Appendix Material for "Fiscal Foundations of Inflation: Imperfect Knowledge"

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Abstract

This appendix provides detailed information on various aspects of the manuscipt "Fiscal Foundations of Inflation: Imperfect Knowledge".

Contents

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1 Proof of the Proposition

The dynamics of inflation and debt are given by the expressions

$$\pi_t = -\frac{\beta - \phi_\pi^{-1}}{1-\beta}\omega_{t-1}^{\pi} - \frac{\sigma}{\phi_\pi}\hat{y}_t +$$

$$+\sigma\left(\beta^{-1}-1\right)\delta\left[\frac{\beta-\phi_\pi^{-1}}{1-\beta} - \frac{(1-\rho)\,\beta\rho}{1-\beta\rho} - \frac{\beta\rho^2}{1-\beta\rho}\right]\omega_{t-1}^{\pi} - \frac{\sigma}{\phi_\pi}\left(\omega_{t-1}^{b} + \omega_{t-1}^{\tau}\right)$$

and

$$\tilde{b}_t^m = \lambda_b \tilde{b}_{t-1}^m - \kappa_b \pi_t + \frac{(1-\rho)\,\beta\rho}{1-\beta\rho}\delta\phi_\pi\omega_{t-1}^{\pi} - \bar{\tau}_t.$$

The associated ordinary differential equation is

$$\frac{\partial \omega}{\partial \tau} = (T - I)\,\omega$$

where

$$T - I = \begin{bmatrix} \Gamma_\pi - 1 & -\frac{\sigma}{\phi_\pi} \\ -\left(\beta^{-1} - (1-\rho)\,\phi_\pi\right)\delta\Gamma_\pi + \frac{(1-\rho)\beta\rho}{1-\beta\rho}\delta\phi_\pi & \left(\beta^{-1} - (1-\rho)\,\phi_\pi\right)\delta\frac{\sigma}{\phi_\pi} - 1 \end{bmatrix}$$

and

$$\Gamma_\pi = -\frac{\beta-\phi_\pi^{-1}}{1-\beta} + \sigma\left(\beta^{-1}-1\right)\delta\left(\frac{\beta-\phi_\pi^{-1}}{1-\beta} - \frac{(1-\rho)\,\beta\rho}{1-\beta\rho} - \frac{\beta\rho^2}{1-\beta\rho}\right).$$

The determinant is

$$\frac{1}{\beta\phi_\pi - \phi_\pi}\left(1-\phi_\pi\right)$$

which provides

$$\left(\frac{1-\phi_\pi^{-1}}{1-\beta}\right).$$

This is positive when $\phi_\pi > 1$.

The trace is given by

$$\begin{aligned} &\frac{\sigma\delta}{\phi_\pi}\left(\frac{1}{\beta} - \phi_\pi\,(1-\rho)\right) - \frac{1}{1-\beta}\left(\beta - \frac{1}{\phi_\pi}\right) \\ &+\sigma\delta\left(\frac{1}{\beta}-1\right)\left(\frac{1}{1-\beta}\left(\beta - \frac{1}{\phi_\pi}\right) - \beta\rho\frac{1-\rho}{1-\beta\rho} - \beta\frac{\rho^2}{1-\beta\rho}\right) - 2. \end{aligned}$$

Multiply by ϕ_π provides

$$\begin{aligned} &\sigma\delta\left(\frac{1}{\beta} - \phi_\pi\,(1-\rho)\right) - \frac{1}{1-\beta}\left(\phi_\pi\beta - 1\right) \\ &+\sigma\delta\left(\frac{1}{\beta}-1\right)\left(\frac{1}{1-\beta}\left(\phi_\pi\beta-1\right) - \beta\rho\frac{1-\rho}{1-\beta\rho}\phi_\pi - \beta\frac{\rho^2}{1-\beta\rho}\phi_\pi\right) - 2\phi_\pi. \end{aligned}$$

To simplify, collect terms involving ϕ_π to give

$$\left(-\sigma\delta\left(1-\rho\right)-\frac{\beta}{1-\beta}+\sigma\delta-\sigma\delta\left(\frac{1}{\beta}-1\right)\left(\frac{\rho\beta}{1-\beta\rho}\right)-2\right)\phi_\pi.$$

Simplifying provides

$$\left(\sigma\delta\left(\rho-\left(\frac{1}{\beta}-1\right)\left(\frac{\rho\beta}{1-\beta\rho}\right)\right)-\frac{\beta}{1-\beta}-2\right)\phi_\pi$$

or

$$\phi_\pi\left(\sigma\delta\frac{\left(1-\rho\right)\beta\rho}{1-\rho\beta}-\frac{\beta}{1-\beta}-2\right)$$

Collecting the remaining terms

$$\sigma\delta\frac{1}{\beta}-\frac{1}{1-\beta}\left(-1\right)+\sigma\delta\left(\frac{1}{\beta}-1\right)\left(\frac{1}{1-\beta}\left(-1\right)\right)=\frac{1}{1-\beta}$$

The trace must be negative which provides the condition:

$$\phi_\pi\left(\sigma\delta\frac{\left(1-\rho\right)\beta\rho}{1-\rho\beta}-\frac{\beta}{1-\beta}-2\right)+\frac{1}{1-\beta}<0$$

where the left-hand side is given by the sum of the previous two derived terms. Re-arranging:

$$\phi_\pi\left(-\sigma\delta\frac{\left(1-\rho\right)\beta\rho}{1-\rho\beta}+\frac{\beta}{1-\beta}+2\right)>\frac{1}{1-\beta}.$$

Hence

$$\begin{aligned}\phi_\pi &> \frac{\frac{1}{1-\beta}}{\left(-\sigma\delta\frac{(1-\rho)\beta\rho}{1-\rho\beta}+\frac{\beta}{1-\beta}+2\right)}\\ &= \frac{1}{(2-\beta)-\sigma\delta\frac{(1-\rho)\beta\rho(1-\beta)}{1-\rho\beta}}\\ &= \left(1+(1-\beta)-\sigma\delta\left(1-\beta\right)\frac{\left(1-\rho\right)\rho\beta}{1-\rho\beta}\right)^{-1}\end{aligned}$$

as required.

2 Additional Endowment Economy Results

The following presents impulse response functions for the model for the endowment economy. They are: i) a shock to tax expectations; ii) a shock to interest-rate expectations (when the monetary policy rule is not known); and iii) a shock to inflation expectations when agents know the policy rule. In this case agents make policy-consistent projections of the nominal interest rate using their inflation forecasts in conjunction with the policy rule. Agents' inflation and interest-rate expectations satisfy the Taylor principle as assumed in the analytical results. The initial shock shifts beliefs from steady state to $\omega = 0.01$ for each variable in question.

The figures are presented without detailed comment. The key property in all cases are that perceived wealth effects drive consumption demand, and, therefore, variations in inflation. In each case the stability conditions for convergence are identical. Moreover, forecast errors of the presented discounted value of tax obligations always dominate perceived wealth, as the discounted expected value of returns to debt holdings and the market value of debt effects roughly off-set each other. If the present discount value of taxes are expected to fall, agents feel wealthier, expand consumption demand, and drive up prices. If taxes are perceived to rise, the opposite.

The cases are distinguished in one important respect. The impact effects of each expectations shock depend on the specific variable being considered, and also whether the agents know the policy rule. However, once tax beliefs adjust after the period of the shock — recall they are a predetermined variable — similar mechanics unfold, with the present discounted value of taxes determining whether agents feel wealthier or poorer. In the case of an interest-rate belief shock, or an inflation shock when agents know the monetary policy rule, the dynamics are quite similar — see figures (2) and (3). The market value of debt initially rises, while the preset value of returns on debt fall. Inflation falls because real interest rates are higher in each case, which also leads to a fall on impact in the present value of returns on debt. In subsequent periods, taxes rise more than proportionately to debt, because the market value of debt previously rose, and because of the assumed tax policy rule (taxes move more than one-for-one with the market value of debt). This leads to the belief that taxes have increased permanently, leading to a large rise in the present discounted value of taxes. This leads to a substantial negative wealth effect, offsetting the initial positive wealth effect from the rise in the market value of debt, that leads to persistent decline in inflation.

In the case of a shocks to tax beliefs (figure 1), the impact effect and subsequent dynamics operate in the same direction. On impact, higher expected taxes leads to a negative wealth effect and a decline in inflation. This effect is reinforced subsequently as the decline in inflation raises the real market value of debt, which in turn raises taxes and their expected present discounted value in subsequent periods. Hence the non-Ricardian term persistently weighs down on consumption demand, leading to persistence dynamics in inflation below steady state. This example highlights, that even under lump-sum taxation, shifting beliefs about tax

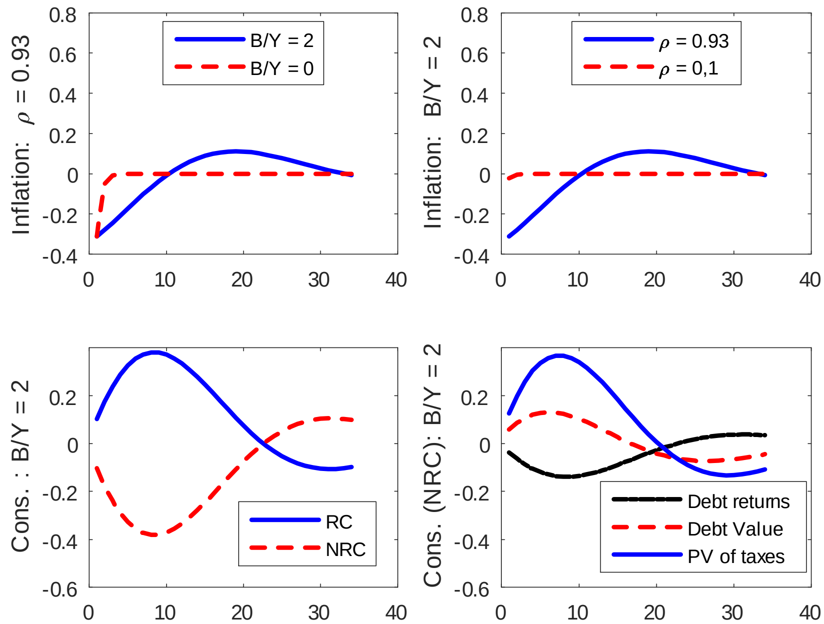


Figure 1: Impulse responses to a tax belief shock.

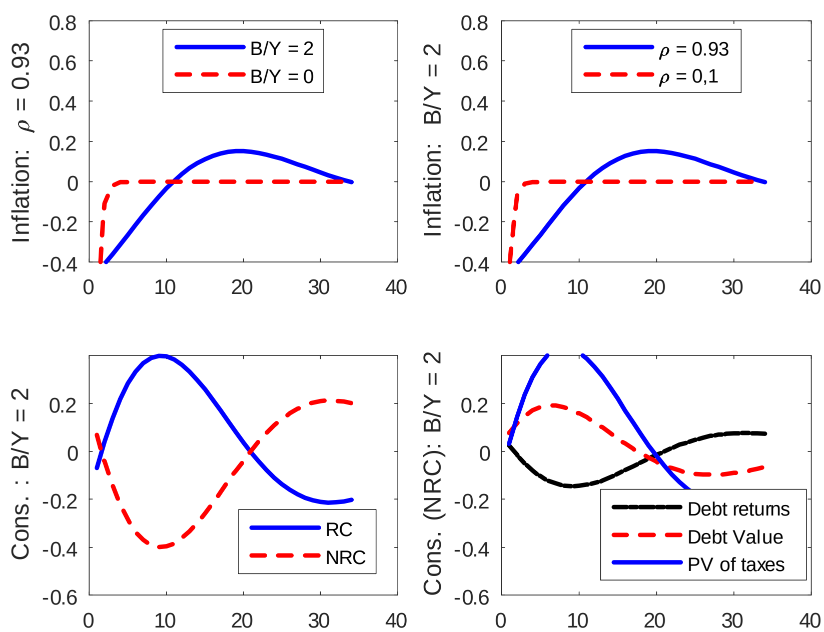


Figure 2: Impulse responses to a interest rate belief shock.

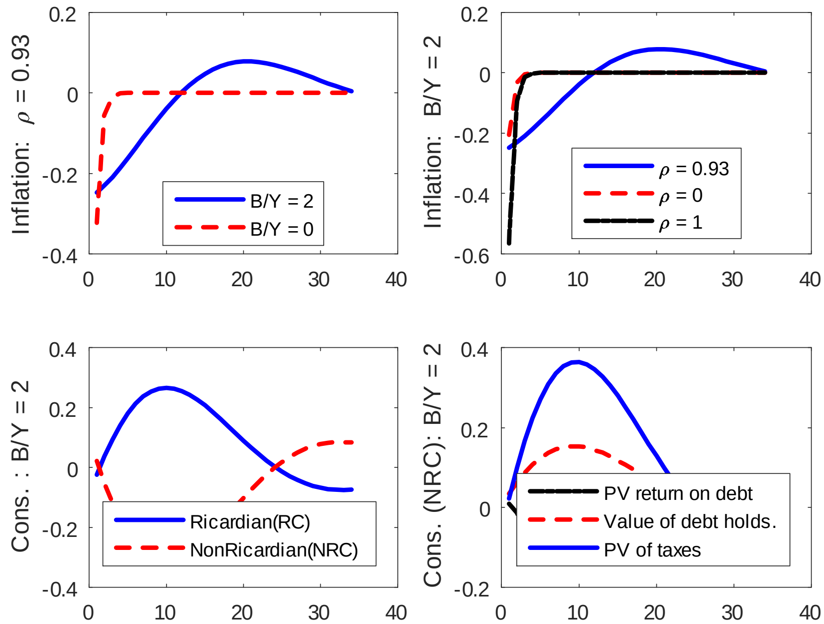


Figure 3: Impulse responses to an inflation belief shock when agents know the interest rate rule and make policy consistent forecasts.

obligations have non-Ricardian effects which influence the evolution of inflation. This is at the heart of our theory.

3 The Model and Solution

This section shows how to derive the optimal decision rules under arbitrary beliefs. The full model is then described and written in compact form

$$A_0 Z_t = \sum_{s=1}^{4} A_s \left(\hat{E}_t \sum_{T=t}^{\infty} \beta_s^{T-t} Z_{T+1} \right) + A_5 Z_{t-1} + A_6 \epsilon_t \tag{1}$$

where Z_t is a vector of size n_Z, ϵ_t a vector of primitive exogenous disturbances, and the A_i matrices of conformable dimension. The task is to compute the matrices A_i and the set of discount factors β_i $(i = 1...j)$ for the DSGE model described in section 5 of the paper. Given the system (1), and an assumption on how beliefs are formed, we derive the state-space representation of model used in estimation.

3.1 The Steady State

Assume in steady state prices are stable so that $\pi = 1$. The steady-state real interest rate is

$$R = \beta^{-1}\gamma.$$

The price of long-term bonds

$$P^m = \frac{1}{R-\rho}.$$

Surplus-output ratio:

$$\frac{s}{y} = \left(\beta^{-1} - 1\right)\frac{P^m b^m}{y}.$$

The marginal cost is

$$S = \frac{\theta_p - 1}{\theta_p}.$$

The real wage, in efficiency units (divided by Z_t), is

$$w^r = S.$$

Output, in efficiency units

$$y = N$$

implying the profit/output ratio

$$\begin{aligned}\Gamma^f &= y\left(1-S\right)\\ \frac{\Gamma^f}{y} &= \left(1-S\right)\end{aligned}$$

and consumption-output ratio

$$\frac{c}{y} = 1 - \frac{g}{y}.$$

Wages bill-consumption ratio

$$\frac{w^r N}{c} = \left(\frac{c}{y}\right)^{-1} w^r = \left(\frac{c}{y}\right)^{-1}\frac{\theta_p - 1}{\theta_p}$$

From the labor supply first-order condition we get

$$\xi_\nu = \frac{\nu'\left(N\right)N}{\nu(N)} = \frac{\sigma-1}{1-b}\frac{\theta_w - 1}{\theta_w}\frac{w^r N}{c}.$$

For later use the second derivative satisfies steady-state restriction

$$\xi_{\nu\nu} = \frac{\nu''(N)N}{\nu'(N)} = \phi_n^{-1} + \frac{(\sigma-1)}{\sigma}\frac{\nu'(N)N}{\nu(N)}.$$

Finally, the tax-to-output ratio is obtained by

$$\frac{\tau}{y} = \frac{s}{y} + \frac{g}{y}.$$

3.2 Decision Rules 1 & 2: Households

This section shows how to solve for the optimal consumption and wage decisions as a function of the beliefs about various objects that are exogenous to households' decisions. Analogous to rational expectations solution procedures, we seek to write all endogenous variables as a function of exogenous variables, and beliefs about future realizations of these same variables. In what follows the index i denotes variables under control of the household, while other variables, without the index, are exogenous to household decisions and outside their control. The log-linear approximation of the model is taken as given. The derivations are available on request.

For each household i, optimality is characterized by the consumption Euler equation, a wage determination equation, a flow budget constraint and transversality condition. These expressions are:

Euler equation:

$$\left[\xi_\nu\left[-\theta_w\left[\hat{w}_t^r(i) - \hat{w}_t^r\right] + \hat{N}_t\right] - \sigma\frac{\hat{c}_t(i) - b\hat{c}_{t-1}}{1-b}\right] = \hat{E}_t^i\left(\hat{R}_{t+1}^m - \hat{\pi}_{t+1}\right) +$$

$$+\hat{E}_t^i\left[\xi_\nu\left[-\theta_w\left[\hat{w}_{t+1}^r(i) - \hat{w}_{t+1}^r\right] + \hat{N}_{t+1}\right] - \sigma\frac{\hat{c}_{t+1}(i) - b\hat{c}_t}{1-b}\right]$$

where

$$\xi_\nu = \frac{\nu'(N)N}{\nu(N)} = \frac{\sigma-1}{1-b}\frac{\theta_w - 1}{\theta_w}\frac{w^r N}{c}.$$

Rearrange to give

$$\hat{E}_t^i\left[\xi_\nu\theta_w\hat{w}_{t+1}^r(i) + \sigma\frac{1}{1-b}\hat{c}_{t+1}(i)\right] = \xi_\nu\theta_w\hat{w}_t^r(i) + \sigma\frac{1}{1-b}\hat{c}_t(i) + \delta_t^C$$

where

$$\begin{aligned}\delta_t^C &= -\xi_\nu\theta_w\hat{w}_t^r - \xi_\nu\hat{N}_t + \sigma\frac{b}{1-b}\left(\hat{c}_t - \hat{c}_{t-1}\right) \\ &\quad +\hat{E}_t^i\left(\hat{R}_{t+1}^m - \hat{\pi}_{t+1} + \xi_\nu\theta_w\hat{w}_{t+1}^r + \xi_\nu\hat{N}_{t+1}\right)\end{aligned}$$

collects variables that are *exogenous* to the household's decision.

Wage equation:

$$c_1^w \hat{w}_t^r(i) = -c_3^w \left(\theta_w \hat{w}_t^r + \hat{N}_t\right) + \frac{\hat{c}_t(i) - b\hat{c}_{t-1}}{1-b} + \xi_w^{-1} \hat{w}_{t-1}^r(i) +$$

$$+\xi_w^{-1} \iota_w \hat{\pi}_{t-1} - \xi_w^{-1} \left(1 + \beta \iota_w\right) \hat{\pi}_t + \xi_w^{-1} \beta \hat{E}_t^i \left[\hat{w}_{t+1}^r(i) + \hat{\pi}_{t+1}\right] - \frac{\hat{\theta}_{w,t}}{\theta_w - 1}$$

where

$$\begin{aligned}
c_1^w &= \left[-\left(\xi_\nu - \xi_{\nu\nu}\right) \theta_w + 1 + \xi_w^{-1} + \xi_w^{-1} \beta\right] \\
c_2^w &= \xi_w^{-1} \left(1 + \beta \iota_w\right) \\
c_3^w &= \xi_\nu - \xi_{\nu\nu}
\end{aligned}$$

Rearrange to give

$$-\xi_w^{-1} \beta \hat{E}_t^i \hat{w}_{t+1}^r(i) = -c_1^w \hat{w}_t^r(i) + \frac{1}{1-b} \hat{c}_t(i) + \xi_w^{-1} \hat{w}_{t-1}^r(i) + \delta_t^w$$

where

$$\begin{aligned}
\delta_t^w &= -c_3^w \left(\theta_w \hat{w}_t^r + \hat{N}_t\right) - \frac{b}{1-b} \hat{c}_{t-1} + \xi_w^{-1} \iota_w \hat{\pi}_{t-1} - \xi_w^{-1} \left(1 + \beta \iota_w\right) \hat{\pi}_t \\
&\quad + \xi_w^{-1} \beta \hat{E}_t^i \hat{\pi}_{t+1} - \frac{\hat{\theta}_{w,t}}{\theta_w - 1}
\end{aligned}$$

again collects all variables that are exogenous to the household.

Budget constraint:

$$\frac{y}{c} \hat{d}_t(i) = -\hat{c}_t(i) + \beta^{-1} \frac{y}{c} \hat{d}_{t-1}(i) + \frac{w^r N}{c} \left(1 - \theta_w\right) \hat{w}_t^r(i) + \delta_t^d.$$

where $\hat{d}_t(i)$ is household wealth and where

$$\delta_t^d = \beta^{-1} \frac{P^m b^m}{y} \frac{y}{c} \left(\hat{R}_t^m - \hat{\pi}_t\right) + \frac{w^r N}{c} \left(\theta_w \hat{w}_t^r + \hat{N}_t\right) + \frac{y}{c} \hat{\Gamma}_t - \frac{y}{c} \hat{\tau}_t.$$

These three expressions and the transversality completely describe the optimal behavior of the household.

Note on the budget constraint: We will later re-write the model in terms of a particular state variable. Rather than work with the variable $\hat{d}_t$, the market value of debt holdings of the household, we work with the state variable $\hat{b}_t^m$, the face value of debt issued by the government. Using the fact they are related by

$$\hat{d}_t(i) = P^m \hat{b}_t^m(i) + \frac{P^m b^m}{y} \hat{P}_t^m,$$

and that

$$\hat{R}_T = \hat{E}_t^i \hat{R}_{T+1}^m \text{ for every } T,$$

$$\hat{R}_t^m = \rho\beta\gamma^{-1}\hat{P}_t^m - \hat{P}_{t-1}^m$$

we can re-write flow budget constraint as

$$\frac{y}{c}\left(P^m \hat{b}_t^m(i) + \frac{P^m b^m}{y}\hat{P}_t^m\right) = -\hat{c}_t(i) + \beta^{-1}\frac{y}{c}\left(P^m \hat{b}_{t-1}^m(i) + \frac{P^m b^m}{y}\hat{P}_{t-1}^m\right) + \frac{w^r N}{c}(1-\theta_w)\hat{w}_t^r(i) + \delta_t^d.$$

where

$$\delta_t^d = \beta^{-1}\frac{P^m b^m}{y}\frac{y}{c}\left(\rho\beta\gamma^{-1}\hat{P}_t^m - \hat{P}_{t-1}^m - \hat{\pi}_t\right) + \frac{w^r N}{c}\left(\theta_w \hat{w}_t^r + \hat{N}_t\right) + \frac{y}{c}\hat{\Gamma}_t - \frac{y}{c}\hat{\tau}_t.$$

Because the terms in $\hat{P}_{t-1}^m$ cancel, the flow budget constraint is

$$\frac{y}{c}P^m \hat{b}_t^m(i) = -\hat{c}_t(i) + \beta^{-1}\frac{y}{c}P^m \hat{b}_{t-1}^m(i) + \frac{w^r N}{c}(1-\theta_w)\hat{w}_t^r(i) + \delta_t^d$$

where

$$\delta_t^d = \frac{P^m b^m}{y}\frac{y}{c}\left(\rho\gamma^{-1} - 1\right)\hat{P}_t^m - \beta^{-1}\frac{P^m b^m}{y}\frac{y}{c}\hat{\pi}_t + \frac{w^r N}{c}\left(\theta_w \hat{w}_t^r + \hat{N}_t\right) + \frac{y}{c}\hat{\Gamma}_t - \frac{y}{c}\hat{\tau}_t.$$

Note the decision rule depends on the discounted sum of interest rates, implicit in the bond price, not on the return of that bond. We appeal to this property later.

3.3 Solving the decision rule

The first-order conditions of the household can be written compactly as

$$\Psi_0^{HH}\hat{E}_t^i \begin{bmatrix} \hat{c}_{t+1}(i) \\ \hat{w}_{t+1}^r(i) \\ \hat{w}_t^r(i) \\ \hat{d}_t(i) \end{bmatrix} = \Psi_1^{HH} \begin{bmatrix} \hat{c}_t(i) \\ \hat{w}_t^r(i) \\ \hat{w}_{t-1}^r(i) \\ \hat{d}_{t-1}(i) \end{bmatrix} + \delta_t,$$

where the vector

$$\begin{bmatrix} \hat{c}_{t+1}(i) \\ \hat{w}^r_{t+1}(i) \\ \hat{w}^r_t(i) \\ \hat{d}_t(i) \end{bmatrix}$$

includes variables that under under control of the agent, and where

$$\delta_t = \begin{bmatrix} \delta^C_t \\ \delta^w_t \\ 0 \\ \delta^d_t \end{bmatrix}$$

is a vector collecting all terms outside the control of households. The matrices are:

$$\Psi_0^{HH} = \begin{bmatrix} \frac{\sigma}{1-b} & \xi_\nu \theta_w & 0 & 0 \\ 0 & -\xi_w^{-1}\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{y}{c} \end{bmatrix}; \quad \text{and}$$

$$\Psi_1^{HH} = \begin{bmatrix} \frac{\sigma}{1-b} & \xi_\nu \theta_w & 0 & 0 \\ \frac{1}{1-b} & -c_1^w & \xi_w^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ -1 & \frac{w^r N}{c}(1-\theta_w) & 0 & \beta^{-1}\frac{y}{c} \end{bmatrix}.$$

This system of expectational difference equations can be solved using standard techniques. These calculations are valid for arbitrary forecasts of variables outside the control of households. The solution to this system is given by

$$\begin{bmatrix} \hat{c}_t(i) \\ \hat{w}^r_t(i) \end{bmatrix} = \underset{2\times 2}{\bar{D}_k} \begin{bmatrix} \hat{w}^r_{t-1}(i) \\ \hat{d}_{t-1}(i) \end{bmatrix} + \underset{2\times 2}{\bar{D}_{\delta 1}} \sum_{T=t}^{\infty} \underset{2\times 2}{\Lambda}^{-(T-t)} \underset{2\times 4}{\bar{D}_{\delta 2}} \hat{E}^i_t \delta_T.$$

where the matrices $\underset{2\times 2}{\bar{D}_k}, \underset{2\times 2}{\bar{D}_{\delta 1}}, \underset{2\times 4}{\bar{D}_{\delta 2}}$ have elements that are composites of model primitives. The *third* column of $\bar{D}_{\delta 2}$ should be eliminated as it multiplies zero in every period. The code getdecision.m implements these calculations and takes as inputs $\Psi_0^{HH}, \Psi_1^{HH}, n_k$ where $n_k = 2$ denotes the number of household-level state variables.

To write in the form (1) requires additional manipulation, as the infinite sums depend upon contemporaneous variables that should appear in the vector Z_t. Dropping the household-level index for ease of notation, the optimal decisions can be written as

$$\left[\begin{array}{c}\hat{c}_t\\ \hat{w}_t^r\end{array}\right]=\bar{D}_k\left[\begin{array}{c}\hat{w}_{t-1}^r\\ \hat{d}_{t-1}\end{array}\right]+\underset{2\times 3}{D_{\lambda_1}}\hat{E}_t^i\sum_{T=t}^{\infty}\lambda_1^{-(T-t)}\hat{E}_t^i\bar{\delta}_T+\underset{2\times 3}{D_{\lambda_2}}\hat{E}_t^i\sum_{T=t}^{\infty}\lambda_2^{-(T-t)}\hat{E}_t^i\bar{\delta}_T \tag{2}$$

where λ_1 and λ_2 are the elements of the diagonal matrix Λ; $\bar{\delta}_T=\left(\delta^C,\delta^w,\delta^d\right)'$ is a vector of variables outside the control of the agent (they are solved for below). The new matrices are defined as

$$\underset{2\times 3}{D_{\lambda_1}}=\bar{D}_{\delta 1}\underset{2\times 3}{\left[\begin{array}{c}\bar{D}_{\delta 2}^1\\ 0\end{array}\right]};\underset{2\times 3}{D_{\lambda_2}}=\bar{D}_{\delta 1}\left[\begin{array}{c}0\\ \bar{D}_{\delta 2}^2\end{array}\right],$$

where $\bar{D}_{\delta 2}^i$ denotes the ith row of the matrix $\bar{D}_{\delta 2}$. Now break down the equations one by one, collecting all contemporaneous terms on the left-hand side:

$$\begin{aligned}\hat{c}_t-\sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}}P_t^{0,1}\left(\lambda_i,\delta_j\right) &= \bar{D}_k(1,1)\hat{w}_{t-1}^r+\bar{D}_k(1,2)\hat{d}_{t-1}+\sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}}P_t^{e,1}\left(\lambda_i,\delta_j\right)\\ \hat{w}_t^r-\sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}}P_t^{0,2}\left(\lambda_i,\delta_j\right) &= \bar{D}_k(2,1)\hat{w}_{t-1}^r+\bar{D}_k(2,2)\hat{d}_{t-1}+\sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}}P_t^{e,2}\left(\lambda_i,\delta_j\right),\end{aligned}$$

where $j=\{C,w,d\}$ indexes the collections of exogenous terms δ_j appearing in the original system. The new terms $P_t^{0,l}\left(\lambda_i,\delta_j\right)$ and $P_t^{e,l}\left(\lambda_i,\delta_j\right)$ capture contemporaneous and expectational variables respectively. Note that $l=1,2$ indexes the row of (2) and $i=1,2$ indexes the terms attached to the relevant discount factor λ_i. These terms are now given explicit expression.

Contemporaneous terms

$$P_t^{0,l}\left(\lambda_i,\delta_C\right)=D_{\lambda_i}(l,1)\times\{-\xi_\nu\theta_w\hat{w}_t^r-\xi_\nu\hat{N}_t+\sigma\frac{b}{1-b}\left(1-\lambda_i^{-1}\right)\hat{c}_t-\sigma\frac{b}{1-b}\hat{c}_{t-1}\}$$

where the term $-\sigma\frac{b}{1-b}\lambda_i^{-1}\hat{c}_t$ comes from realizing that

$$\hat{E}_t\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\hat{c}_{T-1}=\hat{c}_{t-1}+\lambda_i^{-1}\hat{c}_t+\lambda_i^{-2}\hat{E}_t\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\hat{c}_{T+1}.$$

This is also used below when deriving the coefficients corresponding to contemporaneous and the expected discounted sums of consumption and other variables that are lagged. Next

$$\begin{aligned}P_t^{0,l}\left(\lambda_i,\delta_w\right) &= D_{\lambda_i}\left(l,2\right)\times\{-c_3^w\left(\theta_w\hat{w}_t^r+\hat{N}_t\right)-\left[\xi_w^{-1}\left(1+\beta\iota_w\right)-\lambda_i^{-1}\xi_w^{-1}\iota_w\right]\hat{\pi}_t\\ &\quad -\frac{\hat{\theta}_{w,t}}{\theta_w-1}-\lambda_i^{-1}\frac{b}{1-b}\hat{c}_t-\frac{b}{1-b}\hat{c}_{t-1}+\xi_w^{-1}\iota_w\hat{\pi}_{t-1}\}.\end{aligned}$$

Finally

$$P_t^{0,l}\left(\lambda_i,\delta_d\right)=D_{\lambda_i}\left(l,3\right)\times\{\beta^{-1}\frac{P^m b^m}{y}\frac{y}{c}\left(\hat{R}_t^m-\hat{\pi}_t\right)+\frac{w^r N}{c}\left(\theta_w\hat{w}_t^r+\hat{N}_t\right)+\frac{y}{c}\hat{\Gamma}_t-\frac{y}{c}\hat{\tau}_t\}.$$

Discounted forecasts:

$$P_t^{e,l}\left(\lambda_i,\delta_C\right)=D_{\lambda_i}(l,1)\times\{$$

$$+\hat{E}_t\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left[\begin{array}{c}+\sigma\frac{b}{1-b}\lambda_i^{-1}\left(1-\lambda_i^{-1}\right)\hat{c}_{T+1}\\+\left(\hat{R}_{T+1}^m-\hat{\pi}_{T+1}+\xi_\nu\theta_w\left(1-\lambda_i^{-1}\right)\hat{w}_{T+1}^r+\left(1-\lambda_i^{-1}\right)\xi_\nu\hat{N}_{T+1}\right)\end{array}\right]\}.$$

$$P_t^{e,l}\left(\lambda_i,\delta_w\right)=D_{\lambda_i}\left(l,2\right)\times\{$$

$$\hat{E}_t\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left[\begin{array}{c}-\lambda_i^{-1}c_3^w\left(\theta_w\hat{w}_{T+1}^r+\hat{N}_{T+1}\right)-\lambda_i^{-2}\frac{b}{1-b}\hat{c}_{T+1}+\\-\left[\lambda_i^{-1}\xi_w^{-1}\left(1+\beta\iota_w\right)-\lambda_i^{-2}\xi_w^{-1}\iota_w-\xi_w^{-1}\beta\right]\hat{\pi}_{T+1}-\lambda_i^{-1}\frac{\hat{\theta}_{w,T+1}}{\theta_w-1}\end{array}\right]\}$$

$$\begin{aligned}P_t^{e,l}\left(\lambda_i,\delta_d\right) &= D_{\lambda_1}\left(l,3\right)\times\{\\ &\quad +\lambda_i^{-1}\hat{E}_t\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left[\begin{array}{c}\beta^{-1}\frac{P^m b^m}{y}\frac{y}{c}\left(\hat{R}_{T+1}^m-\hat{\pi}_{T+1}\right)\\+\frac{w^r N}{c}\left(\theta_w\hat{w}_{T+1}^r+\hat{N}_{T+1}\right)+\frac{y}{c}\hat{\Gamma}_{T+1}-\frac{y}{c}\hat{\tau}_{T+1}\end{array}\right]\}\end{aligned}$$

Recall,

$$\begin{aligned}c_1^w &= \left[-\left(\xi_\nu-\xi_{\nu\nu}\right)\theta_w+1+\xi_w^{-1}+\xi_w^{-1}\beta\right]\\ c_2^w &= \xi_w^{-1}\left(1+\beta\iota_w\right)\\ c_3^w &= \xi_\nu-\xi_{\nu\nu}\end{aligned}$$

Now we need to eliminate the variables $\hat{R}_t^m$ and $\hat{d}_t$ which are not needed (they are not in the final list of equations characterizing model dynamics) in estimation. We proceed as follows. First, recall

$$\begin{aligned}\hat{d}_t(i) &= P^m\hat{b}_t^m(i)+\frac{P^m b^m}{y}\hat{P}_t^m,\\ \hat{R}_T &= \hat{E}_t^i\hat{R}_{T+1}^m \text{ for every } T,\\ \hat{R}_t^m &= \rho\beta\gamma^{-1}\hat{P}_t^m-\hat{P}_{t-1}^m.\end{aligned}$$

Second, use the above to alter the following expressions containing $\hat{R}^m_t$ or future expected $\hat{R}^m_t$ which affects three sets of terms

1. The contemporaneous term

$$P^{0,l}_t(\lambda_i,\delta_d) = D_{\lambda_i}(l,3)\times\{\beta^{-1}\frac{P^m b^m}{y}\frac{y}{c}\left(\hat{R}^m_t-\hat{\pi}_t\right)+\frac{w^r N}{c}\left(\theta_w\hat{w}^r_t+\hat{N}_t\right)+\frac{y}{c}\hat{\Gamma}_t-\frac{y}{c}\hat{\tau}_t\}$$

becomes

$$\begin{aligned} P^{0,l}_t(\lambda_i,\delta_d) &= D_{\lambda_i}(l,3)\times\{\beta^{-1}\frac{P^m b^m}{y}\frac{y}{c}\left(\left[\rho\beta\gamma^{-1}\hat{P}^m_t-\hat{P}^m_{t-1}\right]-\hat{\pi}_t\right) \\ &\quad +\frac{w^r N}{c}\left(\theta_w\hat{w}^r_t+\hat{N}_t\right)+\frac{y}{c}\hat{\Gamma}_t-\frac{y}{c}\hat{\tau}_t\}. \end{aligned}$$

2. The expectational term

$$P^{e,l}_t(\lambda_i,\delta_C) = D_{\lambda_i}(l,1)\times\{$$
$$+E_t\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left[\begin{array}{c} +\sigma\frac{b}{1-b}\lambda_i^{-1}\left(1-\lambda_i^{-1}\right)\hat{c}_{T+1} \\ +\left(\hat{R}^m_{T+1}-\hat{\pi}_{T+1}+\xi_\nu\theta_w\left(1-\lambda_i^{-1}\right)\hat{w}^r_{T+1}+\left(1-\lambda_i^{-1}\right)\xi_\nu\hat{N}_{T+1}\right) \end{array}\right]\}$$

becomes

$$P^{e,l}_t(\lambda_i,\delta_C) = D_{\lambda_i}(l,1)\times\{$$
$$+E_t\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left[\begin{array}{c} +\sigma\frac{b}{1-b}\lambda_i^{-1}\left(1-\lambda_i^{-1}\right)\hat{c}_{T+1} \\ +\left(\hat{R}_{T}-\hat{\pi}_{T+1}+\xi_\nu\theta_w\left(1-\lambda_i^{-1}\right)\hat{w}^r_{T+1}+\left(1-\lambda_i^{-1}\right)\xi_\nu\hat{N}_{T+1}\right) \end{array}\right]\}.$$

Pulling out the first return, $\hat{R}_t$, in the infinite sequence gives

$$P^{e,l}_t(\lambda_i,\delta_C) = D_{\lambda_i}(l,1)\hat{R}_t + D_{\lambda_i}(l,1)\times\{$$
$$+E_t\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left[\begin{array}{c} +\sigma\frac{b}{1-b}\lambda_i^{-1}\left(1-\lambda_i^{-1}\right)\hat{c}_{T+1} \\ +\left(\left[\lambda_i^{-1}\hat{R}_{T+1}\right]-\hat{\pi}_{T+1}+\xi_\nu\theta_w\left(1-\lambda_i^{-1}\right)\hat{w}^r_{T+1}+\left(1-\lambda_i^{-1}\right)\xi_\nu\hat{N}_{T+1}\right) \end{array}\right]\},$$

so that we have (with a slight abuse of notation redefining the terms $P^{0,l}_t(\lambda_i,\delta_C)$ and $P^{e,l}_t(\lambda_i,\delta_C)$)

$$P^{0,l}_t(\lambda_i,\delta_C) = D_{\lambda_i}(l,1)\times\{\hat{R}_t-\xi_\nu\theta_w\hat{w}^r_t-\xi_\nu\hat{N}_t+\sigma\frac{b}{1-b}\left(1-\lambda_i^{-1}\right)\hat{c}_t-\sigma\frac{b}{1-b}\hat{c}_{t-1}\};$$

and

$$P^{e,l}_t(\lambda_i,\delta_C) = D_{\lambda_i}(l,1)\times\{$$
$$+E_t\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left[\begin{array}{c} +\sigma\frac{b}{1-b}\lambda_i^{-1}\left(1-\lambda_i^{-1}\right)\hat{c}_{T+1} \\ +\left(\left[\lambda_i^{-1}\hat{R}_{T+1}\right]-\hat{\pi}_{T+1}+\xi_\nu\theta_w\left(1-\lambda_i^{-1}\right)\hat{w}^r_{T+1}+\left(1-\lambda_i^{-1}\right)\xi_\nu\hat{N}_{T+1}\right) \end{array}\right]\},$$

3. The final term affected

$$P_t^{e,l}\left(\lambda_i,\delta_d\right)=D_{\lambda_i}\left(l,3\right)\times\{$$

$$+\lambda_i^{-1}E_t\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left[\begin{array}{c}\beta^{-1}\frac{P^m b^m}{y}\frac{y}{c}\left(\hat{R}_{T+1}^m-\hat{\pi}_{T+1}\right)\\+\frac{w^r N}{c}\left(\theta_w\hat{w}_{T+1}^r+\hat{N}_{T+1}\right)+\frac{y}{c}\hat{\Gamma}_{T+1}-\frac{y}{c}\hat{\tau}_{T+1}\end{array}\right]\}$$

becomes

$$P_t^{e,l}\left(\lambda_i,\delta_d\right)=D_{\lambda_i}\left(l,3\right)\lambda_i^{-1}\beta^{-1}\frac{P^m b^m}{y}\frac{y}{c}\hat{R}_t+D_{\lambda_i}\left(l,3\right)\times\{$$

$$+\lambda_i^{-1}E_t\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left[\begin{array}{c}\beta^{-1}\frac{P^m b^m}{y}\frac{y}{c}\left(\lambda_i^{-1}\hat{R}_{T+1}-\hat{\pi}_{T+1}\right)\\+\frac{w^r N}{c}\left(\theta_w\hat{w}_{T+1}^r+\hat{N}_{T+1}\right)+\frac{y}{c}\hat{\Gamma}_{T+1}-\frac{y}{c}\hat{\tau}_{T+1}\end{array}\right]\}.$$

Notice that all the coefficients of $\hat{R}_t$ belong to A_0! Finally, we want to eliminate $\hat{P}_{t-1}^m$ because it is not a relevant state variable. To do this, recall:

$$\begin{aligned}\hat{c}_t-\sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}}P_t^{0,1}\left(\lambda_i,\delta_j\right)&=\bar{D}_k(1,1)\hat{w}_{t-1}^r+\bar{D}_k(1,2)\hat{d}_{t-1}+\sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}}P_t^{e,1}\left(\lambda_i,\delta_j\right)\\ \hat{w}_t^r-\sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}}P_t^{0,2}\left(\lambda_i,\delta_j\right)&=\bar{D}_k(2,1)\hat{w}_{t-1}^r+\bar{D}_k(2,2)\hat{d}_{t-1}+\sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}}P_t^{e,2}\left(\lambda_i,\delta_j\right).\end{aligned}$$

substituting for $\hat{d}_{t-1}$ we get

$$\hat{c}_t-\sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}}P_t^{0,1}\left(\lambda_i,\delta_j\right)=\bar{D}_k(1,1)\hat{w}_{t-1}^r+$$

$$+\bar{D}_k(1,2)\left[P^m\hat{b}_{t-1}^m(i)+\frac{P^m b^m}{y}\hat{P}_{t-1}^m\right]+\sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}}P_t^{e,1}\left(\lambda_i,\delta_j\right)$$

$$\hat{w}_t^r-\sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}}P_t^{0,2}\left(\lambda_i,\delta_j\right)=\bar{D}_k(2,1)\hat{w}_{t-1}^r+$$

$$+\bar{D}_k(2,2)\left[P^m\hat{b}_{t-1}^m(i)+\frac{P^m b^m}{y}\hat{P}_{t-1}^m\right]+\sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}}P_t^{e,2}\left(\lambda_i,\delta_j\right).$$

where

$$\left[\bar{D}_k(1,2) - \left(D_{\lambda_1}(1,3) + D_{\lambda_2}(1,3)\right)\beta^{-1}\frac{y}{c}\right]\frac{P^m b^m}{y} = 0.$$

and

$$\left[\bar{D}_k(2,2) - \left(D_{\lambda_1}(2,3) + D_{\lambda_2}(2,3)\right)\beta^{-1}\frac{y}{c}\right]\frac{P^m b^m}{y} = 0.$$

The final two expressions use the modified coefficients in 1. and 2. The final equations can be written as

$$\begin{aligned}
\hat{c}_t - \sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}} P_t^{0,1}(\lambda_i,\delta_j) &= \bar{D}_k(1,1)\hat{w}_{t-1}^r + \bar{D}_k(1,2)P^m\hat{b}_{t-1}^m(i) + \sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}} P_t^{e,1}(\lambda_i,\delta_j) \\
\hat{w}_t^r - \sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}} P_t^{0,2}(\lambda_i,\delta_j) &= \bar{D}_k(2,1)\hat{w}_{t-1}^r + \bar{D}_k(2,2)P^m\hat{b}_{t-1}^m(i) + \sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}} P_t^{e,2}(\lambda_i,\delta_j).
\end{aligned}$$

where

$$\begin{gathered}
P_t^{0,l}(\lambda_i,\delta_d) = D_{\lambda_i}(l,3) \times \{\lambda_i^{-1}\beta^{-1}\frac{P^m b^m}{y}\frac{y}{c}\hat{R}_t + \\
+\beta^{-1}\gamma\frac{P^m b^m}{y}\frac{y}{c}\left(\left[\rho\beta\gamma^{-1}\hat{P}_t^m - \hat{P}_{t-1}^m\right] - \hat{\pi}_t\right) + \frac{w^r N}{c}\left(\theta_w\hat{w}_t^r + \hat{N}_t\right) + \frac{y}{c}\hat{\Gamma}_t - \frac{y}{c}\hat{\tau}_t\}
\end{gathered}$$

becomes

$$\begin{gathered}
P_t^{0,l}(\lambda_i,\delta_d) = D_{\lambda_i}(l,3) \times \{\lambda_i^{-1}\beta^{-1}\frac{P^m b^m}{y}\frac{y}{c}\hat{R}_t + \\
+\beta^{-1}\frac{P^m b^m}{y}\frac{y}{c}\left(\rho\beta\gamma^{-1}\hat{P}_t^m - \hat{\pi}_t\right) + \frac{w^r N}{c}\left(\theta_w\hat{w}_t^r + \hat{N}_t\right) + \frac{y}{c}\hat{\Gamma}_t - \frac{y}{c}\hat{\tau}_t\}
\end{gathered}$$

and

$$P_t^{e,l}(\lambda_i,\delta_d) = D_{\lambda_i}(l,3) \times \{\lambda_i^{-1}E_t\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left[\begin{array}{c} \beta^{-1}\frac{P^m b^m}{y}\frac{y}{c}\left(\lambda_i^{-1}\hat{R}_{T+1} - \hat{\pi}_{T+1}\right) \\ +\frac{w^r N}{c}\left(\theta_w\hat{w}_{T+1}^r + \hat{N}_{T+1}\right) + \frac{y}{c}\hat{\Gamma}_{T+1} - \frac{y}{c}\hat{\tau}_{T+1}\end{array}\right]\}$$

which are all independent of the lagged debt price. This completes the optimal decisions of households under arbitrary beliefs.

3.4 Decision Rule 3: Firms — Intermediate inputs

Similarly to Preston (2005) and Eusepi and Preston (2010), the firms' pricing problem gives the well-known optimal price setting rule

$$(1+\beta\iota_p(1-\gamma_1))\,\hat{\pi}_t=\iota_p\hat{\pi}_{t-1}$$

$$+\tilde{\kappa}\xi_p\left(\hat{w}^r_t-\frac{\hat{\theta}_{p,t}}{\theta_p-1}-\hat{A}_t\right)+\hat{E}_t\sum_{T=t}^{\infty}\left(\lambda_3^{-1}\right)^{T-t}\left[\begin{array}{c}\xi_p\gamma_1\beta\left(\hat{w}^r_{T+1}-\frac{\hat{\theta}_{p,T+1}}{\theta_p-1}-\hat{A}_{T+1}\right)+\\+\beta\left(1-\gamma_1\right)\left(1-\iota_p\gamma_1\beta\right)\right)\hat{\pi}_{T+1}\end{array}\right]$$

where $\lambda_3^{-1}=\gamma_1\beta$ and $0<\gamma_1<1$ is determined by

$$\xi_p=\left(1-\gamma_1\right)\left(1-\gamma_1\beta\right)\gamma_1^{-1}$$

and $\tilde{\kappa}$ denoted real frictions. Notice that γ_1 has the same interpretation as the probability of not resetting the price in a Calvo setup. Also, as mentioned in the text, γ_1 affects the way the firm discounts the future. The slope of the Phillips curve is then defined by $\kappa=\tilde{\kappa}\xi_p$.

3.5 Decision Rule 4: Price of bonds

$$\hat{P}^m_t=-\hat{R}_t-\rho\beta\gamma^{-1}\sum_{T=t}^{\infty}\left(\lambda_4^{-1}\right)^{T-t}\hat{R}_{T+1}$$

where $\lambda_4^{-1}=\rho\beta\gamma^{-1}$.This completes the description of the optimal decision rules.

4 Summary of Model Equations

4.1 Rational expectations

Under rational expectations the model is described by the following equations, which comprise 10 endogenous variables and 6 exogenous variables (shocks):

1. Labor input: $\hat{N}_t$

$$\hat{y}_t=\hat{N}_t+\hat{A}_t$$

2. Central bank interest rate $\hat{R}_t$

$$\hat{R}^{cb}_t=\rho_i\hat{R}_{t-1}+(1-\rho_i)\left[\phi_\pi\hat{\pi}_t+\phi_x\frac{c}{y}\left(\hat{c}_t-\hat{c}^*_t\right)\right]+\hat{m}_t.$$

3. Taxes: $\hat{\tau}_t$

$$\hat{\tau}_t = \rho_\tau \hat{\tau}_{t-1} + (1-\rho_\tau)\left[\phi_{\tau_l}\left(\left(1+\rho P^m\right)\hat{b}^m_{t-1} + \rho\delta \hat{P}^m_t\right)\right] + \bar{\tau}_t.$$

4. Bond price: $\hat{P}^m_t$

$$\hat{P}^m_t = \rho\beta\gamma^{-1} E_t \hat{P}^m_{t+1} - \hat{R}_t.$$

5. Profits: $\hat{\Gamma}_t$

$$\hat{\Gamma}_t = \hat{y}_t - \frac{c}{y}\frac{w^r N}{c}\hat{N}_t - \frac{c}{y}\frac{w^r N}{c}\hat{w}^r_t$$

6. Consumption: $\hat{c}_t$,

$$\epsilon_\nu \hat{N}_t - \sigma\frac{1+b}{1-b}\hat{c}_t + \sigma\frac{b}{1-b}\hat{c}_{t-1} = E_t\left(\hat{R}_t - \hat{\pi}_{t+1}\right) +$$

$$+E_t\left[\xi_\nu \hat{N}_{t+1} - \sigma\frac{1}{1-b}\hat{c}_{t+1}\right]$$

7. Real wages: $\hat{w}^r_t$

$$\left(c^w_1 + \theta_w c^w_3\right)\hat{w}^r_t$$

$$= -c^w_3 \hat{N}_t + \frac{\hat{c}_t - b\hat{c}_{t-1}}{1-b} + \xi_w^{-1}\hat{w}^r_{t-1} +$$

$$+\xi_w^{-1}\left(\iota_w \hat{\pi}_{t-1}\right) - \xi_w^{-1}\left(1+\beta\iota_w\right)\hat{\pi}_t$$

$$+\xi_w^{-1}\beta E_t\left(\hat{w}^r_{t+1} + \hat{\pi}_{t+1}\right) + \frac{c^w_1 - 1 + \theta_w c^w_3}{\xi_w}\tilde{\theta}_{w,t}$$

where

$$\tilde{\theta}_{w,t} = -\frac{1}{c^w_1 - 1 + \theta_w c^w_3}\frac{\hat{\theta}_{w,t}}{\theta_w - 1}.$$

8. Inflation: π_t

$$\left(1+\beta\iota_p\right)\hat{\pi}_t = \kappa\left(\hat{w}^r_t - \hat{A}_t\right) + \iota_p\hat{\pi}_{t-1} + \beta E_t\hat{\pi}_{t+1} + \left(1+\beta\iota_p\right)\tilde{\theta}_{p,t}$$

where

$$\tilde{\theta}_{p,t} = -\frac{\xi_p}{\left(1+\beta\iota_p\right)}\frac{\hat{\theta}_{p,t}}{\theta_p - 1}.$$

9. Real debt: $\hat{b}_t^m$,

$$P^m \hat{b}_t^m = \beta^{-1} P^m \hat{b}_{t-1}^m - \beta^{-1} \delta \hat{\pi}_t +$$

$$+ \left(\rho \gamma^{-1} - 1\right) \delta \hat{P}_t^m - \hat{\tau}_t + \tilde{g}_t.$$

10. Output: $\hat{y}_t$,

$$\hat{y}_t = \frac{c}{y} \hat{c}_t + \tilde{g}_t$$

11. TFP levels

$$\hat{A}_t = \rho_A \hat{A}_{t-1} + \sigma^A \epsilon_t^A$$

12. Wage mark-up

$$\hat{\theta}_{w,t} = \rho_w \hat{\theta}_{w,t-1} + \sigma^{\theta_w} \epsilon_t^{\theta_w}$$

13. Government spending (efficiency units)

$$\hat{g}_t = \rho_G \hat{g}_{t-1} + \sigma^G \epsilon_t^G + g_{ga} \sigma^A \epsilon_t^A.$$

14. Cost-push shock

$$\hat{\theta}_{p,t} = \sigma^{\theta_p} \epsilon_t^{\theta_p}$$

15. Policy shock

$$\hat{m}_t = \sigma^m \epsilon_t^m$$

16. Tax shock

$$\bar{\tau}_t = \rho_{\bar{\tau}} \bar{\tau}_{t-1} + \sigma^{\bar{\tau}} \epsilon_t^{\bar{\tau}}.$$

We also make use of the natural rate of consumption: $\hat{c}_t^*$

$$\left(\frac{1}{1-b} - c_3^w \frac{c}{y}\right) \hat{c}_t^* = \frac{b}{1-b} \hat{c}_{t-1}^* + \left(1 - c_3^w\right) \hat{A}_t + c_3^w \tilde{g}_t.$$

4.2 Arbitrary Beliefs

1. Labor: $\hat{N}_t$

$$\hat{y}_t = \hat{N}_t + \hat{A}_t$$

2. Central Bank Policy rate: $\hat{R}_t^{cb}$

$$\hat{R}_t = \rho_i \hat{R}_{t-1} + (1-\rho_i)\left[\phi_\pi \hat{\pi}_t + \phi_x \frac{c}{y}\left(\hat{c}_t - \hat{c}_t^*\right)\right] + \hat{m}_t.$$

3. Taxes: $\hat{\tau}_t$

$$\hat{\tau}_t = \rho_\tau \hat{\tau}_{t-1} + (1-\rho_\tau)\left[\phi_{\tau_l}\left((1+\rho P^m)\,\hat{b}_{t-1}^m + \rho\delta \hat{P}_t^m\right)\right] + \bar{\tau}_t.$$

4. Bond prices: $\hat{P}_t^m$

$$\hat{P}_t^m = -\hat{R}_t - \rho\beta\gamma^{-1}\sum_{T=t}^{\infty}\left(\lambda_6^{-1}\right)^{T-t}\hat{R}_{T+1}^{cb}.$$

5. Profits: $\hat{\Gamma}_t$,

$$\hat{\Gamma}_t = \hat{y}_t - \frac{c}{y}\frac{w^r N}{c}\hat{N}_t - \frac{c}{y}\frac{w^r N}{c}\hat{w}_t^r.$$

6. Consumption: $\hat{c}_t$,

$$\hat{c}_t - \sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}} P_t^{0,1}\left(\lambda_i, \delta_j\right) = \bar{D}_k(1,1)\hat{w}_{t-1}^r + \bar{D}_k(1,2)P^m \hat{b}_{t-1}^m(i) + \sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}} P_t^{e,1}\left(\lambda_i, \delta_j\right).$$

7. Real Wages: $\hat{w}_t^r$,

$$\hat{w}_t^r - \sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}} P_t^{0,2}\left(\lambda_i, \delta_j\right) = \bar{D}_k(2,1)\hat{w}_{t-1}^r + \bar{D}_k(2,2)P^m \hat{b}_{t-1}^m(i) + \sum_{i=1}^{2}\sum_{j}^{\{C,w,d\}} P_t^{e,2}\left(\lambda_i, \delta_j\right).$$

8. Inflation: π_t,

$$(1+\beta\iota_p)\,\hat{\pi}_t = \kappa\left(\hat{w}_t^r - \hat{A}_t\right) + \iota_p \hat{\pi}_{t-1} + \beta E_t \hat{\pi}_{t+1} + (1+\beta\iota_p)\,\tilde{\theta}_{p,t}$$

where

$$\tilde{\theta}_{p,t} = -\frac{\xi_p}{(1+\beta\iota_p)}\frac{\hat{\theta}_{p,t}}{\theta_p - 1}.$$

9. Real debt: $\hat{b}_t^m$,

$$P^m \hat{b}_t^m = \beta^{-1} P^m \hat{b}_{t-1}^m - \beta^{-1}\delta\hat{\pi}_t +$$

$$+\left(\rho\gamma^{-1} - 1\right)\delta \hat{P}_t^m - \hat{\tau}_t + \tilde{g}_t.$$

10. Output: $\hat{y}_t$,

$$\hat{y}_t = \frac{c}{y}\hat{c}_t + \tilde{g}_t.$$

11. Natural level of consumption $\hat{c}_t^*$

$$\left(\frac{1}{1-b} - c_3^w \frac{c}{y}\right)\hat{c}_t^* = \frac{b}{1-b}\hat{c}_{t-1}^* + \left(1 - c_3^w\right)\hat{A}_t + c_3^w \tilde{g}_t.$$

The shocks are the same as for the rational expectations model.

5 A useful decomposition

The net wealth component of the optimal decision rule is

$$n_{w,t} = \left(\tilde{b}_{t-1}^m - \delta\pi_t + \beta\rho\gamma^{-1}\delta\hat{P}_t^m\right) + \beta\hat{E}_t\sum_{T=t}^{\infty}\beta^{T-t}\left[\delta\left(\hat{R}_T - \pi_{T+1}\right) - \left(\tilde{\tau}_T - \tilde{g}_T\right)\right]$$

where the arbitrage equation is used. Consider now the same variable, computed under model-consistent expectations, denoted $\tilde{E}_t$,

$$\tilde{n}_{w,t} = \left(\tilde{b}_{t-1}^m - \delta\pi_t + \beta\rho\gamma^{-1}\delta\hat{P}_t^m\right) + \beta\tilde{E}_t\sum_{T=t}^{\infty}\beta^{T-t}\left[\delta\left(\hat{R}_T - \pi_{T+1}\right) - \left(\tilde{\tau}_T - \tilde{g}_T\right)\right].$$

We can now write

$$n_{w,t} + \tilde{n}_{w,t} - \tilde{n}_{w,t} =$$

$$\tilde{n}_{w,t} + \beta\hat{E}_t\sum_{T=t}^{\infty}\beta^{T-t}\left[\delta\left(\hat{R}_T - \pi_{T+1}\right) - \tilde{\tau}_T\right] - \beta\tilde{E}_t\sum_{T=t}^{\infty}\beta^{T-t}\left[\delta\left(\hat{R}_{T+1}^m - \pi_{T+1}\right) - \tilde{\tau}_T\right].$$

Using the fact that $\tilde{n}_{w,t} = 0$ in every period and re-arranging we get

$$n_{w,t} = \beta\delta\left[\hat{E}_t\sum_{T=t}^{\infty}\beta^{T-t}\left(\hat{R}_T - \pi_{T+1}\right) - \tilde{E}_t\sum_{T=t}^{\infty}\beta^{T-t}\left(\hat{R}_{T+1}^m - \pi_{T+1}\right)\right] +$$

$$-\beta\left[\hat{E}_t\sum_{T=t}^{\infty}\beta^{T-t}\tilde{\tau}_T - \tilde{E}_t\sum_{T=t}^{\infty}\beta^{T-t}\tilde{\tau}_T\right].$$

By adding and subtracting for model-consistent expectations of the short-term rate we get

$$n_{w,t} = \beta\delta \left[\hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left(\hat{R}_T - \pi_{T+1} \right) - \tilde{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left(\hat{R}_T - \pi_{T+1} \right) \right]$$

$$-\tilde{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left(\hat{R}^m_{T+1} - \hat{R}_T \right) +$$

$$-\beta \left[\hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \tilde{\tau}_T - \tilde{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \tilde{\tau}_T \right],$$

which gives the expression in the main text.

6 The State-Space Representation

This section details beliefs and the state-space representation of the model to be estimated using likelihood-based methods. Agents have the forecasting model:

$$Z_t = S\hat{\omega}_{0,t} + \Phi^* Z_{t-1} + e_t,$$

where S is a selection matrix which determines the subset of intercepts to be estimated by agents. The matrix S includes *ones* for the variables agents forecast and *zeros* for the variables that are not. Also, the matrix Φ^* depends on the structural parameters of the model: $\Phi^* = \Phi^*(\Theta)$. Evaluating expectations in the optimal decisions rules, collected in (1), provides the true data-generating process

$$Z_t = T_0(\Phi^*) S\hat{\omega}_{0,t} + \Phi^* Z_{t-1} + \Phi^*_\epsilon \epsilon_t$$

The following provides calculations. Given the maintained beliefs, forecasts in any future period $T+1$ are computed as:

$$E_t Z_{T+1} = (I_{n_Z} - \Phi^*)^{-1} \left(I_{n_Z} - \Phi^{*,T-t+1} \right) S\omega_{0,t} + \Phi^{*,T-t+1} Z_t.$$

This permits evaluating the discounted expected sum

$$E_t \sum_{T=t}^{\infty} \beta_s^{T-t} Z_{T+1} = F_0^s(\beta_s) S\hat{\omega}_{0,t} + F_1^s(\beta_s) Z_t$$

where

$$F_0^s(\beta_s) = (I_{n_Z} - \Phi^*)^{-1} \left[(1-\beta_s)^{-1} I_{n_Z} - \Phi^* (I_{n_Z} - \beta_s \Phi^*)^{-1} \right]$$

$$F_1^s\left(\beta_s\right) = \Phi^*\left(I_{n_Z} - \beta_s\Phi^*\right)^{-1}$$

for each discount factor β_s.

Evaluating expectations in (1) provides

$$\left(A_0 - \sum_{s=1}^{4} A_s F_1^s\left(\beta_s\right)\right) Z_t = \left(\sum_{s=1}^{4} A_s F_0^s\left(\beta_s\right)\right) S\hat{\omega}_{0,t} + A_5 Z_{t-1} + A_6\epsilon_t.$$

Re-arranging provides

$$\begin{aligned} Z_t &= \left(A_0 - \sum_{s=1}^{4} A_s F_1^s\left(\beta_s\right)\right)^{-1}\left(\sum_{s=1}^{4} A_s F_0^s\left(\beta_s\right)\right) S\hat{\omega}_{0,t} \\ &\quad + \left(A_0 - \sum_{s=1}^{4} A_s F_1^s\left(\beta_s\right)\right)^{-1} A_5 Z_{t-1} + \left(A_0 - \sum_{s=1}^{4} A_s F_1^s\left(\beta_s\right)\right)^{-1} A_6\epsilon_t \\ &= T_0 S\hat{\omega}_{0,t} + \Phi^* Z_{t-1} + \Phi_\epsilon^*\epsilon_t \end{aligned}$$

Define the forecast errors (possibly a subset of Z_t) as

$$\begin{aligned} \mathcal{F}_t &= Z_t - S\hat{\omega}_{0,t} - \Phi^* Z_{t-1} \\ &= \left[T_0(\Phi^*) - I\right] S\hat{\omega}_{0,t} + \Phi_\epsilon^*\epsilon_t \end{aligned}$$

Coefficients are updated using

$$\hat{\omega}_{0,t} = \hat{\omega}_{0,t-1} + \bar{g}\mathcal{F}_{t-1}.$$

The model state space can then be extended as follows

$$\begin{aligned} \mathbb{Z}_t &= \begin{bmatrix} Z_t \\ \hat{\omega}_{0,t} \\ \mathcal{F}_t \end{bmatrix} = \begin{bmatrix} \Phi^* & T_0(\Phi^*)S & T_0(\Phi^*)S\bar{g} \\ 0 & I & I\bar{g} \\ 0 & \left[T_0(\Phi^*) - I\right] S & \left[T_0(\Phi^*) - I\right] S\bar{g} \end{bmatrix} \\ &\quad \times \begin{bmatrix} Z_{t-1} \\ \hat{\omega}_{0,t-1} \\ \mathcal{F}_{t-1} \end{bmatrix} + \begin{bmatrix} \Phi_\epsilon^* \\ 0 \\ \Phi_\epsilon^* \end{bmatrix} \epsilon_t. \end{aligned}$$

This gives, compactly

$$\mathbb{Z}_t = F\left(\Theta\right)\mathbb{Z}_{t-1} + Q\left(\Theta\right)\epsilon_t.$$

Note that discounted sums using model-consistent expectations are computed as

$$\tilde{E}_t \sum_{T=t}^{\infty} \beta_s^{T-t}\mathbb{Z}_{T+1} = F\left(I_{n_Z} - \beta_s F\right)^{-1}\mathbb{Z}_t.$$

6.1 Observation Equation

The observation equation is

$$\mathbb{Y}_t = \mu_t(\Theta) + H_t(\Theta)\,\mathbb{Z}_t + o_t$$

where

$$\mathbb{Y}_t = \begin{bmatrix} \Delta(\text{GDP Deflator})(t) \\ \Delta\left(\text{NFB real real compensation}\right)(t) \\ \text{3-month Tbill }(t) \\ \text{Tax Revenues/GDP}(t) \\ E_t^{SPF,\text{1Q-ahead}}\Delta(\text{GDP Deflator}) \\ E_t^{SPF,\text{1Q-ahead}}(\text{3-month Tbill}) \\ E_t^{SPF,\text{5-10years}}\Delta(\text{GDP Deflator}) \\ E_t^{SPF,\text{5-10years}}(\text{3-month Tbill}) \\ \text{output gap}(t) \end{bmatrix} = \mu + \begin{bmatrix} \hat{\pi}_t \\ \hat{w}_t^r - \hat{w}_{t-1}^r \\ \hat{R}_t \\ \tau_{y,t} \\ \hat{E}_t\hat{\pi}_{t+1} \\ \hat{E}_t\hat{R}_{t+1} \\ \hat{E}_t^{LR,\pi} \\ \hat{E}_t^{LR,R} \\ x_t \end{bmatrix}$$

and

$$\mu = \begin{bmatrix} \pi \\ \gamma \\ \pi + r \\ 100(\frac{\tau}{y} + \frac{g}{y}) \\ \pi \\ \pi + r \\ \pi \\ \pi + r \\ 0 \end{bmatrix}.$$

Finally the following variables are defined as

$$\tau_{y,t} = \tilde{\tau}_t - \frac{\tau}{y}\hat{y}_t$$

$$\hat{E}_t Z_{t+1} = S\hat{\omega}_{0,t} + \Phi^* Z_t$$

and

$$
\begin{aligned}
\hat{E}_t^{LR,Z} &= \frac{1}{40-20} E_t \sum_{T=t+20}^{t+40} Z_{T+1} \\
&= \frac{1}{40-20} \left(I_{n_Z} - \Omega_Z^*\right)^{-1} \left[(40-20)\, I_{n_Z} - \left(I_{n_Z} - \Omega_Z^*\right)^{-1} \left(I_{n_Z} - \Omega_Z^{*(40-20)}\right) \Omega_Z^{*20}\right] S\omega_{0,t} \\
&\quad + \frac{1}{40-20} \left(I_{n_Z} - \Omega_Z^*\right)^{-1} \left(I_{n_Z} - \Omega_Z^{*(40-20)}\right) \Omega_Z^{*20} Z_t.
\end{aligned}
$$

7 Estimation

We use Bayesian methods. We estimate the mode of the posterior distribution by maximizing the log posterior function. The full posterior distribution is obtained using the Metropolis-Hastings algorithm. We use the Hessian from the posterior optimization procedure to define the transition probability function generating new proposed draws. We generated 5 samples of 200000 draws: a step size of 0.25 gave a rejection rate of 0.60 in each sample. We evaluate convergence using the Gelman and Rubin potential scale reduction factor (the factor is well below 1.01 for all estimated parameters). The posterior distribution is obtained by combining the five chains. The model's credible intervals are obtaining using the Carter and Kohn simulation smoother. Model predictions and samples are obtained from 20000 draws from the posterior distribution.

8 Stability under learning

Below we show stability frontiers for the model, using the posterior distribution of the estimated parameters.

8.1 Convergence

For constant-gain economies stability is evaluated by inspecting the eigenvalue of $F(\Theta)$ for each set of parameters. Eigenvalues within the unit circle imply stationarity: agents' beliefs will be centered around rational expectations ($\hat{\omega}_{0,t} = \omega^* = 0$). We now describe the stability conditions for the case of decreasing-gain (least squares) learning, where agents estimate both the intercept and lag coefficients. They estimate the model

$$Z_t = \hat{\omega}_{0,t} + \hat{\Omega}_{Z,t} Z_{t-1} + e_t,$$

where coefficients are updated according to

$$\begin{aligned}
\Xi_t &= \Xi_{t-1} + t^{-1} R_t^{-1} z_{t-1} z_{t-1}' \eta_t' \\
\eta_t &= Z_t - \Xi_{t-1}' z_{t-1} \\
R_t &= R_{t-1} + t^{-1} \left[z_{t-1} z_{t-1}' - R_{t-1} \right]
\end{aligned} \tag{3}$$

First we can write the true data-generation process as

$$Z_t = T_0(\omega_0, \Omega_Z) + T_c(\omega_0, \Omega_Z) Z_t + T_L(\omega_0, \Omega_Z) Z_{t-1} + A_6 \epsilon_t$$

where

$$T_0(\omega_0, \Omega_Z) = A_0^{-1} \sum_{s=1}^{4} A_s F_0^s \left(\beta_s\right)$$

$$T_c(\Omega_Z) = A_0^{-1} \sum_{s=1}^{4} A_s F_1^s \left(\beta_s\right)$$

$$T_L(\Omega_Z) = A_0^{-1} A_5 Z_{t-1}$$

We can then re-arrange as

$$Z_t = \left(I_{n_Z} - T_c(\Omega_Z)\right)^{-1} T_0(\omega_0, \Omega_Z) + \left(I_{n_Z} - T_c(\Omega_Z)\right)^{-1} T_L(\Omega_Z) Z_{t-1} + \left(I_{n_Z} - T_c(\Omega_Z)\right)^{-1} A_6 \epsilon_t.$$

As shown in Evans and Honkapohja (2001), E-Stability can be analyzed by checking the stability of the following ordinary differential equation in notional time

$$\begin{bmatrix} \dot{\omega}_0 \\ vec\left(\dot{\Omega}_Z\right) \end{bmatrix} = \begin{bmatrix} \left(I_{n_{XY}} - T_c(\Omega_Z)\right)^{-1} T_0(\omega_0, \Omega_Z) \\ vec\left(\left(I_{n_{XY}} - T_c(\Omega_Z)\right)^{-1} T_L(\Omega_Z)\right) \end{bmatrix} - \begin{bmatrix} \omega_0 \\ vec\left(\Omega_Z\right) \end{bmatrix}.$$

To evaluate the Jacobian at the rational equilibrium coefficients ($\omega_0 = \omega^* = 0$; $\Omega_Z = \Phi^*$)

$$\begin{bmatrix} \dot{\omega}_0 \\ vec\left(\dot{\Omega}_Z\right) \end{bmatrix} = \begin{bmatrix} N - I_N & \tilde{D} \\ \mathbf{0} & M - I_M \end{bmatrix} \begin{bmatrix} \omega_0 \\ vec\left(\Omega_Z\right) \end{bmatrix}$$

where the evolution of the intercept and coefficients can be analyzed separately by inspecting the eigenvalues of $N - I_N$ and $M - I_M$ — see Evans and Honkapohja, 2001 for more discussion. The 'partitions' related to the intercept are

$$N = d\left(\left(I_{n_Z} - T_c(\Phi^*)\right)^{-1} T_0(\omega_0, \Phi^*)\right) = \left(I_{n_Z} - T_c(\Phi^*)\right)^{-1} \cdot d\left(T_0(\omega_0, \Phi^*)\right)$$

$$= \left(I_{n_Z} - T_c(\Phi^*)\right)^{-1} \cdot A_0^{-1} \sum_{s=1}^{4} A_s \left\{ \left(I_{n_Z} - \Phi^*\right)^{-1} \left[\left(1 - \beta_s\right)^{-1} I_{n_Z} - \Phi^* \left(I_{n_Z} - \beta_s \Phi^*\right)^{-1} \right] \right\}$$

and the partitions related to the matrix of the lag coefficients

$$\begin{aligned} d\left((I_{n_Z} - T_c(\Omega_Z))^{-1} T_L(\Omega_Z)\right) &= d\left((I_{n_Z} - T_c(\Omega_Z))^{-1}\right) T_L(\Phi^*) + \\ &\quad + (I_{n_{XY}} - T_c(\Phi^*))^{-1} d\left(T_L(\Omega_Z)\right) \\ &= d\left((I_{n_Z} - T_c(\Omega_Z))^{-1}\right) T_L(\Phi^*) \end{aligned}$$

where

$$d\left((I_{n_Z} - T_c(\Omega_Z))^{-1}\right) = (I_{n_Z} - T_c(\Phi^*))^{-1} d\left(T_c(\Omega_Z)\right)(I_{n_Z} - T_c(\Phi^*))^{-1}.$$

First, compute $dT_c(\Omega_Z)$

$$dT_c(\Omega_Z) = A_0^{-1} \sum_{s=1}^{4} A_s F_1^s\left(\beta_s\right)$$

$$\begin{aligned} d\left(F_1^s\right) &= d\left(\Omega_Z\left(I_{n_Z} - \beta_s \Omega_Z\right)^{-1}\right) \\ &= d\left(\Omega_Z\right)\left(I_{n_Z} - \beta_s \Phi^*\right)^{-1} + \Phi^* d\left(\left(I_{n_Z} - \beta_s \Omega_Z\right)^{-1}\right) \end{aligned}$$

and

$$\begin{aligned} d\left(\left(I_{n_Z} - \beta_s \Omega_Z\right)^{-1}\right) &= -\left(I_{n_Z} - \beta_s \Phi^*\right)^{-1} d\left(I_{n_Z} - \beta_s \Omega_Z\right)\left(I_{n_Z} - \beta_s \Phi^*\right)^{-1} \\ &= \left(I_{n_Z} - \beta_s \Phi^*\right)^{-1} \beta_s d\left(\Omega_Z\right)\left(I_{n_Z} - \beta_s \Phi^*\right)^{-1} \end{aligned}$$

$$d\left(F_1^s\right) = d\left(\Omega_Z\right)\left(I_{n_{XY}} - \beta_s \Phi^*\right)^{-1} + \Phi^*\left(I_{n_Z} - \beta_s \Phi^*\right)^{-1} \beta_s d\left(\Omega_Z\right)\left(I_{n_Z} - \beta_s \Phi^*\right)^{-1}$$

so

$$dT_c(\Omega_Z) = A_0^{-1} \sum_{s=1}^{4} A_s \left\{ \begin{array}{c} d\left(\Omega_Z\right)\left(I_{n_Z} - \beta_s \Phi^*\right)^{-1} + \\ +\Phi^*\left(I_{n_Z} - \beta_s \Phi^*\right)^{-1} \beta_s d\left(\Omega_Z\right)\left(I_{n_Z} - \beta_s \Phi^*\right)^{-1} \end{array} \right\}.$$

Second, vectorize

$$\begin{aligned} d\left((I_{n_Z} - T_c(\Omega_Z))^{-1}\right) &= -(I_{n_Z} - T_c(\Phi^*))^{-1} d\left(I_{n_Z} - T_c(\Omega_Z)\right)(I_{n_Z} - T_c(\Phi^*))^{-1} \\ &= (I_{n_Z} - T_c(\Phi^*))^{-1} d\left(T_c(\Omega_Z)\right)(I_{n_Z} - T_c(\Phi^*))^{-1} \end{aligned}$$

so that

$$d\left((I_{n_Z}-T_c(\Omega_Z))^{-1}T_L(\Omega_Z)\right)=(I_{n_Z}-T_c(\Phi^*))^{-1}d\left(T_c(\Omega_Z)\right)(I_{n_Z}-T_c(\Phi^*))^{-1}T_L(\Phi^*)$$

which can be broken into

$$dvec((I_{n_Z}-T_c(\Omega_Z))^{-1}T_L(\Omega_Z))=$$

$$vec\left((I_{n_Z}-T_c(\Phi^*))^{-1}A_0^{-1}\sum_{s=1}^{4}A_s d\left(\Omega_Z\right)(I_{n_Z}-\beta_s\Phi^*)^{-1}(I_{n_Z}-T_c(\Phi^*))^{-1}T_L(\Phi^*)\right)+$$
$$vec\left(\begin{array}{c}(I_{n_Z}-T_c(\Phi^*))^{-1}A_0^{-1}\sum_{s=1}^{4}A_s\Phi^*(I_{n_Z}-\beta_s\Phi^*)^{-1}\beta_s\times\\ \times d\left(\Omega_Z\right)(I_{n_Z}-\beta_s\Phi^*)^{-1}(I_{n_Z}-T_c(\Phi^*))^{-1}T_L(\Phi^*)\end{array}\right).$$

We can consider the two terms separately. First,

$$(I_{n_Z}-T_c(\Phi^*))^{-1}\left(A_0^{-1}\sum_{s=1}^{4}A_s d\left(\Omega_Z\right)(I_{n_Z}-\beta_s\Phi^*)^{-1}\right)(I_{n_Z}-T_c(\Phi^*))^{-1}T_L(\Phi^*)$$

re-arranging this term we write

$$\left(\sum_{s=1}^{4}(I_{n_Z}-T_c(\Phi^*))^{-1}A_0^{-1}A_s d\left(\Omega_Z\right)(I_{n_Z}-\beta_s\Phi^*)^{-1}(I_{n_Z}-T_c(\Phi^*))^{-1}T_L(\Phi^*)\right)$$

which then taking derivatives becomes

$$M_1 vec(d\Omega_Z)=\left(\sum_{s=1}^{4}\left(\begin{array}{c}\left[(I_{n_Z}-\beta_s\Phi^*)^{-1}(I_{n_Z}-T_c(\Phi^*))^{-1}T_L(\Phi^*)\right]'\\ \otimes\left[(I_{n_Z}-T_c(\Phi^*))^{-1}A_0^{-1}A_s\right]\end{array}\right)\right)vec(d\Omega_Z).$$

The second term is

$$(I_{n_Z}-T_c(\Phi^*))^{-1}\left(A_0^{-1}\sum_{s=1}^{4}A_s\Phi^*(I_{n_Z}-\beta_s\Phi^*)^{-1}\beta_s d\left(\Omega_Z\right)(I_{n_Z}-\beta_s\Phi^*)^{-1}\right)(I_{n_Z}-T_c(\Phi^*))^{-1}T_L(\Phi^*)$$

re-arranging

$$\left(\sum_{s=1}^{4}(I_{n_Z}-T_c(\Phi^*))^{-1}A_0^{-1}A_s\Phi^*(I_{n_Z}-\beta_s\Phi^*)^{-1}\beta_s d\left(\Omega_Z\right)(I_{n_Z}-\beta_s\Phi^*)^{-1}(I_{n_Z}-T_c(\Phi^*))^{-1}T_L(\Phi^*)\right)$$

and after taking derivatives becomes

$$M_2 vec(d\Omega_Z) = \left(\begin{array}{c} \sum_{s=1}^{4} \left[\left(I_{n_Z} - \beta_s \Phi^*\right)^{-1} \left(I_{n_Z} - T_c(\Phi^*)\right)^{-1} T_L(\Phi^*) \right]' \\ \otimes \left[\left(I_{n_Z} - T_c(\Phi^*)\right)^{-1} A_0^{-1} A_s \Phi^* \left(I_{n_Z} - \beta_s \Phi^*\right)^{-1} \beta_s \right] \end{array} \right) vec(d\Omega_Z)$$

Finally, we can rewrite the matrix M as

$$M = \sum_{i=1}^{2} M_i.$$

8.2 Stability Frontiers

Figure 7 shows four stability frontiers, evaluated at the posterior distribution of the parameters. The 95% percentiles are defined by the grey area, while the dashed line shows the frontier at the mode. The first three panels depict frontiers relating the maximum level of debt consistent with stability, for a range of debt maturities. The top-left panel describes the frontier drawing all parameters from the posterior distribution. The top-right panel assumes no real price rigidities so that

$$\kappa = \left(1 - \gamma_1 \beta\right) \left(1 - \gamma_1\right) / \gamma_1.$$

The bottom left panel assumes $\phi_x = 0$ (while maintaining real rigidities). Finally, the bottom-right panel maps the maximum sustainable debt level for different values of σ, assuming the maturity of debt at baseline. Both baseline and the case of no-real rigidities are considered.

8.3 Monetary Policy Rule Responds to Expected Inflation

It is often argued that monetary policy ought to be specified in terms of a reaction function in which nominal interest rates respond to expectations of next-period inflation rather than realizations of current-period inflation. To this end, in a previous version of this paper we considered a rule of the form

$$\hat{\imath}_t = \phi_\pi \hat{E}_t \hat{\pi}_{t+1}. \tag{4}$$

It is assumed that in implementing this interest-rate rule, the central bank responds to observed private-sector inflation expectations. An alternative, but equivalent assumption, is that the central bank has the same forecasting model of inflation as households and firms.

One additional assumption is required for interesting results: households understand that monetary policy is determined according to equation (4). Absent this assumption, rules of this kind engender considerable instability. Figure 8 plots stability regions for the contemporaneous inflation and inflation expectation-based rules in the simple New Keynesian model described in the earlier version of this paper, Eusepi and Preston (2103). A notable implication arising from expectations-based instrument rules is that the effect of increasing average maturity of debt is monotonic. As before, the intuition relies on the dynamics of real debt. Under the expectations-based rule the low of motion for government debt becomes

$$\tilde{b}_t^m = \left(\beta^{-1} - \phi_b\right) \tilde{b}_{t-1}^m - \bar{\tau}_t \tag{5}$$

$$-\delta \left[\beta^{-1} \pi_t - (1-\rho)\, \phi_\pi \hat{E}_t \hat{\pi}_{t+1} - (1-\rho)\, \rho \beta \hat{E}_t \sum_{T=t}^{\infty} (\beta\rho)^{T-t}\, \hat{\imath}_{T+1} \right]$$

so that feedback effects from beliefs survive even with $\rho = 0$. If anything they are stronger in that case. The region of instability is largest in the case of a debt portfolio comprised only of one-period instruments.

9 Impulse Responses

Figures 4-6 show the impulse responses to wage markup shock, a spending shock and a technology shock respectively. As for the impulse responses in the main text, wealth effects have opposite effects on consumption demand, compared to the traditional channel of intertemporal substitution of consumption.

10 Further Implications and Robustness

Figure 9, top panels, shows model predictions for one-quarter short term survey forecasts (used in the estimation), while the bottom line shows predictions for the four-quarters ahead survey forecasts (not used in the estimation). Figure 10 shows the tax-to-GDP series used in the estimation (top panel) together with predicted and actual path of government debt-to-GDP ratio (bottom panel). The debt-to-GDP ratio series is not used in estimation. Figure 11 shows the model's forecast variance decomposition for a subset of variable and different forecast horizons. Figure 12, describes the high-debt counterfactual with a lower gain $\bar{g} = 0.02$. Finally, Figure 13-14 show model predictions and high debt counterfactuals for the economy with $\beta = 0.995$. The Tables show the estimated parameters for this economy.

11 Alternative Tax Rule

The remaining figures reproduce all results for the empirical model using a tax rule that responds to the face value of debt rather than the market value. The results are largely unchanged. This is to be expected because the change in specification only affects the magnitude of the coefficient on the price of debt in the debt evolution equation — the basic mechanics of the model are unaltered. For this reason, we present the figures without further comment.

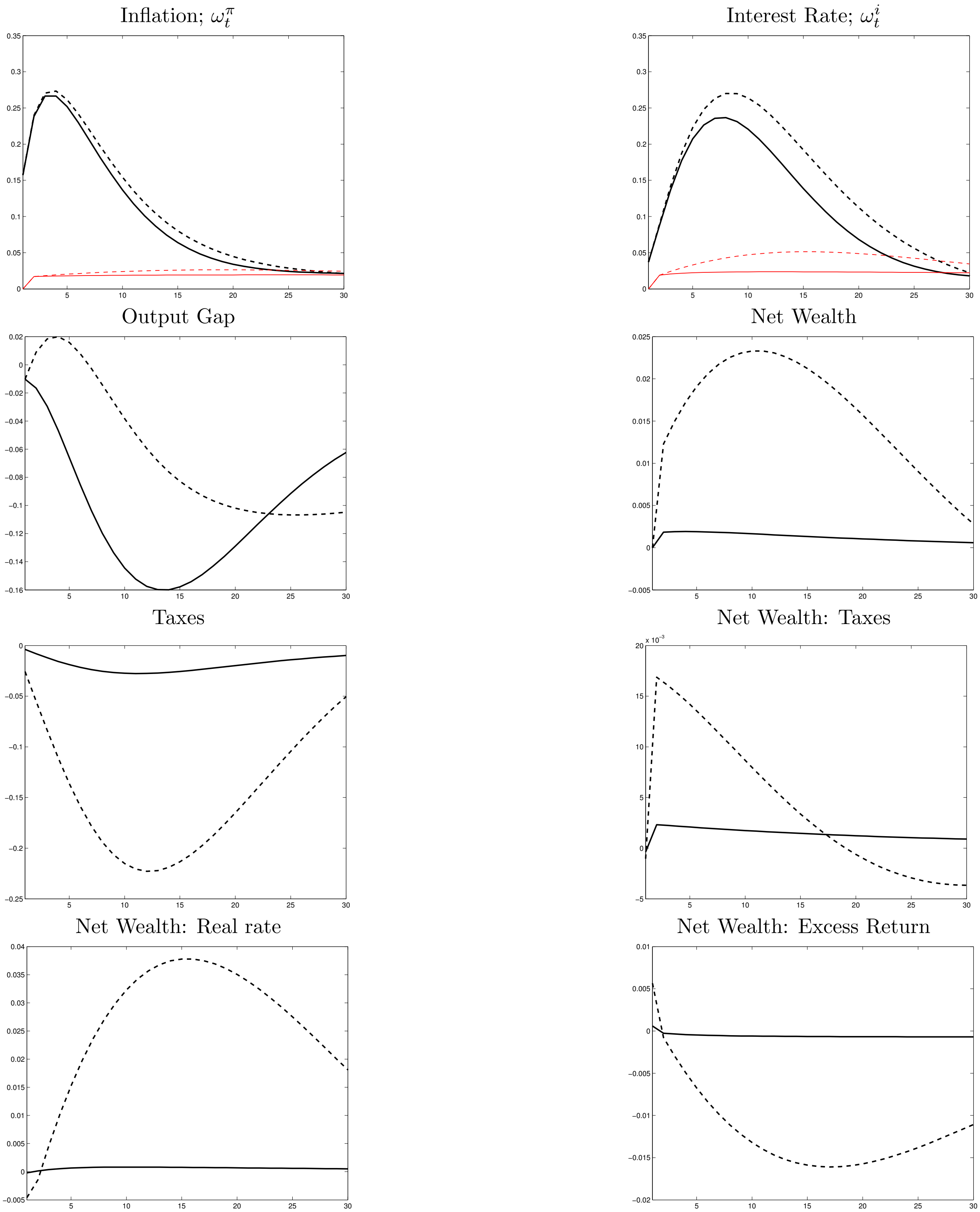


Figure 4: **Mean Impulse Response to a Wage Markup Shock**

Solid black lines denote the baseline economy with Debt-to-output ratio of 30%. Dashed lines correspond to an economy with debt-to-output ratio of 200%. Red lines measure inflation and interest rate estimated drifts.

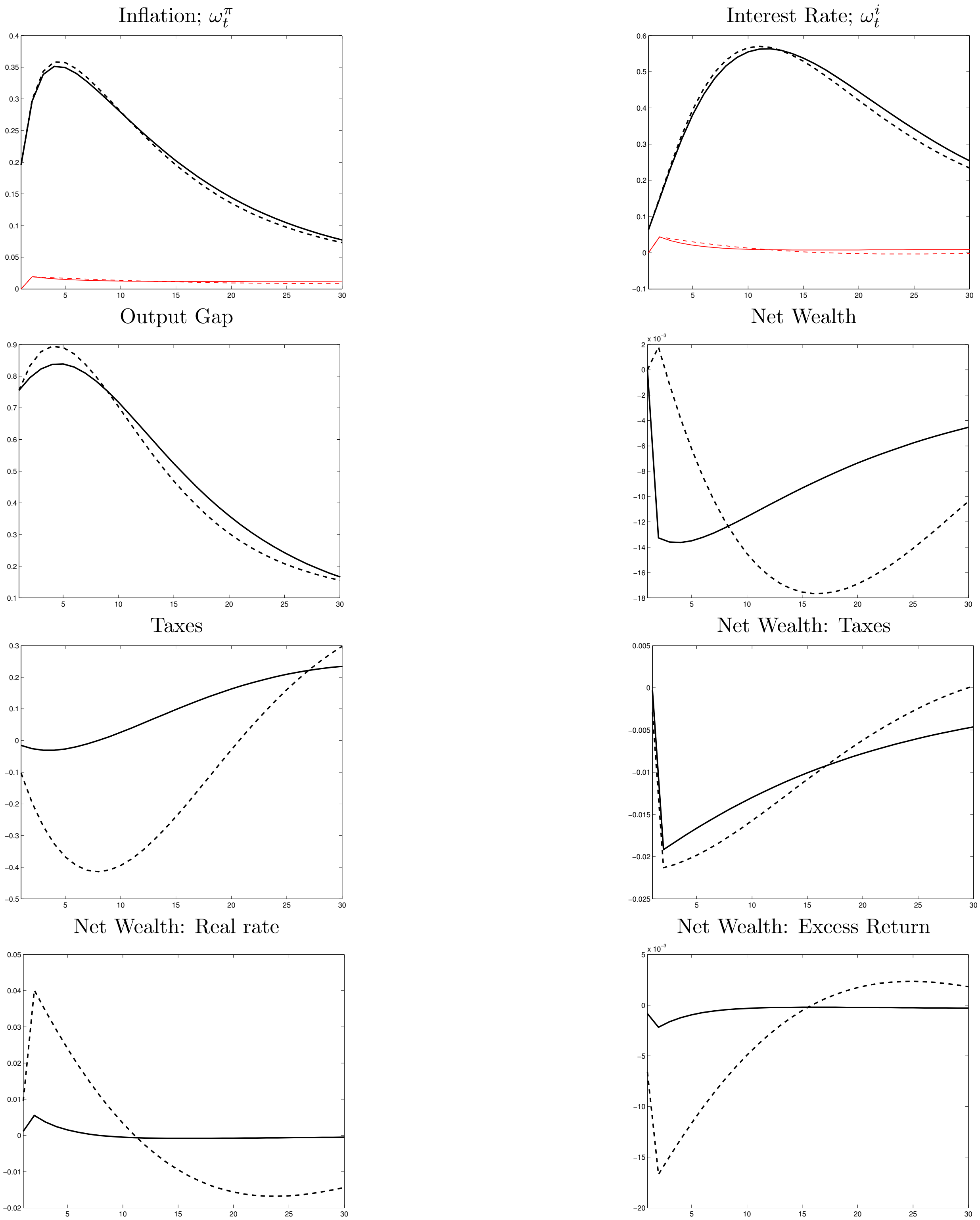


Figure 5: **Mean Impulse Response to a Spending Shock**

Solid black lines denote the baseline economy with Debt-to-output ratio of 30%. Dashed lines correspond to an economy with debt-to-output ratio of 200%. Red lines measure inflation and interest rate estimated drifts.

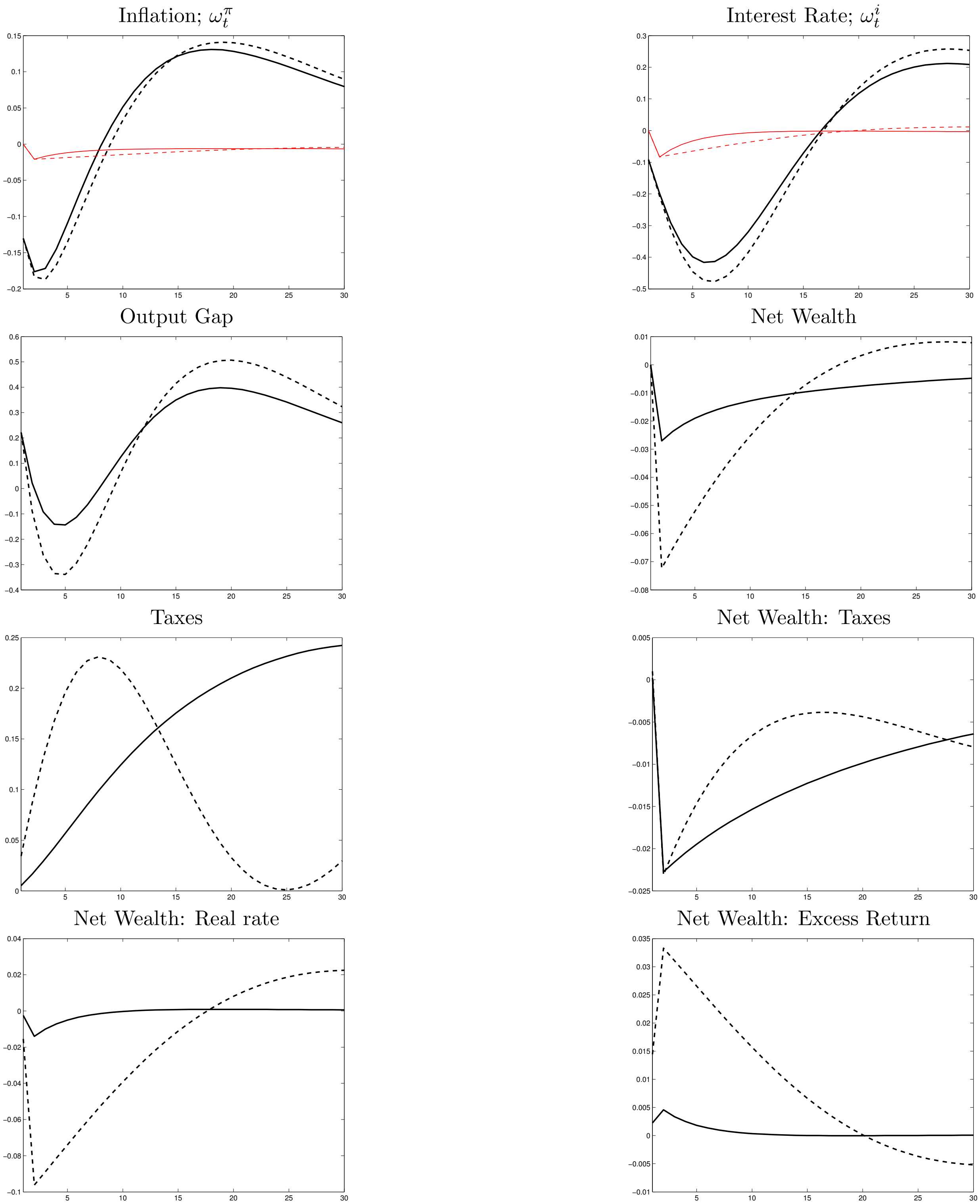


Figure 6: **Mean Impulse Response to a Technology Shock**

Solid black lines denote the baseline economy with Debt-to-output ratio of 30%. Dashed lines correspond to an economy with debt-to-output ratio of 200%. Red lines measure inflation and interest rate estimated drifts.

Figure 7: **Stability Frontiers**

This figure displays stability frontiers computed using the parameters posterior distribution. The top panel shows the stability frontiers under the estimated constant gain. The bottom panel show the stability frontier under a decreasing gain. The red areas denote the model without real rigidities.

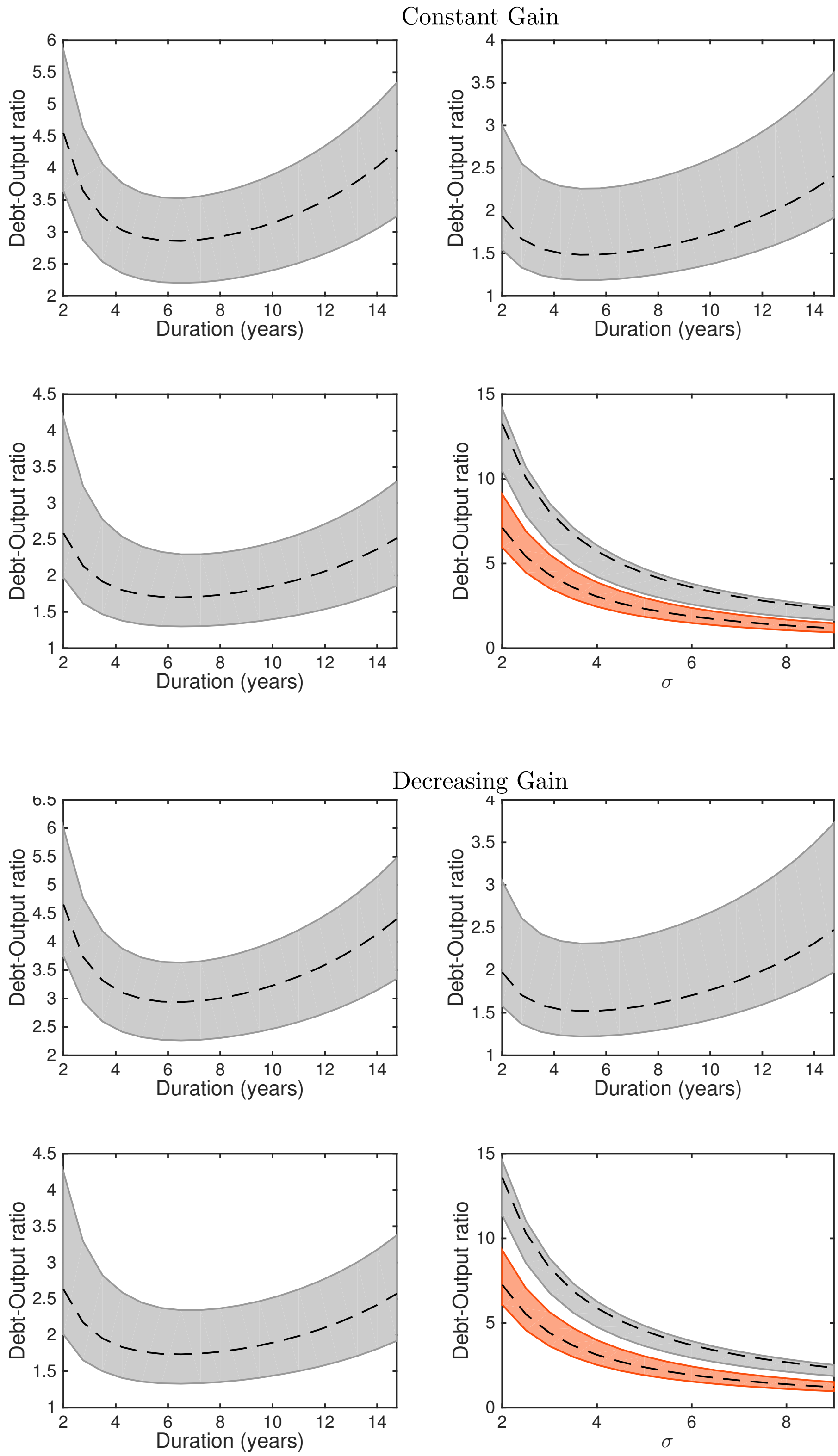


Figure 8: **E-stability: contemporaneous v.s. forward-looking monetary rules**

This figure displays the policy frontiers for a simple New Keynesian model with contemporaneous and forward-looking policy rules. For details see the working paper Eusepi and Preston (2013).

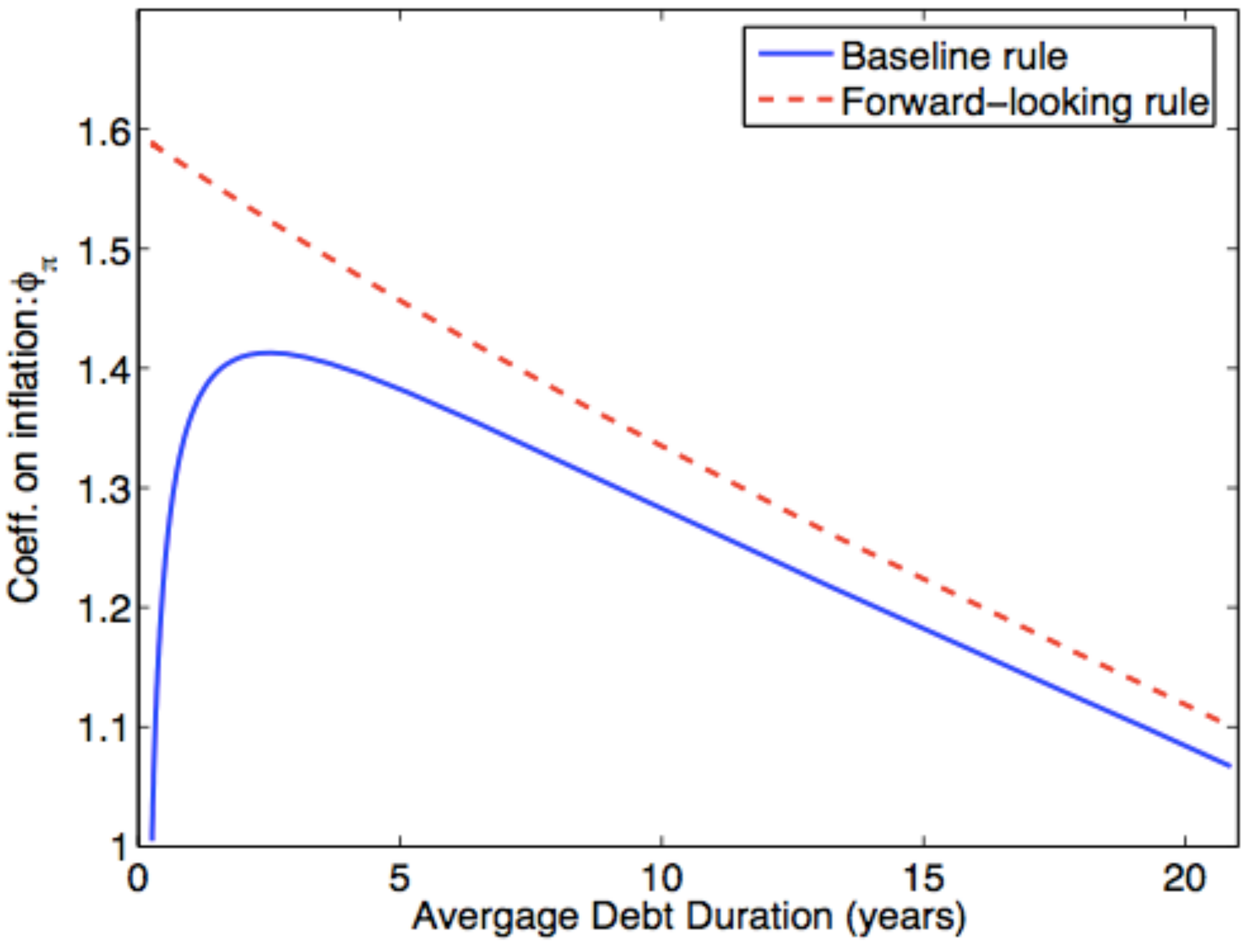


Figure 9: **Short-term Survey Forecasts**

This figure displays model predictions for 1-quarter ahead survey forecast for inflation and interest rate (top panel) and predictions for the four-quarter-ahead forecasts (bottom panel).

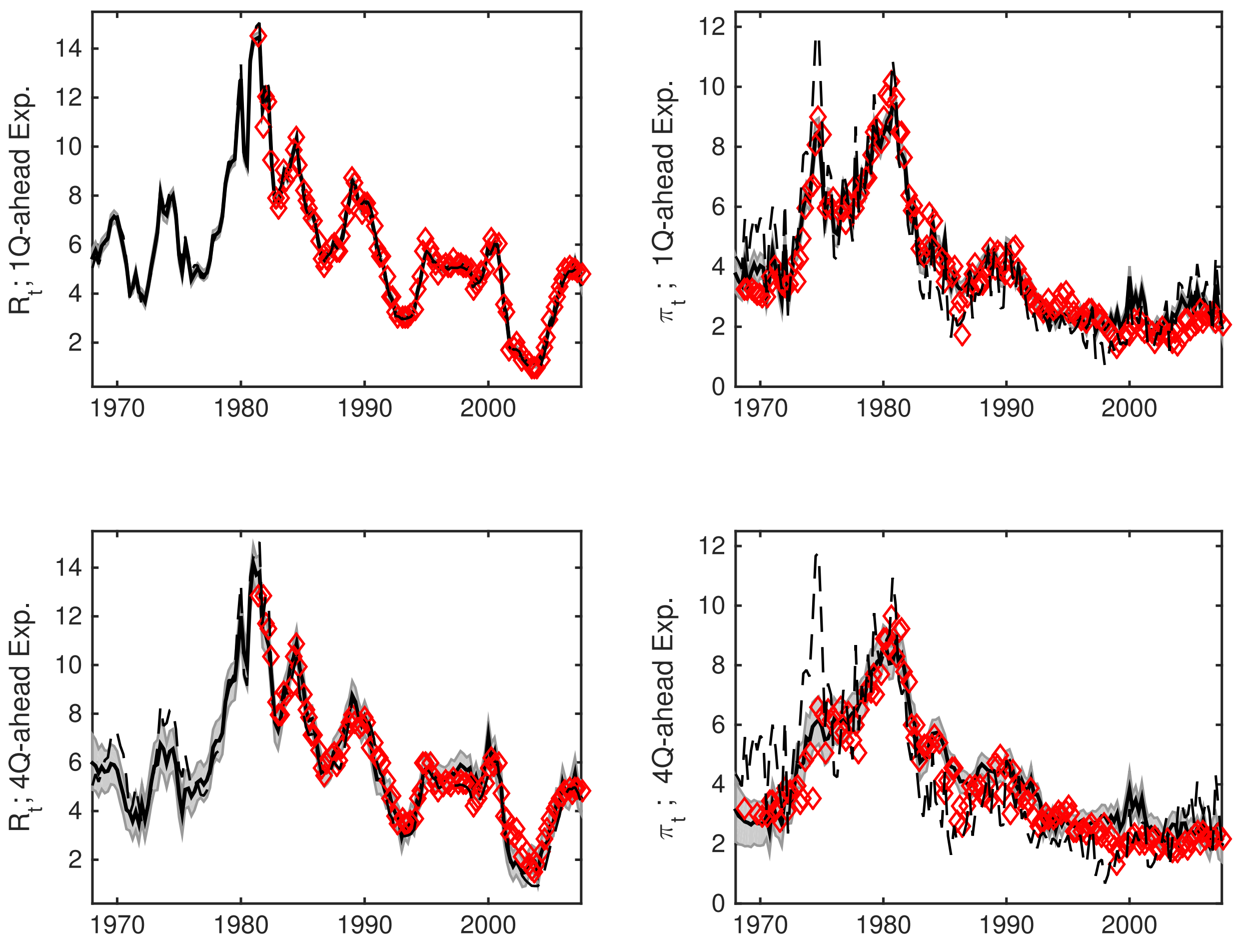
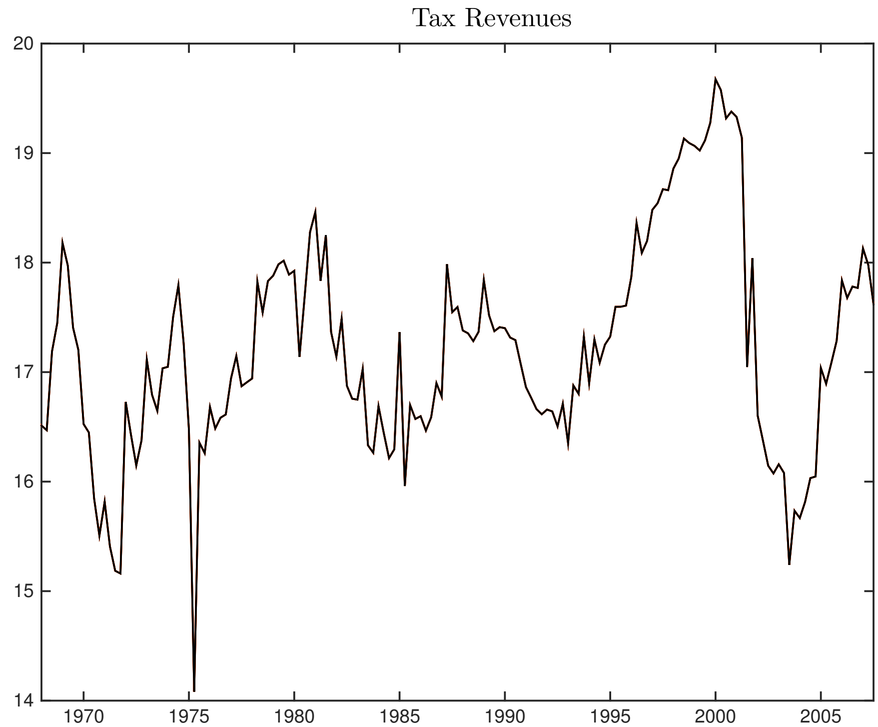
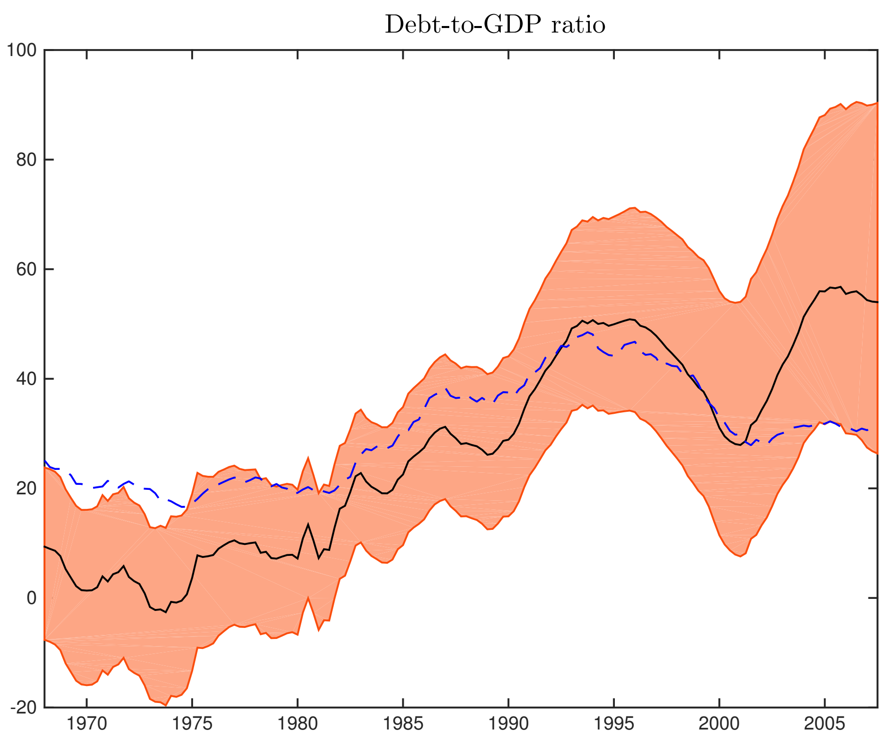


Figure 10: **Debt-to-GDP ratio**

This top figure shows the evolution of tax revenues in the sample. The bottom figure displays the model's prediction for the path of debt-to-output ratio (black solid line, median), compared with the data (blue dashed line). The grey area covers the 95% credible intervals.





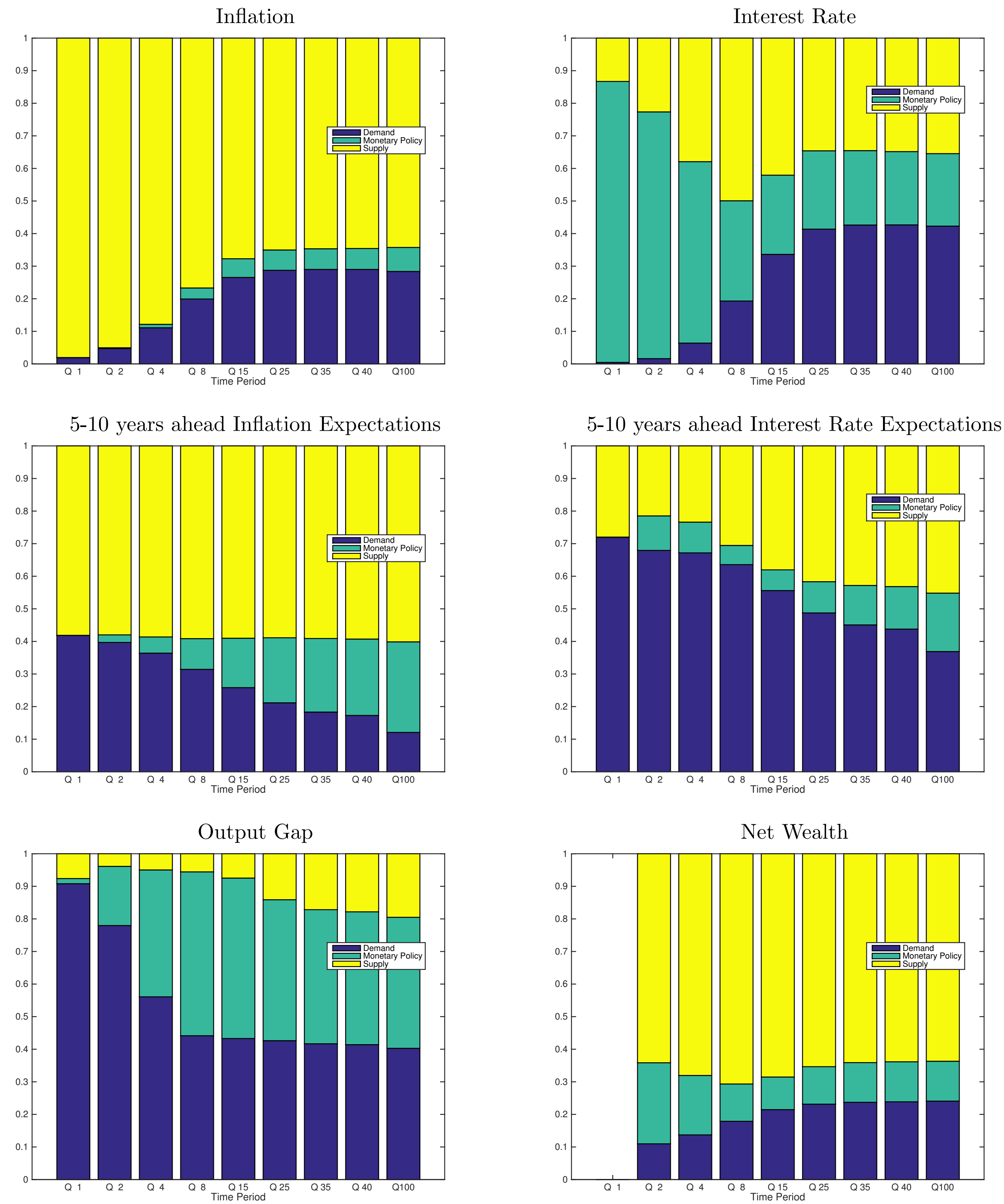
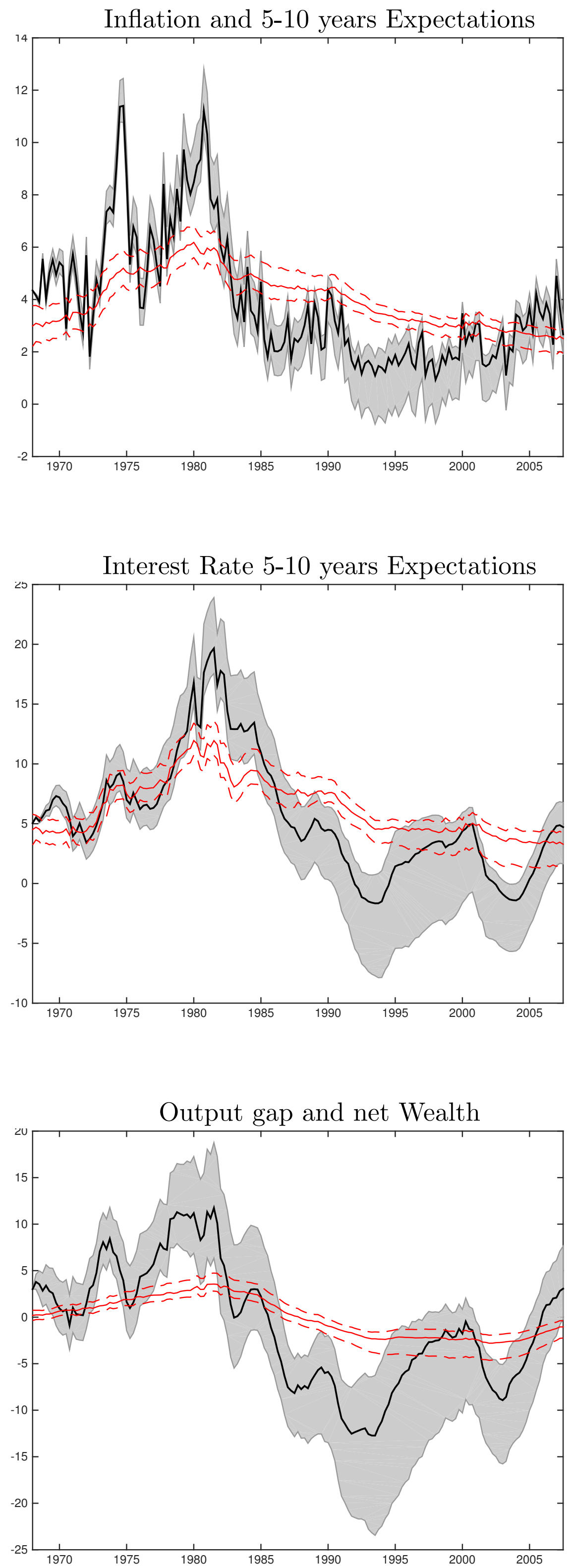


Figure 11: **Variance Decomposition**

The panels show the variance decomposition of selected variables calculated at the posterior mode.

Figure 12: **High Debt Counterfactual with** $\bar{g} = 0.02$

The panels show the evolution of selected variables in a counterfactual economy with 200% debt-to-output ratio and baseline maturity of debt. Inflation, interest rate and output gap are denoted by solid black lines; expectations and net wealth are shown in red solid lines. The 95th percentiles are defined by the grey area and the dashed lines respectively.



	Prior distribution			Posterior distribution			
	Distr.	Mean	St. Dev.	Mode	Mean	5 percent	95 percent
σ	Gamma	2.000	0.600	9.393	8.887	7.146	10.8
b	Beta	0.350	0.100	0.500	0.509	0.462	0.554
ξ_w	Beta	0.500	0.050	0.792	0.808	0.743	0.868
ι_w	Beta	0.500	0.150	0.384	0.468	0.247	0.693
ξ_p	Beta	0.500	0.050	0.751	0.768	0.715	0.818
ι_p	Beta	0.500	0.150	0.503	0.497	0.433	0.562
Θ_p	Normal	1.250	0.120	1.716	1.718	1.579	1.861
κ	Beta	0.300	0.150	0.002	0.002	0.002	0.003
ϕ_π	Normal	1.500	0.150	1.569	1.574	1.460	1.701
ρ_i	Beta	0.500	0.100	0.836	0.839	0.816	0.861
ϕ_x	Normal	0.120	0.050	0.076	0.081	0.065	0.100
ρ_τ	Beta	0.700	0.100	0.840	0.835	0.771	0.894
ϕ_{τ_l}	Gamma	0.070	0.020	0.050	0.054	0.033	0.081
g	Gamma	0.035	0.030	0.038	0.038	0.035	0.042

Table 1: **Prior and Posterior Distribution of Structural Parameters:** $\beta = 0.995$.

Note: The posterior distribution is obtained using the Metropolis-Hastings algorithm.

	Prior distribution			Posterior distribution			
	Distr.	Mean	St. Dev.	Mode	Mean	5 percent	95 percent
ρ_{θ_w}	Beta	0.500	0.200	0.823	0.750	0.524	0.873
ρ_g	Beta	0.500	0.200	0.933	0.934	0.923	0.945
$\rho_{\bar{\tau}}$	Beta	0.500	0.200	0.047	0.072	0.020	0.150
ρ_a	Beta	0.500	0.200	0.851	0.833	0.786	0.869
σ_{θ_w}	InvGamma	0.100	2.000	0.166	0.189	0.148	0.246
σ_{θ_p}	InvGamma	0.100	2.000	0.204	0.205	0.184	0.228
σ_g	InvGamma	0.100	2.000	0.519	0.554	0.463	0.656
σ_m	InvGamma	0.100	2.000	0.197	0.199	0.180	0.219
$\sigma_{\bar{\tau}}$	InvGamma	0.100	2.000	2.029	2.082	1.891	2.289
σ_a	InvGamma	0.100	2.000	1.608	1.715	1.338	2.164
$\sigma_{g\gamma}$	Beta	0.500	0.200	0.475	0.457	0.359	0.561
$\sigma_{o,\pi^{SR}}$	InvGamma	0.100	2.000	0.168	0.170	0.151	0.191
$\sigma_{o,R^{SR}}$	InvGamma	0.100	2.000	0.056	0.057	0.049	0.065
$\sigma_{o,R^{LR}}$	InvGamma	0.100	2.000	0.072	0.081	0.049	0.118
$\sigma_{o,\pi^{LR}}$	InvGamma	0.100	2.000	0.041	0.043	0.035	0.054

Table 2: **Prior and Posterior Distribution of Shock Processes:** $\beta = 0.995$.

Note: The posterior distribution is obtained using the Metropolis-Hastings algorithm.

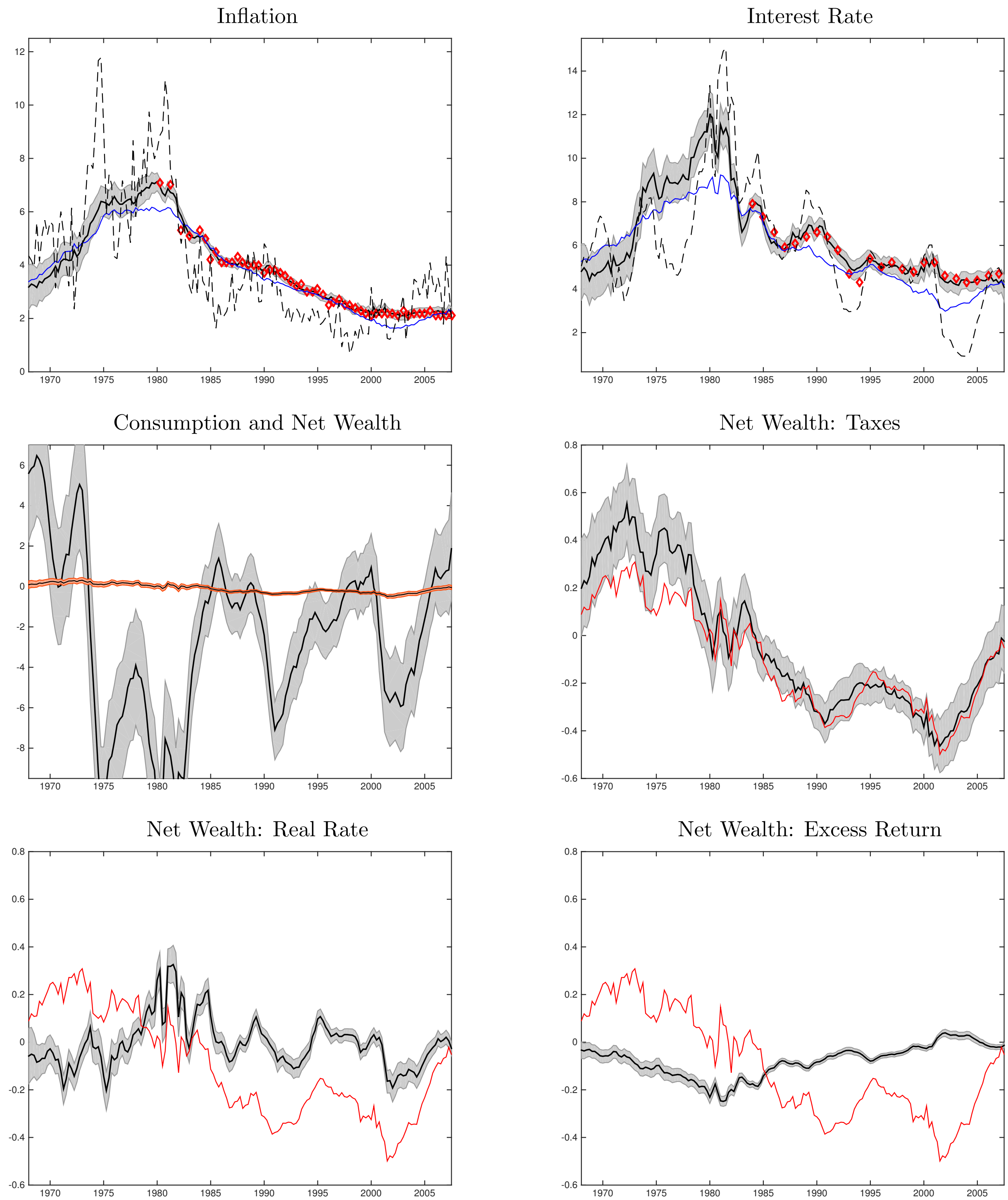
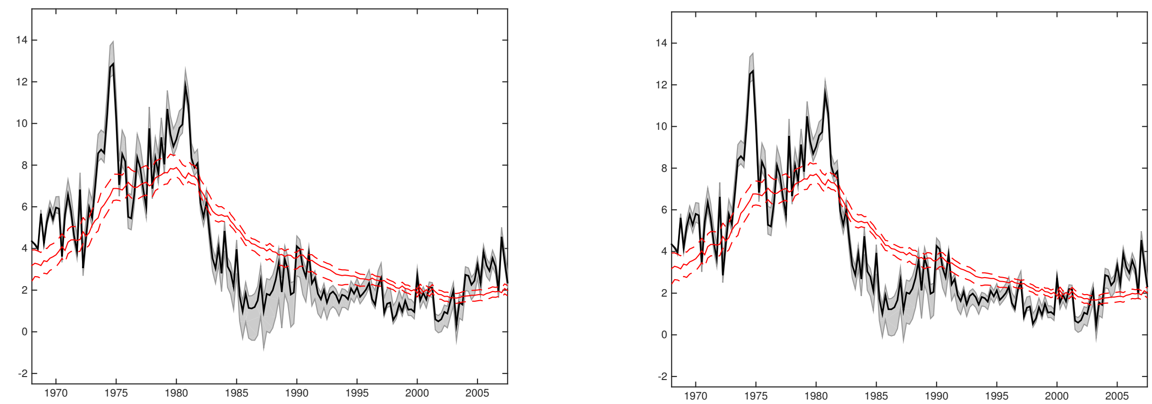


Figure 13: **Model Predictions with** $\beta = 0.995$

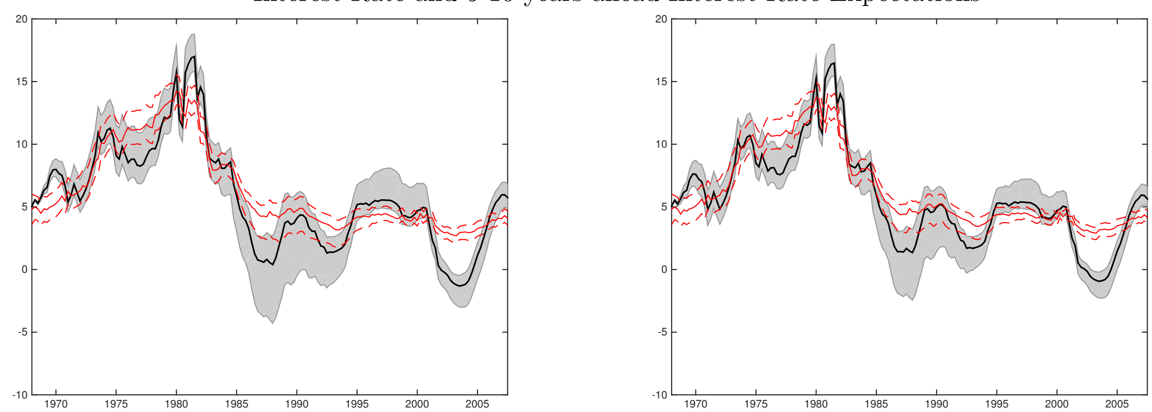
The top figures show the predicted evolution of long-term expectations inflation and nominal interest rate. Median predictions are denoted by the solid black line, while the grey area measures the 95% credible interval; The red diamond denote 5-10 survey forecasts from Blue Chip Economics; the dashed black line denotes actual variables. The other panels show the evolution of consumption (grey), net wealth (red) and its three subcomponents; the red line measures total net wealth.

Debt/GDP=200%; Maturity = 5.5 years **Debt/GDP=200%; Maturity = 14 years**

Inflation and 5-10 years-ahead Inflation Expectations



Interest Rate and 5-10 years-ahead Interest Rate Expectations



Output Gap and Net Wealth

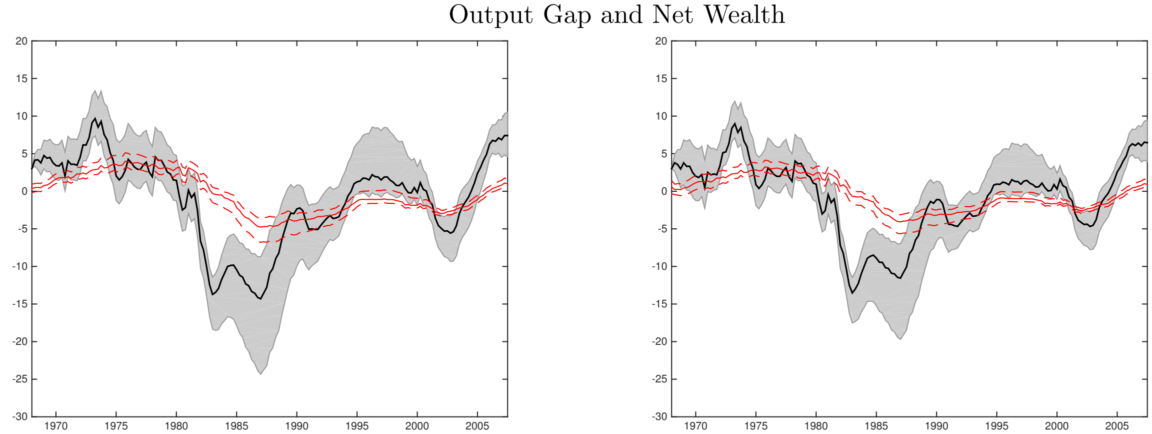


Figure 14: **Counterfactuals: with** $\beta = 0.995$

The panels show the evolution of selected variables in two counterfactual economies. Inflation, interest rate and output gap are denoted by solid black lines; expectations and net wealth are shown in red solid lines. The 95th percentiles are defined by the grey area and the dashed red lines respectively. The left panels show an economy with 200% debt-to-output ratio and baseline average maturity of debt. The right panels show an economy with 200% debt-to-output ratio and 14-years average maturity of debt.

	Prior distribution			Posterior distribution			
	Distr.	Mean	St. Dev.	Mode	Mean	5 percent	95 percent
σ	Gamma	2.000	0.600	7.374	6.912	5.346	8.597
b	Beta	0.350	0.100	0.500	0.528	0.472	0.585
ξ_w	Beta	0.500	0.050	0.771	0.768	0.685	0.856
ι_w	Beta	0.500	0.150	0.321	0.407	0.193	0.664
ξ_p	Beta	0.500	0.050	0.746	0.757	0.699	0.811
ι_p	Beta	0.500	0.150	0.499	0.488	0.422	0.555
Θ_p	Normal	1.250	0.120	1.583	1.621	1.483	1.766
κ	Beta	0.300	0.150	0.002	0.003	0.002	0.004
ϕ_π	Normal	1.500	0.150	1.575	1.622	1.499	1.755
ρ_i	Beta	0.500	0.100	0.842	0.851	0.829	0.871
ϕ_x	Normal	0.120	0.050	0.078	0.097	0.075	0.123
ρ_τ	Beta	0.700	0.100	0.824	0.808	0.734	0.873
ϕ_{τ_l}	Gamma	0.700	0.100	0.648	0.671	0.525	0.833
g	Gamma	0.035	0.030	0.043	0.045	0.039	0.052

Table 3: **Prior and Posterior Distribution of Structural Params: alt. fiscal rule.**

Note: The posterior distribution is obtained using the Metropolis-Hastings algorithm.

	Prior distribution			Posterior distribution			
	Distr.	Mean	St. Dev.	Mode	Mean	5 percent	95 percent
ρ_{θ_w}	Beta	0.500	0.200	0.855	0.800	0.557	0.899
ρ_g	Beta	0.500	0.200	0.940	0.942	0.930	0.955
$\rho_{\bar{\tau}}$	Beta	0.500	0.200	0.051	0.080	0.021	0.166
ρ_a	Beta	0.500	0.200	0.859	0.833	0.780	0.871
σ_{θ_w}	InvGamma	0.100	2.000	0.167	0.196	0.154	0.253
σ_{θ_p}	InvGamma	0.100	2.000	0.205	0.206	0.185	0.230
σ_g	InvGamma	0.100	2.000	0.526	0.580	0.482	0.690
σ_m	InvGamma	0.100	2.000	0.198	0.201	0.182	0.222
$\sigma_{\bar{\tau}}$	InvGamma	0.100	2.000	2.045	2.078	1.889	2.287
σ_a	InvGamma	0.100	2.000	1.429	1.449	1.102	1.888
$\sigma_{g\gamma}$	Beta	0.500	0.200	0.494	0.483	0.355	0.612
$\sigma_{o,\pi^{SR}}$	InvGamma	0.100	2.000	0.166	0.167	0.149	0.188
$\sigma_{o,R^{SR}}$	InvGamma	0.100	2.000	0.057	0.057	0.050	0.066
$\sigma_{o,R^{LR}}$	InvGamma	0.100	2.000	0.069	0.087	0.051	0.128
$\sigma_{o,\pi^{LR}}$	InvGamma	0.100	2.000	0.042	0.044	0.035	0.055

Table 4: **Prior and Posterior Distribution of Shock Processes: alt. fiscal rule.**

Note: The posterior distribution is obtained using the Metropolis-Hastings algorithm.

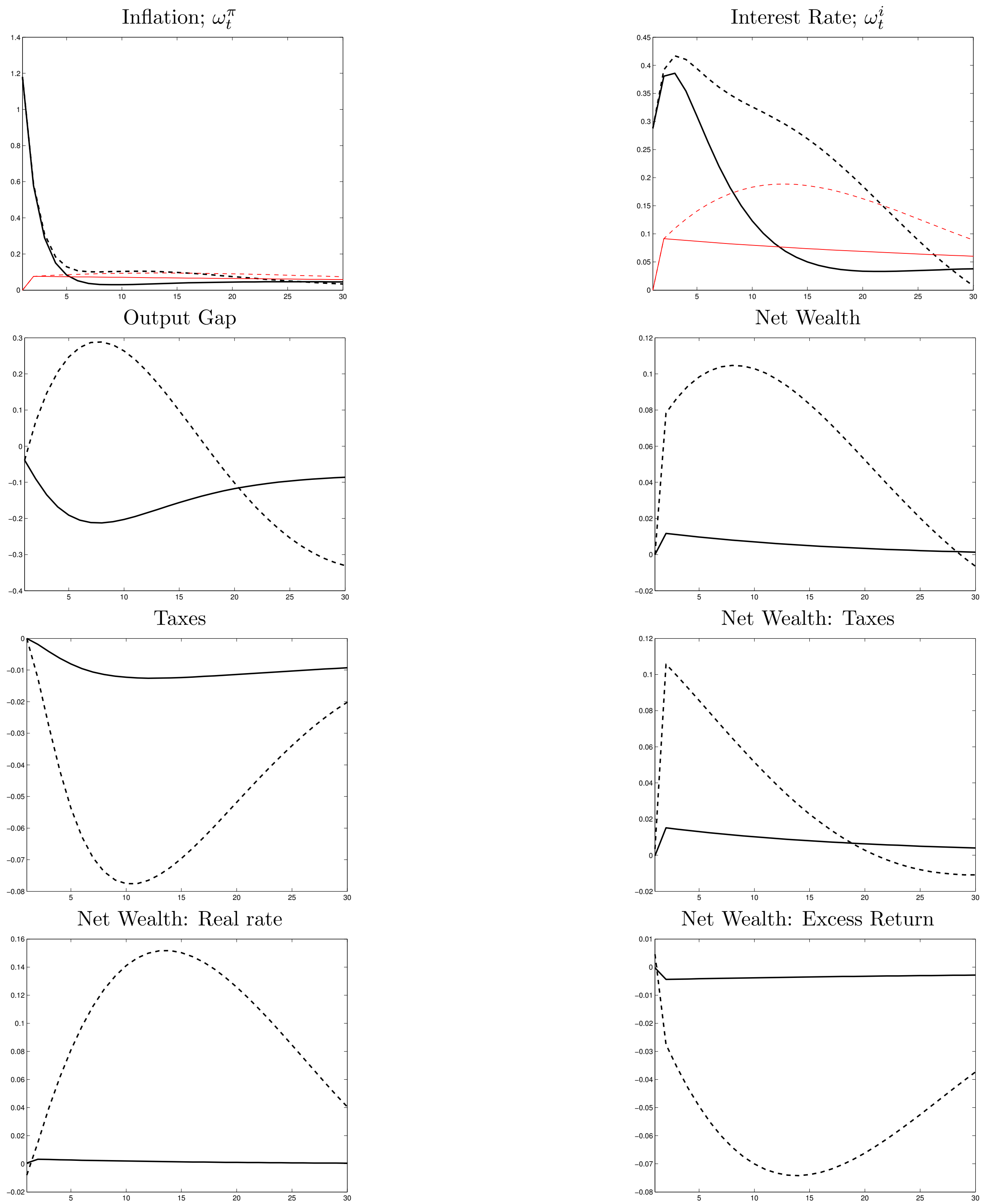


Figure 15: **Mean IR with alt. fiscal rule: Price Markup Shock**

Solid lines denote the baseline economy with Debt-to-output ratio of 30%. Dashed lines correspond to an economy with debt-to-output ratio of 200%. Red lines measure inflation and interest rate estimated drifts.

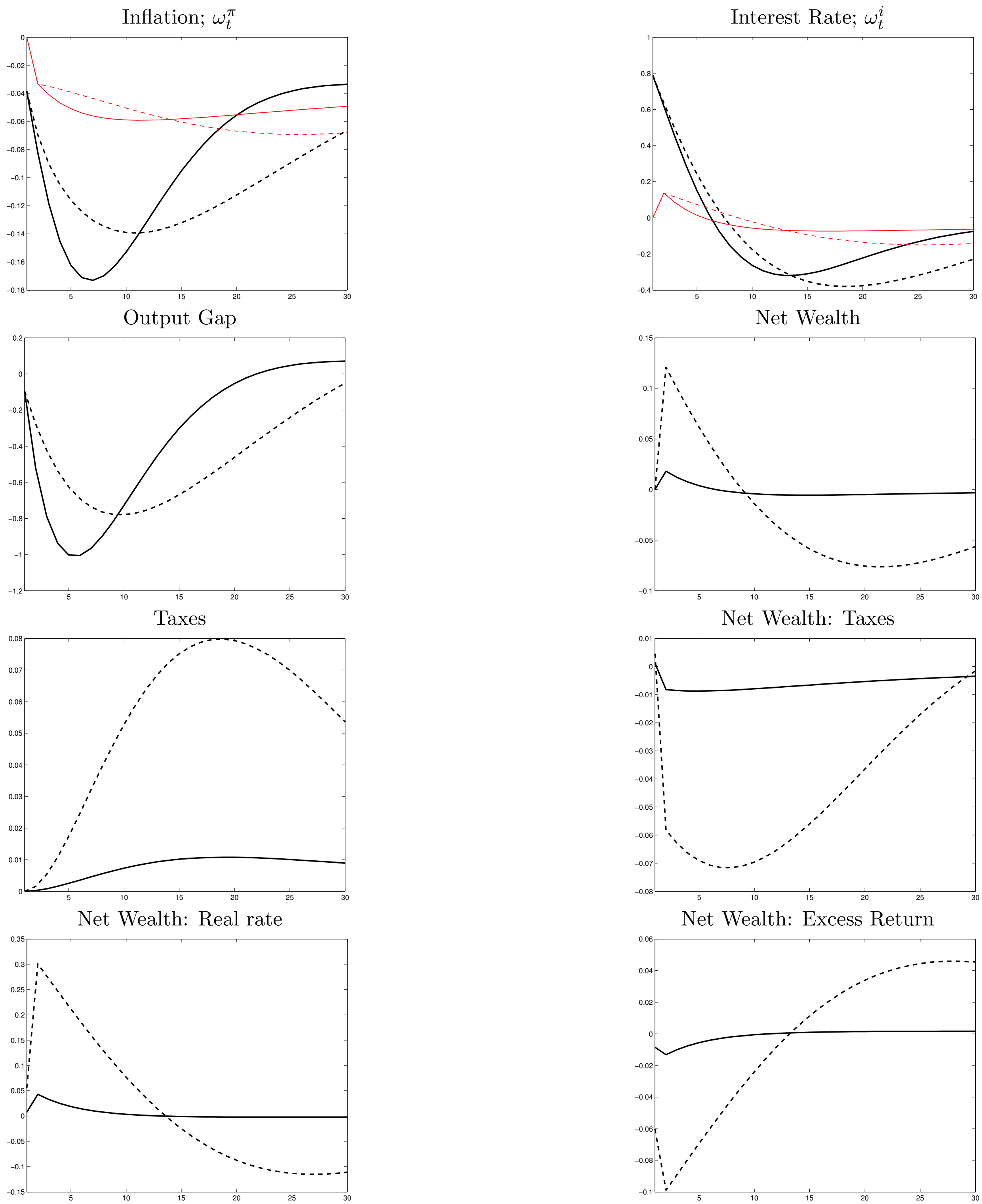


Figure 16: **Mean IR with alt. fiscal rule: Monetary Policy Shock**

Solid lines denote the baseline economy with Debt-to-output ratio of 30%. Dashed lines correspond to an economy with debt-to-output ratio of 200%. Red lines measure inflation and interest rate estimated drifts.

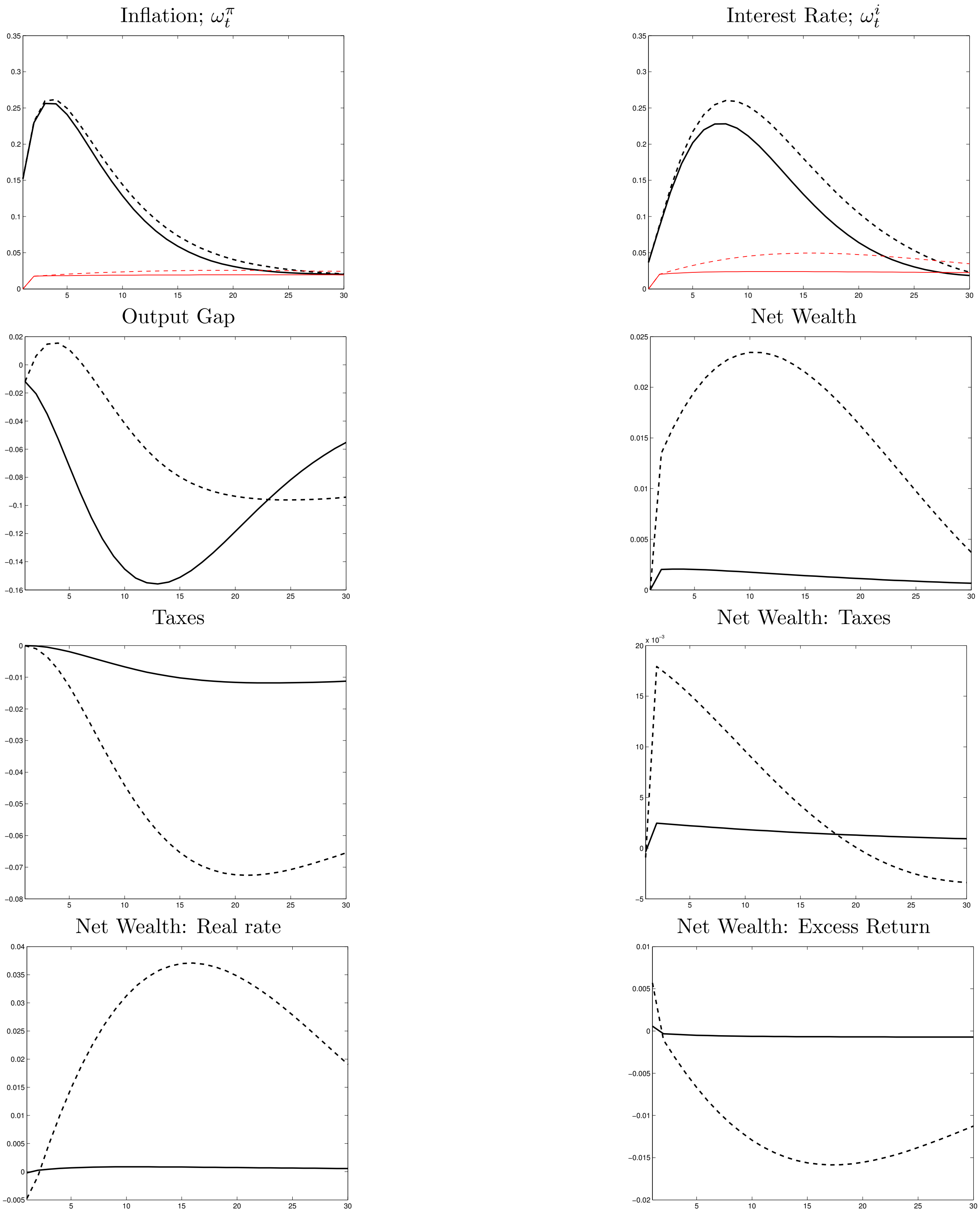


Figure 17: **Mean IR with alt. fiscal rule: Wage Markup Shock**

Solid black lines denote the baseline economy with Debt-to-output ratio of 30%. Dashed lines correspond to an economy with debt-to-output ratio of 200%. Red lines measure inflation and interest rate estimated drifts.

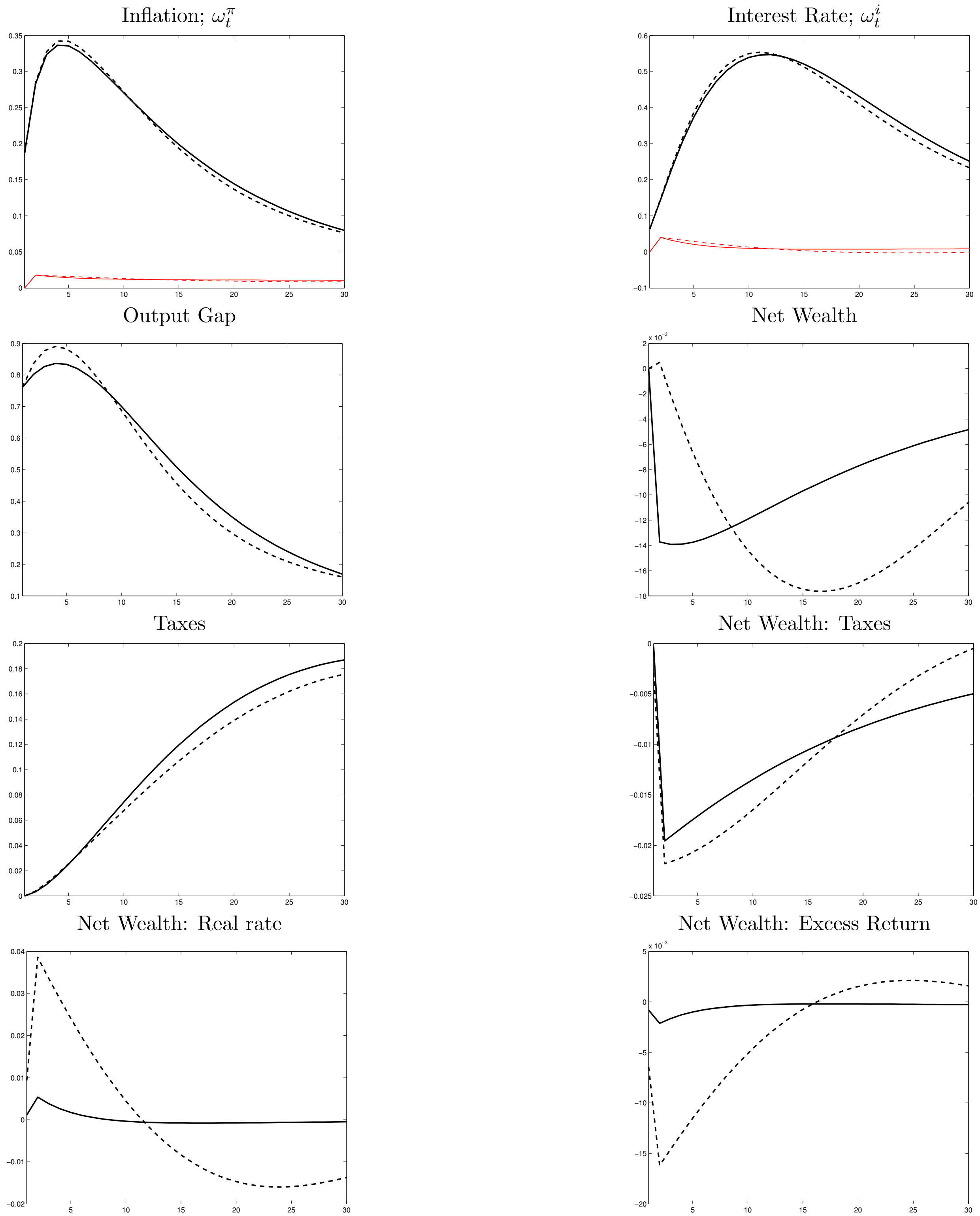


Figure 18: **Mean IR with alt. fiscal rule: Spending Shock**

Solid black lines denote the baseline economy with Debt-to-output ratio of 30%. Dashed lines correspond to an economy with debt-to-output ratio of 200%. Red lines measure inflation and interest rate estimated drifts.

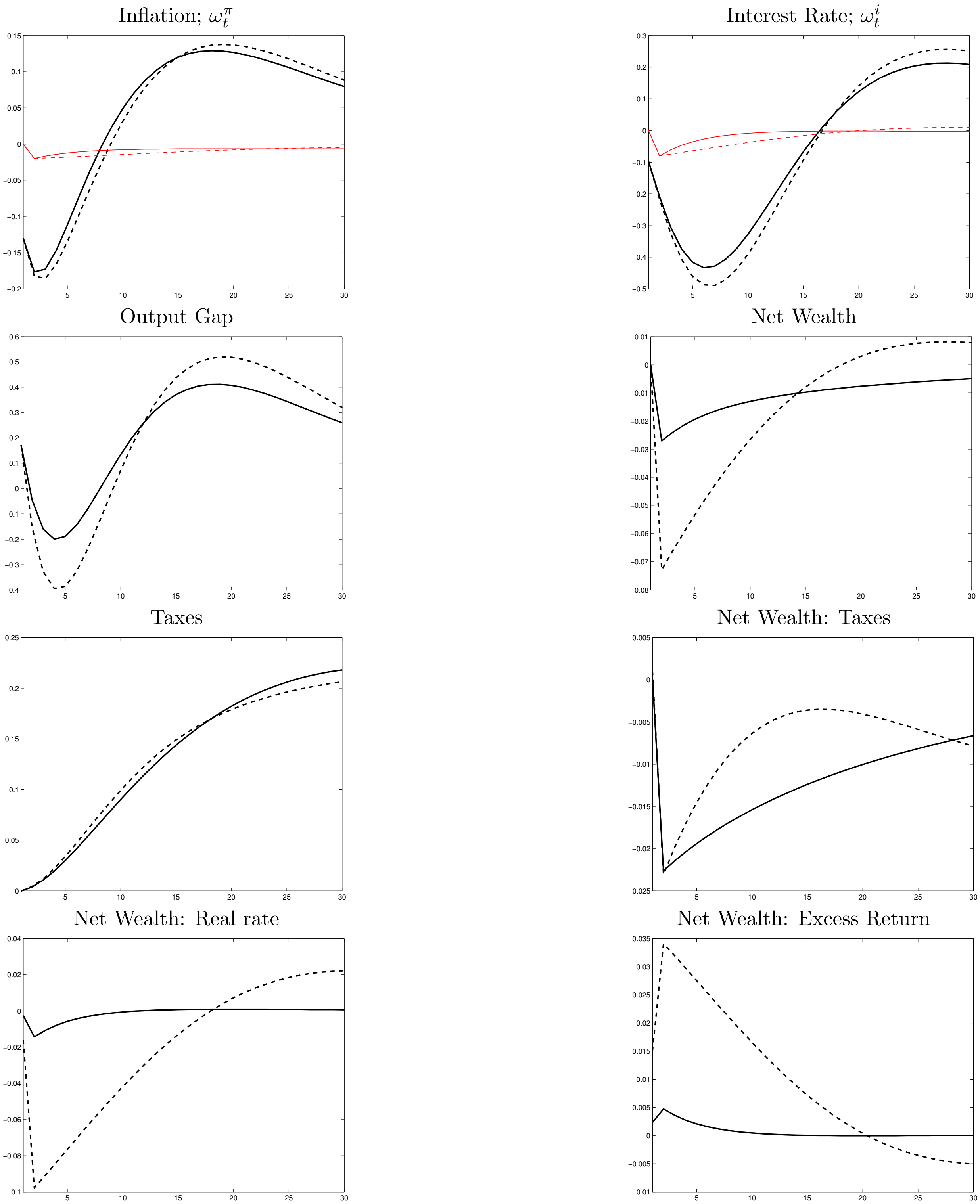


Figure 19: **Mean IR with alt. fiscal rule: Technology Shock**

Solid black lines denote the baseline economy with Debt-to-output ratio of 30%. Dashed lines correspond to an economy with debt-to-output ratio of 200%. Red lines measure inflation and interest rate estimated drifts.

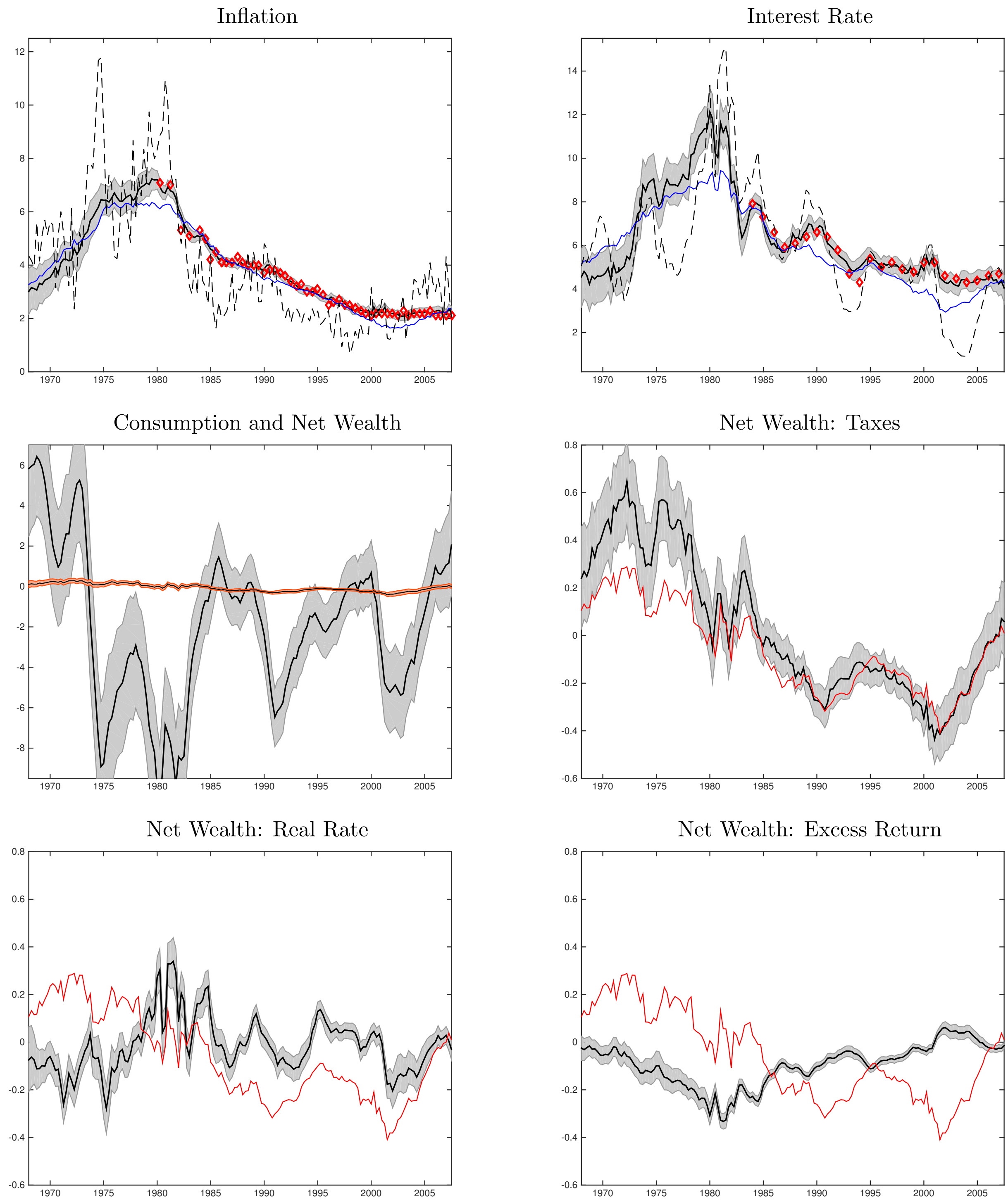


Figure 20: **Model Predictions with alt. fiscal rule**

The top figures show the predicted evolution of long-term expectations inflation and nominal interest rate. Median predictions are denoted by the solid black line, while the grey area measures the 95% credible interval; The red diamond denote 5-10 survey forecasts from Blue Chip Economics; the dashed black line denotes actual variables. The other panels show the evolution of consumption (grey), net wealth (red) and its three subcomponents; the red line measures total net wealth.

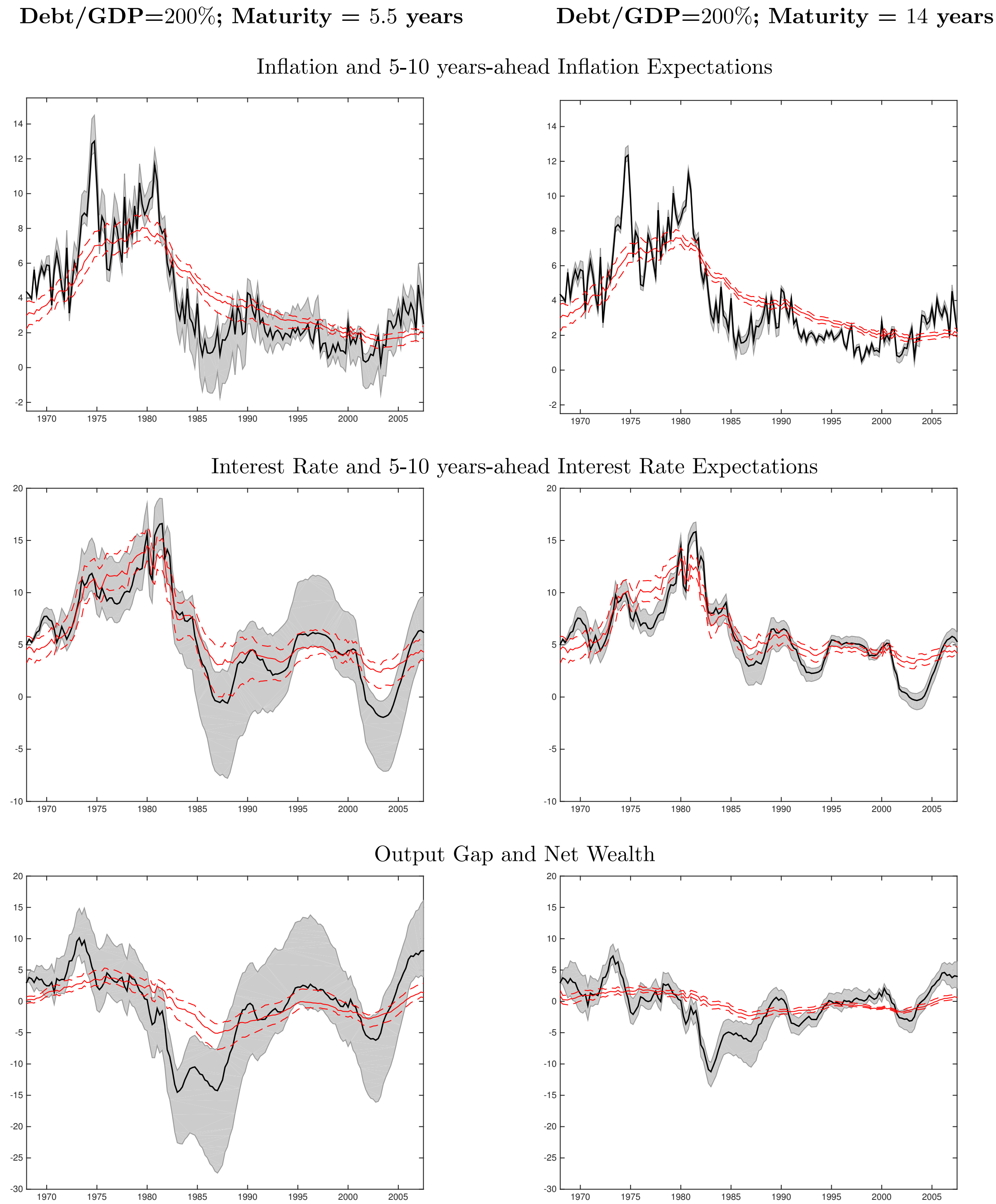


Figure 21: **Counterfactuals: alternative fiscal rule**

The panels show the evolution of selected variables in two counterfactual economies. Inflation, interest rate and output gap are denoted by solid black lines; expectations and net wealth are shown in red solid lines. The 95th percentiles are defined by the grey area and the dashed red lines respectively. The left panels show an economy with 200% debt-to-output ratio and baseline average maturity of debt. The right panels show an economy with 200% debt-to-output ratio and 14-years average maturity of debt.