Online Appendix: Optimal Insurance: Dual Utility, Random Losses and Adverse Selection

Alex Gershkov, Benny Moldovanu, Philipp Strack and Mengxi Zhang

Online Appendix A: Stochastic Mechanisms

We provide an example showing that a stochastic mechanism may be more profitable than the optimal deterministic mechanism. Consider an agent with risk preference represented by $g(x) = x^2$ and assume that $\theta \sim U[0,1]$. Suppose that there is a single deterministic loss level l, and let the agent's type θ be the probability that a loss occurs. The optimal deterministic mechanism consists of full insurance to types $\theta \geq \frac{2}{3}$ at a price $\frac{8}{9}l$, and no insurance for lower types. Using this mechanism, the insurer's profit is $\frac{1}{54}l \approx 0.185l$.

Consider now a stochastic direct mechanism of the form $(t(\theta), p(\theta), l)$ such that: type θ pays a premium $t(\theta)$; in exchange, when a loss occurs, the insurer fully reimburses the agent's loss with (conditional) probability $1 - p(\theta)$. Note that the above class of mechanisms includes the optimal deterministic mechanism.

If type θ reports to be type θ' he receives $-t(\theta') - l$ with probability $p(\theta')\theta$ and receives $-t(\theta')$ otherwise. Thus, in the proposed mechanism, this type of agent has a payoff of

$$\tilde{U}(\theta, \theta') = -l - t(\theta') + g\left(1 - p(\theta')\theta\right) l$$

One can verify that $(t(\theta), p(\theta), l)$ is incentive compatible if and only if p is non-increasing and

$$\tilde{U}(\theta) = \tilde{U}(\underline{\theta}) - l \int_{\underline{\theta}}^{\theta} p(z) g\left(1 - p(z)z\right) dz,$$

where we write $\tilde{U}(\theta) = \tilde{U}(\theta, \theta)$ for short. The above conditions imply that

$$t(\theta) = -l - \tilde{U}(\underline{\theta}) + g\left(1 - p(\theta)\theta\right) l + l \int_{\underline{\theta}}^{\theta} p(z) g'\left(1 - p(z)z\right) dz\,.$$

By using similar arguments to that of Lemma 1, it can be shown that the individual rationality constraint holds if and only if

$$\tilde{U}(\underline{\theta}) \geq -l(1 - g(1 - \underline{\theta}) = 0.$$

From now onward, we only consider mechanism for which $\tilde{U}(\underline{\theta}) = 0$. The insurer's profit is

$$\begin{aligned}
\pi(p,t) &= \int_{\underline{\theta}}^{\bar{\theta}} [t(\theta) - (1 - p(\theta))\theta l] \, f(\theta) d\theta \\
&= -l + \int_{\underline{\theta}}^{\bar{\theta}} \left[g\left(1 - p(\theta)\theta\right) l + l \int_{\underline{\theta}}^{\theta} p(z) g'\left(1 - p(z) z\right) dz - (1 - p(\theta))\theta l \right] f(\theta) d\theta \\
&= l \int_{\underline{\theta}}^{\bar{\theta}} \left[g\left(1 - p(\theta)\theta\right) - (1 - p(\theta))\theta + \frac{1 - F(\theta)}{f(\theta)} p(\theta) g'\left(1 - p(\theta)\theta\right) \right] f(\theta) d\theta - l \\
&= l \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta(3\theta - 2) p^2 - (3\theta - 2) p + 1 - \theta \right] d\theta - l.
\end{aligned}$$

To obtain the this equality, we used integration by parts:

$$\int_{\underline{\theta}}^{\bar{\theta}} [1 - F(\theta)] \, p(\theta) g'\left(1 - p(\theta)\theta\right) d\theta - \int_{\underline{\theta}}^{\bar{\theta}} f(\theta) \left[\int_{\underline{\theta}}^{\theta} p(z) g'\left(1 - p(z) z\right) dz \right] d\theta = 0.$$

The optimal p is then given by by

$$p^*(\theta) = \begin{cases} 1 & \text{if } \theta \leq \frac{1}{2} \\ \frac{1}{2\theta} & \text{if } \frac{1}{2} < \theta < \frac{2}{3} \\ 0 & \text{if } \theta \geq \frac{2}{3}. \end{cases}$$

That is, within the above described class of potentially stochastic mechanisms, it is optimal to offer no insurance to agents with type below $\frac{1}{2}$, to offer unconditional full insurance to those with type above $\frac{2}{3}$, and to offer to reimburse the loss with (conditional) probability $1 - \frac{1}{2\theta}$ to intermediate types in $(\frac{1}{2}, \frac{2}{3})$. This mechanism yields, approximately, an expected profit of $0.188l > 0.185l$, and is thus superior to the optimal deterministic mechanism.

Online Appendix B: Finite Number of Losses

Proof of Proposition 3. It holds that

$$H_\theta(z) = \begin{cases} 1-\theta & \text{if } z < l_1 \\ 1-\theta+\theta\sum_{i=1}^{k-1} p_i & \text{if } l_{k-1} \leq z < l_k \text{ and } k \in \{2,...,n\} \\ 1 & \text{if } z \geq l_n \end{cases}$$

$$\text{and } \frac{\partial H_\theta(z)}{\partial\theta} = \begin{cases} -1 & \text{if } z < l_1 \\ -1+\sum_{i=1}^{k-1} p_i & \text{if } l_{k-1} < z < l_k \text{ and } k \in \{2,...,n\} \\ 0 & \text{if } z > l_n \end{cases}$$

In any incentive compatible mechanism, the menu of deductibles $D(\theta)$ is non-increasing in the probability of accident θ. In particular, $D(\theta)$ is continuous almost everywhere.

Fix such a non-increasing menu, and let $\theta_0 = \overline{\theta}$. Denote by $\theta_1 = \inf\{\theta : D(\theta) \leq l_1\}$. If this set is empty, define $\theta_1 = \theta_0 = \overline{\theta}$. Similarly, for $i \in \{2,...,n\}$ define $\theta_i = \inf\{\theta : D(\theta) \leq l_i\}$ with $\theta_i := \theta_{i-1}$ if the set is empty.

By the monotonicity of $D(\theta)$, it holds that $\underline{\theta} = \theta_n \leq \theta_{n-1} \leq ... \leq \theta_1 \leq \theta_0 = \overline{\theta}$. The insurer's profit becomes then

$$\begin{aligned}
\pi &= \int_{\underline{\theta}}^{\overline{\theta}} \left[-\mathbb{E}[L(\theta)] + \int_0^{D(\theta)} [g(H_\theta(z)) - H_\theta(z) - \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial\theta}] dz \right] f(\theta) d\theta - U(\underline{\theta}) \\
&= -\int_{\underline{\theta}}^{\overline{\theta}} \mathbb{E}[L(\theta)] f(\theta) d\theta + \int_{\theta_1}^{\overline{\theta}} \left[\int_0^{D(\theta)} [g(H_\theta(z)) - H_\theta(z) - \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial\theta}] dz \right] f(\theta) d\theta \\
&+ \sum_{k=2}^{n} \int_{\theta_k}^{\theta_{k-1}} \left[\int_0^{D(\theta)} [g(H_\theta(z)) - H_\theta(z) - \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial\theta}] dz \right] f(\theta) d\theta \\
&+ \int_{\underline{\theta}}^{\theta_n} \left[\int_0^{D(\theta)} [g(H_\theta(z)) - H_\theta(z) - \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial\theta}] dz \right] f(\theta) d\theta - U(\underline{\theta}) \\
&= -\int_{\underline{\theta}}^{\overline{\theta}} \mathbb{E}[L(\theta)] f(\theta) d\theta - U(\underline{\theta}) + \int_{\theta_1}^{\overline{\theta}} \left[\int_0^{D(\theta)} [g(1-\theta) - (1-\theta) + \frac{1-F(\theta)}{f(\theta)} g'(1-\theta)] dz \right] f(\theta) d\theta \\
&+ \sum_{k=2}^{n} \int_{\theta_k}^{\theta_{k-1}} \int_0^{D(\theta)} \left[\begin{array}{c} g(1-\theta+\theta\sum_{i=1}^{k-1} p_i) - \left(1-\theta+\theta\sum_{i=1}^{k-1} p_i\right) \\ +\frac{1-F(\theta)}{f(\theta)} g'(1-\theta+\theta\sum_{i=1}^{k-1} p_i) \left(1-\sum_{i=1}^{k-1} p_i\right) \end{array} \right] dz \ f(\theta) d\theta \\
&= -\int_{\underline{\theta}}^{\overline{\theta}} \mathbb{E}[L(\theta)] f(\theta) d\theta - U(\underline{\theta}) + \int_{\theta_1}^{\overline{\theta}} D(\theta) [g(1-\theta) - (1-\theta) + \frac{1-F(\theta)}{f(\theta)} g'(1-\theta)] f(\theta) d\theta \\
&+ \sum_{k=2}^{n} \int_{\theta_k}^{\theta_{k-1}} D(\theta) \left[\begin{array}{c} g(1-\theta+\theta\sum_{i=1}^{k-1} p_i) - \left(1-\theta+\theta\sum_{i=1}^{k-1} p_i\right) \\ +\frac{1-F(\theta)}{f(\theta)} g'(1-\theta+\theta\sum_{i=1}^{k-1} p_i) \left(1-\sum_{i=1}^{k-1} p_i\right)] \end{array} \right] f(\theta) d\theta.
\end{aligned}$$

By definition, in each interval $[\theta_k, \theta_{k-1}]$, the given deductible $D(\theta)$ belongs to the interval $[l_{k-1}, l_k]$, where we denote $l_0 = 0$. Note that, on each interval $[\theta_k, \theta_{k-1}]$, the obtained expression for profit is linear in D :

$$\int_{\theta_k}^{\theta_{k-1}} D(\theta) \left[\begin{array}{c} g(1-\theta+\theta\sum_{i=1}^{k-1} p_i) - \left(1-\theta+\theta\sum_{i=1}^{k-1} p_i\right) \\ +\frac{1-F(\theta)}{f(\theta)} g'(1-\theta+\theta\sum_{i=1}^{k-1} p_i)\left(1-\sum_{i=1}^{k-1} p_i\right) \end{array} \right] f(\theta) d\theta.$$

Depending on the sign of the integrand, the above expression is maximized with respect to D at an extreme point of the respective feasible set, i.e., either at $D^*(\theta) = l_{k-1}$ or at $D^*(\theta) = l_k$. Thus, the profit from the given mechanism can be increased by changing all deductibles $D(\theta)$ on the interval $[\theta_k, \theta_{k-1}]$ to the value of $D^*(\theta)$ that maximizes the above expression. The obtained D^* is non-increasing by construction, and thus also implementable. Hence, we have shown that the search for an optimal mechanism can be confined to menus consisting of at most $n+1$ deductibles, where each deductible equals either zero, or one of the possible losses. ■

Proof of Corollary 1. Here

$$H_\theta(z) = \begin{cases} 1-\theta \text{ if } z < l \\ 1 \quad\quad \text{if } z \geq l \end{cases} \text{ and } \frac{\partial H_\theta(z)}{\partial\theta} = \begin{cases} -1 \text{ if } z \leq l \\ 0 \text{ if } z \geq l \end{cases}.$$

The insurer's profit becomes:

$$\begin{aligned}
\pi &= \int_{\underline{\theta}}^{\overline{\theta}} \left[-\mathbb{E}[L(\theta)] + \int_0^{D(\theta)} [g(H_\theta(z)) - H_\theta(z) - \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial\theta}] dz \right] f(\theta) d\theta - U(\underline{\theta}) \\
&= -l + \int_{\underline{\theta}}^{\overline{\theta}} \left[\int_0^{D(\theta)} [g(1-\theta) - (1-\theta) + \frac{1-F(\theta)}{f(\theta)} g'(1-\theta)] dz \right] f(\theta) d\theta - U(\underline{\theta}) \\
&= -l + \int_{\underline{\theta}}^{\overline{\theta}} D(\theta) \left[g(1-\theta) - (1-\theta) + \frac{1-F(\theta)}{f(\theta)} g'(1-\theta) \right] f(\theta) d\theta - U(\underline{\theta}).
\end{aligned}$$

The above expression is linear in D, and hence the pointwise maximum in the above expression is attained at an extreme point of the feasible set: it can be either at $D = l$ or at $D = 0$, depending on the sign of the virtual value. ■

Online Appendix C: Binary Lotteries

In this Appendix we document several instances of well-known, non-expected utility formulations that coincide with Yaari's dual utility on the class of binary lotteries (e.g., in an insurance framework with a single, deterministic loss).

1. Gul's [1991] disappointment-averse preferences with linear utility over outcomes:[1]

$$\mathcal{U}(x) = \frac{\alpha}{1+(1-\alpha)\beta}\mathbb{E}[x|x \geq CE(x)] + \frac{(1-\alpha)(1+\beta)}{1+(1-\alpha)\beta}\mathbb{E}[x|x < CE(x)]$$

   where $CE(x)$ is a certainty equivalent of lottery $x \in X$, α is the probability that the outcome of the lottery is above its certainty equivalent, and β is a parameter. For binary lotteries, the above functional form is a special case of Yaari's dual utility with[2]

$$g(p) = \frac{p}{1+(1-p)\beta}.$$

2. Versions of the disappointment aversion theories due to Loomes and Sugden [1986], and Jia et al. [2001] with linear utility over outcomes:

$$\mathcal{U}(x) = \mathbb{E}(x) + (e-d)\mathbb{E}\left[\max\{x-\mathbb{E}(x), 0\}\right],$$

   where $e > 0$, $d > 0$. For binary lotteries, this is a special case of Yaari's dual utility with

$$g(p) = p(1+e-d) + (d-e)p^2.$$

   Risk aversion (either in the weak or strong sense) is obtained when $e < d$.

3. The modified Mean-Variance preferences (see Rockafellar et al. [2006]) with linear utility over outcomes are given by[3]:

$$\mathcal{U}(x) = \mathbb{E}(x) - \frac{1}{2}r\mathbb{E}\left[\mid x - \mathbb{E}(x) \mid\right],$$

   where $r \in [0, 1]$. For binary lotteries this is again a special case of Yaari's preferences where

$$g(p) = p - rp(1-p).$$

[1] This is implicit. See also Cereia-Voglio et al. [2020] for an explicit formulation.

[2] (Weak) risk aversion corresponds then to $\beta > \frac{1}{2}$ and aversion to mean-preserving spreads corresponds to $\beta > 0$.

[3] The modification relative to the standard mean-variance preferences is needed in order to ensure consistency with FOSD.

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