Online Appendix

Making Elections Work: Accountability with Selection *and* Control

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Proof of Lemma 1

As the first-best is implementable with the transfer function $\bar{r} = (\bar{r}_1, \ldots, \bar{r}_n)$, the following must hold for all $i \in C$, $\theta_i \in \Theta_i$, $\hat{\theta} \in \Theta$, and $a \in A$:

$$\pi_i(\theta_i, \hat{\theta}_{-i})\big[B - c(a^*)\big] + \overline{R}_i(\theta_i, \hat{\theta}_{-i}, a^* \mid \theta_i) \geq \pi_i(\hat{\theta})\big[B - c(a)\big] + \overline{R}_i(\hat{\theta}, a \mid \theta_i)$$

or, equivalently,

$$\overline{R}_i(\theta_i, \hat{\theta}_{-i}, a^* \mid \theta_i) - \overline{R}_i(\hat{\theta}, a \mid \theta_i) \geq \pi_i(\hat{\theta})\big[B - c(a)\big] - \pi_i(\theta_i, \hat{\theta}_{-i})\big[B - c(a^*)\big] \ ,$$

where $\overline{R}_i(\hat{\theta}, a \mid \theta_i) \equiv \mathbb{E}_{\tilde{y}}\big[\bar{r}_i(\hat{\theta}, \tilde{y}) \mid \theta_i, a\big]$. Recall from the definition of transfer rules that for each $i \in C$, $\bar{r}_i(\hat{\theta}, y)$ only depends on y if $\pi_i^*(\hat{\theta}) = 1$, so that $\overline{R}_i(\hat{\theta}, a \mid \theta_i)$ only depends on a and θ_i if $\pi_i^*(\hat{\theta}) = 1$. Let $\widehat{\Theta}^i \equiv \big\{\hat{\theta} \in \Theta \colon \pi_i^*(\hat{\theta}) = 1\big\}$, and let $K \equiv \sum_{i \in C} |\widehat{\Theta}^i||A||\Theta_i| + |\Theta \setminus \widehat{\Theta}^i|$, so that $\overline{R} = \big(\overline{R}_i(\hat{\theta}, a \mid \theta_i)\big)_{i \in C, \theta_i \in \Theta_i, \hat{\theta} \in \Theta, a \in A}$ is a vector in $\mathbb{R}^K$. Note that if we change $\overline{R}$ to $\widehat{R}$, where

$$\widehat{R}_i(\hat{\theta}, a \mid \theta_i) \equiv \begin{cases} \overline{R}_i(\hat{\theta}, a \mid \theta_i) + \gamma & \text{if } \pi_i^*(\hat{\theta}) = \pi_i^*(\theta_i, \hat{\theta}_{-i}) = 1 \ \& \ a = a^* \ , \\ \overline{R}_i(\hat{\theta}, a \mid \theta_i) - \gamma & \text{if } \pi_i^*(\hat{\theta}) \neq \pi_i^*(\theta_i, \hat{\theta}_{-i}) = 0 \ , \\ \overline{R}_i(\hat{\theta}, a \mid \theta_i) & \text{otherwise,} \end{cases}$$

for some fixed $\gamma > 0$, then the relevant incentive constraints hold strictly.

Now, let $(Y_1, \ldots, Y_L)$, with $L \gg K$, be a partition of measurable subsets of $[\underline{y}, \overline{y}]$; let $\mu(\cdot \mid \theta_i, a)$ be the probability measure on $[\underline{y}, \overline{y}]$ induced by the c.d.f. $F(\cdot \mid \theta_i, a)$; and let

$M \equiv \sum_{i\in C}\big[L|\widehat{\Theta}_i| + |\Theta \setminus \widehat{\Theta}_i|\big]$. Define the smooth function $h\colon \mathbb{R}^M \to \mathbb{R}^K$ by:

$$h_{i,\theta_i,\hat{\theta},a}\Big(\big(r_{i,l}(\vartheta)\big)_{l=1,\dots,L,\vartheta\in\widehat{\Theta}^i}, \big(s_i(\vartheta)\big)_{\vartheta\notin\widehat{\Theta}^i}\Big)_{i\in C} \equiv$$

$$\begin{cases} \widehat{R}_i(\hat{\theta}, a \mid \theta_i) - \sum_{\ell=1}^{L} \mu(Y_\ell \mid \theta_i, a) r_{i,l}(\hat{\theta}) & \text{if } \pi_i^*(\hat{\theta}) = 1 \ , \\ \widehat{R}_i(\hat{\theta}, a \mid \theta_i) - s_i(\hat{\theta}) & \text{otherwise,} \end{cases}$$

for all $i \in C$, $\theta_i \in \Theta_i$, $\hat{\theta} \in \Theta$, and $a \in A$. Let D be the subset of $\mathbb{R}^K$ that satisfies the following condition: for all $R = \big(R_i(\hat{\theta}, a \mid \theta_i)\big)_{i\in C,\hat{\theta}\in\Theta,a\in A,\theta_i\in\Theta_i} \in D$, $R_i(\hat{\theta}, a \mid \theta_i) \le \widehat{R}_i(\hat{\theta}, a \mid \theta_i)$ if $\hat{\theta}_i \neq \theta_i$ and/or $a \neq a^*$, and $R_i(\hat{\theta}, a \mid \theta_i) \ge \widehat{R}_i(\hat{\theta}, a \mid \theta_i)$ if $\hat{\theta}_i = \theta_i$ and $a = a^*$. An application of Sard's theorem (e.g., Guillemin and Pollack, 1974) implies that for almost all $R \in \mathbb{R}^K$, zero is a regular value of $h(\cdot) + R$. It follows that there exists $R^\circ \in D$ such that the Jacobian of $h(\cdot) + R^\circ$ has full row rank. By the Rouché-Capelli theorem, this in turn implies that the system of K equations $h\big(\big(r_{i,l}(\vartheta)\big), \big(s_i(\vartheta)\big)\big) + R^\circ = 0$ has a solution, $\big(\big(r_l^*(\vartheta)\big), \big(s_i^*(\vartheta)\big)\big)$. Finally, define the transfer function $\rho = (\rho_1, \dots, \rho_n)$ by $\rho_i(\hat{\theta}, y) \equiv r_{i,l}^*(\hat{\theta})$ for all $i \in C$, $\hat{\theta} \in \widehat{\Theta}^i$, $y \in Y_\ell$, and $\rho_i(\hat{\theta}, y) = s_i^*(\hat{\theta})$ for all $i \in C$ and $\hat{\theta} \notin \widehat{\Theta}^i$. By construction, we thus have

$$\begin{aligned} \mathbb{E}_{\tilde{y}}\big[\rho_i(\theta_i, \hat{\theta}_{-i}, \tilde{y}) \mid \theta_i, a^*\big] - \mathbb{E}_{\tilde{y}}\big[\rho_i(\hat{\theta}, \tilde{y}) \mid \theta_i, a\big] &\ge \widehat{R}_i(\theta_i, \hat{\theta}_{-i}, a^* \mid \theta_i) - \widehat{R}_i(\hat{\theta}, a \mid \theta_i) \\ &\ge \overline{R}_i(\theta_i, \hat{\theta}_{-i}, a^* \mid \theta_i) - \overline{R}_i(\hat{\theta}, a \mid \theta_i) + \gamma \\ &> \pi_i(\hat{\theta})\big[B - c(a)\big] - \pi_i(\theta_i, \hat{\theta}_{-i})\big[B - c(a^*)\big] \ , \end{aligned}$$

whenever $\pi_i^*(\theta_i, \hat{\theta}_{-i}) \neq \pi_i^*(\hat{\theta})$, or $\pi_i^*(\hat{\theta}) = 1$ and $a \neq a^*$. Since the candidates' type and action sets are finite, this ensures that there exists a positive number $\beta_i > 0$ such that any deviation from the optimal behavior would reduce i's payoff by at least β_i. Setting $\beta \equiv \max_{i\in C} \beta_i$, we thus obtain the lemma.

Proof of Lemma 2

Fix η and $\lambda \in \mathbb{N}$. For any strategy profile σ, the payoff to candidate $i \in C$ can be written as

$$U_i(\sigma) = (1 - \delta^T)\mathbb{E}_\sigma\left[\sum_{b=1}^{\infty} \delta^{(b-1)T} \mathbf{u}_i^b(\sigma)\right] \ ,$$

where

$$\mathbf{u}_i^b(\sigma) \equiv \frac{1-\delta}{1-\delta^T} \mathbf{1}_{\{e_i^b=0\}} \sum_{t=1}^{T} \delta^{t-1} \left[\pi_i^\eta(\vartheta^{(b-1)T+t}) + \varphi_i^b\right] \left[B - c(a_i^{(b-1)T+t})\right] .$$

Let $\tilde{p}^T$ denote the empirical distribution of types profile in a block of T periods. For any $\gamma > 0$, the following holds for sufficiently large $T \in \mathbb{N}$:

$$\begin{aligned}
\Pr_{\sigma_i^*,\sigma_{-i}}\left[e_i^b = 0 \mid e_i^{b-1} = 0\right] &= \Pr_{\sigma_i^*,\sigma_{-i}} \left\{ \left| \left[B - c(a^*)\right]^{-1} \phi_i^{b-1} \right| < \eta \right\} \\
&= \Pr_{\sigma_i^*,\sigma_{-i}} \left\{ \left| \frac{\mathbb{E}_{\tilde{p}_T,y}\left[r_i(\theta,y) \mid a^*\right] - \mathbb{E}_{\theta,y}\left[r_i(\theta,y) \mid a^*\right]}{\left[B - c(a^*)\right]} \right| < \eta \right\} \\
&> 1 - \gamma ,
\end{aligned}$$

where the second equality follows from the observation that $\mathbb{E}_{\theta,y}\left[r_i(\theta,y) \mid a^*\right] = 0$, and the inequality from Escobar and Toikka's (2013) Lemma 5.1. By the same logic, for any $\gamma > 0$ (independent of b), we have $\left|\mathbb{E}_{\sigma_i^*,\sigma_{-i}}[\varphi_i^b \mid e_i^{b-1} = 0]\right| < \gamma$ if T is sufficiently large. Finally, it is well-known that for any $\gamma > 0$, we have

$$\sup \left\{ \left| \frac{1}{T} \sum_{t=1}^{T} v^t - \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} v^t \right| : (v^1, \ldots, v^T) \in [0,B]^T \right\} < \gamma$$

if we let $T \to \infty$ and $\delta \to 1$.

Now for each $i \in C$, let $v_i^\eta \equiv \mathbb{E}_\theta\Big[\pi_i^\eta(\theta)\left[B - c(a^*)\right]\Big]$. Together with Escobar and Toikka's (2013) Lemma 5.1, the above inequalities imply that for all $\gamma \in (0, v_i^\eta)$, taking T sufficiently large and then δ sufficiently close to one, the following holds for every σ_{-i}:

$$\begin{aligned}
\mathbb{E}\left[\mathbf{u}_i^b(\sigma_i^*,\sigma_{-i}) \mid e_i^b = 0\right] &> \mathbb{E}_\theta\Big[\pi_i^\eta(\theta)\left[B - c(a^*)\right]\Big] - \gamma = v_i^\eta - \gamma \\
&> 0 = \mathbb{E}\left[\mathbf{u}_i^b(\sigma_i^*,\sigma_{-i}) \mid e_i^b > 0\right] .
\end{aligned}$$

Consequently, for all σ_{-i}, $U_i(\sigma_i^*,\sigma_{-i})$ is bounded below by the (fictitious) payoff, denoted $\underline{V}_i$, which candidate i would obtain if it received $v_i^\eta - \gamma$ in every block where she is eligible, 0 in the other blocks, and her probability of becoming ineligible was held constant at γ across blocks. As i (like all the candidates) is eligible at the start of the initial block, we have

$$\underline{V}_i = (1-\delta^T)(v_i^\eta - \gamma) + \delta^T(1 - \gamma + \gamma\delta^{\lambda T})\underline{V}_i$$

or, equivalently,

$$\underline{V}_i = \frac{1-\delta^T}{1-\delta^T(1-\gamma+\gamma\delta^{\lambda T})}(v_i^\eta - \gamma) \ .$$

An application of l'Hôpital's rule shows that

$$\lim_{\delta\to 1} \underline{V}_i = \frac{1}{1+\gamma\lambda}(v_i^\eta - \gamma) \ . \tag{A1}$$

By the same logic as above, for every $\gamma > 0$, if we let $T \to \infty$ and then $\delta \to 1$, then $\mathbb{E}\big[\mathbf{u}_i^b(\sigma_i^*, \sigma_{-i}) \mid e_i^b = 0\big] < v_i^\eta + \gamma$, for all σ_{-i}. As $v_i^\eta + \gamma > 0$, $U_i(\sigma_i^*, \sigma_{-i})$ is bounded above by the payoff candidate i would obtain if she got $v_i^\eta + \gamma$ in every block. Coupled with (A1), this implies that irrespective of σ_{-i}, $U_i(\sigma_i^*, \sigma_{-i})$ approaches v_i^η as $\gamma \to 0$ and, therefore, as $T \to \infty$ and $\delta \to 1$. This proves the first part of Lemma 2. The proof of part (ii) is analogous.

Proof of Lemma 3

As explained above, we prove Lemma 3 in two steps. Step 1 shows that the payoff vector $\big(U_1(\sigma^*), \ldots, U_n(\sigma^*)\big)$ is arbitrarily close to the Pareto frontier of the set of payoffs vectors of $\Gamma(\eta, \lambda, T, \delta)$, so that the payoff to each candidate $i \in C$ must be arbitrarily close to $v_i^\eta \equiv \mathbb{E}_\theta\Big[\pi_i^\eta(\theta)\big[B - c(a^*)\big]\Big]$ in any PBE. Step 2 then uses Step 1 to establish that each candidate i chooses the messages and actions prescribed by σ_i^* arbitrarily often, so that the voter's expected payoff approximates v^* in any PBE (Lemma 2(ii)). Finally, as action sets are finite in $\Gamma(\eta, \lambda, T, \delta)$, it follows from Fudenberg and Levine's (1983) existence theorem that such a PBE exists.

Step 1. Recall from the proof of Lemma 2 that, given any strategy profile σ, the payoff to candidate $i \in C$ in $\Gamma(\eta, \lambda, T, \delta)$ can be written as

$$U_i(\sigma) = (1-\delta^T)\mathbb{E}_\sigma\left[\sum_{b=1}^{\infty} \delta^{(b-1)T}\mathbf{u}_i^b(\sigma)\right] ,$$

where $\mathbf{u}_i^b(\sigma)$ represents i's discounted payoff in block b.

Consider the choice of a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a utilitarian social planner who seeks to maximize $W \equiv \sum_{i\in C} U_i$. For every (arbitrarily small) $\gamma > 0$, let σ^γ be a strategy profile that satisfies $W(\sigma^\gamma) \geq \sup_\sigma W(\sigma) - \gamma$. Now, for any block $b \in \mathbb{N}$, consider a deviation at the start of b (by the social planner) from σ^γ to the strategy profile σ^b, which coincides with $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ in block b, and with σ^γ (taken at the null history, regardless of the previous history of play) from block $b+1$ onward. What is the impact of such a deviation on W? First, the social planner may incur a loss in block b, which is bounded above by $(1-\delta)\sum_{s=1}^{T}\delta^{s-1}\big[B - c(0)\big] = (1-\delta^T)B$. Second, she may incur a loss in block $b+1$ caused by the change in the expected values of the φ_i^{b+1}'s and in the elected candidates' actions. But this loss is also bounded above by $(1-\delta^T)B$. Third, as we saw in the proof of Lemma 2, the probability that all candidates remain eligible at the end of block b under σ^b is arbitrarily close to one. Hence, the social planner may also obtain a gain by increasing the expected number of candidates who remain eligible at the end of block b. More precisely, for every realization of the number of eligible candidates $C^{b+1} \equiv \{i \in C \colon e_i^{b+1} = 0\}$ at the end of block b, this gain is (approximately) bounded below by

$$(1-\delta^T) \sum_{i \in C \setminus C^{b+1}} \sum_{b=1}^{\lambda} \delta^{(b-1)T}\big[v_i^\eta - 0\big] \geq |C \setminus C^{b+1}|(1-\delta^{\lambda T}) \min_i v_i^\eta$$

for sufficiently large T and δ (recall from Lemma 2(i) that each candidate i's payoff is arbitrarily close to v_i^η in every block under σ^*). Dividing this gain by $(1-\delta^T)$ and letting $\delta \to 1$, we obtain $(1-\delta^T)^{-1}(1-\delta^{\lambda T})|C \setminus C^{b+1}| \min_i v_i^\eta \approx \lambda |C \setminus C^{b+1}| \min_i v_i^\eta$; so that, whenever $C^{b+1} \neq C$, the social planner's gain increases without bound with λ. It follows that if we let $\lambda \to \infty$ (and $\gamma \to 0$), the probability that any candidate becomes ineligible at the end of any block b under σ^γ, $\Pr_{\hat{\sigma}}\{C^{b+1} \neq C\}$, must converge to zero — otherwise the deviation would give the social planner a payoff greater than $W(\sigma^\gamma) + \gamma$. Hence, the sum of the expected payoffs induced by σ^γ under the eligibility constraints of $\Gamma(\eta, \lambda, T, \delta)$ is arbitrarily close to the sum of the payoffs which σ^γ would induce if these constraints were ignored (i.e., if all candidates always remained eligible with probability one). Lemma A1 (coupled with the principle of optimality) implies that, in the absence of eligibility constraints, the sum of the payoffs would be maximized by choosing σ^*. Therefore, $W(\sigma^\gamma)$ must be arbitrarily close to $W(\sigma^*) \approx \sum_i v_i^\eta$ or, put differently, $W(\sigma^*)$ must be arbitrarily close to the Pareto frontier. Combined with Lemma A2, this implies that each candidate i's payoff in any PBE of $\Gamma(\eta, \lambda, T, \delta)$ must be arbitrarily close to v_i^η.

Step 2. Take any candidate $i \in C$. We know that for sufficiently large λ, T and δ: (i) $U_i(\sigma)$ must be arbitrarily close to v_i^η for every PBE σ of $\Gamma(\eta, \lambda, T, \delta)$ (step 1); and (ii) i's equilibrium payoff at the start of every block must be approximately bounded below by v_i^η (otherwise, by Lemma A2(i), she could profitably deviate). Therefore, for every PBE σ, we must have $\mathbb{E}_\sigma\big[\mathbf{u}_i^b(\sigma)\big] \approx v_i^\eta$ for all b. By the same logic as in step 1, this implies that the probability that i becomes ineligible in any block b must be close to zero in equilibrium — otherwise, for arbitrarily large λ, the difference between i's equilibrium payoff (from b onward) and v_i^η would also be arbitrarily large.

Now let $\eta \approx 0$; let $\kappa \equiv \min_{a \neq a^*} \big[B - c(a) - \big(B - c(a^*)\big)\big] = \min_{a \neq a^*} \big[c(a^*) - c(a)\big] > 0$; and let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be any PBE of $\Gamma(\eta, \lambda, T, \delta)$, so that $U_i(\sigma) \approx v_i^\eta$ for every $i \in C$. For every block $b \in \mathbb{N}$, let $\rho_i^b(\sigma)$ be the expected proportion of the periods in block b, among those in which candidate i is elected, where it fails to choose a^*; that is, $\rho_i^b(\sigma)\mathbb{E}_\sigma\big[\sum_{t=(b-1)T+1}^{2T} \pi_i^\eta(\vartheta_i^t)\big]$ is approximately the expected number of periods in which i is elected and does not choose a^* — recall that the φ_i^b's are smaller than η in absolute value. For sufficiently large T and δ, we thus have

$$\begin{aligned}
\mathbb{E}_\sigma\big[\mathbf{u}_i^b(\sigma)\big] &\approx \frac{1}{T}\mathbb{E}_\sigma\left[\sum_{t=(b-1)T+1}^{2T} \pi_i^\eta(\vartheta_i^t)\big[B - c(a_i^t)\big]\right] \\
&\geq \frac{1}{T}\mathbb{E}_\sigma\Bigg[\sum_{t=(b-1)T+1}^{2T} \pi_i^\eta(\vartheta_i^t)\Bigg]\Big[\rho_i^b(\sigma)\big[\kappa + B - c(a^*)\big] + \big[1 - \rho_i^b(\sigma)\big]\big[B - c(a^*)\big]\Big] \\
&\approx \mathbb{E}_\theta\big[\pi_i^\eta(\theta)\big]\big[B - c(a^*) + \rho_i^b(\sigma)\kappa\big] = v_i^\eta + \mathbb{E}\big[\pi_i^\eta(\theta)\big]\rho_i^b(\sigma)\kappa \geq v_i^\eta,
\end{aligned}$$

where the second approximation follows from Escobar and Toikka's (2013) Lemma 5.1(ii). As candidate i must be eligible arbitrarily often and her expected payoff must be arbitrarily close to v_i^η (step 1), we must have $\rho_i^b(\sigma) \approx 0$ for an arbitrarily large proportion of blocks b with a probability arbitrarily close to one. As we saw in the main text, it follows from Lemma A1 that if candidate i expects to play a^* arbitrarily often in block $b+1$ and is highly likely to remain eligible at the end of block b, then it is optimal for her to play in accordance with σ_i^* arbitrarily often in block b. We conclude that in any PBE of $\Gamma(\eta, \lambda, T, \delta)$, each candidate i must play in accordance with σ_i^* in an arbitrarily large proportion of blocks with an arbitrarily high probability. Finally, Fudenberg and Levine's (1983) existence theorem guarantees that such an equilibrium exists, thus completing the proof of Lemma 3.