt} \\ \frac{1}{\sigma_0^2}\sum_{t=1}^{T}\tilde{V}'_{nt}s_2s'_2W_n\tilde{V}_{nt} \\ \vdots \\ \frac{1}{\sigma_0^2}\sum_{t=1}^{T}\tilde{V}'_{nt}s_ns'_nW_n\tilde{V}_{nt} \end{pmatrix} = \frac{\sigma_0^2}{\tilde{\sigma}^2}g_{nT}(\sigma_0^2) \quad (9)$$

where $\tilde{V}_{nt} = V_{nt} - \bar{V}_{nT}$ and $\frac{\sigma_0^2}{\tilde{\sigma}^2} \xrightarrow{p} 1$ by Proposition 1. Thus, the limiting result of $g_{nT}(\tilde{\sigma}^2)$ is the same as that of $g_{nT}(\sigma_0^2)$ under H_0.

To show the asymptotic distribution of $\frac{1}{\sqrt{T}} g_{nT}(\sigma_0^2)$ under H_0, we consider

$$\frac{1}{\sqrt{T}} g_{nT}(\sigma_0^2) = \frac{1}{\sigma_0^2} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \begin{pmatrix} V'_{nt} s_1 s'_1 W_n V_{nt} \\ V'_{nt} s_2 s'_2 W_n V_{nt} \\ \vdots \\ V'_{nt} s_n s'_n W_n V_{nt} \end{pmatrix} - \frac{1}{\sigma_0^2} \sqrt{T} \begin{pmatrix} \bar{V}'_{nT} s_1 s'_1 W_n \bar{V}_{nT} \\ \bar{V}'_{nT} s_2 s'_2 W_n \bar{V}_{nT} \\ \vdots \\ \bar{V}'_{nT} s_n s'_n W_n \bar{V}_{nT} \end{pmatrix} \tag{10}$$

where the first term is denoted as $\frac{1}{\sqrt{T}} \frac{\partial ln L^*_{nT}(0, ..., 0, \sigma_0^2)}{\partial \psi}$ and the second term is $O_p(\frac{1}{\sqrt{T}})$ by Lemmas A.3 and A.4[12]. The mean and variance of the first term in (10) are $\mu_{g,n} = 0$ and

$$\Sigma_{g,n} = \begin{pmatrix} s'_1 W_n W'_n s_1 & s'_1 W_n s_2 s'_2 W_n s_1 & \dots & s'_1 W_n s_n s'_n W_n s_1 \\ s'_2 W_n s_1 s'_1 W_n s_2 & s'_2 W_n W'_n s_2 & \dots & s'_2 W_n s_n s'_n W_n s_2 \\ \vdots & \vdots & \ddots & \vdots \\ s'_n W_n s_1 s'_1 W_n s_n & s'_n W_n s_2 s'_2 W_n s_n & \dots & s'_n W_n W'_n s_n \end{pmatrix} \tag{11}$$

where $\Sigma_{g,n}^{i,i} = s'_i W_n W'_n s_i = \sum_{j=1}^{n} w_{ij}^2$ and $\Sigma_{g,n}^{i,j} = s'_i W_n s_j s'_j W_n s_i = w_{ij} w_{ji}$ for all i, j[13]. Note that $\Sigma_{g,n}$ takes the same form regardless of shapes of ε_{it}.

Under Assumptions 1 and 2, $V'_{nt} s_i s'_i W_n V_{nt}$ is i.i.d. across t with $E(V'_{nt} s_i s'_i W_n V_{nt}) = 0$, $E(V'_{nt} s_i s'_i W_n V_{nt})^2 = \sigma_0^4 s'_i W_n W'_n s_i < \infty$ and $E(V'_{nt} s_i s'_i W_n V_{nt})(V'_{nt} s_j s'_j W_n V_{nt}) = \sigma_0^4 s'_i W_n s_j s'_j W_n s_i$ $< \infty$. By the central limit theorem (Multivariate Lindeberg-Levy CLT), we have $\frac{1}{\sqrt{T}} g_{nT}(\sigma_0^2) \xrightarrow{d} N(0, \Sigma_{g,n})$. To derive the LM test, we need the following assumption on $\Sigma_{g,n}$.

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[12] $\frac{1}{\sqrt{T}} \frac{\partial ln L^*_{nT}(0, ..., 0, \sigma_0^2)}{\partial \psi}$ can also be derived from (B.5) in Appendix B with the restriction $\psi = (0, ..., 0)'$ imposed (under H_0). Also, we have $|\sqrt{T} \bar{V}'_{nT} s_i s'_i W_n \bar{V}_{nT}| \leq |\sqrt{T} \bar{V}'_{nT} s_i s'_i W_n \bar{V}_{nT} - E(\sqrt{T} \bar{V}'_{nT} s_i s'_i W_n \bar{V}_{nT})| + |E(\sqrt{T} \bar{V}'_{nT} s_i s'_i W_n \bar{V}_{nT})| = O_p(\frac{1}{\sqrt{T}})$ for all i by Lemmas A.3 and A.4.

[13] $\Sigma_{g,n}$ can also be derived from (B.8) in Appendix B with the restriction $\psi = (0, ..., 0)'$ imposed (under H_0). Thus, we have $E(\frac{1}{T} \frac{\partial^2 ln L^*_{nT}(0, ..., 0, \sigma_0^2)}{\partial \psi \partial \psi'}) = -\Sigma_{g,n}$.

Assumption 4. *For the elements in $\Sigma_{g,n}$, either (1) or (2) is satisfied:*

(1) $\sum_{j=1}^{n} w_{ij}^2 > \sum_{j=1}^{n} |w_{ij}w_{ji}|$ *for all* i.

(2) $\sum_{j=1}^{n} w_{ij}^2 \geq \sum_{j=1}^{n} |w_{ij}w_{ji}|$ *and* $w_{ij} > 0$ *for all* i, j.

Assumption 4(1) does not hold when W_n is symmetric because of $\sum_{j=1}^{n} w_{ij}^2 = \sum_{j=1}^{n} |w_{ij}w_{ji}|$. However, any symmetric W_n can satisfy Assumption 4(2) as long as all off-diagonal entries of W_n are positive. Under Assumptions 1 and 4(1), $\Sigma_{g,n}$ is a strictly diagonally dominant matrix because $\sum_{j=1}^{n} w_{ij}^2 > \sum_{j=1}^{n} |w_{ij}w_{ji}| \geq 0$ where $\sum_{j=1}^{n} |w_{ij}w_{ji}| = \sum_{j\neq i}^{n} |w_{ij}w_{ji}|$ is each row sum of all off-diagonal entries in absolute value[14]. Under Assumptions 1 and 4(2), $\Sigma_{g,n}$ is a diagonally dominant matrix.

Proposition 2. *Under Assumptions 1 and 4,*

$\Sigma_{g,n}$ *is positive definite.*

Based on the limiting results of $\frac{1}{\sqrt{T}} g_{nT}(\tilde{\sigma}^2)$ under H_0 and the condition on $\Sigma_{g,n}$, we can find the asymptotic distribution of the quadratic form, $\frac{1}{T} g_{nT}(\tilde{\sigma}^2)' \Sigma_{g,n}^{-1} g_{nT}(\tilde{\sigma}^2)$ in Theorem 1.

Theorem 1. *Under H_0 and Assumptions 1, 2 and 4, as $T \to \infty$,*

$\frac{1}{T} g_{nT}(\tilde{\sigma}^2)' \Sigma_{g,n}^{-1} g_{nT}(\tilde{\sigma}^2) \xrightarrow{d} \chi_n^2$.

For the empirical cases where T is notably larger than n, one may use this LM test shown in Theorem 1[15]. Hence, the conclusions on whether a spatial correlation exists or not, when T is large, can be drawn based on the value of $\frac{1}{T} g_{nT}(\tilde{\sigma}^2)' \Sigma_{g,n}^{-1} g_{nT}(\tilde{\sigma}^2)$ where $g_{nT}(\tilde{\sigma}^2) = (\frac{1}{\tilde{\sigma}^2} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \tilde{Y}_{nt}' s_1 s_1' W_n \tilde{Y}_{nt}, ..., \frac{1}{\tilde{\sigma}^2} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \tilde{Y}_{nt}' s_n s_n' W_n \tilde{Y}_{nt})'$.

[14] A square matrix is said to be diagonally dominant if, for every row, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all off-diagonal entries in that row. That is, the matrix $A = (a_{ij})$ is diagonally dominant if $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$. Furthermore, it is strictly diagonally dominant if strict inequality holds for all i.

[15] The advantage of the LM tests over the other approaches, such as the Wald and LR tests, is that it only requires the restricted estimate, $\tilde{\sigma}^2$.

3.2 Local power of the LM test

For the asymptotic local power of the LM test, we consider the following local alternatives:

$$H_{1,T} : \delta_{i0} = \frac{\Delta_i}{T^{1/2}} \qquad \text{for } i = 1, ..., n \tag{12}$$

where Δ_i is a fixed constant ($\Delta_i \neq 0$). Denote $\Delta = (\Delta_1, ..., \Delta_n)'$, an $n \times 1$ vector of constants. To investigate the asymptotic properties of the LM test under $H_{1,T}$, we utilize Le Cam's theory, following Qu and Lee (2013) and Cheng and Lee (2017) for spatial models.

Consider $q_{nT} = lnL_{nT}(\Delta_1/T^{1/2}, ..., \Delta_n/T^{1/2}, \sigma_0^2) - lnL_{nT}(0, ..., 0, \sigma_0^2)$. By the second order Taylor series expansion, we have

$$q_{nT} = \frac{1}{T^{1/2}}\Delta' \frac{\partial lnL_{nT}(0, ..., 0, \sigma_0^2)}{\partial \psi} + \frac{1}{2T}\Delta' \frac{\partial^2 lnL_{nT}(\bar{\Delta}_1, ..., \bar{\Delta}_n, \sigma_0^2)}{\partial \psi \partial \psi'}\Delta \tag{13}$$

where $\bar{\Delta}_i$ lies between $\Delta_i/T^{1/2}$ and 0 for all $i = 1, ..., n$. Thus, $\bar{\Delta}_i \xrightarrow{p} 0$ as $T \to \infty$ for all i. From the previous results in Section 3.1, we have

$$\begin{aligned} q_{nT} &= \frac{1}{T^{1/2}}\Delta' \frac{\partial lnL_{nT}(0, ..., 0, \sigma_0^2)}{\partial \psi} + \frac{1}{2T}\Delta' \frac{\partial^2 lnL_{nT}(\bar{\Delta}_1, ..., \bar{\Delta}_n, \sigma_0^2)}{\partial \psi \partial \psi'}\Delta \\ &= \Delta' \frac{1}{\sqrt{T}} \frac{\partial lnL^*_{nT}(0, ..., 0, \sigma_0^2)}{\partial \psi} + \frac{1}{2}\Delta' E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(0, ..., 0, \sigma_0^2)}{\partial \psi \partial \psi'})\Delta + o_p(1) \end{aligned} \tag{14}$$

where $\frac{1}{\sqrt{T}} \frac{\partial lnL^*_{nT}(0, ..., 0, \sigma_0^2)}{\partial \psi} \xrightarrow{d} N(0, \Sigma_{g,n})$ under H_0 and $E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(0, ..., 0, \sigma_0^2)}{\partial \psi \partial \psi'}) = -\Sigma_{g,n}$, provided that $\frac{1}{T}\frac{\partial^2 lnL_{nT}(\bar{\Delta}_1, ..., \bar{\Delta}_n, \sigma_0^2)}{\partial \psi \partial \psi'} - E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(0, ..., 0, \sigma_0^2)}{\partial \psi \partial \psi'}) = o_p(1)$.

Lemma 1. *Under Assumptions 1-3,*
$\frac{1}{T}\frac{\partial^2 lnL_{nT}(\bar{\Delta}_1, ..., \bar{\Delta}_n, \sigma_0^2)}{\partial \psi \partial \psi'} - E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(0, ..., 0, \sigma_0^2)}{\partial \psi \partial \psi'}) = o_p(1).$

Hence, $q_{nT} \xrightarrow{d} N(-\frac{1}{2}\Delta'\Sigma_{g,n}\Delta, \Delta'\Sigma_{g,n}\Delta)$ under H_0 and this result implies that Le Cam's first lemma holds. Denote $\sigma^{*2} = \Delta'\Sigma_{g,n}\Delta$. From (9), (10) and (14), we obtain the asymptotic covariance of $\frac{1}{\sqrt{T}}g_{nT}(\tilde{\sigma}^2)$ and q_{nT} as

$$\begin{aligned} &Cov\big(\frac{1}{\sqrt{T}}\frac{\partial lnL^*_{nT}(0,...,0,\sigma_0^2)}{\partial\psi},\Delta'\frac{1}{\sqrt{T}}\frac{\partial lnL^*_{nT}(0,...,0,\sigma_0^2)}{\partial\psi}\big) \\ &= Var\big(\frac{1}{\sqrt{T}}\frac{\partial lnL^*_{nT}(0,...,0,\sigma_0^2)}{\partial\psi}\big)\Delta = \Sigma_{g,n}\Delta \end{aligned} \tag{15}$$

Denote $\tau = \Sigma_{g,n}\Delta$. Then, using the Cramer-Wold device, we can find the joint asymptotic distribution of $\frac{1}{\sqrt{T}}g_{nT}(\tilde{\sigma}^2)$ and q_{nT} under H_0 as

$$(\frac{1}{\sqrt{T}}g_{nT}(\tilde{\sigma}^2), q_{nT}) \rightarrow N\Big(\begin{pmatrix} 0 \\ -\frac{1}{2}\sigma^{*2} \end{pmatrix}, \begin{pmatrix} \Sigma_{g,n} & \tau \\ \tau' & \sigma^{*2} \end{pmatrix}\Big) \tag{16}$$

Hence, by Le Cam's third lemma, $\frac{1}{\sqrt{T}}g_{nT}(\tilde{\sigma}^2) \xrightarrow{d} N(\tau, \Sigma_{g,n})$ under $H_{1,T}$.

Theorem 2. *Under $H_{1,T}$ and Assumptions 1-4, as $T \rightarrow \infty$,*
$\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1}g_{nT}(\tilde{\sigma}^2) \xrightarrow{d} \chi^2_n(\mu)$ *where $\mu = \Delta'\Sigma_{g,n}\Delta$ is a noncentrality parameter.*

Theorem 2 implies that the LM test has power against local alternatives if $\Delta_i \neq 0$ for some i due to $\mu > 0$ since $\Sigma_{g,n}$ is positive definite by Proposition 2.

4 Test statistic when both n and T are large

In this section, we propose the S test when both n and T are large based on the quadratic form derived in Section 3. We first derive the limiting distribution of S under H_0 when both n and T tend to infinity jointly in the special case where social interactions or networks are completely non-reciprocal. We then extend our discussion to general interactions or networks. Lastly, we derive the limiting result of the S test under local alternatives.

4.1 S test

We propose a standardized version of the LM test, $S = \frac{1}{\sqrt{2n}}\big(\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1}g_{nT}(\tilde{\sigma}^2) - n\big)$, for (3) when both n and T are large. Under H_0, the proposed S test takes the following form:

$$\begin{aligned} S &= \frac{1}{\sqrt{2n}}\big(\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1}g_{nT}(\tilde{\sigma}^2) - n\big) \\ &= \frac{1}{\sqrt{2n}}\Bigg(\frac{1}{T}\begin{pmatrix} g_{nT,1}(\tilde{\sigma}^2) \\ g_{nT,2}(\tilde{\sigma}^2) \\ \vdots \\ g_{nT,n}(\tilde{\sigma}^2) \end{pmatrix}' \begin{pmatrix} s_1'W_nW_n's_1 & s_1'W_ns_2s_2'W_ns_1 & \dots & s_1'W_ns_ns_n'W_ns_1 \\ s_2'W_ns_1s_1'W_ns_2 & s_2'W_nW_n's_2 & \dots & s_2'W_ns_ns_n'W_ns_2 \\ \vdots & \vdots & \ddots & \vdots \\ s_n'W_ns_1s_1'W_ns_n & s_n'W_ns_2s_2'W_ns_n & \dots & s_n'W_nW_n's_n \end{pmatrix}^{-1} \begin{pmatrix} g_{nT,1}(\tilde{\sigma}^2) \\ g_{nT,2}(\tilde{\sigma}^2) \\ \vdots \\ g_{nT,n}(\tilde{\sigma}^2) \end{pmatrix} - n\Bigg) \end{aligned} \tag{17}$$

where $g_{nT,i}(\tilde{\sigma}^2) = \frac{1}{\tilde{\sigma}^2}\sum_{t=1}^{T}\tilde{V}_{nt}'s_is_i'W_n\tilde{V}_{nt}$.

Theorem 1 shows that $\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1}g_{nT}(\tilde{\sigma}^2) \xrightarrow{d} \chi_n^2$ as $T \to \infty$ under H_0. For sequential asymptotics ($T \to \infty$, then $n \to \infty$), it is readily shown that the S test asymptotically follows a standard normal distribution $N(0,1)$ under H_0 since $S \xrightarrow{d} \frac{1}{\sqrt{2n}}\big(\chi_n^2 - n\big)$ as $n \to \infty$[16]. However, our main interest is to analyze the limiting result of S under joint asymptotics. For the asymptotic properties of the S test when both n and T are large, we need the following assumptions:

Assumption 5. *W_n is uniformly bounded in row and column sums in absolute value.*

Assumption 6. $\sup_n \sup_i \frac{\sum_{j=1}^n |w_{ij}w_{ji}|}{\sum_{j=1}^n w_{ij}^2} < 1.$

Assumption 7. *T is an increasing function of n and n goes to infinity.*

Assumption 5 is the standard regularity condition for W_n used in the spatial econometrics literature. This uniform boundedness of W_n, originated by Kelejian and Prucha (1998), is a condition to limit spatial correlations to a manageable degree (Lee 2004). Under Assumption 5, ith diagonal element of $\Sigma_{g,n}$, $\sum_{j=1}^n w_{ij}^2 \leq \max_{i,j}|w_{ij}|(\max_i \sum_{j=1}^n |w_{ij}|) < \infty$ for all n. Assumption

[16] For χ_n^2, $E(\chi_n^2) = n$ and $Var(\chi_n^2) = 2n$. For more details, see de Jong and Bierens (1994).

6 is a condition for the nonsingularity of $\Sigma_{g,n}$ for all n. Assumption 6 implies that the strictly diagonally dominant property of $\Sigma_{g,n}$ holds uniformly in n because the diagonal element is strictly greater than the sum of all off-diagonal entries in that row for all i and n. Assumption 7 allows one case; $T \to \infty$ as $n \to \infty$ where both n and T are large.

The crucial part of this analysis is to apply the appropriate limit theorems to the proposed S test that contains the n-dimensional quadratic form where n is growing. To this end, we first need the analytical form of the inverse of $\Sigma_{g,n}$ in (11). We note that all elements of $\Sigma_{g,n}$ depend on W_n specified by some social interactions or network structures. In this sense, we begin by discussing the limiting result of S under a special interaction in Section 4.2.

4.2 The limiting result of S in a special case

Consider the non-reciprocal interactions or networks in the form of Assumption 8:

Assumption 8. *The reciprocities $w_{ij}w_{ji}$ of W_n is zero, i.e., $w_{ij}w_{ji} = 0$ for all i and j.*

Assumption 8 implies that the interactions or networks are completely non-reciprocal in the sense that either w_{ij} or w_{ji}, or both is zero for all i and j. Under Assumptions 1 and 8, we have $\frac{w_{ij}w_{ji}}{\sqrt{\sum_{j=1}^{n} w_{ij}^2}\sqrt{\sum_{i=1}^{n} w_{ji}^2}} = 0$ for all i, j and n. Assumption 8 is strong in practice; however, these types of interactions or networks can be found in the econometrics literature. Bramoullé et al. (2009) illustrate a special social network where each unit is influenced only by his or her left-hand friend. An example of their weights matrix G is:

$$W_n = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{18}$$

This is a simple case of the non-reciprocal interactions or networks[17]. In the case of (18), $\Sigma_{g,n} = I_n$. Also, interactions are likely to be non-reciprocal when a small number of units affect many others dominantly. Pesaran and Yang (2021) illustrate the following interaction:

$$W_n = \begin{pmatrix} 0 & w_{12} & 0 & 0 & \dots & 0 & 0 \\ w_{21} & 0 & w_{23} & 0 & \dots & 0 & 0 \\ w_{31} & 0 & 0 & w_{34} & \dots & 0 & 0 \\ w_{41} & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{n\text{-}11} & 0 & 0 & 0 & \dots & 0 & w_{n\text{-}1n} \\ w_{n1} & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{19}$$

where the first unit is the dominant unit[18]. This example can be the non-reciprocal network, additionally assuming $w_{21} = 0$. In the case of (19), $\Sigma_{g,n}$ is a simple block diagonal matrix.

In the remaining subsection, we consider the non-reciprocal interactions or networks but do not specify a particular form of W_n[19]. Then, under Assumption 8, all off-diagonal entries of $\Sigma_{g,n}$ are zero because $s_i' W_n s_j s_j' W_n s_i = w_{ij} w_{ji} = 0$ for all i, j, and $\Sigma_{g,n}$ becomes a block diagonal matrix[20]. Denote $\Sigma_{g,n}^D$, the structure of $\Sigma_{g,n}$ under Assumption 8, as

$$\Sigma_{g,n}^D = \begin{pmatrix} s_1' W_n W_n' s_1 & 0 & 0 & \dots & 0 \\ 0 & s_2' W_n W_n' s_2 & 0 & \dots & 0 \\ 0 & 0 & s_3' W_n W_n' s_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & s_n' W_n W_n' s_n \end{pmatrix} \tag{20}$$

and denote $S^{nr} = \dfrac{1}{\sqrt{2n}}\big(\dfrac{1}{T} g_{nT}(\tilde{\sigma}^2)' \Sigma_{g,n}^{D-1} g_{nT}(\tilde{\sigma}^2) - n\big)$ for S under Assumption 8.

[17] We may think of a generalized version of the Bramoullé et al. (2009). For instance, each unit is influenced by all left-hand friends.

[18] Pesaran and Yang (2021) discuss estimation and inference in spatial models with dominant units.

[19] We consider the conventional weights matrices where neighboring units are only a few adjacent ones.

[20] The advantage of the block diagonal matrix is that its inverse can be easily derived; the inverse of any block-diagonal matrix is given by replacing the diagonal elements with their reciprocals.

Using the property of a block diagonal matrix and $\tilde{V}_{nt} = V_{nt} - \bar{V}_{nT}$, we have

$$\begin{aligned}
S^{nr} &= \frac{1}{\sqrt{2n}}\big(\frac{1}{T} g_{nT}(\tilde{\sigma}^2)' \Sigma_{g,n}^{D-1} g_{nT}(\tilde{\sigma}^2) - n\big) \\
&= \frac{1}{\sqrt{2n}}\big(\frac{1}{T}\begin{pmatrix} g_{nT,1}(\tilde{\sigma}^2) \\ g_{nT,2}(\tilde{\sigma}^2) \\ \vdots \\ g_{nT,n}(\tilde{\sigma}^2)\end{pmatrix}' \begin{pmatrix} (s_1'W_nW_n's_1)^{-1} & 0 & \dots & 0 \\ 0 & (s_2'W_nW_n's_2)^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (s_n'W_nW_n's_n)^{-1} \end{pmatrix} \begin{pmatrix} g_{nT,1}(\tilde{\sigma}^2) \\ g_{nT,2}(\tilde{\sigma}^2) \\ \vdots \\ g_{nT,n}(\tilde{\sigma}^2)\end{pmatrix} - n\big) \\
&= \frac{1}{\sqrt{2n}}\big(\frac{1}{T}\sum_{i=1}^n g_{nT,i}(\tilde{\sigma}^2)(s_i'W_nW_n's_i)^{-1} g_{nT,i}(\tilde{\sigma}^2) - \sum_{i=1}^n 1\big) \\
&= \frac{1}{\sqrt{2n}}\sum_{i=1}^n \big(\frac{1}{T} g_{nT,i}(\tilde{\sigma}^2)(s_i'W_nW_n's_i)^{-1} g_{nT,i}(\tilde{\sigma}^2) - 1\big)
\end{aligned} \tag{21}$$

where $g_{nT,i}(\tilde{\sigma}^2) = \frac{1}{\tilde{\sigma}^2}\sum_{t=1}^T \tilde{V}_{nt}' s_i s_i' W_n \tilde{V}_{nt} = \frac{\sigma_0^2}{\tilde{\sigma}^2}(\frac{1}{\sigma_0^2}\sum_{t=1}^T V_{nt}' s_i s_i' W_n V_{nt} - \frac{1}{\sigma_0^2} T \bar{V}_{nT}' s_i s_i' W_n \bar{V}_{nT})$. This can be rewritten as

$$\begin{aligned}
S^{nr} &= \frac{1}{\sqrt{2n}}\sum_{i=1}^n \big(\frac{(\frac{1}{\sqrt{T}} g_{nT,i}(\tilde{\sigma}^2))^2}{s_i'W_nW_n's_i} - (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 + (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 - 1\big) \\
&= (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \frac{1}{\sqrt{2n}}\sum_{i=1}^n \big(\frac{(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T V_{nt}'s_is_i'W_nV_{nt} - \frac{1}{\sigma_0^2}\sqrt{T}\bar{V}_{nT}'s_is_i'W_n\bar{V}_{nT})^2}{s_i'W_nW_n's_i} - 1\big) + \frac{1}{\sqrt{2n}}\sum_{i=1}^n \big((\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 - 1\big) \\
&= (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \frac{1}{\sqrt{2n}}\sum_{i=1}^n \big(\frac{(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T V_{nt}'s_is_i'W_nV_{nt})^2}{s_i'W_nW_n's_i} - 1\big) + (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \frac{1}{\sqrt{2n}}\sum_{i=1}^n \frac{(\frac{1}{\sigma_0^2}\sqrt{T}\bar{V}_{nT}'s_is_i'W_n\bar{V}_{nT})^2}{s_i'W_nW_n's_i} \\
&\quad - (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \frac{2}{\sqrt{2n}}\sum_{i=1}^n \frac{(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T V_{nt}'s_is_i'W_nV_{nt})(\frac{1}{\sigma_0^2}\sqrt{T}\bar{V}_{nT}'s_is_i'W_n\bar{V}_{nT})}{s_i'W_nW_n's_i} + \frac{1}{\sqrt{2n}}\sum_{i=1}^n \big((\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 - 1\big) \\
&= (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \frac{1}{\sqrt{2n}}\sum_{i=1}^n \big((\frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T V_{nt}'s_is_i'W_nV_{nt}}{\sqrt{s_i'W_nW_n's_i}})^2 - 1\big) + \frac{1}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \frac{\sqrt{n}}{T}\frac{1}{n}\sum_{i=1}^n (\frac{\frac{1}{\sigma_0^2}T\bar{V}_{nT}'s_is_i'W_n\bar{V}_{nT}}{\sqrt{s_i'W_nW_n's_i}})^2 \\
&\quad - \frac{2}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \sqrt{\frac{n}{T}}\frac{1}{n}\sum_{i=1}^n (\frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T V_{nt}'s_is_i'W_nV_{nt}}{\sqrt{s_i'W_nW_n's_i}})(\frac{\frac{1}{\sigma_0^2}T\bar{V}_{nT}'s_is_i'W_n\bar{V}_{nT}}{\sqrt{s_i'W_nW_n's_i}}) + \frac{1}{\sqrt{2n}}\sum_{i=1}^n \big((\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 - 1\big)
\end{aligned} \tag{22}$$

Proposition 3. *Under H_0 and Assumptions 2 and 7,*
$\tilde{\sigma}^2 \xrightarrow{p} \sigma_0^2$ *and hence* $\frac{\sigma_0^2}{\tilde{\sigma}^2} \xrightarrow{p} 1$*, and* $\frac{\sigma_0^2}{\tilde{\sigma}^2} - 1 = O_P(\frac{1}{\sqrt{nT}})$.

Define the random variable $z_{i,nT}$ and $r_{i,nT}$ over $i = 1, ..., n$ as

$$z_{i,nT} \equiv \left(\frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} V'_{nt} s_i s'_i W_n V_{nt}}{\sqrt{s'_i W_n W'_n s_i}}\right)^2 = \left(\frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\sum_{j=1}^{n} w_{ij}\varepsilon_{jt}}{\sqrt{\sum_{j=1}^{n} w_{ij}^2}}\right)^2 \tag{23}$$

$$r_{i,nT} \equiv \left(\frac{\frac{1}{\sigma_0^2} T \bar{V}'_{nT} s_i s'_i W_n \bar{V}_{nT}}{\sqrt{s'_i W_n W'_n s_i}}\right)^2 = \left(\frac{\frac{1}{\sigma_0^2} T \bar{\epsilon}_{iT}\sum_{j=1}^{n} w_{ij}\bar{\epsilon}_{jT}}{\sqrt{\sum_{j=1}^{n} w_{ij}^2}}\right)^2 \tag{24}$$

where $\bar{\epsilon}_{iT} = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}$. Let $\tilde{z}_{i,nT} = \frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} V'_{nt} s_i s'_i W_n V_{nt}}{\sqrt{s'_i W_n W'_n s_i}}$ and $\tilde{r}_{i,nT} = \frac{\frac{1}{\sigma_0^2} T \bar{V}'_{nT} s_i s'_i W_n \bar{V}_{nT}}{\sqrt{s'_i W_n W'_n s_i}}$. Thus, $z_{i,nT} = (\tilde{z}_{i,nT})^2$ and $r_{i,nT} = (\tilde{r}_{i,nT})^2$. Under Assumptions 1 and 2, $z_{i,nT}$, $r_{i,nT}$ and $\tilde{z}_{i,nT}\tilde{r}_{i,nT}$ are spatially correlated (dependent heterogeneous) random variables with finite means and variances. That is, as shown in Appendix E, we have

$$E(z_{i,nT}) = 1 \tag{25}$$

$$Var(z_{i,nT}) = 2 + \frac{1}{\sigma_0^8}\frac{1}{T}\frac{\mu_4(\mu_4 - 3\sigma_0^4)\sum_{j=1}^{n} w_{ij}^4}{(\sum_{j=1}^{n} w_{ij}^2)^2} + \frac{3(\mu_4 - \sigma_0^4)}{\sigma_0^4}\frac{1}{T} \tag{26}$$

$$\begin{aligned}
Cov(z_{i,nT}, z_{j,nT}) = {} & \frac{(\mu_4)^2 - 2\mu_4\sigma_0^4 + (\sigma_0^4)^2}{\sigma_0^8}\frac{1}{T}\frac{w_{ij}^2 w_{ji}^2}{(\sum_{j=1}^{n} w_{ij}^2)(\sum_{i=1}^{n} w_{ji}^2)} \\
& + \frac{\mu_4 - \sigma_0^4}{\sigma_0^4}\frac{1}{T}\frac{w_{ji}^2}{\sum_{i=1}^{n} w_{ji}^2} + \frac{\mu_4 - \sigma_0^4}{\sigma_0^4}\frac{1}{T}\frac{w_{ij}^2}{\sum_{j=1}^{n} w_{ij}^2} \\
& + \frac{2}{T}\frac{(\sum_{l=1}^{n} w_{il}w_{jl})^2}{(\sum_{j=1}^{n} w_{ij}^2)(\sum_{i=1}^{n} w_{ji}^2)} + \frac{\mu_4 - 3\sigma_0^4}{\sigma_0^4}\frac{1}{T}\frac{\sum_{l=1}^{n} w_{il}^2 w_{jl}^2}{(\sum_{j=1}^{n} w_{ij}^2)(\sum_{i=1}^{n} w_{ji}^2)} \\
& + \frac{4(\mu_3)^2}{\sigma_0^6}\frac{1}{T}\frac{w_{ij}w_{ji}(\sum_{l=1}^{n} w_{il}w_{jl})}{(\sum_{j=1}^{n} w_{ij}^2)(\sum_{i=1}^{n} w_{ji}^2)} + \frac{2(\mu_3)^2}{\sigma_0^6}\frac{1}{T}\frac{w_{ji}(\sum_{l=1}^{n} w_{il}^2 w_{jl})}{(\sum_{j=1}^{n} w_{ij}^2)(\sum_{i=1}^{n} w_{ji}^2)} \\
& + \frac{2(\mu_3)^2}{\sigma_0^6}\frac{1}{T}\frac{w_{ij}(\sum_{l=1}^{n} w_{jl}^2 w_{il})}{(\sum_{j=1}^{n} w_{ij}^2)(\sum_{i=1}^{n} w_{ji}^2)}
\end{aligned} \tag{27}$$

$$E(r_{i,nT}) = 1 \tag{28}$$

$$E(\tilde{z}_{i,nT}\tilde{r}_{i,nT}) = \frac{1}{\sqrt{T}} \tag{29}$$

where $\mu_s = E(\varepsilon_{it}^s)$ for $s = 3, 4$. Note that Assumption 8 is not needed to obtain the results above. Thus, (25)-(29) hold for any W_n.

Denote $Q_{nT} = \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} \big(z_{i,nT} - E(z_{i,nT})\big)$, $P_{nT} = \frac{1}{n} \sum_{i=1}^{n} \big(r_{i,nT} - E(r_{i,nT})\big)$ and $U_{nT} = \frac{1}{n} \sum_{i=1}^{n} \big(\tilde{z}_{i,nT}\tilde{r}_{i,nT} - E(\tilde{z}_{i,nT}\tilde{r}_{i,nT})\big)$. Then, the mean and variance of Q_{nT}, $\mu_{Q_{nT}} = E(Q_{nT})$ and $\Sigma_{Q_{nT}} = Var(Q_{nT})$ can be found in Proposition 4.

Proposition 4. *Under Assumptions 1, 2, 5 and 7,*
$\mu_{Q_{nT}} = 0$ *and* $\Sigma_{Q_{nT}} = 1 + O(\frac{1}{T})$.

Assumption 8 is not needed to obtain the result of Proposition 4. Hence, Proposition 4 holds for any weights matrix W_n. Using (22), (25), (28) and (29), we have

$$\begin{aligned} S^{nr} &= (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 Q_{nT} + \frac{1}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \frac{\sqrt{n}}{T} P_{nT} + \frac{1}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \frac{\sqrt{n}}{T} \frac{1}{n} \sum_{i=1}^{n} E(r_{i,nT}) - \frac{2}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \sqrt{\frac{n}{T}} U_{nT} \\ &\quad - \frac{2}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \sqrt{\frac{n}{T}} \frac{1}{n} \sum_{i=1}^{n} E(\tilde{z}_{i,nT}\tilde{r}_{i,nT}) + \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} \big((\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 - 1\big) \\ &= (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 Q_{nT} + \frac{1}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \frac{\sqrt{n}}{T} P_{nT} - \frac{2}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \sqrt{\frac{n}{T}} U_{nT} - \frac{1}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \frac{\sqrt{n}}{T} + \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} \big((\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 - 1\big) \end{aligned} \tag{30}$$

We now apply limit theorems to Q_{nT}, P_{nT}, and U_{nT} in (30). We note that the established CLT and LLN for linear-quadratic forms are not applicable because $z_{i,nT}$, $r_{i,nT}$ and $\tilde{z}_{i,nT}\tilde{r}_{i,nT}$ are the nonlinear transformation or product of $\tilde{z}_{i,nT}$ and $\tilde{r}_{i,nT}$. In this sense, we will employ the CLT and LLN under near-epoch dependence, established by Jenish and Prucha (2012)[21]. We first show that $\tilde{z}_{i,nT}$ and $\tilde{r}_{i,nT}$ are L_2-near-epoch dependent (NED)[22]. Let $D \subset \mathbb{R}^d$ ($d \geq 1$) be a lattice of unevenly placed locations in $\mathbb{R}^d$. Assume that each unit i has its fixed location in $\mathbb{R}^d$ over time periods $t = 1, ..., T$. Define the location function $\mathbf{l} : i = \{1, ..., n\} \to D_n \subseteq D \subset \mathbb{R}^d$ by $\mathbf{l}(i) = (\mathbf{l}_1(i), ..., \mathbf{l}_d(i))$. Assume $|D_n| = n$ where $|A|$ denotes the cardinality of A. The distance between $\mathbf{l}(i)$ and $\mathbf{l}(j)$ is defined as $\rho\big(\mathbf{l}(i), \mathbf{l}(j)\big) = \max_{1 \leq k \leq d} \{|\mathbf{l}_k(i) - \mathbf{l}_k(j)|\}$[23].

[21] Jenish and Prucha (2012) extend the concept of near-epoch dependent (NED) processes used in the time series literature to spatial processes.

[22] An attractive feature of NED processes is that the NED property is preserved under transformations (Jenish and Prucha, 2012).

[23] We refer to Jeong and Lee (2021) for this setting.

In the remaining subsection, we use $i = \mathbf{l}(i)$ and $j = \mathbf{l}(j)$ as a location, and $\rho(i,j) = \rho\big(\mathbf{l}(i), \mathbf{l}(j)\big)$ as a distance for simplicity. Let $\xi = \{\varepsilon_{i1}, ..., \varepsilon_{iT}, i \in T_n, n \geq 1\}$ be a random field for all time periods ($\forall t = 1, ..., T$) where $D_n \subseteq T_n \subseteq D$. Consider the following σ-field as $\mathcal{F}_{i,nT}(s) = \sigma\big(\varepsilon_{j1}, ..., \varepsilon_{jT}; j \in T_n, \rho(i,j) \leq s\big)$ generated by the random variables located in the s-neighborhood of i. We need the following assumption to follow the approach of Jenish and Prucha (2012) for the increasing domain asymptotics.

Assumption 9. *The lattice $D \subset \mathbb{R}^d$ ($d \geq 1$) is infinitely countable. All elements in D are located at distances of at least $\rho_0 > 0$ from each other, i.e., $\rho(i,j) \geq \rho_0$ for all $i, j \in D_n$. Without loss of generality, we assume that $\rho_0 > 1$.*

Let $\tilde{Z} = \{\tilde{z}_{i,nT}, i \in D_n, n \geq 1\}$ be a random field with $||\tilde{z}_{i,nT}||_p < \infty$ ($p \geq 1$) and let $\tilde{d} = \{\tilde{d}_{i,n}, i \in D_n, n \geq 1\}$ be an array of finite positive constants[24]. Then, the random field $\tilde{Z}$ is said to be L_2-NED on $\xi = \{\varepsilon_{i1}, ..., \varepsilon_{iT}, i \in T_n, n \geq 1\}$ if $||\tilde{z}_{i,nT} - E(\tilde{z}_{i,nT}|\mathcal{F}_{i,nT}(s))||_2 \leq \tilde{d}_{i,n}\tilde{\gamma}(s)$ where $\tilde{\gamma}(s) \to 0$ as $s \to \infty$. If $\sup_n \sup_{i \in D_n} \tilde{d}_{i,n} < \infty$, then $\tilde{Z}$ is said to be uniformly L_2-NED on ξ. Similarly, define a random field for $\tilde{R} = \{\tilde{r}_{i,nT}, i \in D_n, n \geq 1\}$.

Lemma 2. *Under Assumptions 1 and 2,*
$\tilde{z}_{i,nT}$ and $\tilde{r}_{i,nT}$ are uniformly L_p bounded where $p = 4 + \eta$, i.e., $\sup_n \sup_{i \in D_n} ||\tilde{z}_{i,nT}||_{4+\eta} < \infty$ and $\sup_n \sup_{i \in D_n} ||\tilde{r}_{i,nT}||_{4+\eta} < \infty$.

Proposition 5. *Under Assumptions 1, 2, and 9,*
$\tilde{Z} = \{\tilde{z}_{i,nT}, i \in D_n, n \geq 1\}$ and $\tilde{R} = \{\tilde{r}_{i,nT}, i \in D_n, n \geq 1\}$ are uniformly L_2-NED on ξ with $\tilde{\gamma}(s) = \sup_n \sup_{i \in D_n} \sqrt{\frac{\sum_{j=1}^n w_{ij}^2 \mathbf{1}\big(\rho(i,j) > s\big)}{\sum_{j=1}^n w_{ij}^2}}$[25].

The NED property is preserved under summation and multiplication (Jenish and Prucha, 2012; Xu and Lee, 2015). In this sense, the next step is to consider $z_{i,nT} = (\tilde{z}_{i,nT})^2$, $r_{i,nT} = (\tilde{r}_{i,nT})^2$ and $\tilde{z}_{i,nT}\tilde{r}_{i,nT}$.

[24] For any random variable Y, let $||Y||_p = (E|Y|^p)^{1/p}$, $p \geq 1$.

[25] $\mathbf{1}\big(\rho(i,j) > s\big)$ is an indicator function where $\mathbf{1}\big(\rho(i,j) > s\big) = 0$ if the distance between i and j is equal to or less than s. As s gets larger, $\mathbf{1}\big(\rho(i,j) > s\big)$ goes to zero.

Let $Z = \{z_{i,nT}, i \in D_n, n \geq 1\}$ be a random field with $||z_{i,nT}||_p < \infty$ ($p \geq 1$) and $d = \{d_{i,n}, i \in D_n, n \geq 1\}$ be an array of finite positive constants. Then, the random field Z is said to be L_2-NED on ξ if $||z_{i,nT} - E(z_{i,nT}|\mathcal{F}_{i,nT}(s))||_2 \leq d_{i,n}\gamma(s)$ where $\gamma(s) \to 0$ as $s \to \infty$. If $\sup_n \sup_{i \in D_n} d_{i,n} < \infty$, then Z is said to be uniformly L_2-NED on ξ. Similarly, define random fields for $R = \{r_{i,nT}, i \in D_n, n \geq 1\}$ and $K = \{\tilde{z}_{i,nT}\tilde{r}_{i,nT}, i \in D_n, n \geq 1\}$.

Proposition 6. *Under Assumptions 1, 2, and 9,*
$Z = \{z_{i,nT}, i \in D_n, n \geq 1\}$, $R = \{r_{i,nT}, i \in D_n, n \geq 1\}$ *and* $K = \{\tilde{z}_{i,nT}\tilde{r}_{i,nT}, i \in D_n, n \geq 1\}$ *are uniformly* L_2*-NED on* ξ *with* $\gamma(s) = \sup_n \sup_{i \in D_n} \Big(\frac{\sum_{j=1}^n w_{ij}^2 \mathbf{1}(\rho(i,j) > s)}{\sum_{j=1}^n w_{ij}^2} \Big)^{\frac{\eta}{8+4\eta}}$.

We apply the CLT under near-epoch dependence to Q_{nT} and the LLN under near-epoch dependence to P_{nT} and U_{nT} in (30). Following Xu and Lee (2015), in addition to Assumption 1, we assume the following conditions.

Assumption 10. *The weights* w_{ij} *in* W_n *satisfy at least one of the following conditions:*
(1) Only individuals whose distances are less than or equal to some specific constant may affect each other directly. Without loss of generality, we set it as $\bar{\rho}_0 > 1$*. That is to say,* w_{ij} *can be nonzero only if* $\rho(i,j) \leq \bar{\rho}_0$*.*
(2) There exists an $\alpha > d \geq 1$ *and a constant* $C_0 > 0$ *such that* $|w_{ij}| \leq C_0/\rho(i,j)^\alpha$*.*

Assumption 11. $\alpha > d \cdot (1.5 + 2\eta^{-1})$

As discussed in Xu and Lee (2015), Assumption 10(1) is stronger than Assumption 10(2) in the sense that Assumption 10(2) allows an interaction even if two locations are far away from each other. In our case, it requires the strength to decline with $\rho(i,j)$ in the power of α determined by Assumption 11 to make the strength of spatial dependence decay sufficiently fast. If the elements of W_n are specified by a function of the spatial distance in some space, such as $w_{ij} = C_0/\rho(i,j)^\alpha$, we can have $w_{ij} > 0$ for all i, j.

Proposition 7. *Under Assumptions 1, 2, 7 and 9-11,*
$$Q_{nT} = \frac{1}{\sqrt{2n}} \sum_{i=1}^n \big(z_{i,nT} - E(z_{i,nT})\big) \xrightarrow{d} N(0,1).$$

Proposition 8. *Under Assumptions 1, 2, 7 and 9,*
$P_{nT} = \frac{1}{n}\sum_{i=1}^{n}\big(r_{i,nT} - E(r_{i,nT})\big) \xrightarrow{p} 0$ *and* $U_{nT} = \frac{1}{n}\sum_{i=1}^{n}\big(\tilde{z}_{i,nT}\tilde{r}_{i,nT} - E(\tilde{z}_{i,nT}\tilde{r}_{i,nT})\big) \xrightarrow{p} 0.$

Hence, as we analyze the statistics in (30), we can find the asymptotic distribution of the proposed S test in a special case under Assumption 8 in Theorem 3.

Theorem 3. *Under* H_0*, Assumptions 1, 2, 5, 7-11, and* $\frac{n}{T} \to k$ *where* $0 \leq k < \infty$*,*
$S = \frac{1}{\sqrt{2n}}\big(\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1}g_{nT}(\tilde{\sigma}^2) - n\big) \xrightarrow{d} N(0,1).$

Theorem 3 imposes a restriction on the relative expansion rates of n and T such that $\frac{n}{T} \to k$ where $0 \leq k < \infty$. Hence, in the case of the completely non-reciprocal interactions, the conclusions on whether a spatial correlation exists or not, when n is asymptotically proportional to T or when T grows faster than n, can be drawn based on the value of the proposed test statistic, $S^{nr} = \frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\big(\frac{(\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'s_i s_i' W_n \tilde{Y}_{nt})^2}{s_i' W_n W_n' s_i} - 1\big)$.

4.3 The limiting result of S in general interactions

Assumption 8 (non-reciprocal interactions or networks) is too strong in practice; we now allow the reciprocities ($w_{ij}w_{ji} \neq 0$) in a network as $\Sigma_{g,n}$ in (11). To analyze the asymptotic properties of S in general interactions, we need the analytical form of $\Sigma_{g,n}^{-1}$. The challenge here is to derive the inverse of an $n \times n$ matrix $\Sigma_{g,n}$ in which all off-diagonal entries are not necessarily zero with large n. We note that under Assumption 6, $\Sigma_{g,n}$ is a strictly diagonally dominant matrix uniformly in n; we can obtain $\Sigma_{g,n}^{-1}$ using its property.

$\Sigma_{g,n}$ can be expressed as $\Sigma_{g,n} = \Sigma_{g,n}^D - \Sigma_{g,n}^D B_n = \Sigma_{g,n}^D(I_n - B_n)$ where $\Sigma_{g,n}^D$ in (20) and

$$B_n = -\begin{pmatrix} 0 & \frac{s_1'W_n s_2 s_2' W_n s_1}{s_1' W_n W_n' s_1} & \frac{s_1'W_n s_3 s_3' W_n s_1}{s_1' W_n W_n' s_1} & \cdots & \frac{s_1'W_n s_n s_n' W_n s_1}{s_1' W_n W_n' s_1} \\ \frac{s_2'W_n s_1 s_1' W_n s_2}{s_2' W_n W_n' s_2} & 0 & \frac{s_2'W_n s_3 s_3' W_n s_2}{s_2' W_n W_n' s_2} & \cdots & \frac{s_2'W_n s_n s_n' W_n s_2}{s_2' W_n W_n' s_2} \\ \frac{s_3'W_n s_1 s_1' W_n s_3}{s_3' W_n W_n' s_3} & \frac{s_3'W_n s_2 s_2' W_n s_3}{s_3' W_n W_n' s_3} & 0 & \cdots & \frac{s_3'W_n s_n s_n' W_n s_3}{s_3' W_n W_n' s_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{s_n'W_n s_1 s_1' W_n s_n}{s_n' W_n W_n' s_n} & \frac{s_n'W_n s_2 s_2' W_n s_n}{s_n' W_n W_n' s_n} & \frac{s_n'W_n s_3 s_3' W_n s_n}{s_n' W_n W_n' s_n} & \cdots & 0 \end{pmatrix} \quad (31)$$

where $||B_n||_\infty < 1$[26]. Since the spectral radius of B_n, $\rho(B_n) \leq ||B_n||_\infty < 1$, we can derive the inverse of $\Sigma_{g,n} = \Sigma^D_{g,n}(I_n - B_n)$ as

$$\begin{aligned}\Sigma^{-1}_{g,n} = (I_n - B_n)^{-1}\Sigma^{D-1}_{g,n} &= (I_n + B_n + \sum_{k=2}^{\infty} B^k_n)\Sigma^{D-1}_{g,n} \\ &= \Sigma^{D-1}_{g,n} + B_n\Sigma^{D-1}_{g,n} + \sum_{k=2}^{\infty} B^k_n\Sigma^{D-1}_{g,n}\end{aligned} \tag{32}$$

where $\sum_{k=0}^{\infty} B^k_n < \infty$[27].

Finally, using the result of (32), we obtain the proposed S test:

$$\begin{aligned} S &= \frac{1}{\sqrt{2n}}\big(\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma^{-1}_{g,n}g_{nT}(\tilde{\sigma}^2) - n\big) \\ &= \frac{1}{\sqrt{2n}}\big(\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma^{D-1}_{g,n}g_{nT}(\tilde{\sigma}^2) - n\big) + \frac{1}{\sqrt{2n}}\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'B_n\Sigma^{D-1}_{g,n}g_{nT}(\tilde{\sigma}^2) \\ &\quad + \sum_{k=2}^{\infty}\frac{1}{\sqrt{2n}}\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'B^k_n\Sigma^{D-1}_{g,n}g_{nT}(\tilde{\sigma}^2)\end{aligned} \tag{33}$$

where the first term $\frac{1}{\sqrt{2n}}\big(\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma^{D-1}_{g,n}g_{nT}(\tilde{\sigma}^2) - n\big)$ is the same as S^{nr} in (21). Therefore, Theorem 3 is directly applicable to the first term. If all other remaining terms converge to zero, we can construct a standard normal test.

Proposition 9. *Under Assumptions 6 and 7,*
If $\frac{1}{\sqrt{2n}}\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'B_n\Sigma^{D-1}_{g,n}g_{nT}(\tilde{\sigma}^2)$ *is* $o_p(1)$, *then* $\sum_{k=2}^{\infty}\frac{1}{\sqrt{2n}}\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'B^k_n\Sigma^{D-1}_{g,n}g_{nT}(\tilde{\sigma}^2)$ *is* $o_p(1)$.

Proposition 9 implies that it suffices to show $\frac{1}{\sqrt{2n}}\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'B_n\Sigma^{D-1}_{g,n}g_{nT}(\tilde{\sigma}^2) \xrightarrow{p} 0$ for $\sum_{k=1}^{\infty}\frac{1}{\sqrt{2n}}\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'B^k_n\Sigma^{D-1}_{g,n}g_{nT}(\tilde{\sigma}^2) \xrightarrow{p} 0$.

[26]The ith row sum of B_n is $\frac{\sum_{j\neq i}^n |s_i'W_ns_js_j'W_ns_i|}{s_i'W_nW_n's_i} = \frac{\sum_{j=1}^n |w_{ij}w_{ji}|}{\sum_{j=1}^n w_{ij}^2} < 1$ for all i and n by Assumptions 1 and 6.

[27]$\lim_{k\to\infty}(I_n - (I_n - B_n))^k = \lim_{k\to\infty} B^k_n = 0$ due to $\rho(B_n) < 1$.

Consider $\frac{1}{\sqrt{2n}}\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'B_n\Sigma_{g,n}^{D-1}g_{nT}(\tilde{\sigma}^2)$ where

$$B_n\Sigma_{g,n}^{D-1} = -\begin{pmatrix} 0 & * & * & \dots & * \\ \frac{s_2'W_ns_1s_1'W_ns_2}{s_2'W_nW_n's_2s_1'W_nW_n's_1} & 0 & * & \dots & * \\ \frac{s_3'W_ns_1s_1'W_ns_3}{s_3'W_nW_n's_3s_1'W_nW_n's_1} & \frac{s_3'W_ns_2s_2'W_ns_3}{s_3'W_nW_n's_3s_2'W_nW_n's_2} & 0 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{s_n'W_ns_1s_1'W_ns_n}{s_n'W_nW_n's_ns_1'W_nW_n's_1} & \frac{s_n'W_ns_2s_2'W_ns_n}{s_n'W_nW_n's_ns_2'W_nW_n's_2} & \frac{s_n'W_ns_3s_3'W_ns_n}{s_n'W_nW_n's_ns_3'W_nW_n's_3} & \dots & 0 \end{pmatrix} \tag{34}$$

Using $\frac{\frac{1}{\sqrt{T}}g_{nT,i}(\tilde{\sigma}^2)}{\sqrt{s_i'W_nW_n's_i}} = \frac{\sigma_0^2}{\tilde{\sigma}^2}(\tilde{z}_{i,nT} - \frac{1}{\sqrt{T}}\tilde{r}_{i,nT})$ with the symmetric property of $B_n\Sigma_{g,n}^{D-1}$, we have

$$\begin{aligned} &\frac{1}{\sqrt{2n}}\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'B_n\Sigma_{g,n}^{D-1}g_{nT}(\tilde{\sigma}^2) = -\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\frac{\frac{1}{\sqrt{T}}g_{nT,i}(\tilde{\sigma}^2)}{\sqrt{s_i'W_nW_n's_i}}\sum_{j=1}^{n}\frac{s_i'W_ns_js_j'W_ns_i}{\sqrt{s_i'W_nW_n's_i}\sqrt{s_j'W_nW_n's_j}}\frac{\frac{1}{\sqrt{T}}g_{nT,j}(\tilde{\sigma}^2)}{\sqrt{s_j'W_nW_n's_j}} \\ &= -(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}(\tilde{z}_{i,nT} - \frac{1}{\sqrt{T}}\tilde{r}_{i,nT})\sum_{j\neq i}^{n}\frac{s_i'W_ns_js_j'W_ns_i}{\sqrt{s_i'W_nW_n's_i}\sqrt{s_j'W_nW_n's_j}}(\tilde{z}_{j,nT} - \frac{1}{\sqrt{T}}\tilde{r}_{j,nT}) \\ &= -\frac{1}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\tilde{z}_{i,nT}\sum_{j\neq i}^{n}\frac{s_i'W_ns_js_j'W_ns_i}{\sqrt{s_i'W_nW_n's_i}\sqrt{s_j'W_nW_n's_j}}\tilde{z}_{j,nT} \\ &\quad + \frac{2}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2\sqrt{\frac{n}{T}}\frac{1}{n}\sum_{i=1}^{n}\tilde{r}_{i,nT}\sum_{j\neq i}^{n}\frac{s_i'W_ns_js_j'W_ns_i}{\sqrt{s_i'W_nW_n's_i}\sqrt{s_j'W_nW_n's_j}}\tilde{z}_{j,nT} \\ &\quad - \frac{1}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2\frac{\sqrt{n}}{T}\frac{1}{n}\sum_{i=1}^{n}\tilde{r}_{i,nT}\sum_{j\neq i}^{n}\frac{s_i'W_ns_js_j'W_ns_i}{\sqrt{s_i'W_nW_n's_i}\sqrt{s_j'W_nW_n's_j}}\tilde{r}_{j,nT} \end{aligned} \tag{35}$$

Proposition 10. *Under Assumptions 1, 2, 6 and 7,*
If $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\tilde{z}_{i,nT}\sum_{j\neq i}^{n}\frac{s_i'W_ns_js_j'W_ns_i}{\sqrt{s_i'W_nW_n's_i}\sqrt{s_j'W_nW_n's_j}}\tilde{z}_{j,nT}$ *is* $o_p(1)$*, then* $\frac{1}{\sqrt{2n}}\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'B_n\Sigma_{g,n}^{D-1}g_{nT}(\tilde{\sigma}^2)$ *is* $o_p(1)$*.*

Propositions 9 and 10 imply that it suffices to show $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\tilde{z}_{i,nT}\sum_{j\neq i}^{n}\frac{s_i'W_ns_js_j'W_ns_i}{\sqrt{s_i'W_nW_n's_i}\sqrt{s_j'W_nW_n's_j}}\tilde{z}_{j,nT}$ $\xrightarrow{p} 0$ for the convergence of $\sum_{k=1}^{\infty}\frac{1}{\sqrt{2n}}\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'B_n^k\Sigma_{g,n}^{D-1}g_{nT}(\tilde{\sigma}^2)$ in (33).

Define the random variables $g_{i,nT}$ over $i = 1, ..., n$ as

$$g_{i,nT} \equiv \tilde{z}_{i,nT} \sum_{j \neq i}^{n} \frac{s_i' W_n s_j s_j' W_n s_i}{\sqrt{s_i' W_n W_n' s_i} \sqrt{s_j' W_n W_n' s_j}} \tilde{z}_{j,nT} = \tilde{z}_{i,nT} \sum_{j \neq i}^{n} \frac{w_{ij} w_{ji}}{\sqrt{\sum_{j=1}^{n} w_{ij}^2} \sqrt{\sum_{i=1}^{n} w_{ji}^2}} \tilde{z}_{j,nT} \quad (36)$$

Denote $G_{nT} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_{i,nT}$. We note that the convergence of G_{nT} depends on $\frac{w_{ij}}{\sqrt{\sum_{j=1}^{n} w_{ij}^2}}$.

For some social interactions or economic activities, each unit can be influenced by a significant portion of units, $\sum_{j=1} |w_{ij}|$ (Lee, 2002). In these cases, the weights matrices are row-normalized, and the elements $\bar{w}_{ij} = \frac{w_{ij}}{\sum_{j=1}^{n} |w_{ij}|}$ can depend on n as $\bar{w}_{ij} = O(\frac{1}{h_n})$ where h_n is divergent uniformly in all i, j (Lee, 2004). Similarly, we now introduce the conditions for $\frac{w_{ij}}{\sqrt{\sum_{j=1}^{n} w_{ij}^2}}$; each unit is influenced by a portion of units, measured by $\sqrt{\sum_{j=1}^{n} w_{ij}^2}$. Consider the following interactions or networks in the form of Assumptions 12 and 13:

Assumption 12. *The elements* $w_{ij}^* = \frac{w_{ij}}{\sqrt{\sum_{j=1}^{n} w_{ij}^2}}$ *are at most of order* h_n^{*-1}, $w_{ij}^* = O(\frac{1}{h_n^*})$ *where the rate sequence* h_n^* *is divergent, uniformly in all* i, j.

Assumption 13. $\lim_{n\to\infty} \frac{h_n^*}{n^{1/4}} = \infty$ *and* $\lim_{n\to\infty} \frac{h_n^*}{n^{1/2}} = 0$.

Assumptions 12 and 13 imply that the social interactions or networks are asymptotically small reciprocal in the sense that $\frac{w_{ij} w_{ji}}{\sqrt{\sum_{j=1}^{n} w_{ij}^2} \sqrt{\sum_{i=1}^{n} w_{ji}^2}} = O(\frac{1}{h_n^{*2}})$ which goes to zero. The completely non-reciprocal interaction in Assumption 8, $\frac{w_{ij} w_{ji}}{\sqrt{\sum_{j=1}^{n} w_{ij}^2} \sqrt{\sum_{i=1}^{n} w_{ji}^2}} = 0$, is the special case of Assumptions 12 and 13. The social interactions or networks in the form of Assumptions 12 and 13 can cover much more empirical cases. Also, Assumptions 12 and 13 can include empirical examples where W_n is row-normalized but exclude the case where all units are neighbors of each other (equal weight, $w_{ij} = 1/(n-1)$ for all i, j).

Remark 1. Using w^*_{ij} in Assumption 12, we can rewrite $\tilde{z}_{i,nT}$ and $\tilde{r}_{i,nT}$ as

$$\tilde{z}_{i,nT} = \frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\sum_{j=1}^{n}w_{ij}\varepsilon_{jt}}{\sqrt{\sum_{j=1}^{n}w_{ij}^2}} = \frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\sum_{j=1}^{n}w^*_{ij}\varepsilon_{jt} \tag{37}$$

$$\tilde{r}_{i,nT} = \frac{\frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^{n}w_{ij}\bar{\epsilon}_{jT}}{\sqrt{\sum_{j=1}^{n}w_{ij}^2}} = \frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^{n}w^*_{ij}\bar{\epsilon}_{jT} \tag{38}$$

For both $||\tilde{z}_{i,nT} - E(\tilde{z}_{i,nT}|\mathcal{F}_{i,nT}(s))||_2$ and $||\tilde{r}_{i,nT} - E(\tilde{r}_{i,nT}|\mathcal{F}_{i,nT}(s))||_2$, we can take the same $\sup_n \sup_{i\in D_n} \tilde{d}_{i,n} = 1$ and $\tilde{\gamma}(s) = \sup_n \sup_{i\in D_n} \sqrt{\sum_{j=1}^{n} w^{*2}_{ij}\mathbf{1}\big(\rho(i,j) > s\big)}$ where $\sum_{j=1}^{n} w^{*2}_{ij} = 1$. Also, $\Sigma_{Q_{nT}} \to 1$ as long as $\frac{n}{T} \to k < \infty$. Hence, all results in Section 4.2 hold.

In the remaining subsection, we use Assumptions 12 and 13 instead of Assumptions 8. We show the convergence of G_{nT} using Chebyshev's inequality in Proposition 11.

Proposition 11. *Under Assumptions 1, 2, 5-7, 9, 12 and 13,*
$G_{nT} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} g_{i,nT} \xrightarrow{p} 0.$

Hence, as we analyze the statistics in (33), we can find the asymptotic distribution of the proposed S test in more general interaction under Assumptions 12 and 13 in Theorem 4.

Theorem 4. *Under H_0, Assumptions 1, 2, 5-7 and 9-13, and $\frac{n}{T} \to k$ where $0 \le k < \infty$,*
$S = \frac{1}{\sqrt{2n}}\big(\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1}g_{nT}(\tilde{\sigma}^2) - n\big) \xrightarrow{d} N(0,1).$

Theorem 4 imposes a restriction on the relative expansion rates of n and T such that $\frac{n}{T} \to k$ where $0 \le k < \infty$. Also, Theorem 4 implies that under Assumptions 12 and 13, S is asymptotically equivalent to S^{nr}, $|S - S^{nr}| = o_p(1)$. Hence, in the case of small reciprocal interactions, the conclusions on whether a spatial correlation exists or not, when n is asymptotically proportional to T or when T grows faster than n, can be drawn based on the value of the proposed test statistic, $S = \frac{1}{\sqrt{2n}}\big(\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1}g_{nT}(\tilde{\sigma}^2) - n\big)$ where $g_{nT}(\tilde{\sigma}^2) = (\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}'_{nt}s_1s_1'W_n\tilde{Y}_{nt}, ..., \frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}'_{nt}s_ns_n'W_n\tilde{Y}_{nt})'$, or the asymptotically equivalent form S^{nr}.

4.4 Local power of the proposed S test

For the asymptotic local power of the S test, we adopt the following local alternatives:

$$H_{1,nT} : \delta_{i0} = \frac{\Delta_i}{n^{1/4}T^{1/2}} \qquad \text{for } i = 1, ..., n \tag{39}$$

where Δ_i is a fixed constant ($\Delta_i \neq 0$). Denote $\Delta^D = diag(\Delta_1, ..., \Delta_n)$ and $\Psi^{H_1} = \frac{1}{n^{1/4}T^{1/2}}\Delta^D$. Under Assumptions 12 and 13, S^{nr} is asymptotically equivalent to S in Theorem 4. Thus, we analyze the asymptotic result of S^{nr} under the local alternatives. Under $H_{1,nT}$, S^{nr} takes the following form:

$$S^{nr} = \frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\Big(\frac{(\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{V}_{nt}'(I_n - \Psi^{H_1}W_n)^{-1\prime}s_is_i'W_n(I_n - \Psi^{H_1}W_n)^{-1}\tilde{V}_{nt})^2}{s_i'W_nW_n's_i} - 1\Big) \tag{40}$$

where $(I_n - \Psi^{H_1}W_n)^{-1} = I_n + \sum_{k=1}^{\infty}(\frac{1}{n^{1/4}T^{1/2}}\Delta^D W_n)^k < \infty$ for large enough n by Assumption 3. This can be rewritten as

$$\begin{aligned}
S^{nr} &= \frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\Big(\frac{(\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{V}_{nt}'(I_n + \sum_{k=1}^{\infty}(\frac{1}{n^{1/4}T^{1/2}}\Delta^D W_n)^k)'s_is_i'W_n(I_n + \sum_{k=1}^{\infty}(\frac{1}{n^{1/4}T^{1/2}}\Delta^D W_n)^k)\tilde{V}_{nt})^2}{s_i'W_nW_n's_i} - 1\Big) \\
&= \frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\Big(\frac{(\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{V}_{nt}'(I_n + \frac{1}{n^{1/4}T^{1/2}}\Delta^D W_n)'s_is_i'W_n(I_n + \frac{1}{n^{1/4}T^{1/2}}\Delta^D W_n)\tilde{V}_{nt})^2}{s_i'W_nW_n's_i} - 1\Big) + o_p(1) \\
&= \frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\Big(\frac{(\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{V}_{nt}'s_is_i'W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i} - 1\Big) + \frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\frac{(\frac{1}{n^{1/4}T^{1/2}}\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{V}_{nt}'W_n'\Delta^D s_is_i'W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i} \\
&\quad + \frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\frac{(\frac{1}{n^{1/4}T^{1/2}}\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{V}_{nt}'s_is_i'W_n\Delta^D W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i} + o_p(1) \\
&= \frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\Big(\frac{(\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{V}_{nt}'s_is_i'W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i} - 1\Big) + \frac{1}{\sqrt{2}}\frac{1}{n}\sum_{i=1}^{n}\frac{(\frac{1}{\tilde{\sigma}^2}\frac{1}{T}\sum_{t=1}^{T}\tilde{V}_{nt}'W_n'\Delta^D s_is_i'W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i} \\
&\quad + \frac{1}{\sqrt{2}}\frac{1}{n}\sum_{i=1}^{n}\frac{(\frac{1}{\tilde{\sigma}^2}\frac{1}{T}\sum_{t=1}^{T}\tilde{V}_{nt}'s_is_i'W_n\Delta^D W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i} + o_p(1)
\end{aligned} \tag{41}$$

since $|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{V}_{nt}'(\Delta^D W_n)^{k\prime}s_is_i'W_n(\Delta^D W_n)^k\tilde{V}_{nt}|$ is $O_p(\sqrt{T})$ by Lemmas A.3 and A.4 for any finite k. We show that $\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\big(\frac{(\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{V}_{nt}'s_is_i'W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i}-1\big)\xrightarrow{d}N(0,1)$ in Theorem 3. Therefore, the asymptotic power depends on the limit of

$$\Phi_{nT}=\frac{1}{n}\sum_{i=1}^{n}\frac{(\frac{1}{\tilde{\sigma}^2}\frac{1}{T}\sum_{t=1}^{T}\tilde{V}_{nt}'W_n'\Delta^D s_is_i'W_n\tilde{V}_{nt})^2+(\frac{1}{\tilde{\sigma}^2}\frac{1}{T}\sum_{t=1}^{T}\tilde{V}_{nt}'s_is_i'W_n\Delta^D W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i} \tag{42}$$

Denote $\Phi=\lim_{n\to\infty}\Phi_{nT}$.

Theorem 5. *Under $H_{1,nT}$, Assumptions 1-3 and 5-13, and $\frac{n}{T}\to k$ where $0\le k<\infty$, $S=\frac{1}{\sqrt{2n}}\big(\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1}g_{nT}(\tilde{\sigma}^2)-n\big)\xrightarrow{d}N(\frac{\Phi}{\sqrt{2}},1)$.*

Theorem 5 implies that the S test has power against local alternatives if $\Delta_i\neq 0$ for a non-zero fraction of units in the limit due to $\Phi=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta_i^2(\sum_{j=1}^{n}w_{ij}^2)^2+(\sum_{j=1}^{n}\Delta_jw_{ij}w_{ji})^2}{\sum_{j=1}^{n}w_{ij}^2}>0$. Under Assumption 8, $\Phi=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\Delta_i^2\sum_{j=1}^{n}w_{ij}^2>0$ because $w_{ij}w_{ji}=0$ for all i,j.

5 Properties of the S test

5.1 Power properties of S by comparison with M

As discussed in the previous sections, the proposed S test is based on $\frac{\partial lnL_{nT}(0,...,0,\tilde{\sigma}^2)}{\partial\delta_i}=\frac{1}{\tilde{\sigma}^2}\sum_{t=1}^{T}\tilde{Y}_{nt}'s_is_i'W_n\tilde{Y}_{nt}$ for all i; we construct the $n\times 1$ vector as

$$\frac{1}{\sqrt{T}}g_{nT}(\tilde{\sigma}^2)=\frac{1}{\sqrt{T}}\frac{\partial lnL_{nT}(0,...,0,\tilde{\sigma}^2)}{\partial\psi}=\left(\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'s_1s_1'W_n\tilde{Y}_{nt},\dots,\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'s_ns_n'W_n\tilde{Y}_{nt}\right)' \tag{43}$$

and each element measures the distance away from zero at the points where the function is maximized subject to the restriction. If the restriction is negligible (under H_0, $\tilde{Y}_{nt}=\tilde{V}_{nt}$), the values of the distance should not differ from zero by more than errors. Using the asymptotic variance and a standardized formulation of the LM test, we propose the S test as

$$\begin{aligned} S &= \frac{1}{\sqrt{2n}}\Big(\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1}g_{nT}(\tilde{\sigma}^2) - n\Big) \\ &= \frac{1}{\sqrt{2n}}\Bigg(\begin{pmatrix} \frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'s_1s_1'W_n\tilde{Y}_{nt} \\ \frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'s_2s_2'W_n\tilde{Y}_{nt} \\ \vdots \\ \frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'s_ns_n'W_n\tilde{Y}_{nt} \end{pmatrix}' \Sigma_{g,n}^{-1} \begin{pmatrix} \frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'s_1s_1'W_n\tilde{Y}_{nt} \\ \frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'s_2s_2'W_n\tilde{Y}_{nt} \\ \vdots \\ \frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'s_ns_n'W_n\tilde{Y}_{nt} \end{pmatrix} - n\Bigg) \end{aligned} \tag{17}$$

where $\Sigma_{g,n}$ is a normalization factor. We note that S adds up squared values of each distance $(\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'s_is_i'W_n\tilde{Y}_{nt})^2 \geq 0$ for all i due to the quadratic structure.

We now consider the summation of the entries of the vector in (43) over i as

$$\sum_{i=1}^{n}\frac{1}{\sqrt{T}}g_{nT,i}(\tilde{\sigma}^2) = \sum_{i=1}^{n}\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'s_is_i'W_n\tilde{Y}_{nt} = \frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'W_n\tilde{Y}_{nt} \tag{44}$$

It is noteworthy that $\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'W_n\tilde{Y}_{nt}$ is the numerator of $M = \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'W_n\tilde{Y}_{nt}}{\sqrt{tr(W_n'W_n + W_n^2)}}$ where $\sqrt{tr(W_n'W_n + W_n^2)}$ is a normalization factor. It turns out that the traditional M test adds up the values of each distance, which can be positive or negative. Hence, the power of M may be low when they cancel each other out, even if the restriction is not negligible.

As shown in Theorems 5 and D.2 under Assumption 8, the power of S depends on $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\Delta_i^2\sum_{j=1}^{n}w_{ij}^2$, while the power of M depends on $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\Delta_i\sum_{j=1}^{n}w_{ij}^2$. If the sign of Δ_i is different across i, the power of the traditional M test may decrease in general, or vanish under certain circumstances. On the contrary, even in that case, the power of the proposed S test remains as long as $\Delta_i \neq 0$ for a non-zero fraction of units in the limit. In sum, the low power of M can happen when spatial lag coefficients are heterogeneous in nature and can be more severe when the sample size is small. This analysis implies that the traditional M test does not behave consistently across all potential alternative hypotheses.

5.2 Finite sample properties of S by Monte Carlo experiments

5.2.1 Design

To investigate the performance of the proposed D test, we conduct Monte Carlo experiments. We consider the following Data Generating Process (DGP 1), defined by (3).

DGP 1 (The SAR panel data model with fully heterogeneous spatial lag coefficients)**.**

$$Y_{nt} = (I_n - \Psi_0 W_n)^{-1}(\mathbf{c}_{n0} + V_{nt}) \tag{3}$$

where $\Psi_0 = diag(\delta_{10}, ..., \delta_{n0})$*,* W_n *is an* $n \times n$ *spatial weights matrix,* $\mathbf{c}_{n0}$ *is an* $n \times 1$ *vector of individual fixed effects, and* V_{nt} *is an* $n \times 1$ *vector of i.i.d. disturbances with zero mean and finite variance* σ_0^2*.*

In these experiments, we consider the weights matrix W_n by using the distance based measure such as $w_{ij} = \dfrac{1}{|i-j+1|^4}$ for all $i \neq j$ and $w_{ii} = 0$ for all i, and row-normalize the matrix. We set $c_{i0} = 0.1 \cdot U[0,1]$ for the fixed effects. For the test power analysis, we consider two scenarios: (1) all positive spatial effects with chi-square distributed heterogeneity, $\Psi_0 = diag(\delta_{10}, ..., \delta_{n0})$ where $\delta_{i0} = \lambda + \delta_h \cdot (\chi_1^2 - 1)/\sqrt{2}$ where $\lambda = 0.1$ and $\delta_h = 0.14$ (2) mixed signs of spatial effects with normally distributed heterogeneity, $\Psi_0 = diag(\delta_{10}, ..., \delta_{n0})$ where $\delta_{i0} = \lambda + \delta_h \cdot N(0,1)$ where $\lambda = 0.05$ and $\delta_h = 0.14$[28]. Figure (1) illustrates the spatial lag coefficients of both alternatives in the case of $n = 75$ used in the simulation.

Finally, we consider three distributions of the disturbances: normal ($N(0,1)$), uniform ($U[-\sqrt{3}, \sqrt{3}]$) and chi-square ($(\chi_5^2 - 5)/\sqrt{10}$) distributions with zero mean and $\sigma_0^2 = 1$. We use $1,000$ replications for the size and power with Ψ_0, W_n and $\mathbf{c}_{n0}$ fixed, and then redraw randomly V_{nt} in each replication.

[28] λ implies the magnitude of spatial dependence and δ_h implies the magnitude of heterogeneity across units. When $\delta_h = 0$, the spatial coefficients become homogenous.

(a) Scenario (1) (b) Scenario (2)

Figure 1: Heterogeneous spatial lag coefficients in DGP 1 ($n = 75$)

5.2.2 Results

We report the size of the S test for all $n \in \{25, 50, 75\}$ and $T \in \{25, 50, 75\}$ combinations in Table 1. We show size properties using $N(0, 1)$, $U[-\sqrt{3}, \sqrt{3}]$, and $(\chi_5^2 - 5)/\sqrt{10}$ disturbances at the 5% significance level, respectively. All cases reject the null hypothesis (H_0) at higher rates than the theoretical value 5%, regardless of the forms of disturbances. When T grows faster than n or n is asymptotically proportional to T, the size goes around the theoretical value (0.050), as discussed in Section 4.2 and 4.3. On the other hand, as n becomes notably larger than T, the size distortion appears.

Table 1: Size of the proposed S test

		$T = 25$	$T = 50$	$T = 75$
$\varepsilon_{it} \sim N(0, 1)$	$n = 25$	0.079	0.070	0.074
	$n = 50$	0.097	0.074	0.064
	$n = 75$	0.092	0.071	0.064
$\varepsilon_{it} \sim U[-\sqrt{3}, \sqrt{3}]$	$n = 25$	0.078	0.067	0.062
	$n = 50$	0.074	0.066	0.063
	$n = 75$	0.081	0.077	0.078
$\varepsilon_{it} \sim (\chi_5^2 - 5)/\sqrt{10}$	$n = 25$	0.107	0.087	0.074
	$n = 50$	0.111	0.084	0.075
	$n = 75$	0.125	0.068	0.063

Note: 1,000 Monte Carlo replications

In the power analysis, we report the power of both the proposed S test and the traditional M test for all $n \in \{25, 50, 75\}$ and $T \in \{25, 50, 75\}$ combinations. Table 2 shows the power of the S test for both Scenario (1) and (2). As predicted in Section 4.4, the test has good power properties for all combinations. The power of Scenario (2) is lower than that of (1) because the overall spatial dependence is weaker ($\lambda = 0.05$ in the case of Scenario (2)). The power reaches around the theoretical value (1.000) as the sample size gets large. Also, the power is robust to the shapes of the disturbances.

Table 3 reports the power of the M test with the same alternatives (Ψ_0). By comparing Table 2 with Table 3, we can observe how power changes if we use the traditional test when the spatial processes are heterogeneous in nature. Even when all spatial lag coefficients are positive, such as in Scenario (1), the power of M can be lower than that of S, as expected in Section 5.1. This gap is more apparent when the sample size is small. In particular, power may be reduced if spatial effects have different signs across units when n increases, as shown in Scenario (2), even though there are spatial correlations in a network. These results hold regardless of the shape of error terms. Overall, the proposed S test has satisfactory finite sample properties and better power over the traditional one in these types of networks when the sample size is small.

Table 2: Power of the proposed S test

		(1)				(2)		
		$T=25$	$T=50$	$T=75$		$T=25$	$T=50$	$T=75$
$\varepsilon_{it} \sim N(0,1)$	$n=25$	0.626	0.909	0.985	$n=25$	0.373	0.655	0.850
	$n=50$	0.672	0.937	0.997	$n=50$	0.644	0.935	0.998
	$n=75$	0.786	0.985	1.000	$n=75$	0.726	0.963	0.998
$\varepsilon_{it} \sim U[-\sqrt{3}, \sqrt{3}]$	$n=25$	0.610	0.918	0.990	$n=25$	0.363	0.629	0.851
	$n=50$	0.638	0.944	0.998	$n=50$	0.645	0.942	0.998
	$n=75$	0.807	0.990	1.000	$n=75$	0.695	0.969	1.000
$\varepsilon_{it} \sim (\chi^2_5 - 5)/\sqrt{10}$	$n=25$	0.609	0.883	0.976	$n=25$	0.415	0.647	0.857
	$n=50$	0.626	0.914	0.990	$n=50$	0.655	0.932	0.999
	$n=75$	0.796	0.979	1.000	$n=75$	0.727	0.963	0.999

Note: (1) $\Psi_0 = diag(\delta_{10}, ..., \delta_{n0})$ where $\delta_{i0} = \lambda + \delta_h \cdot (\chi^2_1 - 1)/\sqrt{2}$ with $\lambda = 0.1$ and $\delta_h = 0.14$
(2) $\Psi_0 = diag(\delta_{10}, ..., \delta_{n0})$ where $\delta_{i0} = \lambda + \delta_h \cdot N(0,1)$ with $\lambda = 0.05$ and $\delta_h = 0.14$

Table 3: Power of the traditional M test

		(1)				(2)		
		$T=25$	$T=50$	$T=75$		$T=25$	$T=50$	$T=75$
$\varepsilon_{it} \sim N(0,1)$	$n=25$	0.472	0.733	0.896	$n=25$	0.187	0.324	0.449
	$n=50$	0.601	0.883	0.972	$n=50$	0.374	0.663	0.794
	$n=75$	0.804	0.971	0.996	$n=75$	0.317	0.534	0.695
$\varepsilon_{it} \sim U[-\sqrt{3},\sqrt{3}]$	$n=25$	0.485	0.726	0.891	$n=25$	0.174	0.309	0.427
	$n=50$	0.599	0.883	0.977	$n=50$	0.376	0.632	0.812
	$n=75$	0.796	0.972	0.999	$n=75$	0.294	0.519	0.700
$\varepsilon_{it} \sim (\chi^2_5-5)/\sqrt{10}$	$n=25$	0.472	0.755	0.896	$n=25$	0.200	0.318	0.456
	$n=50$	0.595	0.867	0.972	$n=50$	0.370	0.631	0.799
	$n=75$	0.787	0.969	0.998	$n=75$	0.335	0.557	0.712

Note: (1) $\Psi_0 = diag(\delta_{10},...,\delta_{n0})$ where $\delta_{i0} = \lambda + \delta_h \cdot (\chi^2_1 - 1)/\sqrt{2}$ with $\lambda = 0.1$ and $\delta_h = 0.14$
(2) $\Psi_0 = diag(\delta_{10},...,\delta_{n0})$ where $\delta_{i0} = \lambda + \delta_h \cdot N(0,1)$ with $\lambda = 0.05$ and $\delta_h = 0.14$

6 An empirical illustration

6.1 Motivation

To illustrate the practicality of the proposed S test, we present a simple empirical application in the international knowledge spillover. Research on international knowledge spillovers has made progress, especially since the seminal contribution of Coe and Helpman (1995)[29]. They consider the following specification for innovation-driven growth:

$$log(F_i) = \alpha_i^0 + \alpha_i^d log(S_i^d) + \alpha_i^f log(S_i^f) \tag{45}$$

where i is a country index, F_i is the total factor productivity (TFP), S_i^d represents the domestic $R\&D$ capital stock and S_i^f represents the foreign $R\&D$ capital stock defined as the import-share-weighted average of $R\&D$ capital stock of its trade partners. They show that a country's TFP depends not only on domestic R&D capital but also on foreign R&D capital and the foreign side is more substantial as an economy is more open to trade[30].

[29]For the survey of the early contribution on the knowledge/R&D spillovers, see Coe et al. (2009).

[30]While it is still under discussion whether the knowledge is transmitted through trade or FDI, Keller (2022) points out that over recent decades a number of advances have produced robust evidence that both trade and FDI lead to sizable knowledge spillovers.

In addition to the breakthrough in terms of theory, our understanding of knowledge spillovers has been improved through a combination of advances in econometric methodology, new sources of data, and appropriate empirical work (Keller, 2022). In particular, spatial econometric models may effectively investigate knowledge spillovers and interactions using technological or economic proximity, as surveyed in Autant-Bernard (2012). For the analysis of country-level spillover and transmission using spatial models, bilateral trade data is widely used to construct the weights matrices based on the theory of the trade channel[31]. In the spatial econometric framework, the commonly used approach to test spatial correlation is to formulate a hypothesis on a homogeneous spatial lag coefficient. However, before going into estimation and inference for panel data, one might be interested in testing whether the spatial dependence in knowledge production exists or not in the heterogeneous setting as (45). We consider the following model using the spatial autoregressive term with heterogeneous coefficients:

$$y_{it} = \delta_{i0} \sum_{j=1}^{n} w_{ij} y_{jt} + c_{i0} + \varepsilon_{it}, \qquad i = 1, ..., n, \quad t = 1, ..., T \tag{46}$$

where y_{it} is an innovation output, w_{ij} is the weight specified by bilateral import flows, c_{i0} is a country-specific fixed effect and ε_{it} is an error term. The hypothesis formulated in (46) is $H_0 : \delta_{i0} = 0$ for all $i = 1, ..., n$ against $H_1 : \delta_{i0} \neq 0$ for a non-zero fraction of units.

To calculate the test statistic, one can use a patent indicator as a proxy for innovation or knowledge production (output) following the existing literature[32]. However, the country-level patent data is publicly available on an annual basis. In hypothesis testing, n should be notably smaller than the total time periods T in order to apply the existing testing procedure (e.g., the LM test for seven innovative countries). Therefore, we employ the proposed S test since it is valid for large n when testing the hypothesis formulated in (46). One may use the traditional M test to conclude whether knowledge spillovers exist using the same data. We will compare the results between our proposed S and traditional M tests.

[31] See Ho et al. (2013)., Ho et al. (2018), and Elhorst et al. (2021), among others.

[32] See Bottazzi and Peri (2007), Mancusi (2008), Ho et al. (2018), Drivas et al. (2022), Eugster et al. (2022), and among others.

6.2 Data

We use a balanced panel of 27 innovative countries over the period of 1985-2021[33]. The innovation output y_{it} is the annual growth rate of triadic patent applications (Δlog patent applications)[34]. We use the triadic patent families (OECD MSTI), following Drivas et al. (2022)[35]. The most widely used patent indicators refer to the counts of patent applications to a single patent office[36]. While the richness and strength of those indicators are broadly recognized, they are affected by home advantage bias, and the quality and international comparability of indicators based on the patent families are improved by reducing the weaknesses associated with indicators from a single patent office (Dernis and Khan, 2004)[37]. Thus, this triadic patent indicator allows us to compare knowledge production across countries better. Finally, the average of bilateral import flows over the period of 1998-2016 (IMF Direction of Trade Statistics, DOTS) is used to construct the weights matrix.

6.3 Results

Table 4 reports the results of the proposed S and traditional M tests. The S test provides strong evidence against the hypothesis of no spatial correlation; the null hypothesis is rejected at the 1% significance level. However, the same hypothesis is not rejected at the 10% level when the M test is employed. This contrast implies that the traditional test may draw an erroneous conclusion on spatial correlation and the traditional testing procedures should be reconsidered, especially when the spatial processes are heterogeneous in nature, as discussed in Section 5.1 and shown by simulations in Section 5.2.

[33]See Appendix G for the list of sample countries. These countries account for 96% of the world's innovation activity in 2021.

[34]The number of total observations is 972 ($n = 27$ and $T = 36$).

[35]Triadic patent families are a set of patents filed at three major patent offices, such as the European Patent Office (EPO), the Japan Patent Office (JPO) and the United States Patent and Trademark Office (USPTO), to protect the same innovation.

[36]For example, Bottazzi and Peri (2007), Ho et al. (2018), and Hovhannisyan and Sedgley (2019) use the USPTO patent data, while Mancusi (2008) uses the EPO patent data.

[37]Considering the costs of protection at different offices, triadic patent families would eliminate home advantage biases and capture the more valuable inventions.

Table 4: Results of the test statistics

	The S test	The M test
$N(0,1)$ test	2.8184***	1.5798

Note: $^{***}p < 0.01$, $^{**}p < 0.05$, and $^{*}p < 0.1$

7 Conclusion

In this paper, we propose the test for spatial correlation in spatial panel data models with fully heterogeneous spatial lag coefficients when both n and T are large. We first derive the LM test for large T asymptotics so as not to encounter the issues of identification and dimensionality. We then propose the S test, a standardized version of the LM test, and derive its limiting distributions under the null hypothesis and local alternatives when both n and T tend to infinity jointly. We use limit theorems under near-epoch dependence to show the main asymptotic results.

Furthermore, we show that the traditional M test may lose power when spatial effects are heterogeneous. This analysis implies that the traditional test does not behave consistently across all potential alternative hypotheses. Monte Carlo results show that the S test has satisfactory finite sample properties and is more powerful than the traditional test in these types of networks. Finally, we apply our approach to an empirical example of international knowledge spillovers. The test results imply that the traditional testing procedures may draw erroneous conclusions on spatial correlation or dependence under heterogeneous spatial effects.

In future studies, our approach can be extended to testing spatial lag homogeneity for panel data models in large panels. The test evaluates the null hypothesis of homogeneous spatial lag coefficients against an alternative that allows for heterogeneous coefficients. This specification test can be seen as a generalized version of our approach in this paper. Also, the identification and estimation in the heterogeneous version of spatial panel data models when both n and T are large would be an interesting topic of future research.

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Appendix A Some basic lemmas

We provide some basic properties and the law of large numbers which are useful for showing the asymptotic results of our statistics.

Assumption A1. *The disturbances* ε_{it}*,* $i = 1, ..., n$ *and* $t = 1, ..., T$*, are i.i.d. across* i *and* t *with zero mean, finite variance* $\sigma_0^2 > 0$*, and* $E|\varepsilon_{it}|^{4+\eta} < \infty$ *for some* $\eta > 0$*.*

Assumption A2. *The spatial weights matrix* W_n *is time-invariant and its diagonal elements satisfy* $w_{ii} = 0$ *for* $i = 1, ..., n$*.*

Assumption A3. W_n *is uniformly bounded in row and column sums in absolute value.*

Assumption A4. *The elements* $w^*_{ij} = \dfrac{w_{ij}}{\sqrt{\sum_{j=1}^n w_{ij}^2}}$ *are at most of order* h_n^{*-1}*,* $w^*_{ij} = O(\dfrac{1}{h_n^*})$ *where the rate sequence* h_n^* *is divergent, uniformly in all* i, j*.*

Assumption A5. n *is a non-decreasing function of* T *and* T *goes to infinity.*

Assumption A5 allows two cases: (i) $n \to \infty$ as $T \to \infty$; (ii) n is fixed as $T \to \infty$. Thus, our analysis applies to large T asymptotics. Denote $\tilde{V}_{nt} = V_{nt} - \bar{V}_{nT}$ where $\bar{V}_{nT} = \dfrac{1}{T}\sum_{t=1}^{T} V_{nt}$ with $V_{nt} = (\varepsilon_{1t}, ..., \varepsilon_{nt})'$. Define s_i as an $n \times 1$ vector of zeros, except for one at the ith element for $i = 1, ..., n$[38]. Suppose that an $n \times n$ nonstochastic matrix B_n is a multiplication of $s_i s_i'$ and W_n. For example, $B_n = s_i s_i' W_n$ or $B_n = W_n' s_i s_i' W_n$. Also, define the $n \times n$ nonstochastic matrix $A_n = (a_{n,ij})$ where $a_{n,ij} = \dfrac{s_i' W_n s_j s_j' W_n s_i}{\sqrt{s_i' W_n W_n' s_i}\sqrt{s_j' W_n W_n' s_j}} = \dfrac{w_{ij} w_{ji}}{\sqrt{\sum_{j=1}^n w_{ij}^2}\sqrt{\sum_{i=1}^n w_{ji}^2}}$.

Lemma A.1. *Under Assumptions A2 and A4, for any matrix* A_n*,* $tr(A_n^2) = \sum_{i=1}^n \sum_{j=1}^n a_{n,ij} a_{n,ji}$ *and* $tr(A_n A_n') = \sum_{i=1}^n \sum_{j=1}^n a_{n,ij}^2$ *are* $O(\dfrac{n}{h_n^{*2}})$*.*

Lemma A.2. *Under Assumptions A2 and A3, for any matrix* B_n*,* $tr(B_n) = O(1)$*,* $tr(B_n^2) = O(1)$ *and* $tr(B_n B_n') = O(1)$*.*

[38] Then, $s_i s_i'$ is an $n \times n$ matrix of zeros, except for one at the (i,i)th element.

Lemma A.3. *Under Assumptions A1-A3 and A5, for any matrix* B_n*,*
$E(\frac{1}{T}\sum_{t=1}^{T} V_{nt}' B_n V_{nt}) = O(1),$
$E(\bar{V}_{nT}' B_n \bar{V}_{nT}) = O(\frac{1}{T}),$
$E(\frac{1}{T}\sum_{t=1}^{T} \tilde{V}_{nt}' B_n \tilde{V}_{nt}) = O(1).$

Lemma A.4. *Under Assumptions A1-A3 and A5, for any matrix* B_n*,*
$\frac{1}{T}\sum_{t=1}^{T} V_{nt}' B_n V_{nt} - E(\frac{1}{T}\sum_{t=1}^{T} V_{nt}' B_n V_{nt}) = O_p(\frac{1}{\sqrt{T}}),$
$\bar{V}_{nT}' B_n \bar{V}_{nT} - E(\bar{V}_{nT}' B_n \bar{V}_{nT}) = O_p(\frac{1}{T}),$
$\frac{1}{T}\sum_{t=1}^{T} \tilde{V}_{nt}' B_n \tilde{V}_{nt} - E(\frac{1}{T}\sum_{t=1}^{T} \tilde{V}_{nt}' B_n \tilde{V}_{nt}) = O_p(\frac{1}{\sqrt{T}}).$

Lemmas A.2-A.4 hold when $B_n = W_n^{k\prime} s_i s_i' W_n^k$ for any finite k because similar arguments can be applicable to the matrix.

Proof of Lemma A.1 Since $\sup_{i,j} |w_{ij}^*| = O(\frac{1}{h_n^*})$ by Assumption A4, we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{n,ij}a_{n,ji} &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{w_{ij}w_{ji}}{\sqrt{\sum_{j=1}^{n} w_{ij}^2}\sqrt{\sum_{i=1}^{n} w_{ji}^2}} \frac{w_{ji}w_{ij}}{\sqrt{\sum_{i=1}^{n} w_{ji}^2}\sqrt{\sum_{j=1}^{n} w_{ij}^2}} \\ &\leq (\max_{i,j} \frac{|w_{ij}|}{\sqrt{\sum_{j=1}^{n} w_{ij}^2}})^2 \sum_{i=1}^{n} \frac{\sum_{j=1}^{n} w_{ij}^2}{\sum_{j=1}^{n} w_{ij}^2} = O(\frac{n}{h_n^{*2}})\end{aligned} \tag{A.1}$$

Similarly, we have $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{n,ij}^2 = O(\frac{n}{h_n^{*2}})$ because A_n is symmetric.

Proof of Lemma A.2 For the maximum row sum or column sum norm $|| \quad ||$, $||W_n|| \leq c$ for all n by Assumption A3. By its submultiplicative property, $||W_n|| \leq ||W_n|| ||W_n|| \leq c^2$. Thus, any matrix product of W_n is uniformly bounded in row and column sums in absolute value (for short, UB). Then, for any matrix B_n, $tr(B_n)$ can be written as $tr(s_i' M_n s_i)$ where M_n is UB for any i. Since any elements of M_n are uniformly bounded and $tr(s_i' M_n s_i) = s_i' M_n s_i$ is the (i,i)th element of M_n, we have $tr(B_n) = O(1)$. Similar arguments can be applied to show $tr(B_n^2) = O(1)$ and $tr(B_n B_n') = O(1)$.

Proof of Lemma A.3 First, $E(\frac{1}{T}\sum_{t=1}^{T}V_{nt}'B_nV_{nt}) = \sigma_0^2 tr(B_n) = O(1)$ and $E(\bar{V}_{nT}'B_n\bar{V}_{nT}) = \frac{1}{T}\sigma_0^2 tr(B_n) = O(\frac{1}{T})$ by Lemma A.2. Using these results, we can show $E(\frac{1}{T}\sum_{t=1}^{T}\tilde{V}_{nt}'B_n\tilde{V}_{nt}) = E(\frac{1}{T}\sum_{t=1}^{T}V_{nt}'B_nV_{nt} - \bar{V}_{nT}'B_n\bar{V}_{nT}) = \sigma_0^2 tr(B_n) - \frac{1}{T}\sigma_0^2 tr(B_n) = O(1)$.

Proof of Lemma A.4 The proof is given in the supplementary material.

Appendix B Derivatives of the likelihood function

Denote $\theta = (\psi', \sigma^2)'$ where $\psi = (\delta_1, ..., \delta_n)'$. The concentrated likelihood function of (4) is

$$lnL_{nT}(\theta) = -\frac{nT}{2}ln(2\pi) - \frac{nT}{2}ln(\sigma^2) + Tln|S_n(\psi)| - \frac{1}{2\sigma^2}\sum_{t=1}^{T}\tilde{V}_{nt}(\psi)'\tilde{V}_{nt}(\psi) \tag{B.1}$$

where $\tilde{V}_{nt}(\psi) = (I_n - \Psi W_n)\tilde{Y}_{nt} = S_n(\psi)\tilde{Y}_{nt}$ with $\Psi = \sum_{i=1}^{n}\delta_i s_i s_i'$ and s_i is an $n \times 1$ vector of zeros, except for one at the ith element. Define $G_n(\psi) = W_n(I_n - \Psi W_n)^{-1} = W_n S_n(\psi)^{-1}$.

The first and second order derivatives with respect to θ are:

$$\frac{\partial lnL_{nT}(\theta)}{\partial\theta} = \begin{pmatrix} \frac{\partial lnL_{nT}(\theta)}{\partial\delta_1} \\ \frac{\partial lnL_{nT}(\theta)}{\partial\delta_2} \\ \vdots \\ \frac{\partial lnL_{nT}(\theta)}{\partial\delta_n} \\ \frac{\partial lnL_{nT}(\theta)}{\partial\sigma^2} \end{pmatrix} \tag{B.2}$$

where $\frac{\partial lnL_{nT}(\theta)}{\partial\delta_i} = \frac{1}{\sigma^2}\sum_{t=1}^{T}\big(\tilde{V}_{nt}(\psi)'s_i s_i' G_n(\psi)\tilde{V}_{nt}(\psi) - \sigma^2 s_i' G_n(\psi)s_i\big)$ for $i = 1, ..., n$ and $\frac{\partial lnL_{nT}(\theta)}{\partial\sigma^2} = \frac{1}{2\sigma^4}\sum_{t=1}^{T}\big(\tilde{V}_{nt}(\psi)'\tilde{V}_{nt}(\psi) - n\sigma^2\big)$.

$$\frac{\partial^2 lnL_{nT}(\theta)}{\partial\theta\partial\theta'} = \begin{pmatrix} \frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_1^2} & \frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_1\partial\delta_2} & \cdots & \frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_1\partial\delta_n} & \frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_1\partial\sigma^2} \\ \frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_2\partial\delta_1} & \frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_2^2} & \cdots & \frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_2\partial\delta_n} & \frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_2\partial\sigma^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_n\partial\delta_1} & \frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_n\partial\delta_2} & \cdots & \frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_n^2} & \frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_n\partial\sigma^2} \\ \frac{\partial^2 lnL_{nT}(\theta)}{\partial\sigma^2\partial\delta_1} & \frac{\partial^2 lnL_{nT}(\theta)}{\partial\sigma^2\partial\delta_2} & \cdots & \frac{\partial^2 lnL_{nT}(\theta)}{\partial\sigma^2\partial\delta_n} & \frac{\partial^2 lnL_{nT}(\theta)}{\partial(\sigma^2)^2} \end{pmatrix} \tag{B.3}$$

where $\frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_i^2} = -\frac{1}{\sigma^2}\sum_{t=1}^{T}\tilde{V}_{nt}(\psi)'G_n(\psi)'s_is_i'G_n(\psi)\tilde{V}_{nt}(\psi) - T(s_i'G_n(\psi)s_i)^2$ for $i = 1, ..., n$, $\frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_i\partial\sigma^2} = \frac{\partial^2 lnL_{nT}(\theta)}{\partial\sigma^2\partial\delta_i} = -\frac{1}{\sigma^4}\sum_{t=1}^{T}\tilde{V}_{nt}(\psi)'s_is_i'G_n(\psi)\tilde{V}_{nt}(\psi)$ for $i = 1, ..., n$, $\frac{\partial^2 lnL_{nT}(\theta)}{\partial\delta_i\partial\delta_j} = -Ts_i'G_n(\psi)s_js_j'G_n(\psi)s_i$ for $i \neq j$, and $\frac{\partial^2 lnL_{nT}(\theta)}{\partial(\sigma^2)^2} = -\big(-\frac{nT}{2\sigma^4} + \frac{1}{\sigma^6}\sum_{t=1}^{T}\tilde{V}_{nt}(\psi)'\tilde{V}_{nt}(\psi)\big)$[39].

Hence, at $\theta_0 = (\psi_0', \sigma_0^2)'$, we have

$$\frac{1}{\sqrt{T}}\frac{\partial lnL_{nT}(\theta_0)}{\partial\theta} = \begin{pmatrix} \frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\big(\tilde{V}_{nt}'s_1s_1'G_n(\psi_0)\tilde{V}_{nt} - \sigma_0^2s_1'G_n(\psi_0)s_1\big) \\ \frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\big(\tilde{V}_{nt}'s_2s_2'G_n(\psi_0)\tilde{V}_{nt} - \sigma_0^2s_2'G_n(\psi_0)s_2\big) \\ \vdots \\ \frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\big(\tilde{V}_{nt}'s_ns_n'G_n(\psi_0)\tilde{V}_{nt} - \sigma_0^2s_n'G_n(\psi_0)s_n\big) \\ \frac{1}{2\sigma_0^4}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\big(\tilde{V}_{nt}'\tilde{V}_{nt} - n\sigma_0^2\big) \end{pmatrix} \tag{B.4}$$

where $G_n(\psi_0) = W_n(I_n - \Psi_0W_n)^{-1} = W_nS_n(\psi_0)^{-1}$ and $\tilde{V}_{nt} = V_{nt} - \bar{V}_{nT}$ with $\bar{V}_{nT} = \frac{1}{T}\sum_{t=1}^{T}V_{nt}$.

[39]Detailed derivations are available in the supplementary material.

From (B.4), we have $\frac{1}{\sqrt{T}}\frac{\partial lnL_{nT}(\theta_0)}{\partial\theta} = \frac{1}{\sqrt{T}}\frac{\partial lnL^*_{nT}(\theta_0)}{\partial\theta} - \varphi_{nT}$ where

$$\frac{1}{\sqrt{T}}\frac{\partial lnL^*_{nT}(\theta_0)}{\partial\theta} = \begin{pmatrix} \frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(V'_{nt}s_1s'_1G_n(\psi_0)V_{nt} - \sigma_0^2 s'_1G_n(\psi_0)s_1\right) \\ \frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(V'_{nt}s_2s'_2G_n(\psi_0)V_{nt} - \sigma_0^2 s'_2G_n(\psi_0)s_2\right) \\ \vdots \\ \frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(V'_{nt}s_ns'_nG_n(\psi_0)V_{nt} - \sigma_0^2 s'_nG_n(\psi_0)s_n\right) \\ \frac{1}{2\sigma_0^4}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(V'_{nt}V_{nt} - n\sigma_0^2\right) \end{pmatrix} \tag{B.5}$$

and

$$\varphi_{nT} = \begin{pmatrix} \frac{1}{\sigma_0^2}\sqrt{T}\bar{V}'_{nT}s_1s'_1G_n(\psi_0)\bar{V}_{nT} \\ \frac{1}{\sigma_0^2}\sqrt{T}\bar{V}'_{nT}s_2s'_2G_n(\psi_0)\bar{V}_{nT} \\ \vdots \\ \frac{1}{\sigma_0^2}\sqrt{T}\bar{V}'_{nT}s_ns'_nG_n(\psi_0)\bar{V}_{nT} \\ \frac{1}{2\sigma_0^4}\sqrt{T}\bar{V}'_{nT}\bar{V}_{nT} \end{pmatrix} \tag{B.6}$$

And, its corresponding information matrix ($\Sigma_{\theta_0,nT} = -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\theta\partial\theta'})$) is

$$\Sigma_{\theta_0,nT} = \begin{pmatrix} -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_1^2}) & -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_1\partial\delta_2}) & \dots & -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_1\partial\delta_n}) & -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_1\partial\sigma^2}) \\ -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_2\partial\delta_1}) & -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_2^2}) & \dots & -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_2\partial\delta_n}) & -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_2\partial\sigma^2}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_n\partial\delta_1}) & -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_n\partial\delta_2}) & \dots & -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_n^2}) & -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_n\partial\sigma^2}) \\ -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\sigma^2\partial\delta_1}) & -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\sigma^2\partial\delta_2}) & \dots & -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\sigma^2\partial\delta_n}) & -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial(\sigma^2)^2}) \end{pmatrix} \tag{B.7}$$

where $-E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_i^2}) = s'_iG_n(\psi_0)G_n(\psi_0)'s_i + (s'_iG_n(\psi_0)s_i)^2$ for $i = 1, ..., n$, $-E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_i\partial\sigma^2}) = -E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\sigma^2\partial\delta_i}) = \frac{1}{\sigma_0^2}s'_iG_n(\psi_0)s_i$ for $i = 1, ..., n$, $-E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\delta_i\partial\delta_j}) = s'_iG_n(\psi_0)s_js'_jG_n(\psi_0)s_i$

for $i \neq j$, and $-E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial(\sigma^2)^2}) = \frac{n}{2\sigma_0^4}$.

For the variance of $\frac{1}{\sqrt{T}}\frac{\partial lnL^*_{nT}(\theta_0)}{\partial\theta}$, we have the following equation:

$$E(\frac{1}{\sqrt{T}}\frac{\partial lnL^*_{nT}(\theta_0)}{\partial\theta}\frac{1}{\sqrt{T}}\frac{\partial lnL^*_{nT}(\theta_0)}{\partial\theta'}) = \Sigma_{\theta_0,nT} + \Omega_{\theta_0,nT} \tag{B.8}$$

where $\Omega_{\theta_0,nT} = \frac{\mu_4 - 3\sigma_0^4}{\sigma_0^4}\begin{pmatrix} diag\big(G^2_{n,11}(\psi_0),...,G^2_{n,nn}(\psi_0)\big) & \frac{1}{2\sigma_0^2}\big(G_{n,11}(\psi_0),...,G_{n,nn}(\psi_0)\big)' \\ \frac{1}{2\sigma_0^2}\big(G_{n,11}(\psi_0),...,G_{n,nn}(\psi_0)\big) & \frac{n}{4\sigma_0^4} \end{pmatrix}$
with $\mu_4 = E(\varepsilon_{it}^4)$ and $G_{n,ii}(\psi_0)$ is the (i,i)th entry of $G_n(\psi_0)$.

When V_{nt} are normally distributed, $\Omega_{\theta_0,nT} = 0_{(n+1)\times(n+1)}$ since $\mu_4 - 3\sigma_0^4 = 0$. Using $E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\theta\partial\theta'}) + E(\frac{1}{\sqrt{T}}\frac{\partial lnL^*_{nT}(\theta_0)}{\partial\theta}\frac{1}{\sqrt{T}}\frac{\partial lnL^*_{nT}(\theta_0)}{\partial\theta'}) = 0$ at θ_0 for normally distributed errors, we have $-E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(\theta_0)}{\partial\theta\partial\theta'}) = \Sigma_{\theta_0,nT}$.

Appendix C Proofs

C.1 Proof of Proposition 1

From $\tilde{V}_{nt} = V_{nt} - \bar{V}_{nT}$ where $\bar{V}_{nT} = \frac{1}{T}\sum_{t=1}^{T} V_{nt}$ and $V_{nt} = (\varepsilon_{1t},...,\varepsilon_{nt})'$, the restricted QML estimator can be rewritten as

$$\begin{aligned}\tilde{\sigma}^2 = \frac{1}{nT}\sum_{t=1}^{T}\tilde{V}'_{nt}\tilde{V}_{nt} &= \frac{1}{nT}\sum_{t=1}^{T}V'_{nt}V_{nt} - \frac{1}{n}\bar{V}'_{nT}\bar{V}_{nT} \\ &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2 - \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it})^2\end{aligned} \tag{C.1}$$

under H_0. Note that ε_{it}^2 and ε_{it} are i.i.d. across $t = 1,...,T$ with $E(\varepsilon_{it}^2) = \sigma_0^2$, $E(\varepsilon_{it}) = 0$ and $E|\varepsilon_{it}|^{4+\eta} < \infty$ for some $\eta > 0$ under Assumption 2. By Kolmogorov's Strong LLN, we have $\tilde{\sigma}^2 \xrightarrow{a.s.} \sigma_0^2$. Therefore, with Slutsky's theorem, we have $\frac{\sigma_0^2}{\tilde{\sigma}^2} \xrightarrow{p} 1$.

C.2 Proof of Proposition 2

Denote $\Sigma_{g,n}^{i,j}$ as the (i,j)th entry of $\Sigma_{g,n}$ for all i,j. For any nonzero vector c,

$$\begin{aligned} c'\Sigma_{g,n}c = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i \Sigma_{g,n}^{i,j} c_j &= \sum_{i=1}^{n} c_i^2 \Sigma_{g,n}^{i,i} + \sum_{i=1}^{n}\sum_{j\neq i}^{n} c_i \Sigma_{g,n}^{i,j} c_j \\ &= \sum_{i=1}^{n} c_i^2 \sum_{j=1}^{n} w_{ij}^2 + \sum_{i=1}^{n}\sum_{j\neq i}^{n} c_i w_{ij} w_{ji} c_j \\ &= \sum_{i=1}^{n} c_i^2 \sum_{j=1}^{n} w_{ij}^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} c_i w_{ij} w_{ji} c_j \end{aligned} \tag{C.2}$$

Then, by Assumptions 1 ($w_{ii} = 0$ for all i) and 4(1) ($\sum_{j=1}^{n} w_{ij}^2 > \sum_{j=1}^{n} |w_{ij} w_{ji}|$ for all i), we have

$$\begin{aligned} c'\Sigma_{g,n}c = \sum_{i=1}^{n} c_i^2 \sum_{j=1}^{n} w_{ij}^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} c_i w_{ij} w_{ji} c_j &> \sum_{i=1}^{n} c_i^2 \sum_{j=1}^{n} w_{ij} w_{ji} + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} c_i w_{ij} w_{ji} c_j \\ &= \sum_{i=1}^{n}\sum_{j\neq i}^{n} c_i^2 w_{ij} w_{ji} + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} c_i w_{ij} w_{ji} c_j \\ &= \sum_{i=1}^{n}\sum_{j=i+1}^{n} w_{ij} w_{ji} (c_i + c_j)^2 \geq 0 \end{aligned} \tag{C.3}$$

which implies that $\Sigma_{g,n}$ is positive definite.

For Assumptions 1 and 4(2), $\Sigma_{g,n}$ is symmetric and diagonally dominant with all positive entries. Hillar et al. (2012) investigate the special property of the inverse of symmetric and diagonally dominant matrices with all positive entries. Therefore, Lemma 7.1 in Hillar et al. (2012) is applicable to $\Sigma_{g,n}$, and we have the result that the minimum eigenvalue of $\Sigma_{g,n}$ is positive under Assumptions 1 and 4(2). Hence, $\Sigma_{g,n}$ is positive definite.

C.3 Proof of Theorem 1

From the results under H_0, we have $\frac{1}{\sqrt{T}}g_{nT}(\tilde{\sigma}^2) \xrightarrow{d} N(0,\Sigma_{g,n})$ where $\Sigma_{g,n}$ is positive definite and symmetric. Also, there exists $\Sigma_{g,n}^{\frac{1}{2}}$ which is invertible such that $\Sigma_{g,n} = \Sigma_{g,n}^{\frac{1}{2}}\Sigma_{g,n}^{\frac{1}{2}}$ since $\Sigma_{g,n}$ is positive definite and symmetric, so we have $\Sigma_{g,n}^{-1} = \Sigma_{g,n}^{-\frac{1}{2}}\Sigma_{g,n}^{-\frac{1}{2}}$. Then, the quadratic form, $\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1}g_{nT}(\tilde{\sigma}^2)$ can be written as

$$\begin{aligned}\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1}g_{nT}(\tilde{\sigma}^2) &= \frac{1}{\sqrt{T}}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-\frac{1}{2}}\Sigma_{g,n}^{-\frac{1}{2}}\frac{1}{\sqrt{T}}g_{nT}(\tilde{\sigma}^2) \\ &= (\Sigma_{g,n}^{-\frac{1}{2}}\frac{1}{\sqrt{T}}g_{nT}(\tilde{\sigma}^2))'(\Sigma_{g,n}^{-\frac{1}{2}}\frac{1}{\sqrt{T}}g_{nT}(\tilde{\sigma}^2))\end{aligned} \tag{C.4}$$

where $\Sigma_{g,n}^{-\frac{1}{2}}\frac{1}{\sqrt{T}}g_{nT}(\tilde{\sigma}^2) \xrightarrow{d} N(0,I_n)$. Hence, we have $\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1}g_{nT}(\tilde{\sigma}^2) \xrightarrow{d} \chi_n^2$ under H_0.

C.4 Proof of Lemma 1

Consider (i,j)th entry of the difference as

$$|\frac{1}{T}\frac{\partial^2 lnL_{nT}(\bar{\Delta}_1,...,\bar{\Delta}_n,\sigma_0^2)}{\partial\delta_i\partial\delta_j} - E(\frac{1}{T}\frac{\partial^2 lnL_{nT}^*(0,...,0,\sigma_0^2)}{\partial\delta_i\partial\delta_j})| \tag{C.5}$$

By the triangle inequality, we have

$$\begin{aligned}&|\frac{1}{T}\frac{\partial^2 lnL_{nT}(\bar{\Delta}_1,...,\bar{\Delta}_n,\sigma_0^2)}{\partial\delta_i\partial\delta_j} - E(\frac{1}{T}\frac{\partial^2 lnL_{nT}^*(0,...,0,\sigma_0^2)}{\partial\delta_i\partial\delta_j})| \\ &\leq |\frac{1}{T}\frac{\partial^2 lnL_{nT}(0,...,0,\sigma_0^2)}{\partial\delta_i\partial\delta_j} - E(\frac{1}{T}\frac{\partial^2 lnL_{nT}^*(0,...,0,\sigma_0^2)}{\partial\delta_i\partial\delta_j})| \\ &\quad + |\frac{1}{T}\frac{\partial^2 lnL_{nT}(\bar{\Delta}_1,...,\bar{\Delta}_n,\sigma_0^2)}{\partial\delta_i\partial\delta_j} - \frac{1}{T}\frac{\partial^2 lnL_{nT}(0,...,0,\sigma_0^2)}{\partial\delta_i\partial\delta_j}| \\ &\leq |\frac{1}{T}\frac{\partial^2 lnL_{nT}(0,...,0,\sigma_0^2)}{\partial\delta_i\partial\delta_j} - E(\frac{1}{T}\frac{\partial^2 lnL_{nT}^*(0,...,0,\sigma_0^2)}{\partial\delta_i\partial\delta_j})| + \sup_{\psi\in\Theta^\Delta}||\frac{1}{T}\frac{\partial^3 lnL_{nT}(\delta_1,...,\delta_n,\sigma_0^2)}{\partial\delta_i\partial\delta_j\partial\psi}||\cdot||\bar{\Delta}-0_{n\times 1}||\end{aligned} \tag{C.6}$$

where $\bar{\Delta} = (\bar{\Delta}_1,...,\bar{\Delta}_n)' \xrightarrow{p} 0_{n\times 1}$ as $T\to\infty$.

For the first term when $i \neq j$, we have $|\frac{1}{T}\frac{\partial^2 lnL_{nT}(0,...,0,\sigma_0^2)}{\partial\delta_i\partial\delta_j} - E(\frac{1}{T}\frac{\partial^2 lnL_{nT}^*(0,...,0,\sigma_0^2)}{\partial\delta_i\partial\delta_j})| =$

$-s_i'W_ns_js_j'W_ns_i + s_i'W_ns_js_j'W_ns_i = 0$ derived from (B.3) and (B.7) with $\psi = (0, ..., 0)'$. For the first term when $i = j$, we consider the following equation derived from (B.3) and (B.7) with $\psi = (0, ..., 0)'$:

$$\begin{aligned}
&\frac{1}{T}\frac{\partial^2 lnL_{nT}(0,...,0,\sigma_0^2)}{\partial\delta_i^2} - E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(0,...,0,\sigma_0^2)}{\partial\delta_i^2}) \\
&= -\frac{1}{\sigma_0^2}\frac{1}{T}\sum_{t=1}^{T}\tilde{V}'_{nt}W'_ns_is_i'W_n\tilde{V}_{nt} + s_i'W_nW_n's_i \\
&= -\big(\frac{1}{\sigma_0^2}\frac{1}{T}\sum_{t=1}^{T}\tilde{V}'_{nt}W'_ns_is_i'W_n\tilde{V}_{nt} - E(\frac{1}{\sigma_0^2}\frac{1}{T}\sum_{t=1}^{T}\tilde{V}'_{nt}W'_ns_is_i'W_n\tilde{V}_{nt})\big) + \frac{1}{T}s_i'W_nW_n's_i
\end{aligned} \tag{C.7}$$

because $E(\frac{1}{\sigma_0^2}\frac{1}{T}\sum_{t=1}^{T}\tilde{V}'_{nt}W'_ns_is_i'W_n\tilde{V}_{nt}) = (1-\frac{1}{T})tr(W_n's_is_i'W_n) = (1-\frac{1}{T})s_i'W_nW_n's_i$ shown in Lemma A.3. By Lemmas A.3 and A.4, $|\frac{1}{T}\frac{\partial^2 lnL_{nT}(0,...,0,\sigma_0^2)}{\partial\delta_i^2} - E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(0,...,0,\sigma_0^2)}{\partial\delta_i^2})| = O_p(\frac{1}{\sqrt{T}})$.

For the second term, it suffices to show that $\sup_{\psi\in\theta^\Delta}|\frac{1}{T}\frac{\partial^3 lnL_{nT}(\delta_1,...,\delta_n,\sigma_0^2)}{\partial\delta_i\partial\delta_j\partial\delta_l}| < \infty$. The third order derivative evaluated at σ_0^2 is

$$\begin{aligned}
\frac{1}{T}\frac{\partial^3 lnL_{nT}(\delta_1,...,\delta_n,\sigma_0^2)}{\partial\delta_i\partial\delta_j\partial\delta_l} &= -s_i'W_n(I_n-\Psi W_n)^{-1}s_ls_l'W_n(I_n-\Psi W_n)^{-1}s_js_j'W_n(I_n-\Psi W_n)^{-1}s_i \\
&\quad - s_i'W_n(I_n-\Psi W_n)^{-1}s_js_j'W_n(I_n-\Psi W_n)^{-1}s_ls_l'W_n(I_n-\Psi W_n)^{-1}s_i
\end{aligned} \tag{C.8}$$

where $\Psi = diag(\psi)$ with $\psi = (\delta_1, ..., \delta_n)'$. Note that $(I_n - \Psi W_n)^{-1}$ is invertible for all ψ where Θ^Δ is bounded within a small neighborhood around zero by Assumption 3, and $s_i'W_n(I_n - \Psi W_n)^{-1}s_j$ is the (i,j)th entry of $W_n(I_n-\Psi W_n)^{-1}$. Therefore, $\sup_{\psi\in\theta^\Delta}|\frac{1}{T}\frac{\partial^3 lnL_{nT}(\delta_1,...,\delta_n,\sigma_0^2)}{\partial\delta_i\partial\delta_j\partial\delta_l}| = O(1)$ for all i, j, l.

Similar arguments can be applied to all other entries of $\frac{1}{T}\frac{\partial^2 lnL_{nT}(\bar{\Delta}_1,...,\bar{\Delta}_n,\sigma_0^2)}{\partial\psi\partial\psi'} - E(\frac{1}{T}\frac{\partial^2 lnL^*_{nT}(0,...,0,\sigma_0^2)}{\partial\psi\partial\psi'})$.

C.5 Proof of Theorem 2

From the previous results, we have $\frac{1}{\sqrt{T}} g_{nT}(\tilde{\sigma}^2) \xrightarrow{d} N(\tau, \Sigma_{g,n})$ under $H_{1,T}$ where $\tau = \Sigma_{g,n}\Delta$, $\Sigma_{g,n} = \Sigma_{g,n}^{\frac{1}{2}}\Sigma_{g,n}^{\frac{1}{2}}$ and $\Sigma_{g,n}^{-1} = \Sigma_{g,n}^{-\frac{1}{2}}\Sigma_{g,n}^{-\frac{1}{2}}$. Then, the quadratic form, $\frac{1}{T} g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1} g_{nT}(\tilde{\sigma}^2)$ can be written as

$$\frac{1}{T} g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1} g_{nT}(\tilde{\sigma}^2) = (\Sigma_{g,n}^{-\frac{1}{2}} \frac{1}{\sqrt{T}} g_{nT}(\tilde{\sigma}^2))'(\Sigma_{g,n}^{-\frac{1}{2}} \frac{1}{\sqrt{T}} g_{nT}(\tilde{\sigma}^2)) \tag{C.9}$$

where $\Sigma_{g,n}^{-\frac{1}{2}} \frac{1}{\sqrt{T}} g_{nT}(\tilde{\sigma}^2) \xrightarrow{d} N(\Sigma_{g,n}^{-\frac{1}{2}}\tau, I_n)$. Hence, under $H_{1,T}$, the LM test has a noncentral chi-square distribution with n degrees of freedom and noncentrality of $\mu = \tau'\Sigma_{g,n}^{-\frac{1}{2}}\Sigma_{g,n}^{-\frac{1}{2}}\tau = \Delta'\Sigma_{g,n}\Delta$.

C.6 Proof of Proposition 3

From (C.1), $\tilde{V}_{nt} = V_{nt} - \bar{V}_{nT}$ where $\bar{V}_{nT} = \frac{1}{T}\sum_{t=1}^{T} V_{nt}$ and $V_{nt} = (\varepsilon_{1t}, ..., \varepsilon_{nt})'$, the restricted QML estimator can be rewritten as

$$\begin{aligned}
\tilde{\sigma}^2 = \frac{1}{nT}\sum_{t=1}^{T}\tilde{V}_{nt}'\tilde{V}_{nt} &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2 - \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it})^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2 - \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2 - \frac{2}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varepsilon_{it}\varepsilon_{is}
\end{aligned} \tag{C.10}$$

under H_0. For the first and second terms, ε_{it}^2 is i.i.d. across $i = 1, .., n$ and $t = 1, ..., T$ with $E(\varepsilon_{it}^2) = \sigma_0^2$ and $E|\varepsilon_{it}|^{4+\eta} < \infty$ for some $\eta > 0$ under Assumption 2. By Kolmogorov's Strong LLN, we have $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2 \xrightarrow{a.s.} \sigma_0^2$, and hence $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2 \xrightarrow{a.s.} 0$. For the last term, $\varepsilon_{it}\varepsilon_{is}$ is uncorrelated over i and t, so $Var(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varepsilon_{it}\varepsilon_{is}) = \frac{1}{n^2T^4}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}Var(\varepsilon_{it}\varepsilon_{is}) = O(\frac{1}{nT^2})$. By Chebyshev's inequality, $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varepsilon_{it}\varepsilon_{is} = o_p(1)$. Hence, with Slutsky's theorem, we have $\frac{\sigma_0^2}{\tilde{\sigma}^2} \xrightarrow{p} 1$.

From (C.10), we next consider $\dfrac{\sigma_0^2}{\tilde{\sigma}^2} - 1 = \dfrac{1}{\tilde{\sigma}^2}\big(\sigma_0^2 - \tilde{\sigma}^2\big)$ as

$$\begin{aligned}\frac{\sigma_0^2}{\tilde{\sigma}^2} - 1 &= \frac{1}{\tilde{\sigma}^2}\big(\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\sigma_0^2 - \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2 + \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2 + 2\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varepsilon_{it}\varepsilon_{is}\big)\\ &= \frac{\sigma_0^2}{\tilde{\sigma}^2}\big(\frac{1}{\sigma_0^2}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(\sigma_0^2 - \varepsilon_{it}^2) + \frac{1}{\sigma_0^2}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2 + \frac{2}{\sigma_0^2}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varepsilon_{it}\varepsilon_{is}\big)\end{aligned} \tag{C.11}$$

For each term in (C.11), $Var(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(\sigma_0^2-\varepsilon_{it}^2)) = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}Var(\varepsilon_{it}^2)$, $Var(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2) = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}Var(\varepsilon_{it}^2)$ and $Var(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varepsilon_{it}\varepsilon_{is}) = \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}Var(\varepsilon_{it}\varepsilon_{is})$ are $O(1)$. Hence, along with $\frac{\sigma_0^2}{\tilde{\sigma}^2} \xrightarrow{p} 1$, we have $\frac{\sigma_0^2}{\tilde{\sigma}^2} - 1 = O_P(\frac{1}{\sqrt{nT}})$.

C.7 Proof of Proposition 4

First, consider $\mu_{Q_{nT}} = E(Q_{nT}) = E(\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\big(z_{i,nT} - E(z_{i,nT})\big)) = 0$. Next, consider $\Sigma_{Q_{nT}} = Var(Q_{nT}) = Var(\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\big(z_{i,nT} - E(z_{i,nT})\big)$ as

$$\begin{aligned}\Sigma_{Q_{nT}} = \frac{1}{2n}Var(\sum_{i=1}^{n}z_{i,nT}) &= \frac{1}{2n}\big(\sum_{i=1}^{n}Var(z_{i,nT}) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}Cov(z_{i,nT}, z_{j,nT})\big)\\ &= \frac{1}{2n}\sum_{i=1}^{n}Var(z_{i,nT}) + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n}Cov(z_{i,nT}, z_{j,nT})\end{aligned} \tag{C.12}$$

For the first term in (C.12), from (E.6), we have

$$\begin{aligned}\frac{1}{2n}\sum_{i=1}^{n}Var(z_{i,nT}) &= \frac{1}{2n}\sum_{i=1}^{n}\big(2 + \frac{1}{\sigma_0^8}\frac{1}{T}\frac{\mu_4(\mu_4 - 3\sigma_0^4)\sum_{j=1}^{n}w_{ij}^4}{(\sum_{j=1}^{n}w_{ij}^2)^2} + \frac{3(\mu_4 - \sigma_0^4)}{\sigma_0^4}\frac{1}{T}\big)\\ &= 1 + \frac{1}{T}\big(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2\sigma_0^8}\frac{\mu_4(\mu_4 - 3\sigma_0^4)\sum_{j=1}^{n}w_{ij}^4}{(\sum_{j=1}^{n}w_{ij}^2)^2} + \frac{3(\mu_4 - \sigma_0^4)}{2\sigma_0^4}\big)\end{aligned} \tag{C.13}$$

which implies that $\frac{1}{2n}\sum_{i=1}^{n} Var(z_{i,nT}) = 1 + O(\frac{1}{T})$ since $\sum_{j=1}^{n} w_{ij}^4 \leq (\sum_{j=1}^{n} w_{ij}^2)^2$ for all i and n. For the second term in (C.12), from (E.12), we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n} Cov(z_{i,nT}, z_{j,nT}) \\
&= \frac{1}{T}\Big(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n} \frac{(\mu_4)^2 - 2\mu_4\sigma_0^4 + (\sigma_0^4)^2}{\sigma_0^8} \frac{w_{ij}^2 w_{ji}^2}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n} (\frac{\mu_4 - \sigma_0^4}{\sigma_0^4}\frac{w_{ji}^2}{\sum_{i=1}^n w_{ji}^2} + \frac{\mu_4 - \sigma_0^4}{\sigma_0^4}\frac{w_{ij}^2}{\sum_{j=1}^n w_{ij}^2} + \frac{2(\sum_{l=1}^n w_{il}w_{jl})^2}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)}) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n} (\frac{\mu_4 - 3\sigma_0^4}{\sigma_0^4}\frac{\sum_{l=1}^n w_{il}^2 w_{jl}^2}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} + \frac{4(\mu_3)^2}{\sigma_0^6}\frac{w_{ij}w_{ji}(\sum_{l=1}^n w_{il}w_{jl})}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)}) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n} (\frac{2(\mu_3)^2}{\sigma_0^6}\frac{w_{ji}(\sum_{l=1}^n w_{il}^2 w_{jl})}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} + \frac{2(\mu_3)^2}{\sigma_0^6}\frac{w_{ij}(\sum_{l=1}^n w_{jl}^2 w_{il})}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)})\Big)
\end{aligned} \tag{C.14}$$

Then

$$\sum_{j=i+1}^{n} \frac{w_{ij}^2 w_{ji}^2}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} \leq \max_{i,j} \frac{w_{ji}^2}{\sum_{i=1}^n w_{ji}^2}\frac{\sum_{j=i+1}^n w_{ij}^2}{\sum_{j=1}^n w_{ij}^2} = O(1) \tag{C.15}$$

$$\sum_{j=i+1}^{n} \frac{w_{ji}^2}{\sum_{i=1}^n w_{ji}^2} \leq \max_{i} \frac{\sum_{j=i+1}^n w_{ji}^2}{\min_j \sum_{i=1}^n w_{ji}^2} = O(1) \tag{C.16}$$

$$\sum_{j=i+1}^{n} \frac{w_{ij}^2}{\sum_{j=1}^n w_{ij}^2} \leq \max_{i} \frac{\sum_{j=i+1}^n w_{ij}^2}{\sum_{j=1}^n w_{ij}^2} = O(1) \tag{C.17}$$

$$\begin{aligned}
\sum_{j=i+1}^{n} \frac{(\sum_{l=1}^n w_{il}w_{jl})^2}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} &\leq \frac{(\sum_{j=i+1}^n \sum_{l=1}^n w_{il}w_{jl})^2}{(\sum_{j=1}^n w_{ij}^2)(\min_j \sum_{i=1}^n w_{ji}^2)} \\
&= \frac{(\sum_{l=1}^n w_{il}\sum_{j=i+1}^n w_{jl})^2}{(\sum_{j=1}^n w_{ij}^2)(\min_j \sum_{i=1}^n w_{ji}^2)} \\
&\leq \max_i \frac{(\sum_{l=1}^n |w_{il}|\sum_{j=i+1}^n |w_{jl}|)^2}{(\sum_{l=1}^n w_{il}^2)(\min_j \sum_{i=1}^n w_{ji}^2)} = O(1)
\end{aligned} \tag{C.18}$$

$$\sum_{j=i+1}^{n}\frac{\sum_{l=1}^{n}w_{il}^2w_{jl}^2}{(\sum_{j=1}^{n}w_{ij}^2)(\sum_{i=1}^{n}w_{ji}^2)}\leq\frac{\sum_{l=1}^{n}w_{il}^2\sum_{j=i+1}^{n}w_{jl}^2}{(\sum_{j=1}^{n}w_{ij}^2)(\min_j\sum_{i=1}^{n}w_{ji}^2)}$$

$$\leq\frac{(\max_l|w_{il}|)(\max_{j,l}|w_{jl}|)(\sum_{l=1}^{n}|w_{il}|\sum_{j=i+1}^{n}|w_{jl}|)}{(\sum_{j=1}^{n}w_{ij}^2)(\min_j\sum_{i=1}^{n}w_{ji}^2)} \tag{C.19}$$

$$\leq\max_{i,j}\frac{|w_{ij}||w_{ji}|(\sum_{l=1}^{n}|w_{il}|\sum_{j=i+1}^{n}|w_{jl}|)}{(\sum_{j=1}^{n}w_{ij}^2)(\min_j\sum_{i=1}^{n}w_{ji}^2)}=O(1)$$

$$\sum_{j=i+1}^{n}\frac{w_{ij}w_{ji}(\sum_{l=1}^{n}w_{il}w_{jl})}{(\sum_{j=1}^{n}w_{ij}^2)(\sum_{i=1}^{n}w_{ji}^2)}\leq\frac{(\max_{j,l}|w_{jl}|)(\sum_{l=1}^{n}|w_{il}|)\sum_{j=i+1}^{n}|w_{ij}||w_{ji}|}{(\sum_{j=1}^{n}w_{ij}^2)(\min_j\sum_{i=1}^{n}w_{ji}^2)}$$

$$\leq\max_{i,j}\frac{(|w_{ji}|\sum_{l=1}^{n}|w_{il}|)(|w_{ji}|\sum_{j=i+1}^{n}|w_{ij}|)}{(\sum_{j=1}^{n}w_{ij}^2)(\min_j\sum_{i=1}^{n}w_{ji}^2)}=O(1) \tag{C.20}$$

$$\sum_{j=i+1}^{n}\frac{w_{ji}(\sum_{l=1}^{n}w_{il}^2w_{jl})}{(\sum_{j=1}^{n}w_{ij}^2)(\sum_{i=1}^{n}w_{ji}^2)}\leq\sum_{j=i+1}^{n}\frac{(\max_l|w_{jl}|)|w_{ji}|(\sum_{l=1}^{n}w_{il}^2)}{(\sum_{j=1}^{n}w_{ij}^2)(\sum_{i=1}^{n}w_{ji}^2)}$$

$$=\sum_{j=i+1}^{n}\frac{(\max_l|w_{jl}|)|w_{ji}|}{\sum_{i=1}^{n}w_{ji}^2} \tag{C.21}$$

$$\leq\max_{i,j}\frac{|w_{ji}|(\sum_{j=i+1}^{n}|w_{ji}|)}{\min_j\sum_{i=1}^{n}w_{ji}^2}=O(1)$$

$$\sum_{j=i+1}^{n}\frac{w_{ij}(\sum_{l=1}^{n}w_{jl}^2w_{il})}{(\sum_{j=1}^{n}w_{ij}^2)(\sum_{i=1}^{n}w_{ji}^2)}\leq\sum_{j=i+1}^{n}\frac{(\max_l|w_{il}|)|w_{ij}|(\sum_{l=1}^{n}w_{jl}^2)}{(\sum_{j=1}^{n}w_{ij}^2)(\sum_{i=1}^{n}w_{ji}^2)}$$

$$=\sum_{j=i+1}^{n}\frac{(\max_l|w_{il}|)|w_{ij}|}{\sum_{j=1}^{n}w_{ij}^2} \tag{C.22}$$

$$\leq\max_{i,j}\frac{|w_{ij}|(\sum_{j=i+1}^{n}|w_{ij}|)}{\sum_{j=1}^{n}w_{ij}^2}=O(1)$$

which hold for all n by Assumptions 1 and 5. Therefore, $\frac{1}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n}Cov(z_{i,nT},z_{j,nT})=O(\frac{1}{T})$, and we have $\Sigma_{Q_{nT}}=1+O(\frac{1}{T})$.

C.8 Proof of Lemma 2

We first show that $\tilde{z}_{i,nT}$ is uniformly L_p bounded where $p=4+\eta$, i.e., $\sup_n\sup_{i\in D_n}||\tilde{z}_{i,nT}||_{4+\eta}<\infty$, or equivalently, $\sup_n\sup_{i\in D_n}||\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\sum_{j=1}^{n}w_{ij}^*\varepsilon_{jt}||_{4+\eta}<\infty$ where $w_{ij}^*=\frac{w_{ij}}{\sqrt{\sum_{j=1}^{n}w_{ij}^2}}$.

Consider $\sum_{t=1}^{T}\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\varepsilon_{it}\sum_{j=1}^{n}w_{ij}^*\varepsilon_{jt}$ as a zero-mean martingale over $t=1,...,T$. Then, Burkholder inequality implies that

$$E|\tilde{z}_{i,nT}|^{4+\eta} = E|\sum_{t=1}^{T}\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\varepsilon_{it}\sum_{j=1}^{n}w_{ij}^*\varepsilon_{jt}|^{4+\eta} \leq C_2 E|\sum_{t=1}^{T}\frac{1}{\sigma_0^4}\frac{1}{T}(\varepsilon_{it}\sum_{j=1}^{n}w_{ij}^*\varepsilon_{jt})^2|^{2+\frac{\eta}{2}} \quad \text{(C.23)}$$

where $C_2=(18pq^{1/2})^p$ with $p^{-1}+q^{-1}=1$. By the triangle inequality, we have

$$\begin{aligned}
||\tilde{z}_{i,nT}||_{4+\eta} = (E|\tilde{z}_{i,nT}|^{4+\eta})^{1/(4+\eta)} &\leq C_2^{1/(4+\eta)}\big(E|\sum_{t=1}^{T}\frac{1}{\sigma_0^4}\frac{1}{T}(\varepsilon_{it}\sum_{j=1}^{n}w_{ij}^*\varepsilon_{jt})^2|^{2+\frac{\eta}{2}}\big)^{1/(4+\eta)} \\
&= C_1\Big(\big(E|\sum_{t=1}^{T}\frac{1}{\sigma_0^4}\frac{1}{T}(\varepsilon_{it}\sum_{j=1}^{n}w_{ij}^*\varepsilon_{jt})^2|^{2+\frac{\eta}{2}}\big)^{1/(2+\frac{\eta}{2})}\Big)^{1/2} \\
&= C_1\big(||\frac{1}{\sigma_0^4}\frac{1}{T}\sum_{t=1}^{T}(\varepsilon_{it}\sum_{j=1}^{n}w_{ij}^*\varepsilon_{jt})^2||_{2+\frac{\eta}{2}}\big)^{1/2} \\
&\leq C_1\big(\frac{1}{\sigma_0^4}\frac{1}{T}\sum_{t=1}^{T}||(\varepsilon_{it}\sum_{j=1}^{n}w_{ij}^*\varepsilon_{jt})^2||_{2+\frac{\eta}{2}}\big)^{1/2}
\end{aligned} \quad \text{(C.24)}$$

where $C_1=C_2^{1/(4+\eta)}$. Thus, it remains to show $||(\varepsilon_{it}\sum_{j=1}^{n}w_{ij}^*\varepsilon_{jt})^2||_{2+\frac{\eta}{2}}<\infty$ or $E|(\varepsilon_{it}\sum_{j=1}^{n}w_{ij}^*\varepsilon_{jt})^2|^{2+\frac{\eta}{2}} = E|\varepsilon_{it}\sum_{j=1}^{n}w_{ij}^*\varepsilon_{jt}|^{4+\eta}<\infty$ for all i, t and n.

Under Assumptions 1 ($w_{ii}=0$ for all i) and 2 (ε_{it} are i.i.d. across i and t), $E|\varepsilon_{it}\sum_{j=1}^{n}w_{ij}^*\varepsilon_{jt}|^{4+\eta}$ can be written as

$$\begin{aligned}
E|\varepsilon_{it}\sum_{j=1}^{n}w_{ij}^*\varepsilon_{jt}|^{4+\eta} &= E|\varepsilon_{it}\sum_{j\neq i}^{n}w_{ij}^*\varepsilon_{jt}|^{4+\eta} \\
&= E|\varepsilon_{it}|^{4+\eta}\times E|\sum_{j\neq i}^{n}w_{ij}^*\varepsilon_{jt}|^{4+\eta}
\end{aligned} \quad \text{(C.25)}$$

Consider $\sum_{j\neq i}^{n}w_{ij}^*\varepsilon_{jt}$ as a zero-mean martingale over $j=1,...,n$. Then, by the Burkholder and triangular inequalities, we have

$$
\begin{aligned}
E|\varepsilon_{it}\sum_{j=1}^{n} w_{ij}^{*}\varepsilon_{jt}|^{4+\eta} &= E|\varepsilon_{it}|^{4+\eta}\times E|\sum_{j\neq i}^{n} w_{ij}^{*}\varepsilon_{jt}|^{4+\eta} \\
&\le E|\varepsilon_{it}|^{4+\eta}\times C_2 E|\sum_{j\neq i}^{n}(w_{ij}^{*}\varepsilon_{jt})^2|^{2+\frac{\eta}{2}} \\
&= E|\varepsilon_{it}|^{4+\eta}\times C_2\big((E|\sum_{j\neq i}^{n} w_{ij}^{*2}\varepsilon_{jt}^2|^{2+\frac{\eta}{2}})^{1/(2+\frac{\eta}{2})}\big)^{2+\frac{\eta}{2}} \qquad \text{(C.26)} \\
&\le E|\varepsilon_{it}|^{4+\eta}\times C_2\big(||\sum_{j\neq i}^{n} w_{ij}^{*2}\varepsilon_{jt}^2||_{2+\frac{\eta}{2}}\big)^{2+\frac{\eta}{2}} \\
&\le \sup_{n}\sup_{i,t} E|\varepsilon_{it}|^{4+\eta}\times C_2\big(\sum_{j\neq i}^{n} w_{ij}^{*2}||\varepsilon_{jt}^2||_{2+\frac{\eta}{2}}\big)^{2+\frac{\eta}{2}} < \infty
\end{aligned}
$$

because $\sup_{i,t} E|\varepsilon_{it}|^{4+\eta} < \infty$ and $\sum_{j\neq i}^{n} w_{ij}^{*2} = \sum_{j=1}^{n} w_{ij}^{*2} = 1$ for all i and n under Assumptions 1 and 2.

We next show that $\tilde{r}_{i,nT}$ is uniformly L_p bounded where $p = 4+\eta$, i.e., $\sup_{n}\sup_{i\in D_n} ||\tilde{r}_{i,nT}||_{4+\eta} < \infty$, or equivalently, $\sup_{n}\sup_{i\in D_n} ||\frac{1}{\sigma_0^2} T\bar{\epsilon}_{iT}\sum_{j=1}^{n} w_{ij}^{*}\bar{\epsilon}_{jT}||_{4+\eta} < \infty$ where $\bar{\epsilon}_{iT} = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}$. Consider $\sqrt{T}\bar{\epsilon}_{iT} = \sum_{t=1}^{T}\frac{1}{\sqrt{T}}\varepsilon_{it}$ as a zero-mean martingale over $t = 1, ..., T$. Then, Burkholder inequality implies

$$
E|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}|^{4+\eta} = E|\sum_{t=1}^{T}\frac{1}{\sqrt{T}}\varepsilon_{it}|^{4+\eta} \le C_2 E|\sum_{t=1}^{T}\frac{1}{T}(\varepsilon_{it})^2|^{2+\frac{\eta}{2}} \qquad \text{(C.27)}
$$

By the triangle inequality and Assumption 2, we have

$$
\begin{aligned}
(E|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}|^{4+\eta})^{1/(4+\eta)} &\le C_2^{1/(4+\eta)}\Big(\big(E|\sum_{t=1}^{T}\frac{1}{T}(\varepsilon_{it})^2|^{2+\frac{\eta}{2}}\big)^{1/(2+\frac{\eta}{2})}\Big)^{1/2} \\
&= C_1\Big(||\sum_{t=1}^{T}\frac{1}{T}(\varepsilon_{it})^2||_{2+\frac{\eta}{2}}\Big)^{1/2} \qquad \text{(C.28)} \\
&\le \sup_{i,t} C_1\Big(\frac{1}{T}\sum_{t=1}^{T}||(\varepsilon_{it})^2||_{2+\frac{\eta}{2}}\Big)^{1/2} < \infty
\end{aligned}
$$

which implies that $||\sqrt{T}\bar{\epsilon}_{iT}||_{4+\eta} < \infty$ or $E|\sqrt{T}\bar{\epsilon}_{iT}|^{4+\eta} < \infty$ for all i and n.

Since $\bar{\epsilon}_{iT}$ is independent across i and $w_{ii} = 0$ for all i, we have

$$E|\frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^{n} w_{ij}^{*}\bar{\epsilon}_{jT}|^{4+\eta} = E|\frac{1}{\sigma_0^2}\sqrt{T}\bar{\epsilon}_{iT}|^{4+\eta} \times E|\sum_{j\neq i}^{n} w_{ij}^{*}\sqrt{T}\bar{\epsilon}_{jT}|^{4+\eta} \tag{C.29}$$

Consider $\sum_{j\neq i}^{n} w_{ij}^{*}\sqrt{T}\bar{\epsilon}_{jT}$ as a zero-mean martingale over $j = 1, ..., n$. Then, by the Burkholder and triangular inequalities with the results above, we have

$$\begin{aligned}
E|\frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^{n} w_{ij}\bar{\epsilon}_{jT}|^{4+\eta} &= E|\frac{1}{\sigma_0^2}\sqrt{T}\bar{\epsilon}_{iT}|^{4+\eta} \times E|\sum_{j\neq i}^{n}(w_{ij}\sqrt{T}\bar{\epsilon}_{jT})^2|^{2+\frac{\eta}{2}} \\
&= E|\frac{1}{\sigma_0^2}\sqrt{T}\bar{\epsilon}_{iT}|^{4+\eta} \times \big((E|\sum_{j\neq i}^{n}(w_{ij}\sqrt{T}\bar{\epsilon}_{jT})^2|^{2+\frac{\eta}{2}})^{1/(2+\frac{\eta}{2})}\big)^{2+\frac{\eta}{2}} \\
&= E|\frac{1}{\sigma_0^2}\sqrt{T}\bar{\epsilon}_{iT}|^{4+\eta} \times \big(||\sum_{j\neq i}^{n} w_{ij}^2(\sqrt{T}\bar{\epsilon}_{jT})^2||_{2+\frac{\eta}{2}}\big)^{2+\frac{\eta}{2}} \\
&\leq \sup_n \sup_i E|\frac{1}{\sigma_0^2}\sqrt{T}\bar{\epsilon}_{iT}|^{4+\eta} \times \big(\sum_{j\neq i}^{n} w_{ij}^2||(\sqrt{T}\bar{\epsilon}_{jT})^2||_{2+\frac{\eta}{2}}\big)^{2+\frac{\eta}{2}} < \infty
\end{aligned} \tag{C.30}$$

because $\sup_n \sup_i E|\sqrt{T}\bar{\epsilon}_{iT}|^{4+\eta} < \infty$ and $\sum_{j\neq i}^{n} w_{ij}^{*2} = 1$ for all i and n.

C.9 Proof of Proposition 5

Consider $||\tilde{z}_{i,nT} - E(\tilde{z}_{i,nT}|\mathcal{F}_{i,nT}(s))||_2$ where $\tilde{z}_{i,nT} = \frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\sum_{j=1}^{n} w_{ij}\varepsilon_{jt}}{\sqrt{\sum_{j=1}^{n} w_{ij}^2}}$. Define an indicator function $\mathbf{1}\big(\rho(i,j) > s\big)$. Note that $\mathbf{1}\big(\rho(i,j) > s\big) = 0$ if the distance between i and j is equal to or less than s. As s gets larger, $\mathbf{1}\big(\rho(i,j) > s\big)$ goes to zero for all i, j. Then,

$\tilde{z}_{i,nT} - E(\tilde{z}_{i,nT}|\mathcal{F}_{i,nT}(s))$ can be derived as

$$
\begin{aligned}
&\tilde{z}_{i,nT} - E(\tilde{z}_{i,nT}|\mathcal{F}_{i,nT}(s)) = \frac{1}{\sqrt{\sum_{j=1}^n w_{ij}^2}}\Big(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\varepsilon_{jt} - E\big(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\varepsilon_{jt}|\mathcal{F}_{i,nT}(s)\big)\Big)\\
&= \frac{1}{\sqrt{\sum_{j=1}^n w_{ij}^2}}\Big(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j)>s\big)\varepsilon_{jt} + \frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j)\le s\big)\varepsilon_{jt}\\
&\quad - E\big(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j)>s\big)\varepsilon_{jt} + \frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j)\le s\big)\varepsilon_{jt}|\mathcal{F}_{i,nT}(s)\big)\Big)\\
&= \frac{1}{\sqrt{\sum_{j=1}^n w_{ij}^2}}\Big(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j)>s\big)\varepsilon_{jt} + \frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j)\le s\big)\varepsilon_{jt}\\
&\quad - \frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}E\big(\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j)>s\big)\varepsilon_{jt}\big) - \frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j)\le s\big)\varepsilon_{jt}\Big)\\
&= \frac{1}{\sqrt{\sum_{j=1}^n w_{ij}^2}}\big(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j)>s\big)\varepsilon_{jt}\big)
\end{aligned}
\tag{C.31}
$$

Then

$$
\begin{aligned}
&E|\tilde{z}_{i,nT} - E(\tilde{z}_{i,nT}|\mathcal{F}_{i,nT}(s))|^2 = \frac{1}{\sum_{j=1}^n w_{ij}^2}\frac{1}{\sigma_0^4}\frac{1}{T}E\big(\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j)>s\big)\varepsilon_{jt}\big)^2\\
&= \frac{1}{\sum_{j=1}^n w_{ij}^2}\frac{1}{\sigma_0^4}\frac{1}{T}\sum_{t=1}^T E\big(\varepsilon_{it}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j)>s\big)\varepsilon_{jt}\big)^2\\
&= \frac{1}{\sum_{j=1}^n w_{ij}^2}\frac{1}{\sigma_0^4}E\big(\varepsilon_{it}\sum_{j\neq i}^n w_{ij}\mathbf{1}\big(\rho(i,j)>s\big)\varepsilon_{jt}\big)^2\\
&= \frac{1}{\sum_{j=1}^n w_{ij}^2}\frac{1}{\sigma_0^4}E(\varepsilon_{it}^2)E\big(\sum_{j\neq i}^n w_{ij}\mathbf{1}\big(\rho(i,j)>s\big)\varepsilon_{jt}\big)^2\\
&= \frac{1}{\sum_{j=1}^n w_{ij}^2}\frac{1}{\sigma_0^4}E(\varepsilon_{it}^2)\sum_{j=1}^n w_{ij}^2\mathbf{1}\big(\rho(i,j)>s\big)E(\varepsilon_{jt}^2) = \frac{\sum_{j=1}^n w_{ij}^2\mathbf{1}\big(\rho(i,j)>s\big)}{\sum_{j=1}^n w_{ij}^2}
\end{aligned}
\tag{C.32}
$$

Therefore, for $||\tilde{z}_{i,nT} - E(\tilde{z}_{i,nT}|\mathcal{F}_{i,nT}(s))||_2$, we can take $\tilde{d}_{i,n} = 1$ for all i and n, and $\tilde{\gamma}(s) = \sup_n \sup_{i\in D_n}\sqrt{\frac{\sum_{j=1}^n w_{ij}^2\mathbf{1}\big(\rho(i,j)>s\big)}{\sum_{j=1}^n w_{ij}^2}}$ which goes to zero as $s\to\infty$.

Consider $||\tilde{r}_{i,nT} - E(\tilde{r}_{i,nT}|\mathcal{F}_{i,nT}(s))||_2$ where $\tilde{r}_{i,nT} = \dfrac{\frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^n w_{ij}\bar{\epsilon}_{jT}}{\sqrt{\sum_{j=1}^n w_{ij}^2}}$ with $\bar{\epsilon}_{iT} = \dfrac{1}{T}\sum_{t=1}^T \varepsilon_{it}$. Then, $\tilde{r}_{i,nT} - E(\tilde{r}_{i,nT}|\mathcal{F}_{i,nT}(s))$ can be written as

$$
\begin{aligned}
&\tilde{r}_{i,nT} - E(\tilde{r}_{i,nT}|\mathcal{F}_{i,nT}(s)) = \frac{1}{\sqrt{\sum_{j=1}^n w_{ij}^2}}\Big(\frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^n w_{ij}\bar{\epsilon}_{jT} - E\big(\frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^n w_{ij}\bar{\epsilon}_{jT}|\mathcal{F}_{i,nT}(s)\big)\Big)\\
&= \frac{1}{\sqrt{\sum_{j=1}^n w_{ij}^2}}\Big(\frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j) > s\big)\bar{\epsilon}_{jT} + \frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j) \le s\big)\bar{\epsilon}_{jT}\\
&\quad - E\big(\frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j) > s\big)\bar{\epsilon}_{jT} + \frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j) \le s\big)\bar{\epsilon}_{jT}|\mathcal{F}_{i,nT}(s)\big)\Big)\\
&= \frac{1}{\sqrt{\sum_{j=1}^n w_{ij}^2}}\Big(\frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j) > s\big)\bar{\epsilon}_{jT} + \frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j) \le s\big)\bar{\epsilon}_{jT}\\
&\quad - \bar{\epsilon}_{iT}E\big(\frac{1}{\sigma_0^2}T\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j) > s\big)\bar{\epsilon}_{jT}\big) - \frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j) \le s\big)\bar{\epsilon}_{jT}\Big)\\
&= \frac{1}{\sqrt{\sum_{j=1}^n w_{ij}^2}}\big(\frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j) > s\big)\bar{\epsilon}_{jT}\big)
\end{aligned}
\tag{C.33}
$$

where $\bar{\epsilon}_{iT} = \dfrac{1}{T}\sum_{t=1}^T \varepsilon_{it}$ is independent across i with $E(\bar{\epsilon}_{iT})^2 = \dfrac{1}{T^2}\sum_{t=1}^T E(\varepsilon_{it})^2 = \dfrac{1}{T}\sigma_0^2$. Then

$$
\begin{aligned}
&E|\tilde{r}_{i,nT} - E(\tilde{r}_{i,nT}|\mathcal{F}_{i,nT}(s))|^2 = \frac{1}{\sum_{j=1}^n w_{ij}^2}E\big(\frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j=1}^n w_{ij}\mathbf{1}\big(\rho(i,j) > s\big)\bar{\epsilon}_{jT}\big)^2\\
&= \frac{1}{\sum_{j=1}^n w_{ij}^2}E\big(\frac{1}{\sigma_0^2}T\bar{\epsilon}_{iT}\sum_{j\neq i}^n w_{ij}\mathbf{1}\big(\rho(i,j) > s\big)\bar{\epsilon}_{jT}\big)^2\\
&= \frac{1}{\sum_{j=1}^n w_{ij}^2}\frac{1}{\sigma_0^4}T^2E(\bar{\epsilon}_{iT})^2E\big(\sum_{j\neq i}^n w_{ij}\mathbf{1}\big(\rho(i,j) > s\big)\bar{\epsilon}_{jT}\big)^2\\
&= \frac{1}{\sum_{j=1}^n w_{ij}^2}\frac{1}{\sigma_0^4}T^2E(\bar{\epsilon}_{iT})^2\sum_{j=1}^n w_{ij}^2\mathbf{1}\big(\rho(i,j) > s\big)E(\bar{\epsilon}_{jT})^2 = \frac{\sum_{j=1}^n w_{ij}^2\mathbf{1}\big(\rho(i,j) > s\big)}{\sum_{j=1}^n w_{ij}^2}
\end{aligned}
\tag{C.34}
$$

Therefore, for $||\tilde{r}_{i,nT} - E(\tilde{r}_{i,nT}|\mathcal{F}_{i,nT}(s))||_2$, we can take $\tilde{d}_{i,n} = 1$ for all i and n, and $\tilde{\gamma}(s) = \sup_n \sup_{i\in D_n} \sqrt{\dfrac{\sum_{j=1}^n w_{ij}^2\mathbf{1}\big(\rho(i,j) > s\big)}{\sum_{j=1}^n w_{ij}^2}}$ which goes to zero as $s \to \infty$. Since $\sup_n \sup_{i\in D_n} \tilde{d}_{i,n} = 1 < \infty$, $\tilde{Z} = \{\tilde{z}_{i,nT}, i \in D_n, n \geq 1\}$ and $\tilde{R} = \{\tilde{r}_{i,nT}, i \in D_n, n \geq 1\}$ are uniformly L_2-NED on ξ.

C.10 Proof of Proposition 6

The NED property is kept under summation, product (Lemma A.2) and Lipschitz transformations (Xu and Lee, 2015). Using the result of Proposition 5 with Lemma A.2 in Xu and Lee (2015) and Lemma 2 ($\sup_n \sup_{i\in D_n} ||\tilde{z}_{i,nT}||_{4+\eta} < \infty$), we have

$$\begin{aligned} ||z_{i,nT} - E(z_{i,nT}|\mathcal{F}_{i,nT}(s))||_2 &= ||(\tilde{z}_{i,nT})^2 - E((\tilde{z}_{i,nT})^2|\mathcal{F}_{i,nT}(s))||_2 \\ &\leq d_{i,n}\gamma(s) \end{aligned} \tag{C.35}$$

where $\gamma(s) = \tilde{\gamma}(s)^{\frac{\eta}{4+2\eta}}$ for some $\eta > 0$ and $\sup_n \sup_{i\in D_n} d_{i,n} < \infty$. Thus, for $||z_{i,nT} - E(z_{i,nT}|\mathcal{F}_{i,nT}(s))||_2$, we take $\gamma(s) = \sup_n \sup_{i\in D_n} \Big(\frac{\sum_{j=1}^n w_{ij}^2 \mathbf{1}\big(\rho(i,j) > s\big)}{\sum_{j=1}^n w_{ij}^2}\Big)^{\frac{\eta}{8+4\eta}}$ where $0 < \frac{\eta}{8+4\eta} \leq \frac{1}{4}$. Because $\tilde{z}_{i,nT}$ and $\tilde{r}_{i,nT}$ have the same NED coefficient and scaling factor, similar arguments can be applied to $r_{i,nT} = (\tilde{r}_{i,nT})^2$ and $\tilde{z}_{i,nT}\tilde{r}_{i,nT}$. Since $\sup_n \sup_{i\in D_n} d_{i,n} < \infty$ by Lemma A.2 in Xu and Lee (2015), $Z = \{z_{i,nT}, i \in D_n, n \geq 1\}$, $R = \{r_{i,nT}, i \in D_n, n \geq 1\}$ and $K = \{\tilde{z}_{i,nT}\tilde{r}_{i,nT}, i \in D_n, n \geq 1\}$ are uniformly L_2-NED on ξ.

C.11 Proof of Proposition 7

By Proposition 6, $Z = \{z_{i,nT}, i \in D_n, n \geq 1\}$ is uniformly L_2-NED on ξ with $\gamma(s) = \sup_n \sup_{i\in D_n} \Big(\frac{\sum_{j=1}^n w_{ij}^2 \mathbf{1}\big(\rho(i,j) > s\big)}{\sum_{j=1}^n w_{ij}^2}\Big)^{\frac{\eta}{8+4\eta}}$. Following Jenish and Prucha (2012), the sufficient conditions for $\sigma_{Q_{nT}}^{-1} \sum_{i=1}^n \big(z_{i,nT} - E(z_{i,nT})\big) \xrightarrow{d} N(0,1)$ where $\sigma_{Q_{nT}}^2 = Var\Big(\sum_{i=1}^n \big(z_{i,nT} - E(z_{i,nT})\big)\Big)$ in the case of $|D_n| = n$ and $\sup_n \sup_{i\in D_n} c_{i,n} = \sup_n \sup_{i\in D_n} d_{i,n} < \infty$ are:

(i) $z_{i,nT}$ are uniformly L_p-bounded for $p > 2 + \delta$ for some $\delta > 0$

(ii) $\liminf_{n\to\infty} |D_n|^{-1}\sigma_{Q_{nT}}^2 = \frac{1}{n}\sigma_{Q_{nT}}^2 > 0$

(iii) $\gamma(s) = \sup_n \sup_{i\in D_n} \Big(\frac{\sum_{j=1}^n w_{ij}^2 \mathbf{1}\big(\rho(i,j) > s\big)}{\sum_{j=1}^n w_{ij}^2}\Big)^{\frac{\eta}{8+4\eta}}$ satisfies $\sum_{s=1}^{\infty} s^{d-1}\gamma(s) < \infty$ for some $\eta > 0$.

First, condition (i) is satisfied by Lemma 2. Next, condition (ii) is shown in Proposition 4 since $\Sigma_{Q_{nT}} = Var(\frac{1}{\sqrt{2n}} \sum_{i=1}^n z_{i,nT}) = \frac{1}{2n}\sigma_{Q_{nT}}^2 = 1 + O(\frac{1}{T})$. Hence, it remains to check

$\sum_{s=1}^{\infty} s^{d-1}\gamma(s) < \infty$. Consider $\sum_{s=1}^{\infty} s^{d-1}\gamma(s)$ as

$$\sum_{s=1}^{\infty} s^{d-1}\gamma(s) = \sup_n \sup_{i\in D_n} \sum_{s=1}^{\infty} s^{d-1}\Big(\frac{\sum_{j=1}^{n} w_{ij}^2 \mathbf{1}\big(\rho(i,j) > s\big)}{\sum_{j=1}^{n} w_{ij}^2}\Big)^{\frac{\eta}{8+4\eta}} \tag{C.36}$$

Under Assumption 10(1), w_{ij} can be non-zero only if $\rho(i,j) \leq \bar{\rho}_0$. Since $w_{ij}^2\mathbf{1}\big(\rho(i,j) > s\big) = 0$ for any $s \geq [\bar{\rho}_0] + 1$, we have

$$\sup_n \sup_{i\in D_n} \sum_{s=1}^{\infty} s^{d-1}\Big(\sum_{j=1}^{n} w_{ij}^2 \mathbf{1}\big(\rho(i,j) > s\big)\Big)^{\frac{\eta}{8+4\eta}} = \sup_n \sup_{i\in D_n} \sum_{s=1}^{[\bar{\rho}_0]} s^{d-1}\Big(\sum_{j=1}^{n} w_{ij}^2 \mathbf{1}\big(\rho(i,j) > s\big)\Big)^{\frac{\eta}{4+2\eta}} < \infty \tag{C.37}$$

since the term becomes a sum of finite and bounded series under Assumption 10(1).

Alternatively, under Assumption 10(2) ($|w_{ij}| \leq C_0/\rho(i,j)^{\alpha}$), we have $w_{ij}^2 \leq \tilde{C}_0/\rho(i,j)^{\tilde{\alpha}}$ where $\tilde{\alpha} = 2\alpha$ and $\tilde{C}_0 = C_0^2$. By Lemma A.1 in Jenish and Prucha (2009), $|\{m : x \leq \rho(i,j) < x+1\}| \leq Cx^{d-1}$ for some constant $C > 0$ when $x \geq 1$. Then, we have

$$\begin{aligned}
\sup_n \sup_{i\in D_n} \sum_{s=1}^{\infty} s^{d-1}\Big(\sum_{j=1}^{n} w_{ij}^2 \mathbf{1}\big(\rho(i,j) > s\big)\Big)^{\frac{\eta}{8+4\eta}} &\leq \sup_n \sup_{i\in D_n} \sum_{s=1}^{\infty} s^{d-1}\Big(\sum_{x=[s]}^{\infty} \sum_{x\leq\rho(i,j)<x+1} \tilde{C}_0\rho(i,j)^{-\tilde{\alpha}}\Big)^{\frac{\eta}{4+2\eta}} \\
&\leq \sum_{s=1}^{\infty} s^{d-1}\Big(\sum_{x=[s]}^{\infty} Cx^{d-1}\tilde{C}_0 x^{-\tilde{\alpha}}\Big)^{\frac{\eta}{4+2\eta}} \\
&\leq \sum_{s=1}^{\infty} s^{d-1}\Big(\sum_{x=[s]}^{\infty} C\tilde{C}_0(x+1)^{d-1}[(x+1)/2]^{-\tilde{\alpha}}\Big)^{\frac{\eta}{4+2\eta}} \\
&\leq \sum_{s=1}^{\infty} s^{d-1}\Big(C\tilde{C}_0 2^{\tilde{\alpha}} \int_x^{\infty} u^{-\tilde{\alpha}+d-1}\, du\Big)^{\frac{\eta}{4+2\eta}} \\
&= \sum_{s=1}^{\infty} s^{d-1}\Big(C\tilde{C}_0 2^{\tilde{\alpha}}(\tilde{\alpha}-d)^{-1}s^{d-\tilde{\alpha}}\Big)^{\frac{\eta}{4+2\eta}} \\
&= \sum_{s=1}^{\infty} s^{d-1+\frac{(d-\tilde{\alpha})\eta}{4+2\eta}}\Big(C\tilde{C}_0 2^{\tilde{\alpha}}(\tilde{\alpha}-d)^{-1}\Big)^{\frac{\eta}{4+2\eta}}
\end{aligned} \tag{C.38}$$

where $\left(C\tilde{C}_0 2^{\tilde{\alpha}}(\tilde{\alpha}-d)^{-1}\right)^{\frac{\eta}{4+2\eta}} < \infty$. Thus, the infinite series $\sum_{s=1}^{\infty} s^{d-1+\frac{(d-\tilde{\alpha})\eta}{4+2\eta}}$ converges only if $d-1+\frac{(d-\tilde{\alpha})\eta}{4+2\eta} < -1$. Hence, we have $\sum_{s=1}^{\infty} s^{d-1}\gamma(s) < \infty$ so long as $\tilde{\alpha} > d\cdot(3+4\eta^{-1})$, which implies that $\alpha > d\cdot(1.5+2\eta^{-1})$ (Assumption 11) because $\tilde{\alpha}=2\alpha$.

Finally, using the result $(\frac{1}{\sqrt{2n}}\sigma_{Q_{nT}})^{-1}\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\big(z_{i,nT}-E(z_{i,nT})\big) \xrightarrow{d} N(0,1)$, we have $Q_{nT}=\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\big(z_{i,nT}-E(z_{i,nT})\big) \xrightarrow{d} N(0,1)$.

C.12 Proof of Proposition 8

First, the LLN does not require any restrictions on the NED coefficient. By Proposition 6, $R=\{r_{i,nT}, i\in D_n, n\geq 1\}$ is uniformly L_2-NED on ξ with $\gamma(s)=\sup_n\sup_{i\in D_n}\Big(\frac{\sum_{j=1}^n w_{ij}^2\mathbf{1}\big(\rho(i,j)>s\big)}{\sum_{j=1}^n w_{ij}^2}\Big)^{\frac{\eta}{8+4\eta}}$. Following Jenish and Prucha (2012), the sufficient condition for $P_{nT}=\frac{1}{n}\sum_{i=1}^{n}\big(r_{i,nT}-E(r_{i,nT})\big)\xrightarrow{p} 0$ in the case of $|D_n|=n$ and $\sup_n\sup_{i\in D_n} c_{i,n}=\sup_n\sup_{i\in D_n} d_{i,n}<\infty$ is:

(i) $r_{i,nT}$ are uniformly L_p-bounded for $p>1+\delta$ for some $\delta>0$. Condition (i) is satisfied by Lemma 2. Hence, we have $P_{nT}=\frac{1}{n}\sum_{i=1}^{n}\big(r_{i,nT}-E(r_{i,nT})\big)\xrightarrow{p} 0$.

Similarly, by Propositions 5 and 6, $\tilde{z}_{i,nT}\tilde{r}_{i,nT}$ is uniformly L_2-NED on ξ with $\gamma(s)=\sup_n\sup_{i\in D_n}\Big(\frac{\sum_{j=1}^n w_{ij}^2\mathbf{1}\big(\rho(i,j)>s\big)}{\sum_{j=1}^n w_{ij}^2}\Big)^{\frac{\eta}{8+4\eta}}$. Following Jenish and Prucha (2012), the sufficient condition for in the case of $|D_n|=n$ and $\sup_n\sup_{i\in D_n} c_{i,n}=\sup_n\sup_{i\in D_n} d_{i,n}<\infty$ is:

(i) $\tilde{z}_{i,nT}\tilde{r}_{i,nT}$ are uniformly L_p-bounded for $p>1+\delta$ for some $\delta>0$. For Condition (i), by Holder inequality when $p=2$ and $q=2$, we have

$$\begin{aligned} E|(\tilde{z}_{i,nT})^{2+\frac{\eta}{2}}(\tilde{r}_{i,nT})^{2+\frac{\eta}{2}}| &\leq ||(\tilde{z}_{i,nT})^{2+\frac{\eta}{2}}||_2||(\tilde{r}_{i,nT})^{2+\frac{\eta}{2}}||_2 \\ &\leq \big(E|\tilde{z}_{i,nT}|^{4+\eta}\cdot E|\tilde{r}_{i,nT}|^{4+\eta}\big)^{1/2}<\infty \end{aligned} \tag{C.39}$$

since $\sup_n\sup_{i\in D_n}||\tilde{z}_{i,nT}||_{4+\eta}<\infty$ and $\sup_n\sup_{i\in D_n}||\tilde{r}_{i,nT}||_{4+\eta}<\infty$ by Lemma 2. Hence, Condition (i) is satisfied, and we have $U_{nT}=\frac{1}{n}\sum_{i=1}^{n}\big(\tilde{z}_{i,nT}\tilde{r}_{i,nT}-E(\tilde{z}_{i,nT}\tilde{r}_{i,nT})\big)\xrightarrow{p} 0$.

C.13 Proof of Theorem 3

Under H_0 and Assumption 8, $S = S^{nr}$ takes the following form:

$$S = (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 Q_{nT} + \frac{1}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \frac{\sqrt{n}}{T} P_{nT} - \frac{2}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \sqrt{\frac{n}{T}} U_{nT} - \frac{1}{\sqrt{2}}(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 \frac{\sqrt{n}}{T} + \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} \big((\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 - 1\big) \tag{30}$$

where $\frac{\sigma_0^2}{\tilde{\sigma}^2} \xrightarrow{p} 1$ by Proposition 3. Proposition 7 is applied to Q_{nT}. The second and third terms converge to zero by Proposition 8 as long as $\frac{n}{T} \to k < \infty$. The fourth term goes to zero as long as $\frac{n}{T} \to k < \infty$. The last term can be rewritten as

$$\begin{aligned} \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} \big((\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 - 1\big) = \frac{\sqrt{n}}{\sqrt{2}}\big((\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 - 1\big) &= \frac{\sqrt{n}}{\sqrt{2}}\big((\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 - 2\frac{\sigma_0^2}{\tilde{\sigma}^2} + 2\frac{\sigma_0^2}{\tilde{\sigma}^2} + 1 - 1 - 1\big) \\ &= \frac{\sqrt{n}}{\sqrt{2}}\big((\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 - 2\frac{\sigma_0^2}{\tilde{\sigma}^2} + 1\big) + \frac{\sqrt{n}}{\sqrt{2}}\big(2\frac{\sigma_0^2}{\tilde{\sigma}^2} - 2\big) \qquad \text{(C.40)} \\ &= \frac{\sqrt{n}}{\sqrt{2}}\big(\frac{\sigma_0^2}{\tilde{\sigma}^2} - 1\big)^2 + \sqrt{2n}\big(\frac{\sigma_0^2}{\tilde{\sigma}^2} - 1\big) \end{aligned}$$

which goes to zero by Proposition 3 $\big(\frac{\sigma_0^2}{\tilde{\sigma}^2} - 1 = O_P(\frac{1}{\sqrt{nT}})\big)$. Therefore, for the non-reciprocal interactions in the form of Assumption 8, $S = \frac{1}{\sqrt{2n}}\big(\frac{1}{T} g_{nT}(\tilde{\sigma}^2)' \Sigma_{g,n}^{-1} g_{nT}(\tilde{\sigma}^2) - n\big) \xrightarrow{d} N(0,1)$ under $\frac{n}{T} \to k$ where $0 \le k < \infty$.

C.14 Proof of Proposition 9

Since $\rho(B_n) \le ||B_n||_\infty < 1$, we have $\lim_{k\to\infty} ||B_n^k||_\infty = 0$, the (i,j) entry of B_n^k, $\lim_{k\to\infty} (B_n^k)_{i,j} = 0$ for all i, j, and $\sum_{k=0}^{\infty} B_n^k < \infty$. If $\frac{1}{\sqrt{2n}} \frac{1}{T} g_{nT}(\tilde{\sigma}^2)' B_n \Sigma_{g,n}^{D-1} g_{nT}(\tilde{\sigma}^2) = o_p(1)$, then it follows that $\frac{1}{\sqrt{2n}} \frac{1}{T} g_{nT}(\tilde{\sigma}^2)' B_n^2 \Sigma_{g,n}^{D-1} g_{nT}(\tilde{\sigma}^2) = o_p(1)$ since the norm of B_n^k decreases as k increases. Similar arguments can be applied to $\frac{1}{\sqrt{2n}} \frac{1}{T} g_{nT}(\tilde{\sigma}^2)' B_n^k \Sigma_{g,n}^{D-1} g_{nT}(\tilde{\sigma}^2)$ for all $k \ge 3$ where $\sum_{k=0}^{\infty} B_n^k < \infty$.

C.15 Proof of Proposition 10

Recall (35) where $\frac{\sigma_0^2}{\tilde{\sigma}^2} \xrightarrow{p} 1$. Denote $a_{n,ij} = \frac{s_i' W_n s_j s_j' W_n s_i}{\sqrt{s_i' W_n W_n' s_i}\sqrt{s_j' W_n W_n' s_j}} = \frac{w_{ij} w_{ji}}{\sqrt{\sum_{j=1}^n w_{ij}^2}\sqrt{\sum_{i=1}^n w_{ji}^2}}$.

Thus, it remains to show that $\frac{1}{\sqrt{nT}}\sum_{i=1}^n \tilde{r}_{i,nT}\sum_{j\neq i}^n a_{n,ij}\tilde{z}_{j,nT}$ and $\frac{1}{\sqrt{nT}}\sum_{i=1}^n \tilde{r}_{i,nT}\sum_{j\neq i}^n a_{n,ij}\tilde{r}_{j,nT}$ are $o_p(1)$, presuming $\frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde{z}_{i,nT}\sum_{j\neq i}^n a_{n,ij}\tilde{z}_{j,nT} = o_p(1)$. Note that

$$\begin{aligned}\tilde{r}_{i,nT} = \frac{\frac{1}{\sigma_0^2} T\bar{\epsilon}_{iT}\sum_{j=1}^n w_{ij}\bar{\epsilon}_{jT}}{\sqrt{\sum_{j=1}^n w_{ij}^2}} &= \frac{\frac{1}{\sigma_0^2}(\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it})\sum_{j=1}^n w_{ij}(\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{jt})}{\sqrt{\sum_{j=1}^n w_{ij}^2}} \\ &= \frac{1}{\sqrt{T}}\tilde{z}_{i,nT} + \frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\frac{1}{\sqrt{T}}\sum_{s\neq t}^T \varepsilon_{js}}{\sqrt{\sum_{j=1}^n w_{ij}^2}}\end{aligned} \tag{C.41}$$

Suppose $\frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde{z}_{i,nT}\sum_{j\neq i}^n a_{n,ij}\tilde{z}_{j,nT} = \frac{1}{\sqrt{n}}\sum_{i=1}^n (\frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\varepsilon_{jt}}{\sqrt{\sum_{j=1}^n w_{ij}^2}})\sum_{j\neq i}^n a_{n,ij}\tilde{z}_{j,nT} = o_p(1)$. Then

$$\begin{aligned}&\frac{1}{\sqrt{nT}}\sum_{i=1}^n \tilde{r}_{i,nT}\sum_{j\neq i}^n a_{n,ij}\tilde{z}_{j,nT} \\ &= \frac{1}{T}\frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde{z}_{i,nT}\sum_{j\neq i}^n a_{n,ij}\tilde{z}_{j,nT} + \frac{1}{\sqrt{nT}}\sum_{i=1}^n (\frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\frac{1}{\sqrt{T}}\sum_{s\neq t}^T \varepsilon_{js}}{\sqrt{\sum_{j=1}^n w_{ij}^2}})\sum_{j\neq i}^n a_{n,ij}\tilde{z}_{j,nT} \\ &= \frac{1}{T}\frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde{z}_{i,nT}\sum_{j\neq i}^n a_{n,ij}\tilde{z}_{j,nT} + \frac{1}{\sqrt{n}}\sum_{i=1}^n (\frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\frac{1}{T}\sum_{s\neq t}^T \varepsilon_{js}}{\sqrt{\sum_{j=1}^n w_{ij}^2}})\sum_{j\neq i}^n a_{n,ij}\tilde{z}_{j,nT} = o_p(1)\end{aligned} \tag{C.42}$$

provided that the second term is $o_p(1)$. For the second term, the inner structure only involves $\frac{1}{T}\sum_{s\neq t}^T \varepsilon_{js}$ rather than ε_{jt}. Since $\frac{1}{T}\sum_{s\neq t}^T \varepsilon_{js}$ does not dominate the overall convergence for large enough T, $\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\frac{1}{T}\sum_{s\neq t}^T \varepsilon_{js}$ and $\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T \varepsilon_{it}\sum_{j=1}^n w_{ij}\varepsilon_{jt}$ behave similarly. Thus, we have the same $o_p(1)$ result. Similar arguments can be applied to $\frac{1}{\sqrt{nT}}\tilde{r}_{i,nT}\sum_{j\neq i}^n a_{n,ij}\tilde{r}_{j,nT}$.

C.16 Proof of Proposition 11

Denote $\tilde{b}_{n,ij} = \frac{s_i' W_n s_j s_j' W_n s_i}{s_i' W_n W_n' s_i s_j' W_n W_n' s_j} = \frac{w_{ij} w_{ji}}{\sum_{j=1}^n w_{ij}^2 \sum_{i=1}^n w_{ji}^2}$. From (23), $G_{nT} = \frac{1}{\sqrt{n}} \sum_{i=1}^n g_{i,nT}$ can be rewritten as

$$\begin{aligned}
G_{nT} &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{z}_{i,nT} \sum_{j \neq i}^n \frac{s_i' W_n s_j s_j' W_n s_i}{\sqrt{s_i' W_n W_n' s_i} \sqrt{s_j' W_n W_n' s_j}} \tilde{z}_{j,nT} \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{\frac{1}{\sigma_0^2} \frac{1}{\sqrt{T}} \sum_{t=1}^T V_{nt}' s_i s_i' W_n V_{nt}}{\sqrt{s_i' W_n W_n' s_i}} \right) \sum_{j \neq i}^n \frac{s_i' W_n s_j s_j' W_n s_i}{\sqrt{s_i' W_n W_n' s_i} \sqrt{s_j' W_n W_n' s_j}} \left(\frac{\frac{1}{\sigma_0^2} \frac{1}{\sqrt{T}} \sum_{t=1}^T V_{nt}' s_j s_j' W_n V_{nt}}{\sqrt{s_j' W_n W_n' s_j}} \right) \\
&= \frac{1}{\sigma_0^4} \frac{1}{\sqrt{n}} \frac{1}{T} \sum_{i=1}^n \sum_{t=1}^T V_{nt}' s_i s_i' W_n V_{nt} \sum_{j \neq i}^n \tilde{b}_{n,ij} V_{nt}' s_j s_j' W_n V_{nt} \\
&\quad + \frac{1}{\sigma_0^4} \frac{1}{\sqrt{n}} \frac{1}{T} \sum_{i=1}^n \sum_{t=1}^T V_{nt}' s_i s_i' W_n V_{nt} \sum_{j \neq i}^n \tilde{b}_{n,ij} \sum_{s=1}^{t-1} V_{ns}' s_j s_j' W_n V_{ns} \\
&\quad + \frac{1}{\sigma_0^4} \frac{1}{\sqrt{n}} \frac{1}{T} \sum_{i=1}^n \sum_{t=1}^T V_{nt}' s_i s_i' W_n V_{nt} \sum_{j \neq i}^n \tilde{b}_{n,ij} \sum_{s=t+1}^{T} V_{ns}' s_j s_j' W_n V_{ns}
\end{aligned} \tag{C.43}$$

We first analyze the moment of each term: From (F.3) and (F.9) and Lemma A.1, we have

$$\begin{aligned}
& E\left(\frac{1}{\sigma_0^4} \frac{1}{\sqrt{n}} \frac{1}{T} \sum_{i=1}^n \sum_{t=1}^T V_{nt}' s_i s_i' W_n V_{nt} \sum_{j \neq i}^n \tilde{b}_{n,ij} V_{nt}' s_j s_j' W_n V_{nt}\right) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \sum_{j \neq i}^n \left(\frac{w_{ij} w_{ji}}{\sqrt{\sum_{j=1}^n w_{ij}^2} \sqrt{\sum_{i=1}^n w_{ji}^2}} \right)^2 = O\left(\frac{\sqrt{n}}{h_n^{*2}}\right) \qquad (C.44)
\end{aligned}$$

$$E\left(\frac{1}{\sigma_0^4} \frac{1}{\sqrt{n}} \frac{1}{T} \sum_{i=1}^n \sum_{t=1}^T V_{nt}' s_i s_i' W_n V_{nt} \sum_{j \neq i}^n \tilde{b}_{n,ij} \sum_{s=1}^{t-1} V_{ns}' s_j s_j' W_n V_{ns}\right) = 0 \tag{C.45}$$

$$E\left(\frac{1}{\sigma_0^4} \frac{1}{\sqrt{n}} \frac{1}{T} \sum_{i=1}^n \sum_{t=1}^T V_{nt}' s_i s_i' W_n V_{nt} \sum_{j \neq i}^n \tilde{b}_{n,ij} \sum_{s=t+1}^{T} V_{ns}' s_j s_j' W_n V_{ns}\right) = 0 \tag{C.46}$$

and $O(\frac{\sqrt{n}}{h_n^{*2}})$ in (C.44) goes to zero under Assumption 13. Thus, by Chebyshev's inequality, it remains to show that the second moment of each term in (C.43) goes to zero to show that $G_{nT} \xrightarrow{p} 0$.

Consider the second moment of the first term in (C.43) as

$$
\begin{aligned}
&E(\frac{1}{\sigma_0^4}\frac{1}{\sqrt{n}}\frac{1}{T}\sum_{i=1}^{n}\sum_{t=1}^{T}V_{nt}'s_is_i'W_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V_{nt}'s_js_j'W_nV_{nt})^2 \\
&=\frac{1}{\sigma_0^8}\frac{1}{n}\frac{1}{T^2}\sum_{t=1}^{T}E(\sum_{i=1}^{n}V_{nt}'s_is_i'W_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V_{nt}'s_js_j'W_nV_{nt})^2 \\
&=\frac{1}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}V_{nt}'s_is_i'W_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V_{nt}'s_js_j'W_nV_{nt})^2 \\
&=\frac{1}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}V_{nt}'s_is_i'W_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V_{nt}'s_js_j'W_nV_{nt}\sum_{l=1}^{n}V_{nt}'s_ls_l'W_nV_{nt}\sum_{m\neq l}^{n}\tilde{b}_{n,lm}V_{nt}'s_ms_m'W_nV_{nt}) \\
&=\frac{1}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}V_{nt}'s_is_i'W_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V_{nt}'s_js_j'W_nV_{nt}V_{nt}'s_is_i'W_nV_{nt}\sum_{m\neq i}^{n}\tilde{b}_{n,im}V_{nt}'s_ms_m'W_nV_{nt}) \\
&\quad+\frac{1}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}V_{nt}'s_is_i'W_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V_{nt}'s_js_j'W_nV_{nt}V_{nt}'s_js_j'W_nV_{nt}\sum_{m\neq j}^{n}\tilde{b}_{n,jm}V_{nt}'s_ms_m'W_nV_{nt}) \\
&\quad+\frac{1}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}V_{nt}'s_is_i'W_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V_{nt}'s_js_j'W_nV_{nt}\sum_{l\neq i,j}^{n}V_{nt}'s_ls_l'W_nV_{nt}\sum_{m\neq l}^{n}\tilde{b}_{n,lm}V_{nt}'s_ms_m'W_nV_{nt}) \\
&=\frac{2}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}(V_{nt}'s_is_i'W_nV_{nt})^2\sum_{j\neq i}^{n}\tilde{b}_{n,ij}(V_{nt}'s_js_j'W_nV_{nt})^2\tilde{b}_{n,ij}) \\
&\quad+\frac{1}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}(V_{nt}'s_is_i'W_nV_{nt})^2\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V_{nt}'s_js_j'W_nV_{nt}\sum_{m\neq i,j}^{n}\tilde{b}_{n,im}V_{nt}'s_ms_m'W_nV_{nt}) \\
&\quad+\frac{1}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}V_{nt}'s_is_i'W_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}(V_{nt}'s_js_j'W_nV_{nt})^2\sum_{m\neq i,j}^{n}\tilde{b}_{n,jm}V_{nt}'s_ms_m'W_nV_{nt}) \\
&\quad+\frac{1}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}(V_{nt}'s_is_i'W_nV_{nt})^2\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V_{nt}'s_js_j'W_nV_{nt}\sum_{l\neq i,j}^{n}V_{nt}'s_ls_l'W_nV_{nt}\tilde{b}_{n,li}) \\
&\quad+\frac{1}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}V_{nt}'s_is_i'W_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}(V_{nt}'s_js_j'W_nV_{nt})^2\sum_{l\neq i,j}^{n}V_{nt}'s_ls_l'W_nV_{nt}\tilde{b}_{n,lj}) \\
&\quad+\frac{1}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}V_{nt}'s_is_i'W_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V_{nt}'s_js_j'W_nV_{nt}\sum_{l\neq i,j}^{n}V_{nt}'s_ls_l'W_nV_{nt}\sum_{m\neq i,j,l}^{n}\tilde{b}_{n,lm}V_{nt}'s_ms_m'W_nV_{nt})
\end{aligned}
\tag{C.47}
$$

For the first term in (C.47), from (F.5), we have

$$
\begin{aligned}
&\frac{1}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}(V_{nt}'s_is_i'W_nV_{nt})^2\sum_{j\neq i}^{n}\tilde{b}_{n,ij}(V_{nt}'s_js_j'W_nV_{nt})^2\tilde{b}_{n,ij})\\
&=\frac{1}{\sigma_0^8}\frac{1}{nT}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}^2E(V_{nt}'s_is_i'W_nV_{nt})^2(V_{nt}'s_js_j'W_nV_{nt})^2\\
&=\frac{1}{\sigma_0^8}\frac{1}{nT}\sum_{i=1}^{n}\sum_{j\neq i}^{n}(\frac{w_{ij}w_{ji}}{\sum_{j=1}^n w_{ij}^2\sum_{i=1}^n w_{ji}^2})^2\big(\sigma_0^8(s_i'W_nW_n's_i)(s_j'W_nW_n's_j)+((\mu_4)^2-2\mu_4\sigma_0^4+(\sigma_0^4)^2)(s_i'W_ns_js_j'W_ns_i)^2\\
&\quad+(\mu_4-\sigma_0^4)\sigma_0^4(s_j'W_ns_i)^2(s_i'W_nW_n's_i)+(\mu_4-\sigma_0^4)\sigma_0^4(s_i'W_ns_j)^2(s_j'W_nW_n's_j)\\
&\quad+2\sigma_0^8(s_i'W_nW_n's_j)^2+(\mu_4-3\sigma_0^4)\sigma_0^4\sum_{l=1}^n w_{il}^2w_{jl}^2+4(\mu_3)^2\sigma_0^2(s_i'W_ns_j)(s_j'W_ns_i)(s_i'W_nW_n's_j)\\
&\quad+2(\mu_3)^2\sigma_0^2(s_j'W_ns_i)\sum_{l=1}^n w_{il}^2w_{jl}+2(\mu_3)^2\sigma_0^2(s_i'W_ns_j)\sum_{l=1}^n w_{jl}^2w_{il}\big)=O(\frac{1}{Th_n^{*2}})
\end{aligned} \tag{C.48}
$$

because the maximum order is $\frac{1}{\sigma_0^8}\frac{1}{nT}\sum_{i=1}^{n}\sum_{j\neq i}^{n}(\frac{w_{ij}w_{ji}}{\sum_{j=1}^n w_{ij}^2\sum_{i=1}^n w_{ji}^2})^2\sigma_0^8(s_i'W_nW_n's_i)(s_j'W_nW_n's_j)=$ $\frac{1}{nT}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{(w_{ij}w_{ji})^2}{\sum_{j=1}^n w_{ij}^2\sum_{i=1}^n w_{ji}^2}\leq(\max_{i,j}\frac{|w_{ji}|}{\sqrt{\sum_{i=1}^n w_{ji}^2}})^2\frac{1}{nT}\sum_{i=1}^{n}\frac{\sum_{j\neq i}^n w_{ij}^2}{\sum_{j=1}^n w_{ij}^2}=O(\frac{1}{Th_n^{*2}})$.

For the second term in (C.47), from (F.6), we have

$$
\begin{aligned}
&\frac{1}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}(V_{nt}'s_is_i'W_nV_{nt})^2\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V_{nt}'s_js_j'W_nV_{nt}\sum_{m\neq i,j}^{n}\tilde{b}_{n,im}V_{nt}'s_ms_m'W_nV_{nt})\\
&=\frac{1}{\sigma_0^8}\frac{1}{nT}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{m\neq i,j}^{n}\tilde{b}_{n,ij}\tilde{b}_{n,im}E(V_{nt}'s_is_i'W_nV_{nt})^2(V_{nt}'s_js_j'W_nV_{nt})(V_{nt}'s_ms_m'W_nV_{nt})\\
&=\frac{1}{\sigma_0^8}\frac{1}{nT}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{m\neq i,j}^{n}\frac{w_{ij}w_{ji}}{\sum_{j=1}^n w_{ij}^2\sum_{i=1}^n w_{ji}^2}\frac{w_{im}w_{mi}}{\sum_{j=1}^n w_{ij}^2\sum_{l=1}^n w_{ml}^2}\big(\sigma_0^8w_{jm}w_{mj}\sum_{l=1}^n w_{il}^2\\
&\quad+2\sigma_0^8w_{ij}w_{jm}\sum_{l=1}^n w_{il}w_{ml}+2\sigma_0^8w_{im}w_{mj}\sum_{l=1}^n w_{il}w_{jl}+2\sigma_0^8w_{ij}w_{im}\sum_{l=1}^n w_{jl}w_{ml}\\
&\quad+2(\mu_4-\sigma_0^4)\sigma_0^4w_{ij}w_{im}w_{ji}w_{mi}+(\mu_4-3\sigma_0^4)\sigma_0^4w_{im}^2w_{jm}w_{mj}+(\mu_4-3\sigma_0^4)\sigma_0^4w_{ij}^2w_{jm}w_{mj}\\
&\quad+2(\mu_3)^2\sigma_0^2w_{ij}w_{im}w_{jm}w_{mi}+2(\mu_3)^2\sigma_0^2w_{ij}w_{im}w_{ji}w_{mj}+2(\mu_3)^2\sigma_0^2w_{ij}w_{im}w_{jm}w_{mj}\\
&\quad+(\mu_3)^2\sigma_0^2w_{im}^2w_{ji}w_{mj}\big)=O(\frac{n}{Th_n^{*4}})
\end{aligned} \tag{C.49}
$$

since the order is $\frac{1}{\sigma_0^8}\frac{1}{nT}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{m\neq i,j}^{n}\frac{w_{ij}w_{ji}}{\sum_{j=1}^n w_{ij}^2\sum_{i=1}^n w_{ji}^2}\frac{w_{im}w_{mi}}{\sum_{j=1}^n w_{ij}^2\sum_{l=1}^n w_{ml}^2}\sigma_0^8 w_{jm}w_{mj}\sum_{l=1}^{n} w_{il}^2$ $=\frac{1}{nT}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{m\neq i,j}^{n}\frac{w_{ij}w_{ji}}{\sum_{i=1}^n w_{ji}^2}\frac{w_{im}w_{mi}}{\sum_{j=1}^n w_{ij}^2}\frac{w_{jm}w_{mj}}{\sum_{l=1}^n w_{ml}^2}\leq(\max_{i,j}\frac{|w_{ij}|}{\sqrt{\sum_{j=1}^n w_{ij}^2}})^4\frac{1}{nT}\sum_{i=1}^{n}\sum_{m\neq i,j}^{n}\frac{\sum_{j=1}^n|w_{ij}w_{ji}|}{\sum_{j=1}^n w_{ij}^2}$ $=O(\frac{n}{Th_n^{*4}})$ by Assumption 6. Similar arguments can be applied to the third-fifth terms in (C.47).

For the last term in (C.47), from (F.7), we have

$$\begin{aligned}
&\frac{1}{\sigma_0^8}\frac{1}{nT}E(\sum_{i=1}^{n}V_{nt}'s_is_i'W_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V_{nt}'s_js_j'W_nV_{nt}\sum_{l\neq i,j}^{n}V_{nt}'s_ls_l'W_nV_{nt}\sum_{m\neq i,j,l}^{n}\tilde{b}_{n,lm}V_{nt}'s_ms_m'W_nV_{nt})\\
&=\frac{1}{\sigma_0^8}\frac{1}{nT}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\sum_{m\neq i,j,l}^{n}\tilde{b}_{n,ij}\tilde{b}_{n,lm}E(V_{nt}'s_is_i'W_nV_{nt}V_{nt}'s_js_j'W_nV_{nt}V_{nt}'s_ls_l'W_nV_{nt}V_{nt}'s_ms_m'W_nV_{nt})\\
&=\frac{1}{\sigma_0^8}\frac{1}{nT}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\sum_{m\neq i,j,l}^{n}\frac{w_{ij}w_{ji}}{\sum_{j=1}^n w_{ij}^2\sum_{i=1}^n w_{ji}^2}\frac{w_{lm}w_{ml}}{\sum_{m=1}^n w_{lm}^2\sum_{l=1}^n w_{ml}^2}\big(4\sigma_0^8w_{ij}w_{ji}w_{lm}w_{ml}\\
&\quad+4\sigma_0^8w_{il}w_{li}w_{mj}w_{jm}+4\sigma_0^8w_{im}w_{mi}w_{lj}w_{jl}+4\sigma_0^8w_{ij}w_{jl}w_{lm}w_{mi}+4\sigma_0^8w_{ij}w_{jm}w_{li}w_{ml}\\
&\quad+4\sigma_0^8w_{il}w_{lj}w_{mi}w_{jm}+4\sigma_0^8w_{il}w_{lm}w_{ji}w_{mj}+4\sigma_0^8w_{im}w_{mj}w_{li}w_{jl}+4\sigma_0^8w_{im}w_{ml}w_{ji}w_{lj}\big)=O(\frac{n}{Th_n^{*4}})
\end{aligned}\tag{C.50}$$

since the order is $\frac{1}{\sigma_0^8}\frac{1}{nT}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\sum_{m\neq i,j,l}^{n}\frac{w_{ij}w_{ji}}{\sum_{j=1}^n w_{ij}^2\sum_{i=1}^n w_{ji}^2}\frac{w_{lm}w_{ml}}{\sum_{m=1}^n w_{lm}^2\sum_{l=1}^n w_{ml}^2}4\sigma_0^8w_{il}w_{li}w_{mj}w_{jm}$ $\leq(\max_{i,j}\frac{|w_{ij}|}{\sqrt{\sum_{j=1}^n w_{ij}^2}})^4\frac{4}{nT}\sum_{i=1}^{n}\sum_{l\neq i,j}^{n}\frac{\sum_{j=1}^n|w_{ij}w_{ji}|}{\sum_{j=1}^n w_{ij}^2}\frac{\sum_{m=1}^n|w_{lm}w_{ml}|}{\sum_{m=1}^n w_{lm}^2}=O(\frac{n}{Th_n^{*4}})$ by Assumption 6. The results of (C.48), (C.49) and (C.50) imply that the second moment of the first term in (C.43) is $O(\frac{n}{Th_n^{*4}})=o_p(1)$ under Assumption 13.

For the second moment of the second term in (C.43), we have

$$
\begin{aligned}
& E(\frac{1}{\sigma_0^4}\frac{1}{\sqrt{n}}\frac{1}{T}\sum_{i=1}^{n}\sum_{t=1}^{T}V'_{nt}s_is'_iW_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}\sum_{s=1}^{t-1}V'_{ns}s_js'_jW_nV_{ns})^2 \\
& = \frac{1}{\sigma_0^8}\frac{1}{n}\frac{1}{T^2}\sum_{t=1}^{T}E(\sum_{i=1}^{n}V'_{nt}s_is'_iW_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}\sum_{s=1}^{t-1}V'_{ns}s_js'_jW_nV_{ns})^2 \\
& = \frac{1}{\sigma_0^8}\frac{1}{n}\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{t-1}E(\sum_{i=1}^{n}V'_{nt}s_is'_iW_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V'_{ns}s_js'_jW_nV_{ns})^2 \\
& = \frac{1}{\sigma_0^8}\frac{1}{n}\frac{T(T-1)}{2T^2}E(\sum_{i=1}^{n}V'_{nt}s_is'_iW_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V'_{ns}s_js'_jW_nV_{ns})^2 \\
& = \frac{1}{\sigma_0^8}\frac{1}{n}\frac{T(T-1)}{2T^2}E(\sum_{i=1}^{n}V'_{nt}s_is'_iW_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V'_{ns}s_js'_jW_nV_{ns}\sum_{l=1}^{n}V'_{nt}s_ls'_lW_nV_{nt}\sum_{m\neq l}^{n}\tilde{b}_{n,lm}V'_{ns}s_ms'_mW_nV_{ns})
\end{aligned}
\tag{C.51}
$$

Note that $E(V'_{nt}s_is'_iW_nV_{nt}V'_{ns}s_js'_jW_nV_{ns}V'_{nt}s_ls'_lW_nV_{nt}V'_{ns}s_ms'_mW_nV_{ns})$ will not vanish only when $l=i$ and $m=j$ for all t and s. Thus, by Lemma A.1, we have

$$
\begin{aligned}
& E(\frac{1}{\sigma_0^4}\frac{1}{\sqrt{n}}\frac{1}{T}\sum_{i=1}^{n}\sum_{t=1}^{T}V'_{nt}s_is'_iW_nV_{nt}\sum_{j\neq i}^{n}\frac{s'_iW_ns_js'_jW_ns_i}{s'_iW_nW'_ns_is'_jW_nW'_ns_j}\sum_{s=1}^{t-1}V'_{ns}s_js'_jW_nV_{ns})^2 \\
& = \frac{1}{\sigma_0^8}\frac{1}{n}\frac{T(T-1)}{2T^2}E(\sum_{i=1}^{n}V'_{nt}s_is'_iW_nV_{nt}\sum_{j\neq i}^{n}\tilde{b}_{n,ij}V'_{ns}s_js'_jW_nV_{ns}V'_{nt}s_is'_iW_nV_{nt}\tilde{b}_{n,ij}V'_{ns}s_js'_jW_nV_{ns}) \\
& = \frac{1}{\sigma_0^8}\frac{1}{n}\frac{T(T-1)}{2T^2}E(\sum_{i=1}^{n}(V'_{nt}s_is'_iW_nV_{nt})^2\sum_{j\neq i}^{n}\tilde{b}^2_{n,ij}(V'_{ns}s_js'_jW_nV_{ns})^2) \\
& = \frac{1}{\sigma_0^8}\frac{1}{n}\frac{T(T-1)}{2T^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\tilde{b}^2_{n,ij}E(V'_{nt}s_is'_iW_nV_{nt})^2(V'_{ns}s_js'_jW_nV_{ns})^2 \\
& = \frac{1}{n}\frac{T(T-1)}{2T^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{(s'_iW_ns_js'_jW_ns_i)^2}{s'_iW_nW'_ns_is'_jW_nW'_ns_j} \\
& = \frac{1}{n}\frac{T(T-1)}{2T^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}(\frac{w_{ij}w_{ji}}{\sqrt{\sum_{j=1}^{n}w_{ij}^2}\sqrt{\sum_{i=1}^{n}w_{ji}^2}})^2 = O(\frac{1}{h_n^{*2}})
\end{aligned}
\tag{C.52}
$$

Thus, the second moment of the second term in (C.43) is $O(\frac{1}{h_n^{*2}}) = o_p(1)$ under Assumption 13. Similar arguments can be applied to the second moment of the third term in (C.43).

C.17 Proof of Theorem 4

Under H_0, S takes the following form:

$$\begin{aligned} S = & \frac{1}{\sqrt{2n}}\big(\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{D-1}g_{nT}(\tilde{\sigma}^2) - n\big) + \frac{1}{\sqrt{2n}}\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'B_n\Sigma_{g,n}^{D-1}g_{nT}(\tilde{\sigma}^2) \\ & + \sum_{k=2}^{\infty}\frac{1}{\sqrt{2n}}\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'B_n^k\Sigma_{g,n}^{D-1}g_{nT}(\tilde{\sigma}^2) \end{aligned} \tag{33}$$

Theorem 3 is applied to the first term, we have $\frac{1}{\sqrt{2n}}\big(\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{D-1}g_{nT}(\tilde{\sigma}^2)-n\big) \xrightarrow{d} N(0,1)$. All remaining terms in (33) converge to zero by Propositions 9 and 10, provided that $G_{nT} \xrightarrow{p} 0$. Finally, $G_{nT} \xrightarrow{p} 0$ by Proposition 11. Therefore, for the small reciprocal interactions in the form of Assumptions 12 and 13, $S = \frac{1}{\sqrt{2n}}\big(\frac{1}{T}g_{nT}(\tilde{\sigma}^2)'\Sigma_{g,n}^{-1}g_{nT}(\tilde{\sigma}^2) - n\big) \xrightarrow{d} N(0,1)$ under $\frac{n}{T} \to k$ where $0 \le k < \infty$.

C.18 Proof of Theorem 5

First, consider the restricted QML estimator as

$$\begin{aligned} \tilde{\sigma}^2 &= \frac{1}{nT}\sum_{t=1}^{T}\tilde{V}_{nt}'(I_n - \Psi^{H_1}W_n)^{-1\prime}(I_n - \Psi^{H_1}W_n)^{-1}\tilde{V}_{nt} \\ &= \frac{1}{nT}\sum_{t=1}^{T}\tilde{V}_{nt}'(I_n + \sum_{k=1}^{\infty}(\frac{1}{n^{1/4}T^{1/2}}\Delta^D W_n)^k)'(I_n + \sum_{k=1}^{\infty}(\frac{1}{n^{1/4}T^{1/2}}\Delta^D W_n)^k)\tilde{V}_{nt} \\ &= \frac{1}{nT}\sum_{t=1}^{T}\tilde{V}_{nt}'(I_n + \frac{1}{n^{1/4}T^{1/2}}\Delta^D W_n)'(I_n + \frac{1}{n^{1/4}T^{1/2}}\Delta^D W_n)\tilde{V}_{nt} + o_p(1) \\ &= \frac{1}{nT}\sum_{t=1}^{T}\tilde{V}_{nt}'\tilde{V}_{nt} + o_p(1) \end{aligned} \tag{C.53}$$

because $(I_n - \Psi^{H_1}W_n)^{-1} = I_n + \sum_{k=1}^{\infty}(\frac{1}{n^{1/4}T^{1/2}}\Delta^D W_n)^k < \infty$ for large enough n by Assumption 3 and $|\frac{1}{nT}\sum_{t=1}^{T}\tilde{V}_{nt}'(\Delta^D W_n)^{k\prime}(\Delta^D W_n)^k\tilde{V}_{nt}| = O(1)$ by Lemma 15 in Yu et al. (2008) for any finite k. Since $\frac{1}{nT}\sum_{t=1}^{T}\tilde{V}_{nt}'\tilde{V}_{nt}\xrightarrow{p}\sigma_0^2$ by Proposition 3, we similarly have $\tilde{\sigma}^2\xrightarrow{p}\sigma_0^2$ and $\frac{\sigma_0^2}{\tilde{\sigma}^2}\xrightarrow{p}1$ under $H_{1,nT}$.

We now need to derive the limit of the term Φ_{nT} as

$$
\begin{aligned}
\Phi_{nT} &= \frac{1}{n}\sum_{i=1}^{n}\frac{(\frac{1}{\tilde{\sigma}^2}\frac{1}{T}\sum_{t=1}^{T}\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})^2+(\frac{1}{\tilde{\sigma}^2}\frac{1}{T}\sum_{t=1}^{T}\tilde{V}_{nt}'s_i s_i'W_n\Delta^D W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i}\\
&= (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2\frac{1}{n}\sum_{i=1}^{n}\frac{(\frac{1}{\sigma_0^2}\frac{1}{T}\sum_{t=1}^{T}\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})^2+(\frac{1}{\sigma_0^2}\frac{1}{T}\sum_{t=1}^{T}\tilde{V}_{nt}'s_i s_i'W_n\Delta^D W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i}\\
&= (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2\frac{1}{\sigma_0^4}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i}+(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2\frac{1}{\sigma_0^4}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{(\tilde{V}_{nt}'s_i s_i'W_n\Delta^D W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i}\\
&\quad+(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2\frac{2}{\sigma_0^4}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})(\tilde{V}_{ns}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{ns})}{s_i'W_nW_n's_i}\\
&\quad+(\frac{\sigma_0^2}{\tilde{\sigma}^2})^2\frac{2}{\sigma_0^4}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'s_i s_i'W_n\Delta^D W_n\tilde{V}_{nt})(\tilde{V}_{ns}'s_i s_i'W_n\Delta^D W_n\tilde{V}_{ns})}{s_i'W_nW_n's_i}
\end{aligned}
\tag{C.54}
$$

where $\frac{\sigma_0^2}{\tilde{\sigma}^2}\xrightarrow{p}1$ under $H_{1,nT}$.

For the first term in (C.54), we have

$$
\begin{aligned}
\frac{1}{\sigma_0^4}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i} &\le \frac{1}{T}\frac{1}{\sigma_0^4}\frac{1}{n}\sum_{i=1}^{n}|\frac{1}{T}\sum_{t=1}^{T}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i}|\\
&\le \sup_i\frac{1}{T}\frac{1}{\sigma_0^4}|\frac{1}{T}\sum_{t=1}^{T}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i}|\\
&= O_p(\frac{1}{T})
\end{aligned}
\tag{C.55}
$$

because $\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i}$ is i.i.d over $t=1,...,T$ with $E(|\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i}|)=E(\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})^2}{s_i'W_nW_n's_i})=\frac{\Delta_i^2E(s_i'W_n\tilde{V}_{nt})^4}{s_i'W_nW_n's_i}<\infty$ for all i and n by Assumptions 2 and 5, which implies that Kolmogorov's LLN holds. Similar arguments can be applied to the second term in (C.54). Thus, the first two terms in (C.54) go to zero in probability.

For the third term in (C.54), we have

$$\begin{aligned}
&\frac{2}{\sigma_0^4}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})(\tilde{V}_{ns}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{ns})}{s_i'W_nW_n's_i}\\
&=\frac{2}{\sigma_0^4}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})(\tilde{V}_{ns}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{ns})}{s_i'W_nW_n's_i}\\
&\quad-E(\frac{2}{\sigma_0^4}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})(\tilde{V}_{ns}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{ns})}{s_i'W_nW_n's_i})\\
&\quad+\frac{1}{n}(\frac{T-1}{T})\sum_{i=1}^{n}\frac{(s_i'W_nW_n'\Delta^D s_i)^2}{s_i'W_nW_n's_i}-\frac{1}{n}(\frac{T-1}{T^3})\sum_{i=1}^{n}\frac{(s_i'W_nW_n'\Delta^D s_i)^2}{s_i'W_nW_n's_i}
\end{aligned}\tag{C.56}$$

where $\frac{1}{n}\sum_{i=1}^{n}\frac{(s_i'W_nW_n'\Delta^D s_i)^2}{s_i'W_nW_n's_i}=O(1)$ by Assumption 5 and

$$\begin{aligned}
&\frac{2}{\sigma_0^4}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})(\tilde{V}_{ns}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{ns})}{s_i'W_nW_n's_i}\\
&\quad-E(\frac{2}{\sigma_0^4}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})(\tilde{V}_{ns}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{ns})}{s_i'W_nW_n's_i})\\
&\leq\frac{2}{\sigma_0^4}\frac{1}{n}\sum_{i=1}^{n}|\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})(\tilde{V}_{ns}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{ns})}{s_i'W_nW_n's_i}\\
&\quad-E(\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})(\tilde{V}_{ns}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{ns})}{s_i'W_nW_n's_i})|\\
&\leq\sup_i\frac{2}{\sigma_0^4}|\frac{1}{T}\sum_{t=1}^{T}(\frac{1}{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})(\tilde{V}_{ns}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{ns})}{s_i'W_nW_n's_i}\\
&\quad-E(\frac{1}{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})(\tilde{V}_{ns}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{ns})}{s_i'W_nW_n's_i}))|=o_p(1)
\end{aligned}\tag{C.57}$$

because $\frac{1}{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{nt})(\tilde{V}_{ns}'W_n'\Delta^D s_i s_i'W_n\tilde{V}_{ns})}{s_i'W_nW_n's_i}$ is an uncorrelated sequence over $t=1,...,T$ with the finite mean and variance by Assumptions 2 and 5. Hence, the third term goes to $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{(s_i'W_nW_n'\Delta^D s_i)^2}{s_i'W_nW_n's_i}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta_i^2(s_i'W_nW_n's_i)^2}{s_i'W_nW_n's_i}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta_i^2(\sum_{j=1}^{n}w_{ij}^2)^2}{\sum_{j=1}^{n}w_{ij}^2}$.

For the fourth term in (C.54), we have

$$\begin{aligned}
&\frac{2}{\sigma_0^4}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'s_is_i'W_n\Delta^DW_n\tilde{V}_{nt})(\tilde{V}_{ns}'s_is_i'W_n\Delta^DW_n\tilde{V}_{ns})}{s_i'W_nW_n's_i} \\
&=\frac{2}{\sigma_0^4}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'s_is_i'W_n\Delta^DW_n\tilde{V}_{nt})(\tilde{V}_{ns}'s_is_i'W_n\Delta^DW_n\tilde{V}_{ns})}{s_i'W_nW_n's_i} \\
&\quad -E(\frac{2}{\sigma_0^4}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\frac{(\tilde{V}_{nt}'s_is_i'W_n\Delta^DW_n\tilde{V}_{nt})(\tilde{V}_{ns}'s_is_i'W_n\Delta^DW_n\tilde{V}_{ns})}{s_i'W_nW_n's_i}) \\
&\quad +\frac{1}{n}(\frac{T-1}{T})\sum_{i=1}^{n}\frac{(s_i'W_n\Delta^DW_ns_i)^2}{s_i'W_nW_n's_i}-\frac{1}{n}(\frac{T-1}{T^3})\sum_{i=1}^{n}\frac{(s_i'W_n\Delta^DW_ns_i)^2}{s_i'W_nW_n's_i}
\end{aligned} \tag{C.58}$$

Similar arguments can be applied to the fourth term. Hence, the fourth term goes to $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{(s_i'W_n\Delta^DW_ns_i)^2}{s_i'W_nW_n's_i}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{(\sum_{j=1}^{n}\Delta_jw_{ij}w_{ji})^2}{\sum_{j=1}^{n}w_{ij}^2}$. Therefore, using all the results above, we have $\Phi=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta_i^2(\sum_{j=1}^{n}w_{ij}^2)^2+(\sum_{j=1}^{n}\Delta_jw_{ij}w_{ji})^2}{\sum_{j=1}^{n}w_{ij}^2}$.

Appendix D M test

D.1 The standard SAR panel data model

Consider the following spatial panel data model with a homogeneous spatial coefficient:

$$Y_{nt}=\lambda_0W_nY_{nt}+\mathbf{c}_{n0}+V_{nt},\qquad t=1,...,T \tag{D.1}$$

where $Y_{nt}=(y_{1t},...,y_{nt})'$ is an $n\times 1$ vector of a dependent variable for all units in period t, W_n is an $n\times n$ spatial weights matrix, $\mathbf{c}_{n0}$ is an $n\times 1$ vector of individual fixed effects, and $V_{nt}=(\varepsilon_{1t},...,\varepsilon_{nt})'$ is an $n\times 1$ vector of disturbance terms[40].

Define $S_n(\lambda)=I_n-\lambda W_n$ for any λ. At the true parameter, $S_n(\lambda_0)=I_n-\lambda_0W_n$. Then, presuming $S_n(\lambda_0)$ is invertible, (D.1) can be rewritten as $Y_{nt}=(I_n-\lambda_0W_n)^{-1}(\mathbf{c}_{n0}+V_{nt})$.

[40] For the elements of V_{nt}, we assume that ε_{it} is i.i.d. across i and t with zero mean and variance σ_0^2.

D.2 The concentrated likelihood function

Denote $\theta = (\lambda, \sigma^2)'$ and $\zeta = (\lambda, \mathbf{c}_n')'$. At the true value, $\theta_0 = (\lambda_0', \sigma_0^2)'$ and $\zeta_0 = (\lambda_0, \mathbf{c}_{n0}')'$. The likelihood function of (D.1) is

$$lnL_{nT}(\theta, \mathbf{c}_n) = -\frac{nT}{2}ln(2\pi) - \frac{nT}{2}ln(\sigma^2) + Tln|S_n(\lambda)| - \frac{1}{2\sigma^2}\sum_{t=1}^{T} V_{nt}(\zeta)'V_{nt}(\zeta) \tag{D.2}$$

where $V_{nt}(\zeta) = (I_n - \lambda W_n)Y_{nt} - \mathbf{c}_n = S_n(\lambda)Y_{nt} - \mathbf{c}_n$. Thus, $V_{nt} = V_{nt}(\zeta_0)$.

For analytical purposes, it is convenient to concentrate $\mathbf{c}_n$ out in (D.2). We define $\tilde{Y}_{nt} = Y_{nt} - \bar{Y}_{nT}$ where $\bar{Y}_{nT} = \frac{1}{T}\sum_{t=1}^{T} Y_{nt}$. Similarly, $\tilde{V}_{nt} = V_{nt} - \bar{V}_{nT}$. Using the first order condition that $\frac{\partial lnL_{nT}(\theta, \mathbf{c}_n)}{\partial \mathbf{c}_n} = \frac{1}{\sigma^2}\sum_{t=1}^{T} V_{nt}(\zeta)$ from (D.2), the concentrated likelihood function is

$$lnL_{nT}(\theta) = -\frac{nT}{2}ln(2\pi) - \frac{nT}{2}ln(\sigma^2) + Tln|S_n(\lambda)| - \frac{1}{2\sigma^2}\sum_{t=1}^{T} \tilde{V}_{nt}(\lambda)'\tilde{V}_{nt}(\lambda) \tag{D.3}$$

where $\tilde{V}_{nt}(\lambda) = (I_n - \lambda W_n)\tilde{Y}_{nt} = S_n(\lambda)\tilde{Y}_{nt}$.

Define $G_n(\lambda) = W_n(I_n - \lambda W_n)^{-1} = W_n S_n(\lambda)^{-1}$ for any λ. From (D.3), the first derivative of the concentrated likelihood function with respect to λ can be derived:

$$\frac{\partial lnL_{nT}(\theta)}{\partial \lambda} = \frac{1}{\sigma^2}\sum_{t=1}^{T}\left(\tilde{V}_{nt}(\lambda)'G_n(\lambda)\tilde{V}_{nt}(\lambda) - \sigma^2 tr(G_n(\lambda))\right) \tag{D.4}$$

D.3 The limiting result of M

To test spatial correlation, one may formulate a hypothesis as a restriction on λ_0 in (D.1). The null hypothesis of interest is $H_0 : \lambda_0 = 0$. To analyze the asymptotic properties of the M test when both n and T are large, we need the following standard assumptions:

Assumption D1. *The spatial weights matrix W_n is time-invariant and its diagonal elements satisfy $w_{ii} = 0$ for $i = 1, ..., n$.*

Assumption D2. *The disturbances ε_{it}, $i = 1, ..., n$ and $t = 1, ..., T$, are i.i.d. across i and t with zero mean, finite variance $\sigma_0^2 > 0$, and $E|\varepsilon_{it}|^{4+\eta} < \infty$ for some $\eta > 0$.*

Assumption D3. *$S_n(\psi) = I_n - diag(\psi)W_n$ is invertible for all $\psi = (\delta_1, ..., \delta_n)'$ in a small neighborhood around zero.*

Assumption D4. *W_n is uniformly bounded in row and column sums in absolute value.*

Assumption D5. *n is an increasing function of T and T goes to infinity.*

Assumptions D1, D2, D4 and D5 are the standard regularity conditions used in Yu et al. (2008). Assumption D3 implies that $S_n(\lambda) = I_n - \lambda W_n$ is also invertible because λW_n is a special case of $diag(\psi)W_n$ where ψ consists of the same value of λ for all elements.

Consider the first order derivative in (D.4). Under H_0, $\tilde{V}_{nt}(\lambda) = (I_n - 0W_n)\tilde{Y}_{nt} = \tilde{Y}_{nt}$ and $G_n(\lambda) = W_n(I_n - 0W_n)^{-1} = W_n$ such that $\lambda = 0$. Therefore, the test statistic is based on

$$\frac{\partial lnL_{nT}(0, \sigma^2)}{\partial \lambda} = \frac{1}{\sigma^2} \sum_{t=1}^{T} \left(\tilde{Y}_{nt}' W_n \tilde{Y}_{nt} - \sigma^2 tr(W_n) \right) \tag{D.5}$$

Let $\tilde{\sigma}^2$ be the restricted QML estimator with the restriction $\lambda = 0$ imposed, so $\tilde{\sigma}^2 = max_\sigma lnL_{nT}^c(\sigma^2)$ where

$$lnL_{nT}^c(\sigma^2) = -\frac{nT}{2} ln(2\pi) - \frac{nT}{2} ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^{T} \tilde{Y}_{nt}' \tilde{Y}_{nt} \tag{D.6}$$

Thus, we derive $\tilde{\sigma}^2 = \dfrac{1}{nT} \sum_{t=1}^{T} \tilde{Y}_{nt}' \tilde{Y}_{nt}$.

Proposition D.1. *Under H_0 and Assumptions D2 and D5,*
$\tilde{\sigma}^2 \xrightarrow{p} \sigma_0^2$, and hence $\dfrac{\sigma_0^2}{\tilde{\sigma}^2} \xrightarrow{p} 1$.

We now investigate the asymptotic distribution of $\dfrac{1}{\sqrt{nT}} \dfrac{\partial lnL_{nT}(0, \tilde{\sigma}^2)}{\partial \lambda}$ under H_0. From

(D.5) evaluated at $\tilde{\sigma}^2$ under H_0 ($\tilde{Y}_{nt} = \tilde{V}_{nt}$) and Assumption D1 ($tr(W_n) = 0$), we have

$$\begin{aligned}
\frac{1}{\sqrt{nT}}\frac{\partial lnL_{nT}(0,\tilde{\sigma}^2)}{\partial\lambda} &= \frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{nT}}\sum_{t=1}^{T}\tilde{V}_{nt}'W_n\tilde{V}_{nt} = \frac{\sigma_0^2}{\tilde{\sigma}^2}(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{nT}}\sum_{t=1}^{T}\tilde{V}_{nt}'W_n\tilde{V}_{nt}) \\
&= \frac{\sigma_0^2}{\tilde{\sigma}^2}(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{nT}}\sum_{t=1}^{T}V_{nt}'W_nV_{nt} - \frac{1}{\sigma_0^2}\sqrt{\frac{T}{n}}\bar{V}_{nT}'W_n\bar{V}_{nT}) \\
&= \frac{\sigma_0^2}{\tilde{\sigma}^2}(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{nT}}\sum_{t=1}^{T}V_{nt}'W_nV_{nt}) - \frac{\sigma_0^2}{\tilde{\sigma}^2}(\frac{1}{\sigma_0^2}\sqrt{\frac{T}{n}}\bar{V}_{nT}'W_n\bar{V}_{nT} - E(\frac{1}{\sigma_0^2}\sqrt{\frac{T}{n}}\bar{V}_{nT}'W_n\bar{V}_{nT})) \\
&\quad - \frac{\sigma_0^2}{\tilde{\sigma}^2}E(\frac{1}{\sigma_0^2}\sqrt{\frac{T}{n}}\bar{V}_{nT}'W_n\bar{V}_{nT})
\end{aligned} \tag{D.7}$$

where $\frac{\sigma_0^2}{\tilde{\sigma}^2} \xrightarrow{p} 1$ by Proposition D.1. Note that $\frac{1}{\sigma_0^2}\sqrt{\frac{T}{n}}\bar{V}_{nT}'W_n\bar{V}_{nT} - E(\frac{1}{\sigma_0^2}\sqrt{\frac{T}{n}}\bar{V}_{nT}'W_n\bar{V}_{nT}) = O_p(\frac{1}{\sqrt{T}})$ and $E(\frac{1}{\sigma_0^2}\sqrt{\frac{T}{n}}\bar{V}_{nT}'W_n\bar{V}_{nT}) = \frac{1}{\sqrt{nT}}tr(W_n) = 0$ by Lemma 9 in Yu et al. (2008) and Assumption D1. Thus, the limiting result of $\frac{1}{\sqrt{nT}}\frac{\partial lnL_{nT}(0,\tilde{\sigma}^2)}{\partial\lambda}$ is the same as that of $\frac{1}{\sigma_0^2}\frac{1}{\sqrt{nT}}\sum_{t=1}^{T}V_{nt}'W_nV_{nt}$ under H_0. Denote $Q_{nT} = \sum_{t=1}^{T}V_{nt}'W_nV_{nt}$. Then, the mean and variance of Q_{nT} are $\mu_{Q_{nT}} = 0$ and $\sigma_{Q_{nT}}^2 = T\sigma_0^4 tr(W_n'W_n + W_n^2)$.

Theorem D.1. *Under H_0 and Assumptions D1-D2 and D4-D5,*
$M = \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}'W_n\tilde{Y}_{nt}}{\sqrt{tr(W_n'W_n + W_n^2)}} \xrightarrow{d} N(0,1).$

Hence, the conclusions on whether a spatial correlation exists or not, when both n and T are large, can be drawn based on the value of M.

D.4 Local power of the M test

For the asymptotic local power of the M test when spatial lag coefficients are heterogeneous, we adopt the following local alternatives:

$$H_{1,nT} : \delta_{i0} = \frac{\Delta_i}{n^{1/2}T^{1/2}} \qquad \text{for } i = 1, ..., n \tag{D.8}$$

where Δ_i is a fixed constant ($\Delta_i \neq 0$). Denote $\Delta^D = diag(\Delta_1, ..., \Delta_n)$ and $\Psi^{H_1} = \frac{1}{n^{1/2}T^{1/2}}\Delta^D$. We investigate the asymptotic result of M under $H_{1,nT}$ as

$$M = \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{nT}}\sum_{t=1}^T \tilde{V}_{nt}'(I_n - \Psi^{H_1}W_n)^{-1\prime}W_n(I_n - \Psi^{H_1}W_n)^{-1}\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} \tag{D.9}$$

where $(I_n - \Psi^{H_1}W_n)^{-1} = I_n + \sum_{k=1}^{\infty}(\frac{1}{n^{1/2}T^{1/2}}\Delta^D W_n)^k < \infty$ for large enough T by Assumption D3. Then,

$$\begin{aligned}
M &= \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{nT}}\sum_{t=1}^T \tilde{V}_{nt}'(I_n - \Psi^{H_1}W_n)^{-1\prime}W_n(I_n - \Psi^{H_1}W_n)^{-1}\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} \\
&= \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{nT}}\sum_{t=1}^T \tilde{V}_{nt}'(I_n + \sum_{k=1}^{\infty}(\frac{1}{n^{1/2}T^{1/2}}\Delta^D W_n)^k)'W_n(I_n + \sum_{k=1}^{\infty}(\frac{1}{n^{1/2}T^{1/2}}\Delta^D W_n)^k)\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} \\
&= \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{nT}}\sum_{t=1}^T \tilde{V}_{nt}'(I_n + \frac{1}{n^{1/2}T^{1/2}}\Delta^D W_n)'W_n(I_n + \frac{1}{n^{1/2}T^{1/2}}\Delta^D W_n)\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} + o_p(1) \\
&= \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{nT}}\sum_{t=1}^T \tilde{V}_{nt}'W_n\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} + \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'(\Delta^D W_n)'W_n\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} + \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'W_n(\Delta^D W_n)\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} + o_p(1)
\end{aligned} \tag{D.10}$$

since $|\frac{1}{\sqrt{nT}}\sum_{t=1}^T \tilde{V}_{nt}'(\Delta^D W_n)^{k\prime}W_n(\Delta^D W_n)^k\tilde{V}_{nt}|$ is $O_p(\sqrt{\frac{T}{n}})$ by Lemma 9 in Yu et al. (2008) for any finite k. In Section D.3, we derive the limiting result of the first term in (D.10). Therefore, the asymptotic power depends on the limit of

$$\phi_{nT} = \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'(\Delta^D W_n)'W_n\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} + \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'W_n(\Delta^D W_n)\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} \tag{D.11}$$

Denote $\phi = \lim_{T\to\infty}\phi_{nT}$.

Theorem D.2. *Under $H_{1,nT}$ and Assumptions D1-D5,*
$$M = \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{nT}}\sum_{t=1}^T \tilde{Y}_{nt}'W_n\tilde{Y}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} \xrightarrow{d} N(\phi, 1).$$

Theorem D.2 implies that the M test may lose power if Δ_i has a different sign across i due to $\phi = \lim_{n\to\infty} \frac{\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \Delta_i w_{ij}^2 + \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \Delta_i w_{ij}w_{ji}}{\sqrt{\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n w_{ij}^2 + \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n w_{ij}w_{ji}}}$. Under Assumption 8, $\phi = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \frac{\Delta_i \sum_{j=1}^n w_{ij}^2}{\sqrt{\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n w_{ij}^2}}$ because $w_{ij}w_{ji} = 0$ for all i, j..

Proof of Proposition D.1 See Proposition 3.

Proof of Theorem D.1 By Proposition D.1 and Equation (D.7), $\frac{1}{\sqrt{nT}}\frac{\partial lnL_{nT}(0,\tilde{\sigma}^2)}{\partial\lambda}$ has the same asymptotic distribution as that of $\frac{1}{\sigma_0^2}\frac{1}{\sqrt{nT}}Q_{nT}$ under H_0. Note that $\frac{1}{\sigma_0^4}\frac{1}{nT}\sigma^2_{Q_{nT}} = \frac{1}{n}tr(W_n'W_n + W_n^2)$ is bounded away from zero by Assumption D4, and all conditions of Lemma 13 in Yu et al. (2008) are satisfied. Hence, the lemma (CLT) applies to Q_{nT}, and we have $\frac{Q_{nT}}{\sigma_{Q_{nT}}} \xrightarrow{d} N(0,1)$. Using the results above, we have $\frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{nT}}\sum_{t=1}^T \tilde{Y}_{nt}'W_n\tilde{Y}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} \xrightarrow{d} N(0,1)$.

Proof of Theorem D.2 We need to derive the limit of ϕ_{nT} where

$$\begin{aligned}
\Phi_{nT} &= \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'(\Delta^D W_n)'W_n\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} + \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'W_n(\Delta^D W_n)\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} \\
&= (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2\Big(\frac{\frac{1}{\sigma_0^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'(\Delta^D W_n)'W_n\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} - E(\frac{\frac{1}{\sigma_0^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'(\Delta^D W_n)'W_n\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}})\Big) \\
&\quad + (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2\Big(\frac{\frac{1}{\sigma_0^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'W_n(\Delta^D W_n)\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} - E(\frac{\frac{1}{\sigma_0^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'W_n(\Delta^D W_n)\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}})\Big) \\
&\quad + (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 E(\frac{\frac{1}{\sigma_0^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'(\Delta^D W_n)'W_n\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}}) + (\frac{\sigma_0^2}{\tilde{\sigma}^2})^2 E(\frac{\frac{1}{\sigma_0^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'W_n(\Delta^D W_n)\tilde{V}_{nt}}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}})
\end{aligned} \tag{D.12}$$

For the first and second terms, $\frac{1}{\sigma_0^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'(\Delta^D W_n)'W_n\tilde{V}_{nt} - E(\frac{1}{\sigma_0^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'(\Delta^D W_n)'W_n\tilde{V}_{nt})$ and $\frac{1}{\sigma_0^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'W_n(\Delta^D W_n)\tilde{V}_{nt} - E(\frac{1}{\sigma_0^2}\frac{1}{nT}\sum_{t=1}^T \tilde{V}_{nt}'W_n(\Delta^D W_n)\tilde{V}_{nt})$ are $O_p(\frac{1}{\sqrt{nT}})$ by Lemma 9 in Yu et al. (2008). Also, $\frac{\sigma_0^2}{\tilde{\sigma}^2} \xrightarrow{p} 1$ under $H_{1,nT}$ shown in Theorem 5. For the last two terms,

$E(\frac{1}{\sigma_0^2}\frac{1}{nT}\sum_{t=1}^{T}\tilde{V}_{nt}'(\Delta^D W_n)'W_n\tilde{V}_{nt}) = \frac{T-1}{nT}tr((\Delta^D W_n)'W_n)$ and $E(\frac{1}{\sigma_0^2}\frac{1}{nT}\sum_{t=1}^{T}\tilde{V}_{nt}'W_n(\Delta^D W_n)\tilde{V}_{nt}) = \frac{T-1}{nT}tr(W_n(\Delta^D W_n))$. Therefore, we have

$$\phi = \lim_{n\to\infty}\frac{\frac{1}{n}tr(\Delta^D W_n W_n') + \frac{1}{n}tr(\Delta^D W_n^2)}{\sqrt{\frac{1}{n}tr(W_n'W_n + W_n^2)}} = \lim_{n\to\infty}\frac{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\Delta_i w_{ij}^2 + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\Delta_i w_{ij}w_{ji}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}w_{ij}^2 + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}w_{ij}w_{ji}}}$$

Appendix E The random variables

Assume that W_n is time-invariant and its diagonal elements satisfy $w_{ii} = 0$ for all i. Also, assume that the disturbances ε_{it} are i.i.d. across i and t with zero mean, finite variance $\sigma_0^2 > 0$ and $E|\varepsilon_{it}|^{4+\eta} < \infty$ for some $\eta > 0$. Denote $\mu_s = E(\varepsilon_{it}^s)$ for $s = 3, 4$.

E.1 Moments of $z_{i,nT}$

$$\begin{aligned} z_{i,nT} &= (\frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}V_{nt}'s_i s_i'W_nV_{nt}}{\sqrt{s_i'W_nW_n's_i}})^2 = \frac{1}{\sigma_0^4}\frac{1}{T}\frac{(\sum_{t=1}^{T}V_{nt}'s_i s_i'W_nV_{nt})^2}{s_i'W_nW_n's_i} \\ &= \frac{1}{\sigma_0^4}\frac{1}{T}\Big(\frac{\sum_{t=1}^{T}(V_{nt}'s_i s_i'W_nV_{nt})^2}{s_i'W_nW_n's_i}\Big) + \frac{1}{\sigma_0^4}\frac{1}{T}\Big(\frac{\sum_{t=1}^{T}\sum_{s=1}^{t-1}2(V_{nt}'s_i s_i'W_nV_{nt})(\epsilon_{ns}'s_i s_i'W_n\epsilon_{ns})}{s_i'W_nW_n's_i}\Big) \end{aligned} \tag{E.1}$$

$$\begin{aligned} z_{i,nT}^2 &= \frac{1}{\sigma_0^8}\frac{1}{T^2}\Big(\frac{\sum_{t=1}^{T}(V_{nt}'s_i s_i'W_nV_{nt})^2}{s_i'W_nW_n's_i}\Big)^2 + \frac{1}{\sigma_0^8}\frac{1}{T^2}\Big(\frac{\sum_{t=1}^{T}\sum_{s=1}^{t-1}2(V_{nt}'s_i s_i'W_nV_{nt})(\epsilon_{ns}'s_i s_i'W_n\epsilon_{ns})}{s_i'W_nW_n's_i}\Big)^2 \\ &+ \frac{2}{\sigma_0^8}\frac{1}{T^2}\Big(\frac{(\sum_{t=1}^{T}(V_{nt}'s_i s_i'W_nV_{nt})^2}{s_i'W_nW_n's_i}\Big)\times\Big(\frac{\sum_{t=1}^{T}\sum_{s=1}^{t-1}2(V_{nt}'s_i s_i'W_nV_{nt})(\epsilon_{ns}'s_i s_i'W_n\epsilon_{ns})}{s_i'W_nW_n's_i}\Big) \end{aligned} \tag{E.2}$$

From (E.1), (F.2) and (F.8), we have

$$\begin{aligned} E(z_{i,nT}) &= \frac{1}{\sigma_0^4}\frac{1}{T}\frac{\sum_{t=1}^{T}E(V_{nt}'s_i s_i'W_nV_{nt})^2}{s_i'W_nW_n's_i} + \frac{1}{\sigma_0^4}\frac{1}{T}\frac{\sum_{t=1}^{T}\sum_{s=1}^{t-1}E\big(2(V_{nt}'s_i s_i'W_nV_{nt})(\epsilon_{ns}'s_i s_i'W_n\epsilon_{ns})\big)}{s_i'W_nW_n's_i} \\ &= \frac{1}{\sigma_0^4}\frac{1}{T}\frac{T\sigma_0^4 s_i'W_nW_n's_i}{s_i'W_nW_n's_i} + 0 = 1 \end{aligned} \tag{E.3}$$

From (E.2), (F.2), (F.3), (F.8) and (F.12), we have

$$
\begin{aligned}
E(z_{i,nT}^2) &= \frac{1}{\sigma_0^8}\frac{1}{T^2}\frac{\sum_{t=1}^T E(V_{nt}' s_i s_i' W_n V_{nt})^4}{(s_i' W_n W_n' s_i)^2} + \frac{1}{\sigma_0^8}\frac{1}{T^2}\frac{\sum_{t=1}^T \sum_{s=1}^{t-1} E\big(2(V_{nt}' s_i s_i' W_n V_{nt})^2 (\epsilon_{ns}' s_i s_i' W_n \epsilon_{ns})^2\big)}{(s_i' W_n W_n' s_i)^2} \\
&+ \frac{1}{\sigma_0^8}\frac{1}{T^2}\frac{\sum_{t=1}^T \sum_{s=1}^{t-1} E\big(4(V_{nt}' s_i s_i' W_n V_{nt})^2 (\epsilon_{ns}' s_i s_i' W_n \epsilon_{ns})^2\big)}{(s_i' W_n W_n' s_i)^2} + 0 \\
&= \frac{1}{\sigma_0^8}\frac{1}{T^2}\frac{T E(V_{nt}' s_i s_i' W_n V_{nt})^4}{(s_i' W_n W_n' s_i)^2} + \frac{1}{\sigma_0^8}\frac{1}{T^2}\frac{\frac{T(T-1)}{2} E\big(2(V_{nt}' s_i s_i' W_n V_{nt})^2 (\epsilon_{ns}' s_i s_i' W_n \epsilon_{ns})^2\big)}{(s_i' W_n W_n' s_i)^2} \\
&+ \frac{1}{\sigma_0^8}\frac{1}{T^2}\frac{\frac{T(T-1)}{2} E\big(4(V_{nt}' s_i s_i' W_n V_{nt})^2 (\epsilon_{ns}' s_i s_i' W_n \epsilon_{ns})^2\big)}{(s_i' W_n W_n' s_i)^2} \\
&= \frac{1}{\sigma_0^8}\frac{1}{T^2}\frac{T E(V_{nt}' s_i s_i' W_n V_{nt})^4}{(s_i' W_n W_n' s_i)^2} + \frac{1}{\sigma_0^8}\frac{1}{T}\frac{(T-1)\sigma_0^8 (s_i' W_n W_n' s_i)^2}{(s_i' W_n W_n' s_i)^2} + \frac{1}{\sigma_0^8}\frac{1}{T}\frac{2(T-1)\sigma_0^8 (s_i' W_n W_n' s_i)^2}{(s_i' W_n W_n' s_i)^2} \\
&= \frac{1}{\sigma_0^8}\frac{1}{T}\frac{c_i^{\mu_4}}{(s_i' W_n W_n' s_i)^2} + \frac{1}{\sigma_0^8}\frac{3}{T}\frac{(T-1)\sigma_0^8 (s_i' W_n W_n' s_i)^2}{(s_i' W_n W_n' s_i)^2} \\
&= \frac{1}{\sigma_0^8}\frac{3}{T}\frac{T\sigma_0^8 (s_i' W_n W_n' s_i)^2}{(s_i' W_n W_n' s_i)^2} + \frac{1}{\sigma_0^8}\frac{1}{T}\frac{c_i^{\mu_4} - 3\sigma_0^8 (s_i' W_n W_n' s_i)^2}{(s_i' W_n W_n' s_i)^2}
\end{aligned}
\tag{E.4}
$$

where $c_i^{\mu_4} = E(V_{nt}' s_i s_i' W_n V_{nt})^4 = \mu_4\big((\mu_4 - 3\sigma_0^4)\sum_{j=1}^n w_{ij}^4 + 3\sigma_0^4 (s_i' W_n W_n' s_i)^2\big)$. Thus, we have

$$
\begin{aligned}
E(z_{i,nT}^2) &= 3 + \frac{1}{\sigma_0^8}\frac{1}{T}\frac{\mu_4(\mu_4 - 3\sigma_0^4)\sum_{j=1}^n w_{ij}^4}{(s_i' W_n W_n' s_i)^2} + \frac{1}{\sigma_0^8}\frac{1}{T}\frac{(\mu_4 - \sigma_0^4) 3\sigma_0^4 (s_i' W_n W_n' s_i)^2}{(s_i' W_n W_n' s_i)^2} \\
&= 3 + \frac{1}{\sigma_0^8}\frac{1}{T}\frac{\mu_4(\mu_4 - 3\sigma_0^4)\sum_{j=1}^n w_{ij}^4}{(\sum_{j=1}^n w_{ij}^2)^2} + \frac{3(\mu_4 - \sigma_0^4)}{\sigma_0^4}\frac{1}{T}
\end{aligned}
\tag{E.5}
$$

Finally, from (E.3) and (E.5), we have

$$
\begin{aligned}
Var(z_{i,nT}) &= E(z_{i,nT}^2) - (E(z_{i,nT}))^2 \\
&= 2 + \frac{1}{\sigma_0^8}\frac{1}{T}\frac{\mu_4(\mu_4 - 3\sigma_0^4)\sum_{j=1}^n w_{ij}^4}{(\sum_{j=1}^n w_{ij}^2)^2} + \frac{3(\mu_4 - \sigma_0^4)}{\sigma_0^4}\frac{1}{T}
\end{aligned}
\tag{E.6}
$$

When ε_{it} are normally distributed, $Var(z_{i,nT}) = 2 + \dfrac{6}{T}$ since $\mu_4 - 3\sigma_0^4 = 0$.

E.2 Cross-moment of $z_{i,nT}$

From (E.1), we consider $Cov(z_{i,nT}, z_{j,nT})$ as

$$Cov(z_{i,nT}, z_{j,nT}) = E(z_{i,nT} z_{j,nT}) - E(z_{i,nT})E(z_{j,nT}) \tag{E.7}$$

where

$$z_{j,nT} = \frac{1}{\sigma_0^4}\frac{1}{T}\Big(\frac{\sum_{t=1}^{T}(V_{nt}' s_j s_j' W_n V_{nt})^2}{s_j' W_n W_n' s_j}\Big) + \frac{1}{\sigma_0^4}\frac{1}{T}\Big(\frac{\sum_{t=1}^{T}\sum_{s=1}^{t-1} 2(V_{nt}' s_j s_j' W_n V_{nt})(\epsilon_{ns}' s_j s_j' W_n \epsilon_{ns})}{s_j' W_n W_n' s_j}\Big) \tag{E.8}$$

and

$$z_{i,nT} z_{j,nT} = \frac{1}{\sigma_0^8}\frac{1}{T^2}\Big(\frac{\sum_{t=1}^{T}(V_{nt}' s_i s_i' W_n V_{nt})^2}{s_i' W_n W_n' s_i}\Big) \times \Big(\frac{\sum_{t=1}^{T}(V_{nt}' s_j s_j' W_n V_{nt})^2}{s_j' W_n W_n' s_j}\Big) + \varphi \tag{E.9}$$

with φ has all other cross-product terms[41].

From (E.9), (F.5), (F.10) and (F.11), we have

$$\begin{aligned}
E(z_{i,nT} z_{j,nT}) &= \frac{1}{\sigma_0^8}\frac{1}{T^2}\frac{\sum_{t=1}^{T} E(V_{nt}' s_i s_i' W_n V_{nt})^2 (V_{nt}' s_j s_j' W_n V_{nt})^2}{(s_i' W_n W_n' s_i)(s_j' W_n W_n' s_j)} \\
&\quad + \frac{1}{\sigma_0^8}\frac{1}{T^2}\frac{\sum_{t=1}^{T}\sum_{s\neq t}^{T} E(V_{nt}' s_i s_i' W_n V_{nt})^2 (\epsilon_{ns}' s_j s_j' W_n \epsilon_{ns})^2}{(s_i' W_n W_n' s_i)(s_j' W_n W_n' s_j)} + E(\varphi) \\
&= \frac{1}{\sigma_0^8}\frac{1}{T^2}\frac{T E(V_{nt}' s_i s_i' W_n V_{nt})^2 (V_{nt}' s_j s_j' W_n V_{nt})^2}{(s_i' W_n W_n' s_i)(s_j' W_n W_n' s_j)} \\
&\quad + \frac{1}{\sigma_0^8}\frac{1}{T^2}\frac{(T^2 - T) E(V_{nt}' s_i s_i' W_n V_{nt})^2 (\epsilon_{ns}' s_j s_j' W_n \epsilon_{ns})^2}{(s_i' W_n W_n' s_i)(s_j' W_n W_n' s_j)} + 0 \\
&= \frac{1}{\sigma_0^8}\frac{1}{T}\frac{c_i^{\mu_{4,3}}}{(s_i' W_n W_n' s_i)(s_j' W_n W_n' s_j)} + \frac{1}{\sigma_0^8}\frac{1}{T}\frac{(T-1)\sigma_0^8 (s_i' W_n W_n' s_i)(s_j' W_n W_n' s_j)}{(s_i' W_n W_n' s_i)(s_j' W_n W_n' s_j)} \\
&= \frac{1}{\sigma_0^8}\frac{1}{T}\frac{T\sigma_0^8 (s_i' W_n W_n' s_i)(s_j' W_n W_n' s_j)}{(s_i' W_n W_n' s_i)(s_j' W_n W_n' s_j)} + \frac{1}{\sigma_0^8}\frac{1}{T}\frac{c_i^{\mu_{4,3}} - \sigma_0^8 (s_i' W_n W_n' s_i)(s_j' W_n W_n' s_j)}{(s_i' W_n W_n' s_i)(s_j' W_n W_n' s_j)}
\end{aligned} \tag{E.10}$$

[41] The term isolates all cross-products with zero expected values ($E(\varphi) = 0$).

where $c_i^{\mu_{4,3}} = E(V_{nt}'s_i s_i' W_n V_{nt})^2 (V_{nt}'s_j s_j' W_n V_{nt})^2 = \sigma_0^8 (s_i' W_n W_n' s_i)(s_j' W_n W_n' s_j) + ((\mu_4)^2 - 2\mu_4\sigma_0^4 + (\sigma_0^4)^2)(s_i' W_n s_j s_j' W_n s_i)^2 + (\mu_4 - \sigma_0^4)\sigma_0^4 (s_j' W_n s_i)^2 (s_i' W_n W_n' s_i) + (\mu_4 - \sigma_0^4)\sigma_0^4 (s_i' W_n s_j)^2 (s_j' W_n W_n' s_j) + 2\sigma_0^8 (s_i' W_n W_n' s_j)^2 + (\mu_4 - 3\sigma_0^4)\sigma_0^4 \sum_{l=1}^n w_{il}^2 w_{jl}^2 + 4(\mu_3)^2 \sigma_0^2 (s_i' W_n s_j)(s_j' W_n s_i)(s_i' W_n W_n' s_j)$
$+ 2(\mu_3)^2 \sigma_0^2 (s_j' W_n s_i) \sum_{l=1}^n w_{il}^2 w_{jl} + 2(\mu_3)^2 \sigma_0^2 (s_i' W_n s_j) \sum_{l=1}^n w_{jl}^2 w_{il}$. Thus, we have

$$\begin{aligned}
E(z_{i,nT} z_{j,nT}) = 1 &+ \frac{(\mu_4)^2 - 2\mu_4\sigma_0^4 + (\sigma_0^4)^2}{\sigma_0^8} \frac{1}{T} \frac{w_{ij}^2 w_{ji}^2}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} \\
&+ \frac{\mu_4 - \sigma_0^4}{\sigma_0^4} \frac{1}{T} \frac{w_{ji}^2}{\sum_{i=1}^n w_{ji}^2} + \frac{\mu_4 - \sigma_0^4}{\sigma_0^4} \frac{1}{T} \frac{w_{ij}^2}{\sum_{j=1}^n w_{ij}^2} \\
&+ \frac{2}{T} \frac{(\sum_{l=1}^n w_{il} w_{jl})^2}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} + \frac{\mu_4 - 3\sigma_0^4}{\sigma_0^4} \frac{1}{T} \frac{\sum_{l=1}^n w_{il}^2 w_{jl}^2}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} \\
&+ \frac{4(\mu_3)^2}{\sigma_0^6} \frac{1}{T} \frac{w_{ij} w_{ji} (\sum_{l=1}^n w_{il} w_{jl})}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} + \frac{2(\mu_3)^2}{\sigma_0^6} \frac{1}{T} \frac{w_{ji} (\sum_{l=1}^n w_{il}^2 w_{jl})}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} \\
&+ \frac{2(\mu_3)^2}{\sigma_0^6} \frac{1}{T} \frac{w_{ij} (\sum_{l=1}^n w_{jl}^2 w_{il})}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)}
\end{aligned} \tag{E.11}$$

Finally, from (E.3) and (E.11), we have

$$\begin{aligned}
Cov(z_{i,nT}, z_{j,nT}) &= E(z_{i,nT} z_{j,nT}) - E(z_{i,nT}) E(z_{j,nT}) \\
&= \frac{(\mu_4)^2 - 2\mu_4\sigma_0^4 + (\sigma_0^4)^2}{\sigma_0^8} \frac{1}{T} \frac{w_{ij}^2 w_{ji}^2}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} \\
&\quad + \frac{\mu_4 - \sigma_0^4}{\sigma_0^4} \frac{1}{T} \frac{w_{ji}^2}{\sum_{i=1}^n w_{ji}^2} + \frac{\mu_4 - \sigma_0^4}{\sigma_0^4} \frac{1}{T} \frac{w_{ij}^2}{\sum_{j=1}^n w_{ij}^2} \\
&\quad + \frac{2}{T} \frac{(\sum_{l=1}^n w_{il} w_{jl})^2}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} + \frac{\mu_4 - 3\sigma_0^4}{\sigma_0^4} \frac{1}{T} \frac{\sum_{l=1}^n w_{il}^2 w_{jl}^2}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} \\
&\quad + \frac{4(\mu_3)^2}{\sigma_0^6} \frac{1}{T} \frac{w_{ij} w_{ji} (\sum_{l=1}^n w_{il} w_{jl})}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} + \frac{2(\mu_3)^2}{\sigma_0^6} \frac{1}{T} \frac{w_{ji} (\sum_{l=1}^n w_{il}^2 w_{jl})}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} \\
&\quad + \frac{2(\mu_3)^2}{\sigma_0^6} \frac{1}{T} \frac{w_{ij} (\sum_{l=1}^n w_{jl}^2 w_{il})}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)}
\end{aligned} \tag{E.12}$$

When ε_{it} are normally distributed, $Cov(z_{i,nT} z_{j,nT}) = \frac{4}{T} \frac{w_{ij}^2 w_{ji}^2}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)} + \frac{2}{T} \frac{w_{ji}^2}{\sum_{i=1}^n w_{ji}^2} + \frac{2}{T} \frac{w_{ij}^2}{\sum_{j=1}^n w_{ij}^2} + \frac{2}{T} \frac{(\sum_{l=1}^n w_{il} w_{jl})^2}{(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2)}$ since $\mu_4 - 3\sigma_0^4 = 0$ and $\mu_3 = 0$.

E.3 Moment of $r_{i,nT}$

$$r_{i,nT} = \left(\frac{\frac{1}{\sigma_0^2} T \bar{V}'_{nT} s_i s_i' W_n \bar{V}_{nT}}{\sqrt{s_i' W_n W_n' s_i}}\right)^2 = \left(\frac{\frac{1}{\sigma_0^2} T \bar{\epsilon}_{iT} \sum_{j=1}^n w_{ij} \bar{\epsilon}_{jT}}{\sqrt{\sum_{j=1}^n w_{ij}^2}}\right)^2 \tag{E.13}$$

where $\bar{\epsilon}_{iT} = \frac{1}{T}\sum_{t=1}^{T} \varepsilon_{it}$. Since $\bar{\epsilon}_{iT}$ is independent across i with $E(\bar{\epsilon}_{iT}^2) = \frac{1}{T}\sigma_0^2$ and $w_{ii} = 0$ for all i, we have

$$\begin{aligned}
E(r_{i,nT}) &= E\left(\frac{\frac{1}{\sigma_0^2} T \bar{V}'_{nT} s_i s_i' W_n \bar{V}_{nT}}{\sqrt{s_i' W_n W_n' s_i}}\right)^2 = E\left(\frac{\frac{1}{\sigma_0^2} T \bar{\epsilon}_{iT} \sum_{j=1}^n w_{ij} \bar{\epsilon}_{jT}}{\sqrt{\sum_{j=1}^n w_{ij}^2}}\right)^2 \\
&= \frac{T^2}{\sigma_0^4} \frac{1}{\sum_{j=1}^n w_{ij}^2} E\left(\bar{\epsilon}_{iT} \sum_{j \neq i}^{n} w_{ij} \bar{\epsilon}_{jT}\right)^2 \\
&= \frac{T^2}{\sigma_0^4} \frac{1}{\sum_{j=1}^n w_{ij}^2} E(\bar{\epsilon}_{iT})^2 \times E\left(\sum_{j \neq i}^{n} w_{ij} \bar{\epsilon}_{jT}\right)^2 \\
&= \frac{T^2}{\sigma_0^4} \frac{1}{\sum_{j=1}^n w_{ij}^2} E(\bar{\epsilon}_{iT})^2 \times \sum_{j=1}^{n} w_{ij}^2 E(\bar{\epsilon}_{jT})^2 = 1
\end{aligned} \tag{E.14}$$

E.4 Moment of $\tilde{z}_{i,nT}\tilde{r}_{i,nT}$

$$\begin{aligned}
\tilde{z}_{i,nT}\tilde{r}_{i,nT} &= \left(\frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}} \sum_{t=1}^T V'_{nt} s_i s_i' W_n V_{nt}}{\sqrt{s_i' W_n W_n' s_i}}\right)\left(\frac{\frac{1}{\sigma_0^2} T \bar{V}'_{nT} s_i s_i' W_n \bar{V}_{nT}}{\sqrt{s_i' W_n W_n' s_i}}\right) \\
&= \left(\frac{\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} \sum_{j=1}^n w_{ij} \varepsilon_{jt}}{\sqrt{\sum_{j=1}^n w_{ij}^2}}\right)\left(\frac{\frac{1}{\sigma_0^2} T \bar{\epsilon}_{iT} \sum_{j=1}^n w_{ij} \bar{\epsilon}_{jT}}{\sqrt{\sum_{j=1}^n w_{ij}^2}}\right)
\end{aligned} \tag{E.15}$$

where $\bar{\epsilon}_{iT} = \sum_{t=1}^{T} \varepsilon_{it}$. Then

$$\begin{aligned}
E(\tilde{z}_{i,nT}\tilde{r}_{i,nT}) &= \frac{1}{\sum_{j=1}^n w_{ij}^2} E\left(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} \sum_{j=1}^n w_{ij} \varepsilon_{jt}\right)\left(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} \sum_{j=1}^n w_{ij} \frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{jt}\right) \\
&= \frac{1}{\sqrt{T}} \frac{1}{\sum_{j=1}^n w_{ij}^2} E\left(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} \sum_{j=1}^n w_{ij} \varepsilon_{jt}\right)^2 \\
&+ \frac{1}{\sqrt{T}} \frac{1}{\sum_{j=1}^n w_{ij}^2} E\left(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} \sum_{j=1}^n w_{ij} \varepsilon_{jt}\right)\left(\frac{1}{\sigma_0^2}\frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} \sum_{j=1}^n w_{ij} \sum_{s \neq t}^{T} \varepsilon_{js}\right) \\
&= \frac{1}{\sqrt{T}} \frac{1}{\sum_{j=1}^n w_{ij}^2} \sum_{j=1}^{n} w_{ij}^2 + 0 = \frac{1}{\sqrt{T}}
\end{aligned} \tag{E.16}$$

Appendix F Moments for products of quadratic forms

Assume that ε_{it} are i.i.d. across i and t with zero mean, finite variance $\sigma_0^2 > 0$ and $E|\varepsilon_{it}|^{4+\eta} < \infty$ for some $\eta > 0$. Denote $\mu_s = E(\varepsilon_{it}^s)$ for $s = 3, 4$. Suppose that W_n is time-invariant and its diagonal elements satisfy $w_{ii} = 0$ for all i. Then, we have the following moments for products of quadratic forms[42]: For $i \neq j \neq l \neq m$,

$$E(V_{nt}' s_i s_i' W_n V_{nt}) = \sigma_0^2 s_i' W_n s_i = 0 \tag{F.1}$$

$$E(V_{nt}' s_i s_i' W_n V_{nt})^2 = \sigma_0^4 s_i' W_n W_n' s_i = \sigma_0^4 \sum_{j=1}^{n} w_{ij}^2 \tag{F.2}$$

$$E(V_{nt}' s_i s_i' W_n V_{nt})(V_{nt}' s_j s_j' W_n V_{nt}) = \sigma_0^4 s_i' W_n s_j s_j' W_n s_i = \sigma_0^4 w_{ij} w_{ji} \tag{F.3}$$

$$E(V_{nt}' s_i s_i' W_n V_{nt})^4 = \mu_4 \big((\mu_4 - 3\sigma_0^4) \sum_{j=1}^{n} w_{ij}^4 + 3\sigma_0^4 (s_i' W_n W_n' s_i)^2 \big) \tag{F.4}$$

$$\begin{aligned}
&E(V_{nt}' s_i s_i' W_n V_{nt})^2 (V_{nt}' s_j s_j' W_n V_{nt})^2 \\
&= \sigma_0^8 (s_i' W_n W_n' s_i)(s_j' W_n W_n' s_j) + ((\mu_4)^2 - 2\mu_4 \sigma_0^4 + (\sigma_0^4)^2)(s_i' W_n s_j s_j' W_n s_i)^2 \\
&\quad + (\mu_4 - \sigma_0^4)\sigma_0^4 (s_j' W_n s_i)^2 (s_i' W_n W_n' s_i) + (\mu_4 - \sigma_0^4)\sigma_0^4 (s_i' W_n s_j)^2 (s_j' W_n W_n' s_j) \\
&\quad + 2\sigma_0^8 (s_i' W_n W_n' s_j)^2 + (\mu_4 - 3\sigma_0^4)\sigma_0^4 \sum_{l=1}^{n} w_{il}^2 w_{jl}^2 + 4(\mu_3)^2 \sigma_0^2 (s_i' W_n s_j)(s_j' W_n s_i)(s_i' W_n W_n' s_j) \\
&\quad + 2(\mu_3)^2 \sigma_0^2 (s_j' W_n s_i) \sum_{l=1}^{n} w_{il}^2 w_{jl} + 2(\mu_3)^2 \sigma_0^2 (s_i' W_n s_j) \sum_{l=1}^{n} w_{jl}^2 w_{il}
\end{aligned} \tag{F.5}$$

[42]The detailed derivations are given in the supplementary material.

$$E(V_{nt}'s_is_i'W_nV_{nt})^2(V_{nt}'s_js_j'W_nV_{nt})(V_{nt}'s_ms_m'W_nV_{nt})$$
$$= \sigma_0^8 w_{jm}w_{mj}\sum_{l=1}^n w_{il}^2 + 2\sigma_0^8 w_{ij}w_{jm}\sum_{l=1}^n w_{il}w_{ml} + 2\sigma_0^8 w_{im}w_{mj}\sum_{l=1}^n w_{il}w_{jl} + 2\sigma_0^8 w_{ij}w_{im}\sum_{l=1}^n w_{jl}w_{ml}$$
$$+ 2(\mu_4-\sigma_0^4)\sigma_0^4 w_{ij}w_{im}w_{ji}w_{mi} + (\mu_4-3\sigma_0^4)\sigma_0^4 w_{im}^2 w_{jm}w_{mj} + (\mu_4-3\sigma_0^4)\sigma_0^4 w_{ij}^2 w_{jm}w_{mj}$$
$$+ 2(\mu_3)^2\sigma_0^2 w_{ij}w_{im}w_{jm}w_{mi} + 2(\mu_3)^2\sigma_0^2 w_{ij}w_{im}w_{ji}w_{mj} + 2(\mu_3)^2\sigma_0^2 w_{ij}w_{im}w_{jm}w_{mj}$$
$$+ (\mu_3)^2\sigma_0^2 w_{im}^2 w_{ji}w_{mj} \tag{F.6}$$

$$= E(V_{nt}'s_is_i'W_nV_{nt})(V_{nt}'s_js_j'W_nV_{nt})(V_{nt}'s_ls_l'W_nV_{nt})(V_{nt}'s_ms_m'W_nV_{nt})$$
$$= 4\sigma_0^8 w_{ij}w_{ji}w_{lm}w_{ml} + 4\sigma_0^8 w_{il}w_{li}w_{mj}w_{jm} + 4\sigma_0^8 w_{im}w_{mi}w_{lj}w_{jl} + 4\sigma_0^8 w_{ij}w_{jl}w_{lm}w_{mi}$$
$$+ 4\sigma_0^8 w_{ij}w_{jm}w_{li}w_{ml} + 4\sigma_0^8 w_{il}w_{lj}w_{mi}w_{jm} + 4\sigma_0^8 w_{il}w_{lm}w_{ji}w_{mj} + 4\sigma_0^8 w_{im}w_{mj}w_{li}w_{jl}$$
$$+ 4\sigma_0^8 w_{im}w_{ml}w_{ji}w_{lj} \tag{F.7}$$

For $i \neq j$ and $t \neq g$,

$$E(V_{nt}'s_is_i'W_nV_{nt})(V_{ns}'s_is_i'W_nV_{ns}) = \sigma_0^4 tr(s_is_i'W_n)\times tr(s_is_i'W_n) = 0 \tag{F.8}$$

$$E(V_{nt}'s_is_i'W_nV_{nt})(V_{ns}'s_js_j'W_nV_{ns}) = \sigma_0^4 tr(s_is_i'W_n)\times tr(s_js_j'W_n) = 0 \tag{F.9}$$

$$E(V_{nt}'s_is_i'W_nV_{nt})^2(V_{ns}'s_is_i'W_nV_{ns})^2 = \sigma_0^8(s_i'W_nW_n's_i)^2 = \sigma_0^8(\sum_{j=1}^n w_{ij}^2)^2 \tag{F.10}$$

$$E(V_{nt}'s_is_i'W_nV_{nt})^2(V_{ns}'s_js_j'W_nV_{ns})^2 = \sigma_0^8(s_i'W_nW_n's_i)(s_j'W_nW_n's_j) = \sigma_0^8(\sum_{j=1}^n w_{ij}^2)(\sum_{i=1}^n w_{ji}^2) \tag{F.11}$$

$$E(V_{nt}'s_is_i'W_nV_{nt})^3(V_{ns}'s_is_i'W_nV_{ns}) = 0 \tag{F.12}$$

Appendix G Sample countries

Table G1: The list of sample countries

Australia	Austria	Belgium	Canada	China	Denmark
Finland	France	Germany	Hungary	Ireland	Israel
Italy	Japan	Korea	Netherlands	New Zealand	Norway
Poland	Portugal	Russia	Singapore	Spain	Sweden
Switzerland	United Kingdom	United States			

Note: These countries account for 96% of the world's innovation activity in 2021.