Online appendix for "The importance of commitment power in games with imperfect evidence"

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1 Proof of Lemma 1

Proof. Take some feasible profile (f, σ). For all $\theta \in \Theta$, let

$$G(\theta) \equiv \{(d, m, a) \in D \times M \times \{0,1\} : \sigma(\theta)(d, m, a) > 0\}$$

and

$$\left(\widehat{d}(\theta), \widehat{m}(\theta), \widehat{a}(\theta)\right) \in \arg \max_{(d,m,a) \in G(\theta)} \left\{ \begin{array}{c} ap(\theta, d)\, v(\theta, f(d, m, 1)) + \\ (1 - ap(\theta, d))\, v(\theta, f(d, m, 0)) \end{array} \right\}.$$

Pure strategy $\left(\widehat{d}(\theta), \widehat{m}(\theta), \widehat{a}(\theta)\right)$ is the principal's preferred pure strategy for type θ among those that type θ prefers. Let λ denote the lowest expected utility of any type under profile (f, σ). Consider the alternative profile (f'', σ''), where

$$\sigma''(\theta)(d, m, a) = \left\{ \begin{array}{c} 1 \text{ if } (d, m, a) = \left(\widehat{d}(\theta), \theta, \widehat{a}(\theta)\right) \\ 0 \text{ otherwise} \end{array} \right.$$

for all $(d, m, a) \in D \times M \times \{0,1\}$ and $\theta \in \Theta$ and

$$f''(d, m, s) = \left\{ \begin{array}{c} f(d, \widehat{m}(m), s) \text{ if } d = \widehat{d}(m) \text{ and } m \in \Theta \\ \lambda \text{ otherwise} \end{array} \right.$$

for all $(d, m, s) \in D \times M \times \{0,1\}$. It follows that, by construction, profile (f'', σ'') is feasible and is such that $V(f, \sigma) \leq V(f'', \sigma'')$.

Finally, consider profile (f', σ'), where $f' = f''$ except that

$$f'\left(\widehat{d}(\theta), \theta, 1\right) = f'\left(\widehat{d}(\theta), \theta, 0\right) = f''\left(\widehat{d}(\theta), \theta, 0\right)$$

for all $\theta \in \Theta$ for which $\widehat{a}(\theta) = 0$, and

$$\sigma'(\theta)(d, m, a) = \begin{cases} 1 \text{ if } (d, m, a) = \left(\widehat{d}(\theta), \theta, 1\right) \\ 0 \text{ otherwise} \end{cases}$$

for all $(d, m, a) \in D \times M \times \{0, 1\}$ and $\theta \in \Theta$. It follows that, by construction, profile (f', σ') is also feasible, because, for all $\theta \in \Theta$ for which $\widehat{a}(\theta) = 0$, $f''\left(\widehat{d}(\theta), \theta, 1\right) \leq f''\left(\widehat{d}(\theta), \theta, 0\right)$. And, again by construction, $V(f', \sigma') = V(f'', \sigma'') \geq V(f, \sigma)$, which completes the proof. ∎

2 Proof of Proposition 1

Take any information structure with perfect evidence.

Lemma 2 *There is an optimal profile (f, σ) such that for all $(d, m, a) \in D \times M \times \{0, 1\}$ and $\theta \in \Theta$,*

$$\sigma(\theta)(d, m, a) = \begin{cases} 1 \text{ if } (d, m, a) = \left(\widehat{d}(\theta), \theta, 1\right) \\ 0 \text{ otherwise} \end{cases}$$

for some mapping $\widehat{d} : \Theta \to D$ with the property that $p\left(\theta, \widehat{d}(\theta)\right) = 1$ for all $\theta \in \Theta$.

Proof. From lemma 1, there is some optimal profile (f', σ') such that for all $(d, m, a) \in D \times M \times \{0, 1\}$ and $\theta \in \Theta$,

$$\sigma'(\theta)(d, m, a) = \begin{cases} 1 \text{ if } (d, m, a) = \left(\widetilde{d}(\theta), \theta, 1\right) \\ 0 \text{ otherwise} \end{cases}$$

for some $\widetilde{d} : \Theta \to D$. Let $\widehat{\Theta} \equiv \left\{\theta \in \Theta : p\left(\theta, \widetilde{d}(\theta)\right) = 0\right\}$ and, for all $\theta \in \widehat{\Theta}$, let $\overline{d}(\theta)$ be such that $p\left(\theta, \overline{d}(\theta)\right) = 1$. Notice that $\overline{d}(\theta)$ exists for all $\theta \in \widehat{\Theta}$ by the definition of what constitutes perfect evidence. For all $\theta \in \Theta$, let

$$\widehat{d}(\theta) = \begin{cases} \overline{d}(\theta) \text{ if } \theta \in \widehat{\Theta} \\ \widetilde{d}(\theta) \text{ if } \theta \notin \widehat{\Theta} \end{cases}.$$

Consider the following profile (f, σ), where, for all $(d, m, s) \in D \times M \times \{0, 1\}$,

$$f(d, m, s) = \begin{cases} f'\left(\widetilde{d}(m), m, 0\right) & \text{if } m \in \widehat{\Theta} \\ f'(d, m, s) & \text{if } m \notin \widehat{\Theta} \end{cases}$$

and, for all $(d, m, a) \in D \times M \times \{0, 1\}$ and $\theta \in \Theta$,

$$\sigma(\theta)(d, m, a) = \begin{cases} 1 \text{ if } (d, m, a) = \left(\widehat{d}(\theta), \theta, 1\right) \\ 0 \text{ otherwise} \end{cases}.$$

By construction, (f, σ) is also optimal, which proves the statement. ■

For convenience, let us consider the following persuasion game: first, the agent chooses $(d, m) \in E^{\theta} \times M$, where $E^{\theta} \equiv \{d \in D : p(\theta, d) = 1\}$; second, the principal observes (d, m) and chooses reward $x = \widehat{f}(d, m) \in \mathbb{R}$. Let $\left(\widehat{f}, \widehat{\sigma}\right)$ denote the principal's preferred sequential equilibrium, where $\widehat{\sigma} : \Theta \to \Delta\left(E^{\theta} \times M\right)$ represents the agent's strategy and $\widehat{f} : D \times M \to \mathbb{R}$ represents the principal's strategy. Take any other profile $\left(\widehat{f}', \widehat{\sigma}'\right)$ such that, for all $\theta \in \Theta$ and $(d, m) \in E^{\theta} \times M$,

$$\widehat{\sigma}'(\theta)(d, m) > 0 \Rightarrow \widehat{f}'(d, m) \geq \widehat{f}'(d', m')$$

for all $(d', m') \in E^{\theta} \times M$. Sher (2011) shows that no such profile is strictly preferred by the principal to $\left(\widehat{f}, \widehat{\sigma}\right)$.

I complete the proof by using $\left(\widehat{f}, \widehat{\sigma}\right)$ to build an optimal credible profile.

Let

$$f^*(d, m, s) = \begin{cases} \widehat{f}(d, m) & \text{if } s = 1 \\ \min_{\theta \in \Theta} \; x^*(\theta) & \text{if } s = 0 \end{cases}$$

for all $(d, m, s) \in D \times M \times \{0, 1\}$ and

$$\sigma^*(\theta)(d, m, a) = \begin{cases} \widehat{\sigma}(\theta)(d, m) & \text{if } a = 1 \\ 0 \text{ if } a = 0 \end{cases}$$

for all $(d, m, a) \in D \times M \times \{0, 1\}$ and $\theta \in \Theta$.

Lemma 3 *Profile (f^*, σ^*) is credible and optimal.*

Proof. The claim that (f^*, σ^*) is credible follows directly from the fact that $\left(\widehat{f}, \widehat{\sigma}\right)$ is a sequential equilibrium of the persuasion game described above and the fact that

$\min_{\theta\in\Theta}\ x^*(\theta)$ is the lowest possible punishment that is justified by some off-the-equilibrium-path belief.

To show that (f^*, σ^*) is optimal, I proceed by contradiction. Suppose not. Then, by lemma 2, there is some other feasible profile (f', σ') such that $V(f', \sigma') > V(f^*, \sigma^*)$ for which

$$\sigma'(\theta)(d,m,a) = \begin{cases} 1 \text{ if } (d,m,a) = \left(\widehat{d}(\theta), \theta, 1\right) \\ 0 \text{ otherwise} \end{cases}$$

for some $\widehat{d}: \Theta \to E^\theta$, for all $(d,m,a) \in D \times M \times \{0,1\}$ and $\theta \in \Theta$.

Let us return to the persuasion game described above and consider profile $\left(\widehat{f}', \widehat{\sigma}'\right)$, where, for all $(d,m) \in D \times M$, $\widehat{f}'(d,m) = f'(d,m,1)$ and $\widehat{\sigma}'(\theta)(d,m) = \sigma'(\theta)(d,m,1)$ for all $\theta \in \Theta$. Because profile (f', σ') is feasible, it follows that, for all $\theta \in \Theta$ and $(d,m) \in E^\theta \times M$,

$$\widehat{\sigma}'(\theta)(d,m) > 0 \Rightarrow \widehat{f}'(d,m) \geq \widehat{f}'(d',m')$$

for all $(d',m') \in E^\theta \times M$, which is a contradiction to Sher (2011), because the principal strictly prefers profile $\left(\widehat{f}', \widehat{\sigma}'\right)$ to $\left(\widehat{f}, \widehat{\sigma}\right)$. ■

3 Proof of Proposition 2

I show the statement by contradiction and assume that there is an optimal profile (f, σ) that is credible. Let $\lambda \in \mathbb{R}$ denote the lowest expected utility of any type given profile (f, σ), and let Θ^λ denote the set of types whose expected utility is given by λ.

For every $\theta \in \Theta$, let $\left(\widehat{d}(\theta), \widehat{m}(\theta), \widehat{a}(\theta)\right)$ be defined as in the proof of lemma 1, i.e., of all the actions that type θ prefers, $\left(\widehat{d}(\theta), \widehat{m}(\theta), \widehat{a}(\theta)\right)$ is the principal's preferred one.

Let profile (f', σ') be defined as follows: for all $(d,m,s) \in D \times M \times \{0,1\}$,

$$f'(d,m,s) = \begin{cases} u^{-1}(\lambda) \text{ if } m \in \Theta^\lambda \\ f(d,m,s) \text{ if } m \notin \Theta^\lambda \text{ and } d = \widehat{d}(m) \\ \min_{\theta\in\Theta}\ x^*(\theta) \text{ otherwise} \end{cases},$$

while, for all $(d,m,a) \in D \times M \times \{0,1\}$ and $\theta \in \Theta$,

$$\sigma'(\theta)(d,m,a) = \begin{cases} 1 \text{ if } d = \widehat{d}(\theta),\ m = \theta \text{ and } a = \widehat{a}(\theta) \\ 0 \text{ otherwise} \end{cases}.$$

Lemma 4 *Profile (f', σ') is optimal.*

Proof. Profile (f', σ') is feasible by construction: any type $\theta \in \Theta^{\lambda}$ does not want to deviate, because $\lambda \geq \min_{\theta \in \Theta} u(x^*(\theta))$, while any type $\theta \notin \Theta^{\lambda}$ does not want to deviate because their expected utility is larger than λ, by the definition of set Θ^{λ}.

To show that profile (f', σ') is optimal, one must show that $V(f', \sigma') \geq V(f, \sigma)$. For that, it is enough to notice that, for all $\theta \in \Theta^{\lambda}$, for all $(d, m, a) \in D \times M \times \{0, 1\}$ for which $\sigma(\theta)(d, m, a) > 0$,

$$v\left(\theta, u^{-1}(\lambda)\right) \geq ap(\theta, d) v(\theta, f(d, m, 1)) + (1 - ap(\theta, d)) v(\theta, f(d, m, 0)),$$

because function $g(\theta, \cdot)$ is concave. ■

Lemma 5 *For all $(d, m) \in D \times M$ for which there is $\theta \in \Theta^{\lambda}$ such that*

$$\sigma(\theta)(d, m, 1) + \sigma(\theta)(d, m, 0) > 0,$$

it follows that

$$f(d, m, 1) \leq f(d, m, 0) = u^{-1}(\lambda) = \arg\max_{x \in \mathbb{R}} E\left(v(\theta, x) | \theta \in \Theta^{\lambda}\right).$$

Proof. Take any $\theta \in \Theta^{\lambda}$ and consider any $(d, m, a) \in D \times M \times \{0, 1\}$ that is sent with positive probability by type θ given σ. If $f(d, m, 1) > f(d, m, 0)$, then $a = 1$, which would imply that

$$p(\theta, d) v(\theta, f(d, m, 1)) + (1 - p(\theta, d)) v(\theta, f(d, m, 0)) < v\left(\theta, u^{-1}(\lambda)\right)$$

because function $g(\theta, \cdot)$ is strictly concave. This would be a contradiction to the optimality of profile (f, σ), because (f', σ') would be better.

As for the second part, notice that, for any message $(d, m) \in D \times M$ that, under σ, is sent with positive probability by some type $\theta \in \Theta^{\lambda}$, there is no other type $\theta \notin \Theta^{\lambda}$ that also sends it with positive probability. Therefore, because profile (f, σ) is credible, it follows that for any such $(d, m) \in D \times M$,

$$u^{-1}(\lambda) = \arg\max_{x \in \mathbb{R}} E^{\sigma}(v(\theta, x) | d, m, s)$$

whenever $s = 0$, and whenever $s = 1$ and there is some type $\theta \in \Theta^{\lambda}$ who chooses

$(d, m, 1)$. Therefore, it follows that

$$u^{-1}(\lambda) = \arg\max_{x \in \mathbb{R}} \; E\left(v(\theta, x) \,|\theta \in \Theta^{\lambda}\right).$$

■

I complete the proof by showing that one can perturb profile (f', σ') and make the principal better off.

Let $\eta \in \mathbb{R}$ denote the second lowest expected utility of any type, given profile (f', σ'), and let Θ^{η} denote the set of types whose expected utility is η (which is not empty because there is no optimal profile that is constant). To be clear, $\eta > \lambda$ and any type $\theta \notin \left\{\Theta^{\lambda} \cup \Theta^{\eta}\right\}$ has an expected utility larger than η under profile (f', σ'). Pick some type $\theta^{\eta} \in \Theta^{\eta}$ and, without loss of generality, assume that $\widehat{d}(\theta^{\eta}) = d'$. Because type $\theta^{\eta} \notin \Theta^{\lambda}$, he does not receive a constant reward, so that $\widehat{a}(\theta^{\eta}) = 1$ and

$$f'(d', \theta^{\eta}, 1) > u^{-1}(\lambda) > f'(d', \theta^{\eta}, 0).$$

Let

$$\theta^{\lambda} \in \arg\max_{\theta \in \Theta^{\lambda}} \; \left\{p(\theta, d')\, u(f'(d', \theta^{\eta}, 1)) + (1 - p(\theta, d'))\, u(f'(d', \theta^{\eta}, 0))\right\},$$

i.e., type θ^{λ} is the type that would be more willing to mimic type θ^{η} of all types in set Θ^{λ}. It follows that $p(\theta^{\eta}, d') > p\left(\theta^{\lambda}, d'\right) \geq p(\theta, d')$ for all $\theta \in \Theta^{\lambda}$.

For any $\varepsilon \geq 0$, consider the following profile $(f^{\varepsilon}, \sigma')$, where, for all $(d, m, s) \in D \times M \times \{0, 1\}$,

$$f^{\varepsilon}(d, m, s) = \begin{cases} f'(d', \theta^{\eta}, 1) - \varepsilon \text{ if } (d, m, s) = (d', \theta^{\eta}, 1) \\ f'(d', \theta^{\eta}, 0) + \delta(\varepsilon) \text{ if } (d, m, s) = (d', \theta^{\eta}, 0) \\ u^{-1}(\lambda) + \xi(\varepsilon) \text{ if } m \in \Theta^{\lambda} \\ f(d, m, z) \text{ if } m \notin \Theta^{\lambda} \cup \{\theta^{\eta}\} \text{ and } d = \widehat{d}(m) \\ \min_{\theta \in \Theta} x^{*}(\theta) \text{ otherwise} \end{cases},$$

and where $\delta(\varepsilon)$ is such that

$$\eta = p(\theta^{\eta}, d')\, u(f^{\varepsilon}(d', \theta^{\eta}, 1)) + (1 - p(\theta^{\eta}, d'))\, u(f^{\varepsilon}(d', \theta^{\eta}, 0))$$

and $\xi(\varepsilon)$ is defined as follows: if

$$\lambda > p\left(\theta^{\lambda}, d'\right) u\left(f'\left(d', \theta^{\eta}, 1\right)\right) + \left(1 - p\left(\theta^{\lambda}, d'\right)\right) u\left(f'\left(d', \theta^{\eta}, 0\right)\right),$$

then $\xi(\varepsilon) = 0$; if not, then $\xi(\varepsilon)$ is such that

$$u\left(u^{-1}(\lambda) + \xi(\varepsilon)\right) = p\left(\theta^{\lambda}, d'\right) u\left(f^{\varepsilon}\left(d', \theta^{\eta}, 1\right)\right) + \left(1 - p\left(\theta^{\lambda}, d'\right)\right) u\left(f^{\varepsilon}\left(d', \theta^{\eta}, 0\right)\right).$$

Lemma 6 *There is some $\overline{\varepsilon} > 0$ such that, for all $\varepsilon \in [0, \overline{\varepsilon}]$, profile $(f^{\varepsilon}, \sigma')$ is feasible.*

Proof. Let $\overline{\varepsilon}$ be sufficiently small so that, for all $\varepsilon \in [0, \overline{\varepsilon}]$,

$$f^{\varepsilon}\left(d', \theta^{\eta}, 1\right) > u^{-1}(\lambda) > f^{\varepsilon}\left(d', \theta^{\eta}, 0\right).$$

Take any type $\theta \in \Theta^{\lambda}$. The only deviation that is better under f^{ε} than under f' is to deviate to choosing $(d, m, a) = (d', \theta^{\eta}, 1)$. However, by construction of $\xi(\varepsilon)$, that deviation does not make type θ^{λ} strictly better off, provided $\overline{\varepsilon}$ is sufficiently small, which implies that no type in Θ^{λ} strictly benefits from such deviation.

Now, take type θ^{η}. By construction of $\delta(\varepsilon)$, his expected utility is $\eta > u\left(u^{-1}(\lambda) + \xi(\varepsilon)\right)$, so that deviations to reporting $m \in \Theta^{\lambda}$ are not strictly beneficial. Any other deviation returns the same expected utility than under f', so that type θ^{η} has no interest in deviating.

Now, take any type $\theta \notin \left\{\Theta^{\lambda} \cup \Theta^{\eta}\right\}$. If he does not deviate, his expected utility is strictly larger than $\eta > \lambda$. Therefore, deviations to reporting $m \in \Theta^{\lambda}$ or $(d, m, a) = (d', \theta^{\eta}, 1)$ are not strictly beneficial, provided $\overline{\varepsilon}$ is sufficiently small. Other deviations would have also been available under f'.

Finally, take any type $\theta \in \Theta^{\eta}$. Once again, type θ does not want to deviate to reporting $m \in \Theta^{\lambda}$, because $\eta > u\left(u^{-1}(\lambda) + \xi(\varepsilon)\right)$. Now, let us consider deviations to choosing $(d, m, a) = (d', \theta^{\eta}, 1)$. If $p(\theta, d') = p(\theta^{\eta}, d')$, then this deviation has an expected utility of η and, so, it is not a strictly beneficial deviation. If, on the other hand, $p(\theta, d') \neq p(\theta^{\eta}, d')$, it must be that $p(\theta, d') < p(\theta^{\eta}, d')$, because $\theta \in \Theta^{\eta}$. Therefore, provided $\overline{\varepsilon}$ is sufficiently small, the expected utility of deviating would be smaller than η. Any other deviations would have also been available under f'. ■

For every $\varepsilon \geq 0$, let $z(\varepsilon) \equiv V(f^{\varepsilon}, \sigma')$ and notice that

$$z(\varepsilon) = z_1(\varepsilon) + z_2(\varepsilon) + z_3,$$

where

$$z_1(\varepsilon) \equiv \sum_{\theta\in\Theta^\lambda} q(\theta)\, v\left(\theta, u^{-1}(\lambda)+\xi(\varepsilon)\right),$$

$$z_2(\varepsilon) \equiv q(\theta^\eta)\begin{pmatrix} p(\theta^\eta, d')\, v(\theta^\eta, f'(d',\theta^\eta,1)-\varepsilon)+ \\ (1-p(\theta^\eta,d'))\, v(\theta^\eta, f'(d',\theta^\eta,0)+\delta(\varepsilon)) \end{pmatrix}$$

and

$$z_3 \equiv \sum_{\theta\notin\Theta^\lambda\cup\{\theta^\eta\}} q(\theta)\begin{pmatrix} p\left(\theta,\widehat{d}(\theta)\right) v\left(\theta, f'\left(\widehat{d}(\theta),\theta,1\right)\right)+ \\ \left(1-p\left(\theta,\widehat{d}(\theta)\right)\right) v\left(\theta, f'\left(\widehat{d}(\theta),\theta,0\right)\right) \end{pmatrix}.$$

By definition, notice that $z(0) = V(f', \sigma')$.

Lemma 7 $z_2'(0) > 0$.

Proof. Recall that, for all $\theta\in\Theta$ and $x\in\mathbb{R}$, there is a strictly concave function $g(\theta,\cdot)$ such that $v(\theta,x) = g(\theta,u(x))$. Therefore,

$$\frac{\partial v(\theta,x)}{\partial x} = \frac{\partial g(\theta,u(x))}{\partial u}u'(x)$$

Notice also that, because $\delta(0)=0$, it follows that

$$\delta'(0) = \frac{p(\theta^\eta,d')}{1-p(\theta^\eta,d')}\frac{u'(f'(d',\theta^\eta,1))}{u'(f'(d',\theta^\eta,0))}.$$

Combining these two results, we get that

$$z_2'(0) = q(\theta^\eta)\, p(\theta^\eta,d')\, u'(f'(d',\theta^\eta,1))\left(\frac{\partial g(\theta^\eta, u(f'(d',\theta^\eta,0)))}{\partial u} - \frac{\partial g(\theta^\eta, u(f'(d',\theta^\eta,1)))}{\partial u}\right) > 0.$$

■

Lemma 8 $z_1'(0) = 0$.

Proof. First, suppose that

$$\lambda > p(\theta^\eta,d')\, u(f'(d',\theta^\eta,1)) + (1-p(\theta^\eta,d'))\, u(f'(d',\theta^\eta,0)).$$

In that case, z_1 is independent of ε, so the statement follows trivially. If, instead,

$$\lambda = p(\theta^\eta,d')\, u(f'(d',\theta^\eta,1)) + (1-p(\theta^\eta,d'))\, u(f'(d',\theta^\eta,0)),$$

then $\xi(\varepsilon)$ is such that

$$u\left(u^{-1}(\lambda)+\xi(\varepsilon)\right)=p(\theta^{\eta},d')\,u\left(f^{\varepsilon}(d',\theta^{\eta},1)\right)+\left(1-p(\theta^{\eta},d')\right)u\left(f^{\varepsilon}(d',\theta^{\eta},0)\right).$$

Notice that

$$\xi'(0)\,u'\left(u^{-1}(\lambda)\right)=-p(\theta^{\eta},d')\,u'\left(f'(d',\theta^{\eta},1)\right)+\left(1-p(\theta^{\eta},d')\right)u'\left(f'(d',\theta^{\eta},0)\right)\delta'(0).$$

After replacing $\delta'(0)$, we get that $\xi'(0)=0$. Hence,

$$z_1'(0)=\sum_{\theta\in\Theta^{\lambda}}q(\theta)\frac{\partial v\left(\theta,u^{-1}(\lambda)\right)}{\partial x}\xi'(0)=0$$

■

Combining the two previous lemmas, we get that there is some $\overline{\varepsilon}$ for which, for all $\varepsilon\in(0,\overline{\varepsilon})$,

$$z(\varepsilon)=V(f^{\varepsilon},\sigma')>V(f',\sigma'),$$

which is a contradiction to the optimality of profile (f',σ').

4 Proof of Proposition 3

Let me start by defining V_p^{OB} for each information structure p: it represents the expected utility of the principal's preferred feasible profile (f,σ) that is bounded (OB stands for "optimal bounded"), where the bounds are such that

$$f(d,m,s)\in\left[\min_{\theta\in\Theta}x^*(\theta),\max_{\theta\in\Theta}x^*(\theta)\right]$$

for all $(d,m,s)\in D\times M\times\{0,1\}$.

Fix any D and $\widehat{p}\in P$. Notice that, by definition, for any sequence $\{p^t\}\to\widehat{p}\in P$, it follows that

$$V_{p_t}^{O}\geq V_{p_t}^{OB}\geq V_{p_t}^{OC}$$

for all t, while

$$V_{\widehat{p}}^{O}=V_{\widehat{p}}^{OB}=V_{\widehat{p}}^{OC}$$

by proposition 1.

Lemma 9 *For any sequence* $\{p^t\} \to \widehat{p} \in P$, $\left\{V_{p^t}^{OB}\right\} \to V_{\widehat{p}}^{OB}$.

Proof. By lemma 1, the problem of finding the optimal bounded profile is a typical mechanism design problem with a compact choice set, so the statement follows by the theorem of the maximum. ■

To complete the proof, it is enough to show that there is some sequence $\{p^t\} \to \widehat{p}$ for which $\left\{V_{p^t}^{O}\right\} \nrightarrow V_{\widehat{p}}^{O}$.

By lemmas 1 and 2, when $p = \widehat{p}$, there is some optimal profile (f^*, σ^*) such that for all $(d, m, s) \in D \times M \times \{0, 1\}$,

$$f^*(d, m, s) = \begin{cases} \widehat{x}(\theta) \text{ if } (d, m, s) = \left(\widehat{d}(\theta), \theta, 1\right) \\ \min_{\theta \in \Theta} x^*(\theta) \text{ otherwise} \end{cases}$$

for some $\left(\widehat{x}(\theta), \widehat{d}(\theta)\right)$ for each $\theta \in \Theta$, and

$$\sigma^*(\theta)(d, m, a) = \begin{cases} 1 \text{ if } (d, m, a) = \left(\widehat{d}(\theta), \theta, 1\right) \\ 0 \text{ otherwise} \end{cases}$$

for all $(d, m, a) \in D \times M \times \{0, 1\}$. Let

$$\Theta^x \equiv \{\theta \in \Theta : \widehat{x}(\theta) = x\}$$

denote the set of types whose expected utility under profile (f^*, σ^*) is given by $u(x)$ when $p = \widehat{p}$.

Lemma 10 *For every* $\theta \in \Theta$,

$$\widehat{x}(\theta) = \arg\max_{x \in \mathbb{R}} E\left(v(\theta', x) | \theta' \in \Theta^{\widehat{x}(\theta)}\right)$$

Proof. Take some $\theta'' \in \Theta$ and suppose that

$$\widehat{x}(\theta'') \neq \arg\max_{x \in \mathbb{R}} E\left(v(\theta', x) | \theta' \in \Theta^{\widehat{x}(\theta'')}\right)$$

Assume that $p = \widehat{p}$ and consider the following alternative profile (f', σ^*), where $f' = f^*$ except that, for all $\theta \in \Theta^{\widehat{x}(\theta'')}$, $f'\left(\widehat{d}(\theta), \theta, 1\right) = \widehat{x}(\theta'') + \varepsilon$. Provided $|\varepsilon| > 0$ is small enough and ε is positive if and only if

$$\widehat{x}(\theta'') < \arg\max_{x\in\mathbb{R}} E\left(v(\theta', x) \,|\theta' \in \Theta^{\widehat{x}(\theta'')}\right),$$

profile (f', σ^*) is feasible and strictly preferred by the principal to profile (f^*, σ^*), because, for all $\theta \in \Theta^{\widehat{x}(\theta'')}$, $g(\theta, \cdot)$ is strictly concave. But that is a contradiction to the optimality of profile (f^*, σ^*). ■

Let me add some additional notation: for any information structure p, let $V^p(f, \sigma)$ denote the expected utility of the principal under profile (f, σ) when the information structure is p.

Notice that, if $V^O_{\widehat{p}} < E(v(\theta, x^*(\theta)))$, there is some set $\Theta^{x'}$ that has more than one element. Let θ' denote the lowest type in set $\Theta^{x'}$ and notice that $x^*(\theta') < x'$. Let $x'' = \max_{\theta:\widehat{x}(\theta)<x'} \{\widehat{x}(\theta)\}$ and define

$$\alpha = \frac{x' + \max\{x'', x^*(\theta')\}}{2}.$$

Consider the following alternative profile (f'', σ^*), where $f'' = f^*$ except that $f''\left(\widehat{d}(\theta'), \theta', 1\right) = \alpha$, and notice that $V^{\widehat{p}}(f'', \sigma^*) > V^{\widehat{p}}(f^*, \sigma^*) = V^O_{\widehat{p}}$.

Consider the following sequence $\{p^t\}$: for each t,

$$p^t(\theta, d) = \begin{cases} \max\{p(\theta, d) - \epsilon_t, 0\} \text{ if } \theta = \theta' \\ p(\theta, d) \text{ if } \theta \neq \theta' \end{cases}$$

for all $(\theta, d) \in \Theta \times D$, where $\epsilon_t \in (0, 1)$ and is such that $\{\epsilon_t\} \to 0$. In words, type θ''s probability of success in each document is reduced by ϵ_t. Notice that, by definition, $\{p^t\} \to \widehat{p}$.

Consider the following profile (f^t, σ^*):

$$f^t(d, m, s) = \begin{cases} \widehat{x}(\theta) \text{ if } (d, m, z) = \left(\widehat{d}(\theta), \theta, 1\right) \text{ and } \theta \neq \theta' \\ \underline{x}(t) \text{ if } (d, m, z) \neq \left(\widehat{d}(\theta), \theta, 1\right) \text{ and } \theta \neq \theta' \\ \alpha \text{ if } (d, m, z) = \left(\widehat{d}(\theta), \theta, 1\right) \text{ and } \theta = \theta' \\ \min_{\theta\in\Theta} x^*(\theta) \text{ if } (d, m, z) \neq \left(\widehat{d}(\theta), \theta, 1\right) \text{ and } \theta = \theta' \end{cases},$$

where, for each t, $\underline{x}(t) \in \mathbb{R}$ is sufficiently small that i) $\underline{x}(t) < \min_{\theta \in \Theta} x^*(\theta)$ and that

$$p^t(\theta', d)\, u\left(f^t(d, \theta, 1)\right) + \left(1 - p^t(\theta', d)\right) u(\underline{x}(t)) < \alpha$$

for all $(d, \theta) \in D \times \Theta$. Because $\lim_{x \to -\infty} u(x) = -\infty$, it follows that such $\underline{x}(t)$ exists for all t. Therefore, it follows that for any p^t, profile (f^t, σ^*) is feasible. Moreover, notice that

$$\left\{V^{p^t}\left(f^t, \sigma^*\right)\right\} \to V^{\widehat{p}}(f'', \sigma^*) > V(f^*, \sigma^*),$$

which implies that $V^O_{p^t} \nrightarrow V^O_{\widehat{p}}$.