Optimal Allocation with Ex-post Verification and Limited Penalties

Tymofiy Mylovanov and Andriy Zapechelnyuk

Online Appendix

Proof of Lemma 2. Consider an allocation $g(x)$ that satisfies (IC) and (F). We construct a monotonic $\tilde{g}(x)$ that preserves constraints (IC) and (F), but increases the principal's payoff.

We have assumed that F has almost everywhere positive density, so F^{-1} exists. Define

$$S(t) = \left|\{y : g(F^{-1}(y)) \le t\}\right|, \quad t \in \mathbb{R}_+.$$

Note that S is weakly increasing and satisfies $S(t) \in [0,1]$ for all t. Define

$$\tilde{g}(x) = S^{-1}(F(x))$$

for all x where $S^{-1}(F(x))$ exists, and extend $\tilde{g}$ to $[a,b]$ by right continuity. Observe that $\tilde{g}$ satisfies (F) by construction. In addition,

$$\sup_{x\in[a,b]} g(x) = \sup_{y\in[0,1]} g(F^{-1}(y)) = S^{-1}(1) = \sup_{y\in[0,1]} \tilde{g}(F^{-1}(y)) = \sup_{x\in[a,b]} \tilde{g}(x),$$

thus $\tilde{g}$ satisfies (IC). Finally, we show that $\tilde{g}$ yields a weakly greater payoff to the principal. By construction,

$$\int_a^z \tilde{g}(x)\mathrm{d}F(x) \le \int_a^z g(x)\mathrm{d}F(x) \quad \text{for all } z \in [a,b],$$

and it holds with equality for $z = b$. Hence, using integration by parts, the expression

$$\int_a^b x(\tilde{g}(x)-g(x))\mathrm{d}F(x) = b\int_a^b (\tilde{g}(x)-g(x))\mathrm{d}F(x) - \int_a^b \left(\int_a^z (\tilde{g}(x)-g(x))\mathrm{d}F(x)\right)\mathrm{d}z$$

is nonnegative. ■

Proof of Corollary 3. Let $Q = \int_a^{z^*} q\mathrm{d}F(x) + \int_{z^*}^b \mathrm{d}F(x)$ be the ex-ante probability to be short-listed, and let A and B be the expected probabilities to be chosen conditional on being shortlisted and conditional on not being short-listed, respectively:

$$A = \sum_{k=1}^n \frac{1}{k}\binom{n-1}{k-1} Q^{k-1}(1-Q)^{n-k} \quad \text{and} \quad B = \frac{1}{n}(1-Q)^{n-1}.$$

The associated reduced-form rule is as follows. An agent's probability $g_i(x)$ to be chosen conditional on $x_i \ge z^*$ and $x_i < z^*$ is given by A and $qA + (1-q)B$, respectively. Hence,

$$g(x) \equiv \sum_i g_i(x) = \begin{cases} n(qA + (1-q)B), & x < z^*, \\ nA, & x \geq z^*. \end{cases} \tag{B1}$$

We now prove that g is identical to g^* whenever q satisfies (15). We have

$$\begin{aligned} Q &= \int_a^{z^*} q \mathrm{d}F(x) + \int_{z^*}^b \mathrm{d}F(x) = \int_a^{z^*} \left(1 - \frac{c}{s}\right) \mathrm{d}F(x) + \int_{z^*}^b \mathrm{d}F(x) \\ &= \left(\int_a^{z^*} \left(\frac{1-c}{s} - \frac{1-s}{s} \right) \mathrm{d}F(x) + \int_{z^*}^b \left(\frac{1}{s} - \frac{1-s}{s} \right) \mathrm{d}F(x) \right) \\ &= \frac{1}{s} \left(\int_a^{z^*} (1-c) \mathrm{d}F(x) + \int_{z^*}^b \mathrm{d}F(x) \right) - \frac{1-s}{s} \\ &= \frac{1/r^*}{s} - \frac{1-s}{s} = \frac{1 - r^* + r^* s}{r^* s}, \end{aligned} \tag{B2}$$

where we used (9). Hence, $1 - Q = \frac{r^*-1}{r^*s}$. Next,

$$\begin{aligned} A &= \sum_{k=1}^n \frac{1}{k} \frac{(n-1)!}{(k-1)!(n-k)!} Q^{k-1} (1-Q)^{n-k} = \frac{1}{nQ} \sum_{k=1}^n \frac{n!}{k!(n-k)!} Q^k (1-Q)^{n-k} \\ &= \frac{1}{nQ} \left(1 - (1-Q)^n\right). \end{aligned}$$

Substituting (B2) into the above yields

$$A = \frac{r^* s}{n(1 - r^* + r^* s)} \left(1 - \frac{(r^*-1)^n}{(r^* s)^n}\right).$$

By (16), after some algebraic transformations,

$$A = \frac{r^* s}{n(1 - r^* + r^* s)} \left(1 - \frac{(r^*-1)^n}{(r^* s)^n}\right) = \frac{r^*}{n}.$$

Also, using (B2) and (16) we obtain

$$B = \frac{1}{n}(1-Q)^{n-1} = \frac{1}{n} \frac{(r^*-1)^{n-1}}{(r^* s)^{n-1}} = \frac{(1-s)r^*}{n}.$$

Substitute A and B into (B1):

$$n(qA + (1-q)B) = \frac{(s-c)nA + cnB}{s} = \frac{(s-c)r^* + c(1-s)r^*}{s} = (1-c)r^*$$

and $nA = r^*$. Hence, $g(x) = g^*(x)$ for all $x \in X$.

It remains to show that, whenever $n \geq \bar{n}$, this shortlisting procedure is feasible and well defined, i.e., $h \geq s$ and the solution of (16) exists and is unique.

Let $n \geq \bar{n}$. Observe that $F(z^*) < 1$, as evident from (8) and the assumption that $c > 0$. Using the definition of r^*, we can rewrite (14) as

$$r^* \leq \frac{1 - F^n(z^*)}{1 - F(z^*)} = 1 + F(z^*) + F^2(z^*) + \ldots + F^{n-1}(z^*) < n.$$

In addition, $1/r^* = (1-c)F(z^*) + 1 - F(z^*) < 1$. Consequently, $\frac{1}{n} < \frac{1}{r^*} < 1$.

Observe that $(1-s)s^{n-1}$ unimodal on $[0, 1]$ with zero at the endpoints and the maximum at $s = \frac{n-1}{n}$. Moreover, it is strictly decreasing on $[\frac{n-1}{n}, 1]$. Since the right-hand side of (16) is strictly between zero and the maximum, there exists a unique solution of (16) on $[\frac{n-1}{n}, 1]$.

Now we prove that $c \leq s$. It is immediate if $c \leq \frac{n-1}{n}$ (since $s \in [\frac{n-1}{n}, 1]$). Assume now that $c > \frac{n-1}{/}n$. Because $n \geq \bar{n}$, condition (14) must hold, which can be written as

$$F^{n-1}(z^*) \leq (1-c)r^*.$$

Thus, the right-hand side of (16) satisfies:

$$\frac{1}{r^*}\left(1 - \frac{1}{r^*}\right)^{n-1} = \frac{\Big(cF(z^*)\Big)^{n-1}}{r^*} \leq (1-c)c^{n-1}.$$

That is, $n \geq \bar{n}$ and (16) entail

$$(1-s)s^{n-1} = \frac{1}{r^*}\left(1 - \frac{1}{r^*}\right)^{n-1} \leq (1-c)c^{n-1}.$$

As $(1-s)s^{n-1}$ is decreasing on $[\frac{n-1}{n}, 1]$ and we have assumed $c > (n-1)/n$, it follows that $c \leq s$. ■

Proof of Proposition 3. We have already established that the solution g must satisfy (21) for some $r \in R = [1, \min\{n, 1/(1-c)\}]$. It remains to show that the optimal r is the unique solution of (22).

Let us first derive how $\overline{x}_r$ and $\underline{x}_r$ change w.r.t. r. From (19) we have

$$(1 - F(\overline{x}_r))\mathrm{d}r - rf(\overline{x}_r)\mathrm{d}\overline{x}_r = -nF^{n-1}(\overline{x}_r)f(\overline{x}_r)\mathrm{d}\overline{x}_r.$$

Hence,

$$\frac{\mathrm{d}\overline{x}_r}{\mathrm{d}r} = \frac{1 - F(\overline{x}_r)}{(r - nF^{n-1}(\overline{x}_r))f(\overline{x}_r)},$$

and thus

$$\overline{x}_r(nF^{n-1}(\overline{x}_r) - r)f(\overline{x}_r)\frac{\mathrm{d}\overline{x}_r}{\mathrm{d}r} = -\overline{x}_r(1 - F(\overline{x}_r)). \tag{B3}$$

Next, if $\underline{x}_r = 0$, then $\frac{\mathrm{d}\underline{x}_r}{\mathrm{d}r} = 0$. Suppose that $\underline{x}_r > 0$. By (20) it satisfies $(1-c)rF(\underline{x}_r) + 1 - F^n(\underline{x}_r) = 1$. Hence,

$$(1-c)F(\underline{x}_r)\mathrm{d}r + (1-c)rf(\underline{x}_r)\mathrm{d}\underline{x}_r - nF^{n-1}(\underline{x}_r)f(\underline{x}_r)\mathrm{d}\underline{x}_r = 0.$$

Hence,

$$\frac{\mathrm{d}\underline{x}_r}{\mathrm{d}r} = \begin{cases} \frac{F(\overline{x}_r)}{(nF^{n-1}(\underline{x}_r)-(1-c)r)f(\underline{x}_r)}, & \text{if } \underline{x}_r > 0, \\ 0, & \text{if } \underline{x}_r = 0. \end{cases}$$

Thus we obtain

$$\underline{x}_r((1-c)r - nF^{n-1}(\underline{x}_r))f(\underline{x}_r)\frac{\mathrm{d}\overline{x}_r}{\mathrm{d}r} = -\underline{x}_rF(\underline{x}_r). \tag{B4}$$

Finally, with $g = g_r$, the principal's objective function is

$$W(r) = \int_a^{\underline{x}_r} x(1-c)r\mathrm{d}F(x) + \int_{\underline{x}_r}^{\overline{x}_r} xnF^{n-1}(x)\mathrm{d}F(x) + \int_{\overline{x}_r}^b xr\mathrm{d}F(x).$$

Taking the derivative w.r.t. r and using (B3) and (B4) we obtain

$$\begin{aligned}\frac{\mathrm{d}W(r)}{\mathrm{d}r} &= \int_a^{\underline{x}_r} x(1-c)\mathrm{d}F(x) + \int_{\overline{x}_r}^b x\mathrm{d}F(x) + \underline{x}_r((1-c)r - nF^{n-1}(\underline{x}_r))f(\underline{x}_r)\frac{\mathrm{d}\overline{x}_r}{\mathrm{d}r} \\ &\quad + \overline{x}_r(nF^{n-1}(\overline{x}_r) - r)f(\overline{x}_r)\frac{\mathrm{d}\overline{x}_r}{\mathrm{d}r} \\ &= \int_a^{\underline{x}_r} x(1-c)\mathrm{d}F(x) + \int_{\overline{x}_r}^b x\mathrm{d}F(x) - \underline{x}_rF(\underline{x}_r) - \overline{x}_r(1-F(\overline{x}_r)) \\ &= \int_a^{\underline{x}_r} (x-\underline{x}_r)(1-c)\mathrm{d}F(x) + \int_{\overline{x}_r}^b (x-\overline{x}_r)\mathrm{d}F(x).\end{aligned}$$

The equation $\frac{\mathrm{d}W(r)}{\mathrm{d}r} = 0$ is exactly (22). To show that it has a unique solution, observe that $\frac{\mathrm{d}\underline{x}_r}{\mathrm{d}r} \geq 0$ and $\frac{\mathrm{d}\underline{x}_r}{\mathrm{d}r} > 0$ (since $g_r(\underline{x}_r) = nF^{n-1}(\underline{x}_r) \geq (1-c)r$ and $g_r(\overline{x}_r) = nF^{n-1}(\overline{x}_r) \leq r$ by (IC)). Consequently, $\frac{\mathrm{d}W(r)}{\mathrm{d}r}$ is strictly decreasing in r. Moreover, for r sufficiently close to 0, we have both $\underline{x}_r$ and $\overline{x}_r$ close to a, in which case $W(r) > 0$, and similarly, for $r = 1/(1-c)$, we have $\overline{x}_r = \underline{x}_r = b$, in which case $W(r) < 0$. ■

Proof of Propositions 5a, 5b, 5c. The points of interest are the optimal principal's payoff z^* and the structure of the optimal allocation mechanism.

First, let us deal with the optimal principal's payoff z^*.

5a: Increasing c affects only the incentive constraint (IC) by making it looser. Optimization on a larger set yields a weakly higher optimal payoff.

5b: Increasing n affects only the feasibility constraint (F) by making it looser. Optimization on a larger set yields a weakly higher optimal payoff. When $n \geq \bar{n}$, the feasibility constraint is not binding and hence has no effect on the optimal payoff.

5c: Let $\tilde{F}(x) \leq F(x)$ for all x. This affects the feasibility constraint (F) by making it looser for all x. Optimization on a larger set yields a weakly higher optimal payoff.

Next, we deal with the structure of the optimal allocation mechanism: threshold $\bar{x}$ of the high pooling interval and threshold $\underline{x}$ of the low pooling interval for the case of $n < \bar{n}$. The interval $[\underline{x}, \overline{x}]$ is the separating interval. There are three cases to consider.

Case 1: $n \geq \bar{n}$. By Proposition 2, the optimal allocation has to satisfy the equation

$$(1-c)\int_a^{z^*}(z^*-x)\mathrm{d}F(x) = \int_{z^*}^b (x-z^*)\mathrm{d}F(x).$$

Integrating by parts, we obtain

$$(1-c)\int_a^{z^*} F(x)\mathrm{d}x = \int_{z^*}^b (1-F(x))\mathrm{d}x. \tag{B5}$$

In this case, the threshold of the high pooling interval and the principal's payoff are the same, $\bar{x} = z^*$. The separating interval is empty.

5a: From (B5) it is immediate that $\frac{\mathrm{d}z^*}{\mathrm{d}c} > 0$. That is, the size of the high pooling interval is decreasing in c.

5b: Equation (B5) is independent of n, so a change in n has no effect (so long as $n \geq \bar{n}$).

5c: Let $\tilde{F}(x) \leq F(x)$ for all x. From (B5) it is immediate that replacing F with $\tilde{F}$ yields a greater solution z^*. That is, the high pooling interval shrinks.

Case 2: $n < \bar{n}$ and $\underline{x} = 0$. By Proposition 3, the optimal allocation has to satisfy equation (22) where we use $\underline{x} = 0$:

$$\int_a^0 (-x)(1-c)\mathrm{d}F(x) = \int_{\overline{x}}^b (x-\overline{x})\mathrm{d}F(x).$$

Integrating by parts, we obtain

$$(1-c)\int_a^0 F(x)\mathrm{d}x = \int_{\overline{x}}^b (1-F(x))\mathrm{d}x. \tag{B6}$$

Note that (19) is satisfied, as it has a free variable r that does not appear in (B6).

Assuming that variations of the parameters are marginal and $\underline{x}$ remains equal to zero, the value of interest is the threshold $\overline{x}$ of the high pooling interval. The change in the length of the separating interval $t = \overline{x} - \underline{x}$ is the same as the change in $\overline{x}$.

5a: From (B6) it is immediate that $\frac{\mathrm{d}\overline{x}}{\mathrm{d}c} > 0$. That is, the high pooling interval is decreasing and the separating interval is increasing in c.

5b: Equation (B6) is independent of n. Hence, a change in n has no effect, so long as $\underline{x} = 0$.

5c: Let $\tilde{F}(x) \leq F(x)$ for all x. From (B6) it is immediate that replacing F by $\tilde{F}$ yields a greater solution $\overline{x}$. That is, the high pooling interval shrinks and the separating interval expands.

Case 3: $n < \bar{n}$ and $\underline{x} > 0$. By Proposition 3, the optimal allocation is described by thee variables, $\overline{x}$, $\underline{x}$, and r, that must satisfy (19), (20), and (22). Combining (19) and (20) to eliminate r, we obtain

$$(1-c)\frac{1-F^n(\bar{x})}{1-F(\bar{x})} = F^{n-1}(\underline{x}). \tag{B7}$$

Also, integrating (22) by parts, we obtain

$$(1-c)\int_a^{\underline{x}} F(x)\mathrm{d}x = \int_{\bar{x}}^b (1-F(x))\mathrm{d}x. \tag{B8}$$

Thus, the structure of the optimal allocation is characterized by $\bar{x}$ and $\underline{x}$ that satisfy (B7) and (B8).

Let us now evaluate $\frac{\mathrm{d}\bar{x}}{\mathrm{d}n}$, $\frac{\mathrm{d}\underline{x}}{\mathrm{d}n}$, $\frac{\mathrm{d}\bar{x}}{\mathrm{d}c}$, and $\frac{\mathrm{d}(\bar{x}-\underline{x})}{\mathrm{d}c}$. After taking the full differential of (B7) and (B8) w.r.t. $\bar{x}$, $\underline{x}$, c, and n, we obtain

$$\begin{aligned} 0 &= L_{\bar{x}}\mathrm{d}\bar{x} - L_{\underline{x}}\mathrm{d}\underline{x} - L_c\mathrm{d}c + L_n\mathrm{d}n, \\ 0 &= M_{\bar{x}}\mathrm{d}\bar{x} + M_{\underline{x}}\mathrm{d}\underline{x} - M_c\mathrm{d}c, \end{aligned} \tag{B9}$$

where

$$\begin{aligned} L_{\bar{x}} &= (1-c)\frac{\mathrm{d}}{\mathrm{d}\bar{x}}(1+F(\bar{x})+F^2(\bar{x})+...+F^{n-1}(\bar{x})) > 0, \\ L_{\underline{x}} &= \frac{\mathrm{d}}{\mathrm{d}\underline{x}}F^{n-1}(\underline{x}) > 0, \\ L_c &= 1+F(\bar{x})+F^2(\bar{x})+...+F^{n-1}(\bar{x}) > 0, \\ L_n &= -\left((1-c)\frac{F^n(\bar{x})}{1-F(\bar{x})}\ln F(\bar{x}) + F^{n-1}(\underline{x})\ln F(\underline{x})\right) > 0, \\ M_{\bar{x}} &= 1-F(\bar{x}) > 0, \\ M_{\underline{x}} &= (1-c)F(\underline{x}) > 0, \\ M_c &= \int_a^{\underline{x}} F(x)\mathrm{d}x > 0, \end{aligned}$$

where we used $c > 0$, $\underline{x} > a$ and $\bar{x} < b$ (i.e., the best payoff is better than random allocation) and that $f(x)$ is everywhere positive.

To evaluate $\frac{\mathrm{d}\bar{x}}{\mathrm{d}n}$ and $\frac{\mathrm{d}\underline{x}}{\mathrm{d}n}$, we set $\mathrm{d}c = 0$ and solve the system of equations (B9),

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}n} = -\frac{L_n M_{\underline{x}}}{L_{\bar{x}} M_{\underline{x}} + L_{\underline{x}} M_{\bar{x}}} < 0,$$
$$\frac{\mathrm{d}\underline{x}}{\mathrm{d}n} = \frac{L_n M_{\bar{x}}}{L_{\bar{x}} M_{\underline{x}} + L_{\underline{x}} M_{\bar{x}}} > 0,$$

and hence $\frac{\mathrm{d}(\bar{x}-\underline{x})}{\mathrm{d}n} < 0$.

To evaluate $\frac{\mathrm{d}\bar{x}}{\mathrm{d}c}$ and $\frac{\mathrm{d}\underline{x}}{\mathrm{d}c}$, we set $\mathrm{d}n = 0$ and solve the system of equations (B9),

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}c} = \frac{L_{\underline{x}} M_c + L_c M_{\underline{x}}}{L_{\bar{x}} M_{\underline{x}} + L_{\underline{x}} M_{\bar{x}}} > 0,$$
$$\frac{\mathrm{d}\underline{x}}{\mathrm{d}c} = \frac{L_{\bar{x}} M_c - L_c M_{\bar{x}}}{L_{\bar{x}} M_{\underline{x}} + L_{\underline{x}} M_{\bar{x}}}.$$

To prove $\frac{\mathrm{d}(\bar{x}-\underline{x})}{\mathrm{d}c} > 0$, it is sufficient to check that $\frac{L_{\underline{x}} - L_{\bar{x}}}{(1-c)L_c} > 0$. By (B7) we have

$$L_c = 1 + F(\bar{x}) + F^2(\bar{x}) + ... + F^{n-1}(\bar{x}) = \frac{1}{1-c} F^{n-1}(\underline{x}).$$

Thus,

$$\begin{aligned}
\frac{L_{\underline{x}} - L_{\bar{x}}}{(1-c)L_c} &= \frac{\frac{\mathrm{d}}{\mathrm{d}\underline{x}} F^{n-1}(\underline{x})}{F^{n-1}(\underline{x})} - \frac{\frac{\mathrm{d}}{\mathrm{d}\bar{x}}(1 + F(\bar{x}) + F^2(\bar{x}) + ... + F^{n-1}(\bar{x}))}{1 + F(\bar{x}) + F^2(\bar{x}) + ... + F^{n-1}(\bar{x})} \\
&= \frac{(n-1)f(\underline{x})}{F(\underline{x})} - \frac{(1 + 2F(\bar{x}) + ... + (n-1)F^{n-2}(\bar{x}))f(\bar{x})}{1 + F(\bar{x}) + F^2(\bar{x}) + ... + F^{n-1}(\bar{x})} \\
&> (n-1)\left(\frac{f(\underline{x})}{F(\underline{x})} - \frac{f(\bar{x})}{F(\bar{x})}\right) \geq 0,
\end{aligned}$$

where we use

$$\frac{(1 + 2x + 3x^2 ... + (n-1)x^{n-2})}{1 + x + x^2 + ... + x^{n-1}} < \frac{n-1}{x}, \quad x \in (0,1),$$

and the hazard rate condition, $F(x)/f(x)$ is increasing.

Lastly, we cannot conclude anything from (B7)-(B8) about how the thresholds change if F is f.o.s.d. improved.

To summarize:

5a: The high pooling interval decreases and, under the hazard rate condition, the separating interval increases in c;

5b: The high pooling interval increases and the separating interval decreases in

n.

5c: The result is ambiguous. If $\tilde{F}(x) \leq F(x)$ for all x, we are unable to make any conclusions about how thresholds $\bar{x}$ and $\underline{x}$ change if F is replaced by $\tilde{F}$. ■