Delegated Expertise, Authority, and Communication
Online Appendix

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Appendix A

Proof of Lemma 1. For quadratic utilities, players' optimal actions are given by the conditional means, $\mathbb{E}\left[\tilde{\omega}\right|\tilde{s}_\omega = s_\omega, \tilde{s}_\eta = s_\eta]$ and $\mathbb{E}\left[\tilde{\eta}\right|\tilde{s}_\omega = s_\omega, \tilde{s}_\eta = s_\eta]$, respectively. We denote $\theta \equiv \mathbb{E}\left[\tilde{\eta}\right|\tilde{s}_\omega = s_\omega, \tilde{s}_\eta = s_\eta]$. For the sender $\mathbb{E}\left[-(y-\tilde{\eta})^2|\tilde{s}_\omega = s_\omega, \tilde{s}_\eta = s_\eta\right] = -y^2 + 2y\mathbb{E}\left[\tilde{\eta}|\tilde{s}_\omega = s_\omega, \tilde{s}_\eta = s_\eta\right] - \mathbb{E}\left[\tilde{\eta}^2|\tilde{s}_\omega = s_\omega, \tilde{s}_\eta = s_\eta\right]$, hence the sender's preferences satisfy the single crossing condition in y and θ.

We now argue that any equilibrium is either fully separating in θ or involves partial pooling in θ everywhere. Partial pooling everywhere follows straightforwardly from single crossing. Suppose there exists an equilibrium where the receiver is fully responsive to θ on some interval $\left[\underline{\theta}, \overline{\theta}\right]$ and involves pooling around the interval. This implies $\mathbb{E}\left[\tilde{\omega}\middle|\tilde{\theta} = \theta\right] = \theta$ on $\left[\underline{\theta}, \overline{\theta}\right]$. Now take some type $\hat{\theta} = \overline{\theta} + \delta$ for some $\delta > 0$ that induces the pooling action y_p^r strictly above $y^r(\overline{\theta}) = \overline{\theta}$. For δ sufficiently small $\hat{\theta} - y^r(\overline{\theta}) < y_p^r - \hat{\theta}$, implying that $\mathbb{E}u^s(y^r(\overline{\theta})) > \mathbb{E}u^s(y_p^r)$. Hence sender types close to $\overline{\theta}$ have an incentive to deviate, so the receiver's choice of actions does not constitute an equilibrium. Since the same argument holds for $\underline{\theta}$, the receiver cannot be fully responsive to θ on a bounded interval. Hence, an equilibrium that involves separation in θ somewhere must involve separation in θ everywhere.

Clearly, these two classes of equilibria can be characterized by communication about θ only. Consider now an equilibrium where some sender types with the same conditional expectation play different strategies. By the single crossing property, types with the same θ are indifferent between at most two distinct actions. It follows immediately from single crossing that this corresponds to the second class of equilibria considered above, where

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the strategies are changed on measure zero sets. Since this does not change the receiver's equilibrium actions, we can characterize such equilibria – up to the strategies of sender types on measure zero sets – by communication about θ only. ∎

Proof of Lemma 2. **i)** The random vector $\tilde{\boldsymbol{\tau}} = (\tilde{\omega}, \tilde{\eta}, \tilde{\varepsilon}_\omega, \tilde{\varepsilon}_\eta)$ follows a joint Laplace distribution. Since the Laplace distribution is a member of the class of elliptically contoured distributions, the following well-known properties apply:

The sender's conditional mean θ can be calculated as $\mathbb{E}\left[\tilde{\eta} | \tilde{s}_\omega = s_\omega, \tilde{s}_\eta = s_\eta\right] = \gamma_\omega s_\omega + \gamma_\eta s_\eta$ with $\gamma_\omega = \frac{\sigma^2_{\varepsilon_\eta}\rho\sigma^2}{(\sigma^2+\sigma^2_{\varepsilon_\omega})(\sigma^2+\sigma^2_{\varepsilon_\eta})-\rho^2\sigma^4}$, and $\gamma_\eta = \frac{\sigma^2\sigma^2_{\varepsilon_\omega}+\sigma^4(1-\rho^2)}{(\sigma^2+\sigma^2_{\varepsilon_\omega})(\sigma^2+\sigma^2_{\varepsilon_\eta})-\rho^2\sigma^4}$; the weights γ_ω and γ_η are constants, independent of the realized signals. The equation follows from the fact that conditional expectations are linear functions for elliptically contoured distributions (see, e.g., Fang et al. (1990) Theorem 2.18).

The random vector $\left(\tilde{\omega}, \tilde{\eta}, \tilde{\theta}\right)$ is Laplace, since affine transformations of random vectors that follows an elliptical distribution with a given characteristic generator follow a distribution with the same characteristic generator (see, e.g., Fang et al. (1990) Theorem 2.16).

The first moment of $\tilde{\theta}$ is zero, because the mean of $\tilde{\boldsymbol{\tau}}$ is the zero vector. Plugging in the weights γ_ω and γ_η, the second moments of $\left(\tilde{\omega}, \tilde{\eta}, \tilde{\theta}\right)$ can straightforwardly be calculated:

$$\begin{aligned}
\sigma^2_\theta &= \gamma^2_\omega Var\left(\tilde{s}_\omega\right) + \gamma^2_\eta Var\left(\tilde{s}_\eta\right) + 2\gamma_\omega\gamma_\eta Cov\left(\tilde{s}_\omega, \tilde{s}_\eta\right) \\
&= \gamma^2_\omega\left(\sigma^2 + \sigma^2_{\varepsilon_\omega}\right) + \gamma^2_\eta\left(\sigma^2 + \sigma^2_{\varepsilon_\eta}\right) + 2\gamma_\omega\gamma_\eta\sigma_{\omega\eta} = \sigma^2\frac{\frac{\sigma^2_{\varepsilon_\omega}}{\sigma^2} + \frac{\sigma^2_{\varepsilon_\eta}}{\sigma^2}\rho^2 + 1 - \rho^2}{\left(1 + \frac{\sigma^2_{\varepsilon_\omega}}{\sigma^2}\right)\left(1 + \frac{\sigma^2_{\varepsilon_\eta}}{\sigma^2}\right) - \rho^2} \\
&= \sigma_{\eta\theta},
\end{aligned}$$

and

$$\sigma_{\omega\theta} = \gamma_\omega\sigma^2 + \gamma_\eta\sigma_{\omega\eta} = \rho\sigma^2\frac{\frac{\sigma^2_{\varepsilon_\omega}}{\sigma^2} + \frac{\sigma^2_{\varepsilon_\eta}}{\sigma^2} + 1 - \rho^2}{\left(1 + \frac{\sigma^2_{\varepsilon_\omega}}{\sigma^2}\right)\left(1 + \frac{\sigma^2_{\varepsilon_\eta}}{\sigma^2}\right) - \rho^2}.$$

ii) Letting $a \equiv \frac{\sigma^2_{\varepsilon_\omega}}{\sigma^2}$ and $b \equiv \frac{\sigma^2_{\varepsilon_\eta}}{\sigma^2}$, we can rewrite $\sigma_{\omega\theta}$ and σ^2_θ as

$$\sigma_{\omega\theta} = \rho\sigma^2\frac{a + b + 1 - \rho^2}{(1+a)(1+b) - \rho^2},$$

and

$$\sigma_{\theta}^{2}=\sigma^{2}\frac{a+b\rho^{2}+1-\rho^{2}}{(1+a)\,(1+b)-\rho^{2}}.$$

Consider first the set of feasible levels of $\sigma_{\omega\theta}=C$. Note that for $a=0$ or $b=0$, the covariance is constant and equal to $\rho\sigma^{2}=\sigma_{\omega\eta}$. Moreover, the covariance is decreasing in a for given b and decreasing in b for given a. By l'Hôpital's rule, we have

$$\lim_{b\to\infty}\frac{a+b+1-\rho^{2}}{(1+a)\,(1+b)-\rho^{2}}=\frac{1}{1+a},$$

and

$$\lim_{a\to\infty}\frac{a+b+1-\rho^{2}}{(1+a)\,(1+b)-\rho^{2}}=\frac{1}{1+b}.$$

So, letting both a and b (in whatever order) go to infinity results in a covariance of zero. By continuity, any $C\in(0,\sigma_{\omega\eta}]$ can be generated by finite levels a,b. Including the case where no signal is observed at all, we can generate all $C\in[0,\sigma_{\omega\eta}]$.

Consider next the set of feasible σ_{θ}^{2} for any given level $\sigma_{\omega\theta}=C$. Distinguish two cases, i) $C=\sigma_{\omega\eta}$ and ii) $C\in[0,\sigma_{\omega\eta})$.

Case i) requires that $a=0$ or $b=0$. If $b=0$, then $\frac{a+b\rho^{2}+1-\rho^{2}}{(1+a)(1+b)-\rho^{2}}=1$ and thus $\sigma_{\theta}^{2}=\sigma^{2}$ for all a. If $a=0$, then

$$\sigma_{\theta}^{2}=\sigma_{\eta}^{2}\frac{b\rho^{2}+1-\rho^{2}}{(1+b)-\rho^{2}}$$

is decreasing in b and attains value $\sigma_{\theta}^{2}=\sigma^{2}$ for $b=0$. Moreover,

$$\lim_{b\to\infty}\frac{b\rho^{2}+1-\rho^{2}}{(1+b)-\rho^{2}}=\rho^{2}.$$

Hence, for $C=\sigma_{\omega\eta}$, $\sigma_{\theta}^{2}\in[\rho^{2}\sigma^{2},\sigma^{2}]$; the lower limit is included because we allow for the case where only one signal is observed.

Case ii) $C\in[0,\sigma_{\omega\eta})$ requires that $a>0$ and $b>0$. Let $\delta\equiv\frac{C}{\sigma_{\omega\eta}}\in[0,1)$. The combinations of a and b that generate C satisfy

$$\frac{a+b+1-\rho^{2}}{(1+a)\,(1+b)-\rho^{2}}=\delta.$$

Solving for a as a function of b, we obtain

$$a\,(b;\delta)=\frac{(1-\delta)\,(1+b-\rho^{2})}{\delta b-(1-\delta)}=\frac{(1+b-\rho^{2})}{\frac{\delta}{1-\delta}b-1}.$$

The function $a(b;\delta)$ is decreasing in b and has the limit

$$\lim_{b\to\infty}\frac{1+b-\rho^2}{\frac{\delta}{1-\delta}b-1}=\frac{1-\delta}{\delta}.$$

In the limit as $b\to\frac{1-\delta}{\delta}$, we obtain $a\to\infty$. Hence, C can be generated for $b>\frac{1-\delta}{\delta}$ and $a=\frac{\left(1+b-\rho^2\right)}{\frac{\delta}{1-\delta}b-1}$. Substituting for $\frac{\left(1+b-\rho^2\right)}{\frac{\delta}{1-\delta}b-1}$ into σ_θ^2, we obtain

$$\sigma_\theta^2\left(b,a\left(b;\delta\right),\delta\right)=\sigma^2\frac{\frac{\left(1+b-\rho^2\right)}{\frac{\delta}{1-\delta}b-1}+b\rho^2+1-\rho^2}{\left(1+\frac{\left(1+b-\rho^2\right)}{\frac{\delta}{1-\delta}b-1}\right)\left(1+b\right)-\rho^2}=\sigma^2\frac{b\delta\rho^2+1-\rho^2}{1+b-\rho^2}.$$

The derivative of this expression in b is $\frac{\left(\delta\rho^2-1\right)\left(1-\rho^2\right)}{\left(1+b-\rho^2\right)^2}<0$, so $Var\left(\tilde{\theta};b,a\left(b;\delta\right),\delta\right)$ is continuous and monotone decreasing in b. In the limit as b tends to infinity, we obtain

$$\lim_{b\to\infty}\sigma^2\frac{b\delta\rho^2+1-\rho^2}{1+b-\rho^2}=\sigma^2\delta\rho^2=\sigma^2\frac{C}{\sigma_{\omega\eta}}\rho^2=\rho C.$$

In the limit as $b\to\frac{1-\delta}{\delta}$, we obtain

$$\lim_{b\to\frac{1-\delta}{\delta}}\sigma^2\frac{b\delta\rho^2+1-\rho^2}{1+b-\rho^2}=\sigma^2\frac{\frac{1-\delta}{\delta}\delta\rho^2+1-\rho^2}{1+\frac{1-\delta}{\delta}-\rho^2}=\delta\sigma^2=\frac{1}{\rho}C.$$

Hence, we have shown that for any given $C\in[0,\sigma_{\omega\eta})$, $\sigma_\theta^2\in\left[\rho C,\frac{1}{\rho}C\right]$. We include the lower limit, because the case where $b\to\infty$ is equivalent to the case with one signal only. ■

Lemma A1 For the Laplace distribution, for $0\leq\underline{\theta}<\overline{\theta}$ we can write

$$\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta}\in\left[\underline{\theta},\overline{\theta}\right]\right]=\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta}\geq 0\right]+\overline{\theta}-g\left(\overline{\theta}-\underline{\theta}\right),\tag{17}$$

where $g(q)=\frac{q}{1-\exp(-\lambda q)}$ and $\frac{1}{\lambda}=\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta}\geq 0\right]$. The function $g(q)$ *satisfies* $\lim_{q\to 0}g(q)=\frac{1}{\lambda}$ *and has limits* $\lim_{q\to\infty}g(q)=\infty$, *and* $\lim_{q\to\infty}(q-g(q))=0$. *Moreover, the function is increasing and convex, with a slope satisfying* $\lim_{q\to 0}g'(q)=\frac{1}{2}$ *and attaining the limit* $\lim_{q\to\infty}g'(q)=1$.

Proof of Lemma A1. Recall that the marginal density of the Laplace distribution is $f_{\theta}(\theta) = \lambda e^{-\lambda|\theta|}$. For the Laplace distribution for $0 \leq \underline{\theta} < \overline{\theta}$, an integration by parts gives

$$\begin{aligned}
\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \in \left[\underline{\theta}, \overline{\theta}\right]\right] &= \int_{\underline{\theta}}^{\overline{\theta}} \theta\lambda \frac{\exp^{-\lambda\theta}}{\exp^{-\lambda\underline{\theta}} - \exp^{-\lambda\overline{\theta}}} d\theta = -\theta \frac{\exp^{-\lambda\theta}}{\exp^{-\lambda\underline{\theta}} - \exp^{-\lambda\overline{\theta}}}\Bigg|_{\underline{\theta}}^{\overline{\theta}} + \int_{\underline{\theta}}^{\overline{\theta}} \frac{\exp^{-\lambda\theta}}{\exp^{-\lambda\underline{\theta}} - \exp^{-\lambda\overline{\theta}}} d\theta. \\
&= \overline{\theta} - \frac{\left(\overline{\theta} - \underline{\theta}\right)}{1 - \exp^{-\lambda\left(\overline{\theta} - \underline{\theta}\right)}} - \frac{1}{\lambda} \frac{\exp^{-\lambda\theta}}{\exp^{-\lambda\underline{\theta}} - \exp^{-\lambda\overline{\theta}}}\Bigg|_{\underline{\theta}}^{\overline{\theta}} \\
&= \frac{1}{\lambda} + \overline{\theta} - \frac{\left(\overline{\theta} - \underline{\theta}\right)}{1 - \exp^{-\lambda\left(\overline{\theta} - \underline{\theta}\right)}}.
\end{aligned}$$

By l'Hôpital's rule, $\lim_{q\to 0} g(q) = \frac{1}{\lambda}$. The limit $\lim_{q\to\infty} 1 - \exp(-\lambda q) = 1$ implies that $\lim_{q\to\infty} g(q) = \infty$. Using $q - g(q) = -\frac{q\exp(-\lambda q)}{1-\exp(-\lambda q)}$ and $\lim_{q\to\infty} q\exp(-\lambda q) = 0$, we have $\lim_{q\to\infty}(q - g(q)) = 0$.

The slope of the function is

$$g'(q) = \frac{\left(1 - (1 + \lambda q) e^{-q\lambda}\right)}{\left(1 - e^{-q\lambda}\right)^2} \geq 0.$$

The inequality is strict for $q > 0$ since $\lim_{q\to 0}(1 + \lambda q) e^{-q\lambda} = 1$ and $\frac{\partial}{\partial q}(1 - (1 + \lambda q) e^{-q\lambda}) = \lambda^2 q e^{-q\lambda} > 0$ for $q > 0$. Applying l'Hôpital's rule twice, one finds that $\lim_{q\to 0} g'(q) = \frac{1}{2}$, and since $\lim_{q\to\infty} \lambda q e^{-q\lambda} = 0$, we have $\lim_{q\to\infty} g'(q) = 1$.

Differentiating $g(q)$ twice, we obtain

$$g''(q) = \lambda \frac{e^{-q\lambda}}{\left(1 - e^{-q\lambda}\right)^3} \left(2e^{-q\lambda} + q\lambda + q\lambda e^{-q\lambda} - 2\right).$$

The sign of the second derivative is equal to the sign of the expression in brackets. At $q = 0$, the expression is zero. The change of the expression is given by

$$\frac{\partial}{\partial q}\left(2e^{-q\lambda} + q\lambda + q\lambda e^{-q\lambda} - 2\right) = \lambda\left(1 - (1 + \lambda q) e^{-q\lambda}\right) \geq 0,$$

by the same argument as given above. Hence, $g(q)$ is convex. ■

Proof of Lemma 3. **i)** Equation (3) follows immediately from applying again Fang et al. (1990) Theorem 2.18.

ii) By the law of iterated expectations,

$$\mathbb{E}\left[\tilde{\omega}|\,\tilde{\theta}\in\left[\underline{\theta},\overline{\theta}\right]\right]=\mathbb{E}\left[\mathbb{E}\left[\tilde{\omega}|\tilde{\theta}=\theta\right]\middle|\,\tilde{\theta}\in\left[\underline{\theta},\overline{\theta}\right]\right]=\mathbb{E}\left[\frac{\sigma_{\omega\theta}}{\sigma_{\theta}^{2}}\tilde{\theta}\middle|\,\tilde{\theta}\in\left[\underline{\theta},\overline{\theta}\right]\right]=\frac{\sigma_{\omega\theta}}{\sigma_{\theta}^{2}}\cdot\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta}\in\left[\underline{\theta},\overline{\theta}\right]\right].$$

iii) The marginal distribution of $\tilde{\theta}$ is a classical Laplace distribution with density of the form $f_{\theta}(\theta)=\lambda e^{-\lambda|\theta|}$ by the same argument as given in Lemma 2 i). Since by Lemma A1 $\mathbb{E}\left[\tilde{\theta}\middle|\,\theta\in\left[\underline{\theta},\overline{\theta}\right]\right]=\frac{1}{\lambda}+\overline{\theta}-\frac{\left(\overline{\theta}-\underline{\theta}\right)}{1-\exp^{-\lambda\left(\overline{\theta}-\underline{\theta}\right)}}=\frac{1}{\lambda}+\frac{\underline{\theta}}{1-\exp^{-\lambda\left(\overline{\theta}-\underline{\theta}\right)}}-\frac{\exp^{-\lambda\left(\overline{\theta}-\underline{\theta}\right)}\overline{\theta}}{1-\exp^{-\lambda\left(\overline{\theta}-\underline{\theta}\right)}}$ and $\lim_{\overline{\theta}\to\infty}\exp^{-\lambda\left(\overline{\theta}-\underline{\theta}\right)}\overline{\theta}=0$, we have

$$\lim_{\overline{\theta}\to\infty}\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta}\in\left[\underline{\theta},\overline{\theta}\right]\right]=\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta}\geq 0\right]+\underline{\theta}.$$

For a discussion of the parameter α see the proof of Proposition 3. ∎

Appendix B

Characterization of partitional equilibria

Partitional equilibria are completely characterized by a sequence of marginal types, a_i, who are indifferent between pooling with types slightly below and with types slightly above them. In our description here, we focus on symmetric equilibria. This is without loss, since for the case $c\leq 1$ symmetric equilibria are the only ones that exist. We do prove their existence, and for logconcave densities, equilibria are unique (see Szalay (2012)). For the case $c>1$, we prove our results also allowing for asymmetric equilibria.

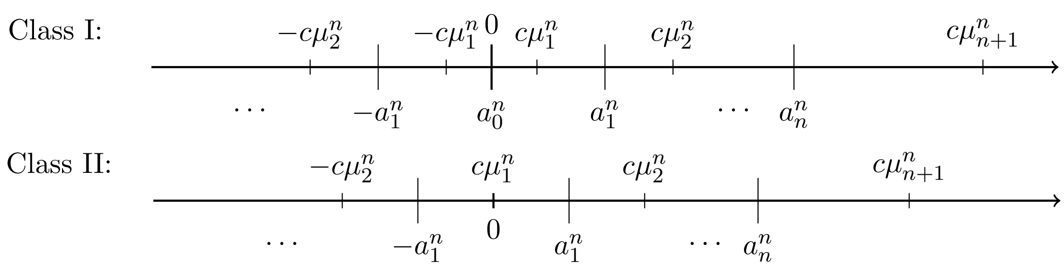

Figure 6: Class I equilibrium and Class II equilibrium.

Symmetric equilibria come in two classes; see Figure 6 for an illustration. Class I has zero as a threshold, $a_0^n = 0$, and in addition $n \geq 0$ thresholds $a_1^n, \ldots, a_n^n$ above the prior mean. By symmetry, types $-a_n^n, \ldots, -a_1^n$ are the threshold types below the prior mean. Such an equilibrium induces $2(n+1)$ actions; superscript n captures the dependence of the equilibrium threshold types on the number of induced actions. Class II has zero as an action taken by the receiver instead of a threshold. In this case, we eliminate a_0^n altogether. Such an equilibrium induces $2n+1$ actions. Consider Class I equilibria first.

For $n \geq 1$, let

$$\mu_i^n \equiv \mathbb{E}\left[\tilde{\theta}\middle|\, \tilde{\theta} \in \left[a_{i-1}^n, a_i^n\right)\right] \qquad \text{for } i = 1, \ldots, n \tag{18}$$

and $\mu_{n+1}^n \equiv \mathbb{E}\left[\tilde{\theta}\middle|\, \tilde{\theta} \geq a_n^n\right]$. By convention, we take all intervals as closed from below and open from above. Clearly, given quadratic loss functions and Part ii) of Lemma 3, the receiver's best reply if sender types in the interval $\left[a_{i-1}^n, a_i^n\right)$ pool is to choose

$$y\left(a_{i-1}^n, a_i^n\right) = c \cdot \mu_i^n \quad \text{ for } i = 1, \ldots, n \tag{19}$$

and $y(a_n^n, \infty) = c \cdot \mu_{n+1}^n$ if sender types with $\theta \geq a_n^n$ pool. Hence, a Class I equilibrium that induces $2(n+1)$ actions by the receiver is completely characterized by the indifference conditions of the marginal types $a_1^n, \ldots, a_n^n$:

$$a_i^n - c \cdot \mu_i^n = c \cdot \mu_{i+1}^n - a_i^n, \qquad \text{for } i = 1, \ldots, n. \tag{20}$$

By symmetry, this system of equations also characterizes the marginal types below the prior mean.

A Class II equilibrium is characterized by the same set of indifference conditions, (20). However, in this case conditions (18) and (19) apply only for $i = 2, \ldots, n$, while we let $\mu_1^n \equiv \mathbb{E}\left[\tilde{\theta}\middle|\, \tilde{\theta} \in \left[a_{-1}^n, a_1^n\right)\right] = 0$ and $y\left(a_{-1}^n, a_1^n\right) = c \cdot \mu_1^n = 0$.

Equation (20) defines a nonlinear difference equation for any given n. The qualitative features of the equilibrium set - in particular, the maximum number n such that a solution to (20) exists - depend crucially on the magnitude of the regression coefficient, c.

For $c \leq 1$, there is no bound on the number of induced actions (see Proposition 1). One way to understand an equilibrium is as a combination of a "forward solution" and a "closure condition". A forward solution starting at zero takes the length of the first interval as given, say x, and computes the "next" threshold, $a_2(x)$, as a function of the preceding two, and likewise for the following thresholds. The closure condition for an equilibrium with n positive thresholds requires that x is such that type $a_n^n(x)$ is exactly indifferent between pooling downwards and upwards. Using this construction, we prove the existence of an equilibrium for arbitrary n and show that the limit as n goes to infinity is an equilibrium. As more and more distinct receiver actions are induced, the length of the interval(s) that are closest to the agreement point, $\theta = 0$, must go to zero. The reason is that the length of all intervals is increasing in the distance from the agreement point to the first threshold. Moreover, the level of the last threshold is bounded from above.

The case $c > 1$ is different in very essential ways, as shown in Proposition 2. Again, any equilibrium must feature intervals that increase in length the farther they are located from the agreement point. This is intuitive, since the extent of disagreement increases in $|\theta|$. However, the forward solution only has this feature if the length of the first interval is bounded away from zero and n is bounded.

Proof of Proposition 1. Before proving Parts i) to iii) of the proposition by a sequence of claims, we make the equilibrium conditions for the Laplace in Claim 0) explicit. Recall the definition of the g function from Lemma A1.

Claim 0) A Class I equilibrium is a set of marginal types that satisfies the conditions

$$cg\left(a_i^n - a_{i-1}^n\right) = 2\frac{c}{\lambda} + c\left(a_{i+1}^n - a_i^n\right) - cg\left(a_{i+1}^n - a_i^n\right) + 2(c-1)\,a_i^n. \tag{21}$$

for $i = 1, \ldots, n-1$ and

$$cg\left(a_n^n - a_{n-1}^n\right) = 2\frac{c}{\lambda} + 2(c-1)\,a_n^n, \tag{22}$$

where $a_0^n = 0$. A Class II equilibrium satisfies

$$a_1^n = \frac{c}{\lambda} + c\left(a_2^n - a_1^n\right) - cg\left(a_2^n - a_1^n\right) - (1-c)\,a_1^n, \tag{23}$$

and in addition (21) for $i = 2, \ldots, n-1$, and (22).

Proof: Recall the proof of Lemma A1; we write the conditional mean for the Laplace $\mu_{i+1} = \mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \in [\theta_i, \theta_{i+1}]\right] = \mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq 0\right] + \theta_{i+1} - g(\theta_{i+1} - \theta_i)$, where $0 \leq \theta_i < \theta_{i+1}$, $g(q) = \frac{q}{1-\exp(-\lambda q)}$, and $\frac{1}{\lambda} = \mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq 0\right]$. In combination with the sender's indifference conditions (20), $a_i^n - c \cdot \mu_i^n = c \cdot \mu_{i+1}^n - a_i^n$, this implies the claim.

Part i) We use the combination of forward solution and condition (22) to show equilibrium existence. Formally, for an initial value x a forward solution $a_2(x)$ is defined as the value of a_2 that solves

$$cg\left(x\right)-\frac{c}{\lambda}+cg\left(a_2-x\right)-c\left(a_2-x\right)-\frac{c}{\lambda}-2\left(c-1\right)x=0.$$

The forward solution for $a_i(x)$ for $i \geq 3$ is defined recursively by

$$cg\left(a_{i-1}\left(x\right)-a_{i-2}\left(x\right)\right)-\frac{c}{\lambda}-\frac{c}{\lambda}-c\left(a_i-a_{i-1}\left(x\right)\right)+cg\left(a_i-a_{i-1}\left(x\right)\right)-2\left(c-1\right)a_{i-1}\left(x\right)=0.$$

We prove existence of Class I equilibria first. The argument is structured as follows. In Claims i.1) to i.3), we investigate the forward solution, addressing first properties of solutions (Claims i.1) and i.2)) and then existence (Claim i.3)). In Claim i.4), we address existence and uniqueness of a fixed point. Finally, in Claim i.5) the extension to the case of Class II equilibria is presented.

Claim i.1) The forward solution features increasing intervals,

$$a_{i+1}^n-a_i^n>a_i^n-a_{i-1}^n.$$

Proof: Consider

$$\Delta\left(a_2-x,x\right)\equiv cg\left(x\right)-\frac{c}{\lambda}+cg\left(a_2-x\right)-c\left(a_2-x\right)-\frac{c}{\lambda}-2\left(c-1\right)x.$$

The forward solution for a_2 given x is the value of a_2 that solves $\Delta\left(a_2-x,x\right)=0$. Take $a_2-x=x$, then

$$\Delta\left(x,x\right)=2\left(cg\left(x\right)-\frac{c}{\lambda}\right)-cx-2\left(c-1\right)x.$$

Since $\lim_{x\to 0}g(x)=\lim_{x\to 0}\frac{x}{1-e^{-\lambda x}}=\frac{1}{\lambda}$, we have $\lim_{x\to 0}\Delta\left(x,x\right)=0$. Moreover,

$$\frac{\partial}{\partial x}\Delta\left(x,x\right)=2cg'\left(x\right)-c-2\left(c-1\right),$$

$$\frac{\partial^2}{\partial x^2}\Delta\left(x,x\right)=2cg''\left(x\right)>0.$$

Observe that

$$\lim_{x\to 0}\frac{\partial}{\partial x}\Delta\left(x,x\right)=c-c-2\left(c-1\right)=-2\left(c-1\right)\geq 0$$

with a strict inequality if $c<1$. It follows that $\Delta(x,x)>0$ for all $x>0$. Since for all finite a_2

$$\frac{\partial}{\partial a_2}\Delta(a_2-x,x)=cg'(a_2-x)-c<0,$$

the forward solution for a_2 given x, satisfies $a_2-x>x$.

Consider the forward solution for a_i

$$cg(a_{i-1}(x)-a_{i-2}(x))-\frac{c}{\lambda}-\frac{c}{\lambda}-c(a_i-a_{i-1}(x))+cg(a_i-a_{i-1}(x))-2(c-1)a_{i-1}(x)=0.$$

Let $z=a_i(x)-a_{i-1}(x)=a_{i-1}(x)-a_{i-2}(x)$. Define

$$\Delta(z,z;a_{i-1})\equiv 2\left(cg(z)-\frac{c}{\lambda}\right)-cz-2(c-1)a_{i-1}(x).$$

Then

$$\lim_{z\to 0}\Delta(z,z;a_{i-1})=-2(c-1)a_{i-1}(x)>0$$

for any $a_{i-1}(x)>0$. Since $2cg'(z)-c\geq 0$ with strict inequality for $z>0$, we have $\Delta(z,z;a_{i-1})>0$ for all $z>0$. Since the left-hand side of the equation defining the forward solution is decreasing in a_i, for any $a_{i-2}(x), a_{i-1}(x)>0$ the solution of the forward equation must satisfy $a_i(x)-a_{i-1}(x)>a_{i-1}(x)-a_{i-2}(x)$.

Claim i.2) The forward solution $a_2(x)$ satisfies $\lim_{x\to 0}(a_2(x)-x)=0$ and $\frac{da_2}{dx}>1$, implying that $a_2(x)-x$ is increasing in x. Moreover, the forward solutions $a_i(x)-a_{i-1}(x)$ for $i=3,\ldots,n$ all satisfy $\lim_{x\to 0}(a_i(x)-a_{i-1}(x))=0$ and $\frac{da_{i+1}(x)}{dx}>\frac{da_i(x)}{dx}$, implying that $a_i(x)-a_{i-1}(x)$ is increasing in x.

Proof: Consider the equation determining the forward solution for $a_2(x)$, that is condition (21) for $i=1$, $a_0=0$, and $a_1=x$; formally, $a_2(x)$ is the value of a_2 that solves

$$cg(x)-\frac{c}{\lambda}=\frac{c}{\lambda}+c(a_2-x)-cg(a_2-x)+2(c-1)x.$$

In the limit as $x\to 0$, we obtain $\lim_{x\to 0}a_2(x)=0$ from the fact that $\lim_{q\to 0}g(q)=\frac{1}{\lambda}$ (Lemma A1). Totally differentiating, we obtain

$$(cg'(x)+c(1-g'(a_2(x)-x))-2(c-1))\,dx-c(1-g'(a_2(x)-x))\,da_2=0,$$

so that

$$\frac{da_2}{dx}=\frac{(cg'(x)+c(1-g'(a_2(x)-x))-2(c-1))}{c(1-g'(a_2(x)-x))}>0.$$

Moreover, $\frac{da_2}{dx} > 1$ by the fact that $cg'(x) - 2(c-1) > 0$ for $c \leq 1$. Hence, we have that $\lim_{x\to 0}(a_2(x) - x) = 0$ and $\frac{d}{dx}(a_2(x) - x) > 0$.

For $i = 2$, consider the forward equation for $a_3(x)$. Formally, $a_3(x)$ is the value of a_3 that solves

$$cg(a_2(x) - x) - \frac{c}{\lambda} = \frac{c}{\lambda} + c(a_3 - a_2(x)) - cg(a_3 - a_2(x)) + 2(c-1)a_2(x).$$

Since $\lim_{x\to 0} a_2(x) = 0$ and $\lim_{x\to 0}(a_2(x) - x) = 0$, we also have $\lim_{x\to 0} a_3(x) = 0$ and $\lim_{x\to 0}(a_3(x) - a_2(x)) = 0$. Totally differentiating, we obtain

$$\frac{da_3(x)}{da_2(x)} = \frac{cg'(a_2(x) - x)\left(\frac{da_2(x)}{dx} - 1\right) + (c(1 - g'(a_3(x) - a_2(x))) - 2(c-1))\frac{da_2(x)}{dx}}{c(1 - g'(a_3(x) - a_2(x)))\frac{da_2(x)}{dx}}.$$

Since $\frac{da_2(x)}{dx} > 1$, we have $\frac{da_3(x)}{da_2(x)} > 0$, and moreover $\frac{da_3(x)}{da_2(x)} > 1$. Finally,

$$\frac{da_3(x)}{dx} = \frac{da_3(x)}{da_2(x)}\frac{da_2(x)}{dx} > \frac{da_2(x)}{dx}.$$

Hence, we have that $\lim_{x\to 0}(a_3(x) - a_2(x)) = 0$ and $\frac{d}{dx}(a_3(x) - a_2(x)) > 0$.

Suppose as an inductive hypothesis that the forward solutions up to and including $a_i(x)$ have the properties that $\lim_{x\to 0}(a_i(x) - a_{i-1}(x)) = 0$, $\lim_{x\to 0} a_i(x) = 0$, and $\frac{da_i(x)}{da_{i-1}(x)} > 1$, so that $a_i(x) - a_{i-1}(x)$ increasing in x. Consider the equation for a_{i+1} with solution $a_{i+1}(x)$,

$$cg(a_i(x) - a_{i-1}(x)) - \frac{c}{\lambda} = \frac{c}{\lambda} + c(a_{i+1} - a_i(x)) - cg(a_{i+1} - a_i(x)) + 2(c-1)a_i(x).$$

The inductive assumptions for $a_i(x)$ and $a_{i-1}(x)$ imply that $\lim_{x\to 0}(a_{i+1}(x) - a_i(x)) = 0$, so that $\lim_{x\to 0} a_{i+1}(x) = 0$. Totally differentiating, we obtain

$$\frac{da_{i+1}(x)}{da_i(x)} = \frac{cg'(a_i(x) - a_{i-1}(x))\left(\frac{da_i(x)}{da_{i-1}(x)} - 1\right) + (c(1 - g'(a_{i+1}(x) - a_i(x))) - 2(c-1))\frac{da_i(x)}{da_{i-1}(x)}}{c(1 - g'(a_{i+1}(x) - a_i(x)))\frac{da_i(x)}{da_{i-1}(x)}}.$$

The assumption $\frac{da_i(x)}{da_{i-1}(x)} > 1$ implies that $\frac{da_{i+1}(x)}{da_i(x)} > 1$. We can conclude that, $a_{i+1}(x) - a_i(x)$ is increasing in x for all $i = 1, \ldots, n$.

Claim i.3) For each $i = 2, \ldots, n$, there is x_i^* such that a unique, finite forward solution for $a_i(x)$ exists for all $x \in [0, x_i^*)$. In the limit as $x \to x_i^*$, $\lim_{x \to x_i^*} a_i(x) = \infty$. Furthermore, $x_{i+1}^* < x_i^*$.

Proof: The forward solution $a_2(x)$ solves

$$cg(x) - \frac{c}{\lambda} = \frac{c}{\lambda} + c(a_2 - x) - cg(a_2 - x) + 2(c-1)x.$$

The left-hand side satisfies $\lim_{x \to 0} cg(x) - \frac{c}{\lambda} = 0$ and increases in x. The right-hand side satisfies

$$\lim_{a_2 \to x} \left\{ \frac{c}{\lambda} + c(a_2 - x) - cg(a_2 - x) + 2(c-1)x \right\} = 2(c-1)x \leq 0,$$

where the inequality is strict for $c < 1$ and $x > 0$. Moreover, the right-hand side is increasing and concave in a_2 with limiting value

$$\lim_{a_2 \to \infty} \left\{ \frac{c}{\lambda} + c(a_2 - x) - cg(a_2 - x) + 2(c-1)x \right\} = \frac{c}{\lambda} + 2(c-1)x.$$

Hence, there exists a finite forward solution $a_2(x)$ if and only if

$$cg(x) - \frac{c}{\lambda} < \frac{c}{\lambda} + 2(c-1)x. \tag{24}$$

Since $cg(x) - \frac{c}{\lambda}$ is nonnegative and increasing in x and $\frac{c}{\lambda} + 2(c-1)x$ is positive for $x = 0$ and nonincreasing in x, there exists a unique value x_2^* such that (24) is satisfied with equality. Hence, a finite forward solution $a_2(x)$ exists for all $x \in [0, x_2^*)$. In the limit as $x \to x_2^*$, we have $\lim_{x \to x_2^*} a_2(x) = \infty$.

Consider now the forward solution for $a_i(x)$ for $i = 3, \ldots, n$. The forward solution a_i solves

$$cg(a_{i-1}(x) - a_{i-2}(x)) - \frac{c}{\lambda} = \frac{c}{\lambda} + c(a_i - a_{i-1}(x)) - cg(a_i - a_{i-1}(x)) + 2(c-1)a_{i-1}(x).$$

The left-hand side satisfies $\lim_{x \to 0} cg(a_{i-1}(x) - a_{i-2}(x)) - \frac{c}{\lambda} = 0$ and is increasing in x. The right-hand side satisfies

$$\lim_{a_i \to a_{i-1}(x)} \frac{c}{\lambda} + c(a_i - a_{i-1}(x)) - cg(a_i - a_{i-1}(x)) + 2(c-1)a_{i-1}(x) = 2(c-1)a_{i-1}(x) \leq 0,$$

with strict inequality for $x > 0$ and $c < 1$. Moreover, the right-hand side is increasing and concave in a_{i-1} with limiting value

$$\lim_{a_i \to \infty} \frac{c}{\lambda} + c(a_i - a_{i-1}(x)) - cg(a_i - a_{i-1}(x)) + 2(c-1)a_{i-1}(x) = \frac{c}{\lambda} + 2(c-1)a_{i-1}(x).$$

Therefore, a unique solution for a_i exists if and only if

$$cg\left(a_{i-1}\left(x\right)-a_{i-2}\left(x\right)\right)-\frac{c}{\lambda}<\frac{c}{\lambda}+2\left(c-1\right)a_{i-1}\left(x\right). \tag{25}$$

Given the derived properties of the forward solution, we have that $cg\left(a_{i-1}\left(x\right)-a_{i-2}\left(x\right)\right)-\frac{c}{\lambda}$ is nonnegative and increasing in x and $\frac{c}{\lambda}+2\left(c-1\right)a_{i-1}\left(x\right)$ is positive for $x=0$ and nonincreasing in x. Therefore, there exists a unique value $x=x_i^*$ such that (25) is satisfied with equality. Hence a finite forward solution $a_i\left(x\right)$ exists for all $x\in\left[0,x_i^*\right)$. In the limit as $x\to x_i^*$, we have $\lim_{x\to x_i^*}a_i\left(x\right)=\infty$.

Define

$$A_i\left(x\right)\equiv cg\left(a_{i-1}\left(x\right)-a_{i-2}\left(x\right)\right)-\frac{c}{\lambda}-\left(\frac{c}{\lambda}+2\left(c-1\right)a_{i-1}\left(x\right)\right),$$

and analogously $A_{i+1}\left(x\right)$. Since $a_i\left(x\right)-a_{i-1}\left(x\right)>a_{i-1}\left(x\right)-a_{i-2}\left(x\right)$ and $a_i\left(x\right)>a_{i-1}\left(x\right)$ for all x, we have $A_{i+1}\left(x\right)>A_i\left(x\right)$. Moreover, both $A_{i+1}\left(x\right)$ and $A_i\left(x\right)$ are increasing in x. Letting x_i^* and x_{i+1}^* denote the values of x such that $A_i\left(x_i^*\right)=0$ and $A_{i+1}\left(x_{i+1}^*\right)=0$, we have $x_{i+1}^*<x_i^*$.

Claim i.4) For any n there exists a unique value of $\tilde{x}_n$ such that condition (22) holds for a_{n-1} and a_n defined as solutions to the forward equation.

Proof: Take the forward solution for $a_i\left(x\right)$ for $i=2,\ldots,n$ and consider the difference between the left-hand and the right-hand side of the condition (22), which we define as

$$\Delta_n\left(x\right)\equiv cg\left(a_n\left(x\right)-a_{n-1}\left(x\right)\right)-\frac{c}{\lambda}-\frac{c}{\lambda}-2\left(c-1\right)a_n\left(x\right).$$

Differentiating $\Delta_n\left(x\right)$ with respect to x we get

$$\begin{aligned}\frac{d\Delta_n\left(x\right)}{dx}&=cg'\left(a_n\left(x\right)-a_{n-1}\left(x\right)\right)\left(\frac{da_n\left(x\right)}{dx}-\frac{da_{n-1}\left(x\right)}{dx}\right)-2\left(c-1\right)\frac{da_n\left(x\right)}{dx}\\&=cg'\left(a_n\left(x\right)-a_{n-1}\left(x\right)\right)\left(\frac{da_n\left(x\right)}{da_{n-1}\left(x\right)}-1\right)\frac{da_{n-1}\left(x\right)}{dx}-2\left(c-1\right)\frac{da_n\left(x\right)}{dx}.\end{aligned}$$

Since $\frac{da_n(x)}{da_{n-1}(x)}>1$, $\Delta_n\left(x\right)$ is strictly monotonic in x. This implies that there is at most one value of x that solves the fixed point equation

$$\Delta_n\left(x\right)=0.$$

Let $\tilde{x}_n$ denote the value of x that satisfies $\Delta_n(\tilde{x}_n) = 0$ for given n, if it exists. To show that a fixed point exists, we need to show that $\tilde{x}_n$ is such that the forward solution for $a_n(\tilde{x}_n)$ exists. To see this is true, note simply that $\Delta_n(\tilde{x}_n) = 0$ for $\tilde{x}_n = x^*_{n+1}$. That is, $\tilde{x}_n$ is the value of x, such that forward solutions for $a_i(x)$ for $i = 2, \ldots, n+1$ exist and are finite for all $x \in [0, \tilde{x}_n)$. Since $x^*_{n+1} < x^*_n$, the forward solutions for $i = 2, \ldots, n$ exist and are finite at $x = \tilde{x}_n$. Hence, this completes the proof that there exists exactly one fixed point, $\tilde{x}_n$. So in equilibrium $a^n_1 = \tilde{x}_n$.

Claim i.5) For all n, there exists a unique Class II equilibrium.

Proof: A Class II equilibrium is characterized by

$$a_1 = \frac{c}{\lambda} + c(a_2 - a_1) - cg(a_2 - a_1) - (1-c)a_1$$

in addition to condition (21) for $i = 2, \ldots, n-1$ and condition (22).

To construct a forward solution, take an arbitrary initial value x for the first threshold as given and compute $a_2(x)$ as the solution to

$$x = \frac{c}{\lambda} + c(a_2(x) - x) - cg(a_2(x) - x) - (1-c)x.$$

We have $\lim_{a_2 \to x}\left(\frac{c}{\lambda} + c(a_2 - x) - cg(a_2 - x) - (1-c)x\right) = -(1-c)x$ and $\lim_{a_2 \to \infty}\left(\frac{c}{\lambda} + c(a_2 - x) - cg(a_2 - x) - (1-c)x\right) = \frac{c}{\lambda} - (1-c)x$. Hence, there is a unique finite forward solution $a_2(x)$ if and only if $x < \frac{c}{\lambda} - (1-c)x$, or equivalently $(2-c)x < \frac{c}{\lambda}$. Since $c \leq 1$, this is equivalent to $x < \frac{c}{\lambda(2-c)}$. We have $\lim_{x \to \frac{c}{\lambda(2-c)}} a_2(x) = \infty$. Likewise, for $x = 0$ we have $a_2(x)|_{x=0} = 0$.

Differentiating totally, we find

$$0 = (c(1 - g'(a_2(x) - x)))\,da_2 - (c(1 - g'(a_2(x) - x)) + (2-c))\,dx,$$

and so

$$\frac{da_2}{dx} = \frac{(c(1 - g'(a_2(x) - x)) + (2-c))}{(c(1 - g'(a_2(x) - x)))} > 1.$$

Since the forward equations for $a_i(x)$ for $i = 3, \ldots, n$ as well as the fixed point condition (22) are unchanged, all the remaining arguments are unchanged.

Part ii) Before analyzing the limits of equilibrium thresholds as $n \to \infty$ in Claims ii.2) and ii.3), claim ii.1) establishes some important monotonicity properties of equilibrium thresholds.

Claim ii.1) The sequence $(a_1^n)_n$ is monotone decreasing, while the sequence $(a_n^n)_n$ is monotone increasing. Moreover, equilibrium thresholds are nested,

$$a_1^{n+1} < a_1^n < a_2^{n+1} < \cdots a_n^{n+1} < a_n^n < a_{n+1}^{n+1} \qquad \forall n. \tag{26}$$

Proof: Using the notation from Part i), since $a_1^n = \tilde{x}_n = x_{n+1}^*$ and $a_1^{n+1} = \tilde{x}_{n+1} = x_{n+2}^*$ it follows immediately from Part i) that $a_i^{n+1} < a_i^n$ for $i = 1, \ldots, n$. In particular, the argument follows from the fact that the solution of the forward equation is monotonic in the initial condition, x. Hence, it suffices to prove that $a_{i+1}^{n+1} > a_i^n$ for $i = 1, \ldots, n$.

We start with two preliminary observations. Firstly, the "next" solution of the forward equation, $a_{i+1}^k(x)$ for $i = 1, \ldots, k-1$, $k = n, n+1$ is monotonic in $a_i^k(x)$, and the length of the previous interval, $a_i^k(x) - a_{i-1}^k(x)$. To see this, note that the forward equations for a_2^k, a_3^k, and a_{i+1}^k, for $i = 3, \ldots, k-1$ and $k = n, n+1$, satisfy:

$$cg(x) - \frac{c}{\lambda} = \frac{c}{\lambda} + c\left(a_2^k - x\right) - cg\left(a_2^k - x\right) + 2(c-1)x,$$

$$cg\left(a_2^k(x) - x\right) - \frac{c}{\lambda} = \frac{c}{\lambda} + c\left(a_3^k - a_2^k(x)\right) - cg\left(a_3^k - a_2^k(x)\right) + 2(c-1)a_2^k(x),$$

and

$$cg\left(a_i^k(x) - a_{i-1}^k(x)\right) - \frac{c}{\lambda} = \frac{c}{\lambda} + c\left(a_{i+1}^k - a_i^k(x)\right) - cg\left(a_{i+1}^k - a_i^k(x)\right) + 2(c-1)a_i^k(x).$$

The conclusion follows from the fact that $a_i^k(x)$ decreases the value of the right-hand side and increases the value of the left-hand side. Moreover, the left-hand side is increasing in $a_i^k(x) - a_{i-1}^k(x)$.

Secondly, it is impossible that $a_{n+1}^{n+1}(\tilde{x}_{n+1}) < a_n^n(\tilde{x}_n)$ and $a_{n+1}^{n+1}(\tilde{x}_{n+1}) - a_n^{n+1}(\tilde{x}_{n+1}) < a_n^n(\tilde{x}_n) - a_{n-1}^n(\tilde{x}_n)$. If these conditions would hold, then one of the fixed point conditions,

$$0 = cg\left(a_n^n(\tilde{x}_n) - a_{n-1}^n(\tilde{x}_n)\right) - \frac{c}{\lambda} - \frac{c}{\lambda} - 2(c-1)a_n^n(\tilde{x}_n)$$

and

$$0 = cg\left(a_{n+1}^{n+1}(\tilde{x}_{n+1}) - a_n^{n+1}(\tilde{x}_{n+1})\right) - \frac{c}{\lambda} - \frac{c}{\lambda} - 2(c-1)a_{n+1}^{n+1}(\tilde{x}_{n+1})$$

would necessarily be violated.

We now show that $a_{j+1}^{n+1} > a_j^n$ for all $j \leq n$. Suppose that this were not true and let the property be violated for the first time at $j = l$.

Suppose $a_{j+1}^{n+1}(\tilde{x}_{n+1}) > a_j^n(\tilde{x}_n)$ for all $j = 1, \dots, l-1$ and $a_{l+1}^{n+1}(\tilde{x}_{n+1}) < a_l^n(\tilde{x}_n)$. Taken together, these inequalities immediately imply that $a_{l+1}^{n+1}(\tilde{x}_{n+1}) - a_l^{n+1}(\tilde{x}_{n+1}) < a_l^n(\tilde{x}_n) - a_{l-1}^n(\tilde{x}_n)$. In turn, the monotonicity property of the next forward solution implies then that $a_{l+2}^{n+1}(\tilde{x}_{n+1}) < a_{l+1}^n(\tilde{x}_n)$.

It also follows then that $a_{l+2}^{n+1}(\tilde{x}_{n+1}) - a_{l+1}^{n+1}(\tilde{x}_{n+1}) < a_{l+1}^n(\tilde{x}_n) - a_l^n(\tilde{x}_n)$. To see this, suppose instead that $a_{l+2}^{n+1}(\tilde{x}_{n+1}) - a_{l+1}^{n+1}(\tilde{x}_{n+1}) \geq a_{l+1}^n(\tilde{x}_n) - a_l^n(\tilde{x}_n)$ or equivalently that $a_{l+2}^{n+1}(\tilde{x}_{n+1}) \geq a_{l+1}^n(\tilde{x}_n) + \left(a_{l+1}^{n+1}(\tilde{x}_{n+1}) - a_l^n(\tilde{x}_n)\right)$. However, this is impossible since both $a_{l+2}^{n+1}(\tilde{x}_{n+1}) < a_{l+1}^n(\tilde{x}_n)$ and $a_{l+1}^{n+1}(\tilde{x}_{n+1}) < a_l^n(\tilde{x}_n)$. Hence, the claim follows.

However, if $a_{l+2}^{n+1}(\tilde{x}_{n+1}) < a_{l+1}^n(\tilde{x}_n)$ and $a_{l+2}^{n+1}(\tilde{x}_{n+1}) - a_{l+1}^{n+1}(\tilde{x}_{n+1}) < a_{l+1}^n(\tilde{x}_n) - a_l^n(\tilde{x}_n)$, then $a_{l+3}^{n+1}(\tilde{x}_{n+1}) < a_{l+2}^n(\tilde{x}_n)$ and so forth. Hence, we would have $a_{j+1}^{n+1}(\tilde{x}_{n+1}) < a_j^n(\tilde{x}_n)$ and $a_{j+1}^{n+1}(\tilde{x}_{n+1}) - a_j^{n+1}(\tilde{x}_{n+1}) < a_j^n(\tilde{x}_n) - a_{j-1}^n(\tilde{x}_n)$ for all $j \geq l$ and in particular for $j = n$, leading to a violation of one of the fixed point conditions.

The same argument can be given for a Class II equilibrium. This is omitted.

Claim ii.2) Equilibrium thresholds converge for $n \to \infty$.

Proof: We know from Part i) that $(a_1^n)_n$ is monotone decreasing in n. Since the sequence is bounded by zero it must converge. Similarly, by Claim ii.1) the sequence $(a_n^n)_n$ is monotone increasing in n. The fixed point condition, (22), implies that it is bounded by $\frac{c}{1-c}\frac{1}{2\lambda}$, hence it converges. Since equilibrium thresholds are nested (cf. (26)) all sequences of thresholds must converge for $n \to \infty$.

Claim ii.3) The limit of the sequences of thresholds and actions is an equilibrium.

Proof: The limit is an equilibrium if $\lim_{n\to\infty} c\mu_i^n \leq \lim_{n\to\infty} a_i^n \leq \lim_{n\to\infty} c\mu_{i+1}^n$. Therefore, we have to show that equilibrium thresholds remain ordered in the limit, $\lim_{n\to\infty} a_i^n < \lim_{n\to\infty} a_{i+1}^n$. For all finite n, thresholds are ordered in equilibrium, $a_i^n < a_{i+1}^n$, since they are ordered for any forward equation. By Claim ii.2) equilibrium thresholds converge; denote the limits by $\overline{a}_i = \lim_{n\to\infty} a_i^n$ for all i. By convergence, for any ε there is a N such that for all $n > N$: $a_i^n \geq \overline{a}_i - \frac{\varepsilon}{2}$ and $a_{i+1}^n \leq \overline{a}_{i+1} + \frac{\varepsilon}{2}$. Suppose for contradiction that $\overline{a}_i \geq \overline{a}_{i+1} + \delta$ for some $\delta > 0$; this implies

$$a_i^n \geq \overline{a}_i - \frac{\varepsilon}{2} \geq \overline{a}_{i+1} + \delta - \frac{\varepsilon}{2} \geq a_{i+1}^n - \frac{\varepsilon}{2} + \delta - \frac{\varepsilon}{2} > a_{i+1}^n,$$

for all $\varepsilon < \delta$. Hence thresholds remain ordered in the limit and the limit is an equilibrium.

Part iii) In the limit as $n \to \infty$, we have $\lim_{n\to\infty} \tilde{x}_n = 0$.

Proof: The fixed point argument in the proof of Part i) implies that the sequence $(\tilde{x}_n)_n$ is monotone decreasing. Since it is bounded from below by zero it converges. As before, we use the notation $a_1^n = \tilde{x}_n = x_{n+1}^*$.

Recall that $x_{n+1}^* < x_n^*$ and that the forward solution for $a_n(x)$ exists for $x \leq x_n^*$, where x_n^* satisfies

$$cg\left(a_{n-1}\left(x_n^*\right) - a_{n-2}\left(x_n^*\right)\right) - \frac{c}{\lambda} = \frac{c}{\lambda} + 2\left(c-1\right)a_{n-1}\left(x_n^*\right).$$

Monotonicity of the forward solutions, $a_k(x) > a_{k-1}(x)$, and increasing length of the intervals, $a_k(x) - a_{k-1}(x) > a_{k-1}(x) - a_{k-2}(x)$, imply for $c \leq 1$ the following. For any $x > 0$ there is a k such that

$$cg\left(a_{k-1}(x) - a_{k-2}(x)\right) - \frac{c}{\lambda} \leq \frac{c}{\lambda} + 2\left(c-1\right)a_{k-1}(x)$$

and

$$cg\left(a_k(x) - a_{k-1}(x)\right) - \frac{c}{\lambda} > \frac{c}{\lambda} + 2\left(c-1\right)a_k(x).$$

Therefore, for a fixed length x of the first interval, the forward equation has a solution only for a finite number of steps. Hence, in an infinite equilibrium we have $\lim_{n\to\infty} x_n^* = 0$, implying that the length of the first interval goes to zero, $\lim_{n\to\infty} \tilde{x}_n = 0$.

The proof for the case of a Class II equilibrium is virtually the same and hence omitted.

■

Proof of Proposition 2. Before proving that actions are bounded away from zero for Class I equilibria in Claim 1) and for Class II equilibria in Claim 2), Claim 0) shows a monotonicity condition. Finally, Claim 3) proves finiteness of equilibria. Recall the definition of the g function from Lemma A1.

Claim 0) If a Class I equilibrium exists, it features increasing intervals for all $i = 1, \ldots, n-1$,

$$a_{i+1}^n - a_i^n > a_i^n - a_{i-1}^n; \tag{27}$$

If a Class II equilibrium exists, it always shares this feature for $i = 2, \ldots, n-1$.

Proof: Consider first Class I equilibria for given $n \geq 2$. For $n < 2$, the question is meaningless. Define

$$z_i^n \equiv a_i^n - a_{i-1}^n \text{ for } i = 1, \ldots, n.$$

For $c \geq 2$, no equilibrium of the considered kind exists this is shown in Claim 1) below. Now take $c \in (1, 2)$. For $n = 2$, the indifference condition of type a_2^n and a_1^n are, in that order,

$$cg(z_2^n) = 2\frac{c}{\lambda} + 2(c-1)(z_1^n + z_2^n),$$

and

$$cg(z_1^n) = 2\frac{c}{\lambda} + c(z_2^n - g(z_2^n)) + 2(c-1)z_1^n.$$

Substituting the former condition into the latter and simplifying, we have

$$z_2^n = \frac{c}{2-c} g(z_1^n).$$

Since $g(z) > z$ and $\frac{c}{2-c} > 1$, we have $z_2^n > z_1^n$.

For $n \geq 3$, the indifference conditions of types a_n^n and a_{n-1}^n, respectively, can be written as

$$cg(z_n^n) = 2\frac{c}{\lambda} + 2(c-1)\sum_{j=1}^{n} z_j^n, \text{ and}$$

$$cg(z_{n-1}^n) = 2\frac{c}{\lambda} + c(z_n^n - g(z_n^n)) + 2(c-1)\sum_{j=1}^{n-1} z_j^n.$$

Adding $-2\frac{c}{\lambda} - 2(c-1)\sum\limits_{j=1}^{n} z_j^n + cg(z_n^n) = 0$ to the indifference condition of type a_{n-1}^n, we get

$$cg(z_{n-1}^n) = (2-c)z_n^n,$$

and hence

$$z_n^n = \frac{c}{2-c} g(z_{n-1}^n).$$

Since $\frac{c}{2-c} > 1$ for $c > 1$ and $g(z) > z$, this implies that $z_n^n > z_{n-1}^n$. By Lemma A1, we therefore have $g(z_n^n) - z_n^n < g(z_{n-1}^n) - z_{n-1}^n$. Hence, we also have

$$\begin{aligned} cg(z_{n-1}^n) &= 2\frac{c}{\lambda} + c(z_n^n - g(z_n^n)) + 2(c-1)\sum_{j=1}^{n-2} z_j^n + 2(c-1)z_{n-1}^n \\ &> 2\frac{c}{\lambda} + c(z_{n-1}^n - g(z_{n-1}^n)) + 2(c-1)\sum_{j=1}^{n-2} z_j^n = cg(z_{n-2}^n), \end{aligned}$$

where the first equality is the indifference condition of type a_{n-1}^n and the second equality the one for type a_{n-2}^n. Hence, we can conclude that $z_{n-2}^n < z_{n-1}^n$.

Likewise, suppose as an inductive hypothesis that $z_i^n < z_{i+1}^n$. Consider the indifference conditions of types a_i^n and a_{i-1}^n, respectively,

$$cg\left(z_i^n\right) = 2\frac{c}{\lambda} + c\left(z_{i+1}^n - g\left(z_{i+1}^n\right)\right) + 2\left(c-1\right)\sum_{j=1}^{i-1} z_j^n + 2\left(c-1\right)z_i^n$$

and

$$cg\left(z_{i-1}^n\right) = 2\frac{c}{\lambda} + c\left(z_i^n - g\left(z_i^n\right)\right) + 2\left(c-1\right)\sum_{j=1}^{i-1} z_j^n.$$

By Lemma A1, the value of the right-hand side of the former equation exceeds the value of the right-hand side of the latter equation, and hence we have shown that $z_{i-1}^n < z_i^n$.

Class II equilibria have the same indifference conditions for the marginal types a_i^n for $i = 2, \ldots, n-1$. Hence, the same argument applies.

Note that we do not invoke symmetry of the equilibrium in any way. Therefore, except for notation, the same argument applies also to asymmetric equilibria.

Claim 1) In any Class I equilibrium the receiver's induced actions are bounded away from zero.

Proof: Any equilibrium must be a solution to the forward equation. This requires that the solution of the forward equation exists and features increasing intervals. This is possible only if the length of the first interval is bounded away from zero.

The forward equation for a_2 is given by

$$cg\left(x\right) - \frac{c}{\lambda} = \frac{c}{\lambda} + c\left(a_2 - x\right) - cg\left(a_2 - x\right) + 2\left(c-1\right)x. \tag{28}$$

The left-hand side satisfies $\lim_{x\to 0} cg\left(x\right) - \frac{c}{\lambda} = 0$ and is increasing and convex in x, with slope between $\frac{c}{2}$ and c. The right-hand side satisfies

$$\lim_{a_2\to x} \frac{c}{\lambda} + c\left(a_2 - x\right) - cg\left(a_2 - x\right) + 2\left(c-1\right)x = 2\left(c-1\right)x \geq 0,$$

where the inequality is strict for $x > 0$. Moreover, the right-hand side is increasing and concave in a_2 with limit

$$\lim_{a_2\to \infty} \frac{c}{\lambda} + c\left(a_2 - x\right) - cg\left(a_2 - x\right) + 2\left(c-1\right)x = \frac{c}{\lambda} + 2\left(c-1\right)x.$$

Hence, there exists a forward solution $a_2(x)$ if and only if

$$2(c-1)x < cg(x) - \frac{c}{\lambda} < \frac{c}{\lambda} + 2(c-1)x.$$

There are three cases to distinguish: i) $c \in \left(1, \frac{4}{3}\right]$, ii) $c \in \left(\frac{4}{3}, 2\right)$, and iii) $c \geq 2$.

i) For $c \in \left(1, \frac{4}{3}\right]$, there exists a solution $a_2(x)$ for $x < \overline{x}$ where $\overline{x}$ is the unique value of x that satisfies $cg(\overline{x}) - \frac{c}{\lambda} = \frac{c}{\lambda} + 2(c-1)\overline{x}$. To see this, note that we have $2(c-1) \leq \frac{c}{2}$ and thus $2(c-1) \leq cg'(x)$ for all x, since $g'(x) \geq \frac{1}{2}$ for all x. Therefore, $2(c-1)x < cg(x) - \frac{c}{\lambda}$ is satisfied for all $x > 0$. $cg(x) - \frac{c}{\lambda} < \frac{c}{\lambda} + 2(c-1)x$ holds for x small since $\lim_{x\to 0} cg(x) - \frac{c}{\lambda} = 0 < \frac{c}{\lambda}$. As x increases, the latter inequality eventually ceases to hold, since $c > 2(c-1)$ and thus $cg'(x) > 2(c-1)$ for x sufficiently large, as $g'(x)$ tends to one as $x \to \infty$.

ii) For $c \in \left(\frac{4}{3}, 2\right)$, there exists a solution $a_2(x)$ for $x \in (\underline{x}, \overline{x})$ where $\underline{x}$ is the uniqe value of x that satisfies $2(c-1)\underline{x} < cg(\underline{x}) - \frac{c}{\lambda}$. Note that for $c \in \left(\frac{4}{3}, 2\right)$ we have $\frac{c}{2} < 2(c-1) < c$. Since $\lim_{x\to 0} g'(x) = \frac{1}{2}$, we have $2(c-1)x \geq cg(x) - \frac{c}{\lambda}$ for x positive and small, so that the former inequality is violated for x small. Thus, no solution for $a_2(x)$ exists if x is close to zero.

iii) For $c \geq 2$ we have $2(c-1) \geq c$ and therefore $2(c-1) \geq cg'(x)$ for all x. Hence, $2(c-1)x \geq cg(x) - \frac{c}{\lambda}$ for all x so that no solution exists for $a_2(x)$. This implies that at most two actions can be induced in equilibrium.

Hence, it follows immediately that x is bounded away from zero for $c > \frac{4}{3}$. Consider therefore the case where $c \in \left(1, \frac{4}{3}\right]$. Since equilibrium thresholds have to satisfy the increasing interval property (27), the solution must satisfy $a_2(x) - x > x$ for any equilibrium. We show that this condition is violated for small x. Suppose that $a_2 - x = x$. We define the difference between the right-hand side and the left-hand side of condition (28) at $a_2 - x = x$ as

$$D(x) \equiv \frac{c}{\lambda} + cx - cg(x) + 2(c-1)x + \frac{c}{\lambda} - cg(x).$$

If $D(x)$ is positive (negative), then a_2 needs to decrease (increase) to satisfy the forward equation, since the right-hand side of (28) is increasing in a_2. We have $\lim_{x\to 0} D(x) = 0$. Moreover, the slope of $D(x)$ at $x = 0$ is $D'(x)|_{x=0} = 2(c-1) > 0$. Hence, for x small, we would get $a_2(x) - x < x$, violating the increasing interval property (27). However, since any equilibrium needs to have this property, x is bounded away from zero.

Note that this argument extends to any equilibrium with zero as a threshold, not just symmetric equilibria.

Claim 2) In any Class II equilibrium all but at most one of the receiver's induced actions are bounded away from zero.

Proof: Given x, $a_2(x)$ is the value of a_2 that solves

$$cg(a_2 - x) - \frac{c}{\lambda} = c(a_2 - x) + (c-2)x. \tag{29}$$

Note first that no solution $a_2(x)$ exists for $c \geq 2$. To see this, note that

$$\lim_{a_2 \to x} \frac{c}{\lambda} + c(a_2(x) - x) - cg(a_2(x) - x) - (2-c)x = -(2-c)x \geq 0$$

for any $c \geq 2$ and any $x \geq 0$. Therefore, we consider $1 < c < 2$ from now on. Equation (29) has a solution for $x < \frac{c}{\lambda(2-c)}$, which satisfies $\lim_{x \to 0} a_2(x) = 0$ and moreover,

$$\frac{da_2}{dx} = \frac{c(1 - g'(a_2 - x)) + (2-c)}{c(1 - g'(a_2 - x))} > 1.$$

Rearranging (29) we can write

$$-2\frac{(c-1)}{(c-2)}\left(cg(a_2(x) - x) - \frac{c}{\lambda} - c(a_2(x) - x)\right) = -2(c-1)x.$$

Given x and $a_2(x)$, $a_3(x)$ is the value of a_3 that solves

$$cg(a_2(x) - x) - \frac{c}{\lambda} = \frac{c}{\lambda} + c(a_3 - a_2(x)) - cg(a_3 - a_2(x)) + 2(c-1)a_2(x). \tag{30}$$

Adding up both equations and rearranging, we can conclude that $a_3(x)$ is the value of a_3 that solves

$$0 = \frac{c}{\lambda} + c(a_3 - a_2(x)) - cg(a_3 - a_2(x)) + 4\frac{c-1}{2-c}(a_2(x) - x) - \frac{c}{2-c}\left(cg(a_2(x) - x) - \frac{c}{\lambda}\right). \tag{31}$$

Note that the right-hand side of this equation is increasing in a_3 and that $a_3(x)$ is the unique value that sets the expression equal to zero. We show that the expression is strictly positive for $a_3 - a_2(x) = a_2(x) - x$, to get $a_3(x) - a_2(x) < a_2(x) - x$, in contradiction to the increasing interval property (27).

Note that the right-hand side of (31) depends only on the differences $a_2(x) - x$ and $a_3 - a_2(x)$. Moreover, note that $a_2(x) - x$ goes to zero as x goes to zero. Let $z = a_2(x) - x$ and evaluate the rhs of (31) at $a_3 - a_2(x) = z$. We obtain

$$F(z) \equiv cz + 4\frac{c-1}{2-c}z + \frac{2}{2-c}\left(\frac{c}{\lambda} - cg(z)\right).$$

$F(z)$ is concave in z. In the limit as x and hence z tends to zero, we find

$$F'(z)|_{z=0} = \frac{5c - c^2 - 4}{2 - c},$$

where we use that $g'(z)|_{z=0} = \frac{1}{2}$. For $c \in (1, 2)$, we have $5c - c^2 - 4 > 0$ and we know that $F(z) > 0$ for z small. Since, the right-hand side of (31) is increasing in a_3, to restore equality with zero, a_3 needs to decrease, which would imply that $a_3(x) - a_2(x) < a_2(x) - x$. However, this contradicts the the increasing interval property (27) of any equilibrium. This implies that x must be bounded away from zero.

Consider now an asymmetric interval around zero. Fix an arbitrary point $a_{-1} = -y < 0$ and an arbitrary point $a_1 = x > 0$. We have $\Pr\left(\tilde{\theta} \in (0, x]\right) = \frac{1}{2}\left(1 - e^{-\lambda x}\right)$ and $\Pr\left(\tilde{\theta} \in (-y, 0]\right) = \Pr\left(\tilde{\theta} \in [0, y)\right) = \frac{1}{2}\left(1 - e^{-\lambda y}\right)$. Let $\delta(x, y) \equiv \frac{\left(1 - e^{-\lambda x}\right)}{\left(1 - e^{-\lambda x}\right) + \left(1 - e^{-\lambda y}\right)}$, then the conditional expectation over the interval $[-y, x]$ is

$$w(x, y) \equiv \delta(x, y)\left(\frac{1}{\lambda} + x - g(x)\right) - (1 - \delta(x, y))\left(\frac{1}{\lambda} + y - g(y)\right).$$

Clearly, $w(x, y) \gtreqqless 0$ for $x \gtreqqless y$. The forward solution $a_2(x, y)$ is the value of a_2 that solves

$$-cw(x, y) = \frac{c}{\lambda} + c(a_2 - x) - cg(a_2 - x) + (c - 2)x. \tag{32}$$

Note first that for $c \geq 2$ necessarily $x < y$. However, we need to have $y < x$ to get a solution for the isomorphic problem on the negative orthant. Hence for $c \geq 2$ the forward solution does not exist in both directions.

Now consider $1 < c < 2$. A solution $a_2(x, y)$ exists if and only if

$$(c - 2)x < -cw(x, y) < \frac{c}{\lambda} + (c - 2)x.$$

Note that this is always satisfied for $x = y$, and hence by continuity also for x close to y. The condition determining a_3 is unchanged,

$$cg(a_2(x, y) - x) - \frac{c}{\lambda} = \frac{c}{\lambda} + c(a_3 - a_2(x, y)) - cg(a_3 - a_2(x, y)) + 2(c - 1)a_2(x, y). \tag{33}$$

Rearranging (32), we can write

$$\frac{2(c - 1)}{(c - 2)}cw(x, y) - \frac{2(c - 1)}{(c - 2)}\left(cg(a_2 - x) - \frac{c}{\lambda} - c(a_2 - x)\right) = -2(c - 1)x.$$

Adding up with (33),

$$\begin{aligned}
&\frac{2(c-1)}{(c-2)}cw(x,y) \\
&= \frac{c}{\lambda} + c(a_3(x,y) - a_2(x,y)) - cg(a_3(x,y) - a_2(x,y)) + 4\frac{c-1}{2-c}(a_2(x,y) - x) \\
&- \frac{c}{2-c}\left(cg(a_2(x,y) - x) - \frac{c}{\lambda}\right).
\end{aligned}$$

For $x > y$, the left-hand side is strictly negative. On the other hand, the right-hand side is strictly positive at $a_3(x,y) - a_2(x,y) = a_2(x,y) - x = z$ for z small. Hence, the argument extends to this case. Note that by symmetry of the distribution, the case $x < y$ causes the isomorphic problem on the negative orthant. Hence, the size of the interval around zero must be bounded away from zero.

To conclude, we have shown that in a Class I equilibrium, $\mu_1^n > 0$, in a Class II equilibrium, $\mu_2^n > 0$ (by definition, we have $\mu_1^n = 0$). Finally, in any asymmetric equilibrium, the lengths of the intervals that are adjacent to the interval containing the prior mean are bounded away from zero.

Claim 3) Only a finite number of distinct receiver actions are induced in equilibrium.

Proof: Consider a Class I equilibrium first. We show that the solution of the forward equation violates the increasing interval property (27) for n large enough.

Consider the forward equation for a_n with length x of the first interval,

$$a_{n-1}(x) - c\left(\frac{1}{\lambda} + a_{n-1}(x) - g(a_{n-1}(x) - a_{n-2}(x))\right) = c\left(\frac{1}{\lambda} + a_n - g(a_n - a_{n-1}(x))\right) - a_{n-1}(x).$$

There is a unique value $a_n(x)$ of a_n that solves this equation. Let a_n be such that $a_n - a_{n-1}(x) = a_{n-1}(x) - a_{n-2}(x) \equiv z$, for some $z > 0$. Let $D(z;x)$ denote the difference between the right-hand side and the left-hand side of the forward equation evaluated at z,

$$D(z;x) = 2(c-1)a_{n-1}(x) + c\left(\frac{2}{\lambda} + z - 2g(z)\right).$$

If $D(z;x) > 0$, then a_n needs to decrease to satisfy the forward equation. Note that $\frac{2}{\lambda} + z - 2g(z)$ is strictly negative for $z > 0$ and $2(c-1)a_{n-1}(x)$ is strictly positive. From the first part of the proposition, we know that x is bounded away from zero. Moreover, x has to satisfy the increasing interval property (27) for $a_2(x) - x > x$. Suppose that the increasing interval property is satisfied up to the interval $a_{n-1}(x) - a_{n-2}(x)$. (If not, then we are done already.) If all intervals up to $a_{n-1}(x) - a_{n-2}(x)$ satisfy the increasing interval property, then $a_{n-1}(x) \geq (n-1)x$. Note that x does not depend on n. Hence, for any finite z, there is a $n(z,x)$ such that $D(z;x) > 0$ for all $n \geq n(z,x)$, implying that the increasing interval property is violated.

For the Class II equilibrium, note that the forward equation for a_n (for $n \geq 3$) is the same as above. The only difference is the value of $a_{n-1}(x)$ and the lower bound on x. However, $a_{n-1}(x) \geq x + (n-2)(a_2(x) - x)$. Note again that x and $a_2(x)$ do not depend on n.

The same argument can be given for the asymmetric case. Hence, the same conclusions obtain. ■

Appendix C

Proof of Lemma 4. We have

$$\mathbb{E}_{\tilde{\mu}\tilde{\omega}} u^r(c\tilde{\mu}, \tilde{\omega}) = -\mathbb{E}_{\tilde{\mu}\tilde{\omega}}\left[(c\tilde{\mu} - \tilde{\omega})^2\right] = -\mathbb{E}_{\tilde{\mu}\tilde{\omega}}\left[c^2\tilde{\mu}^2 - 2c\tilde{\omega}\tilde{\mu} - \tilde{\omega}^2\right] = c^2\mathbb{E}_{\tilde{\mu}}\left[\tilde{\mu}\right]^2 - \sigma^2.$$

The last equality follows from the fact that $\mathbb{E}_{\tilde{\mu}\tilde{\omega}}[\tilde{\omega}\tilde{\mu}] = c\mathbb{E}_{\tilde{\mu}}\left[\tilde{\mu}^2\right]$, which we now demonstrate. Let $j = 1, \ldots, J$ label the partition intervals in the natural order. Let Θ_j denote a generic interval, μ_j the mean over that interval, and define $\Pr(\Theta_j) \equiv \Pr\left(\tilde{\theta} \in \Theta_j\right)$. Moreover, let $f_{\tilde{\omega}\tilde{\theta}}(\omega, \theta)$ denote the joint density of $\tilde{\omega}$ and $\tilde{\theta}$. We can write

$$\begin{aligned}\mathbb{E}_{\tilde{\mu}\tilde{\omega}}[\tilde{\omega}\tilde{\mu}] &= \mathbb{E}_{\tilde{\mu}}\left[\mathbb{E}_{\tilde{\omega}|\tilde{\mu}=\mu}\left[\tilde{\omega}\tilde{\mu}|\, \tilde{\mu} = \mu\right]\right] = \sum_j \Pr(\Theta_j)\left[\mathbb{E}_{\tilde{\omega}|\tilde{\mu}=\mu_j}\left[\tilde{\omega}\tilde{\mu}|\, \tilde{\mu} = \mu_j\right]\right] \\ &= \sum_j \Pr(\Theta_j)\, \mu_j \int \omega f_{\tilde{\omega}|\Theta_j}\left(\omega|\, \tilde{\theta} \in \Theta_j\right) d\omega,\end{aligned}$$

where

$$f_{\tilde{\omega}|\Theta_j}\left(\omega|\, \tilde{\theta} \in \Theta_j\right) = \int_{\Theta_j} \frac{f_{\tilde{\omega}\tilde{\theta}}(\omega, \theta)}{\Pr(\Theta_j)} d\theta.$$

Interchanging the order of integration (Fubini's theorem) gives us,

$$\sum_j \Pr(\Theta_j)\, \mu_j \int \omega \int_{\Theta_j} \frac{f_{\tilde{\omega}\tilde{\theta}}(\omega, \theta)}{\Pr(\Theta_j)} d\theta d\omega = \sum_j \Pr(\Theta_j)\, \mu_j \int_{\Theta_j} \int \omega \frac{f_{\tilde{\omega}\tilde{\theta}}(\omega, \theta)}{\Pr(\Theta_j)} d\omega d\theta.$$

Dividing and multiplying by $f(\theta)$, recognizing that $\frac{f_{\tilde{\omega}\tilde{\theta}}(\omega,\theta)}{f(\theta)} = f_{\tilde{\omega}|\tilde{\theta}=\theta}\left(\omega|\,\tilde{\theta}=\theta\right)$, and applying (4) (Lemma 3 ii)), we have

$$\begin{aligned}
\sum_j \Pr(\Theta_j)\,\mu_j \int_{\Theta_j}\int \omega \frac{f_{\tilde{\omega}\tilde{\theta}}(\omega,\theta)}{\Pr(\Theta_j)} d\omega d\theta &= \sum_j \Pr(\Theta_j)\,\mu_j \int_{\Theta_j}\int \omega f_{\omega|\tilde{\theta}=\theta}\left(\tilde{\omega}|\,\tilde{\theta}=\theta\right) d\omega \frac{f(\theta)}{\Pr(\Theta_j)} d\theta \\
&= \sum_j \Pr(\Theta_j)\,\mu_j \int_{\Theta_j} c\theta \frac{f(\theta)}{\Pr(\Theta_j)} d\theta \\
&= c\sum_j \Pr(\Theta_j)\,\mu_j^2.
\end{aligned}$$

Substituting back and simplifying delivers the result. ■

Proof of Proposition 3. Preliminaries on Probabilities

Recall that $f(\theta)$ and $F(\theta)$ denote the pdf and cdf of $\tilde{\theta}$. For $k = 2,\ldots,n$, define $\hat{p}_{k-1}$ as the probability that $\tilde{\theta} \in [a_{k-2}, a_{k-1}]$ conditional on $\tilde{\theta} \geq a_{k-2}$,

$$\hat{p}_{k-1} \equiv \frac{F(a_{k-1}) - F(a_{k-2})}{1 - F(a_{k-2})}.$$

Accordingly, $1 - \hat{p}_{k-1} = \frac{1-F(a_{k-1})}{1-F(a_{k-2})}$ is the probability that $\tilde{\theta} \geq a_{k-1}$, conditional on $\tilde{\theta} \geq a_{k-2}$. We can write these probabilities as

$$\hat{p}_{k-1} = \frac{\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-1}\right] - \mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-2}\right]}{\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-1}\right] - \mu_{k-1}} \tag{34}$$

and

$$1 - \hat{p}_{k-1} \equiv \frac{\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-2}\right] - \mu_{k-1}}{\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-1}\right] - \mu_{k-1}}.$$

To see this, note that

$$\begin{aligned}
\left(F(a_{k-1}) - F(a_{k-2})\right)\mu_{k-1} &= \int\limits_{a_{k-2}}^{a_{k-1}} \theta f(\theta)\,d\theta = \int\limits_{a_{k-2}}^{\infty} \theta f(\theta)\,d\theta - \int\limits_{a_{k-1}}^{\infty} \theta f(\theta)\,d\theta \\
&= \left(1 - F(a_{k-2})\right)\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-2}\right] - \left(1 - F(a_{k-1})\right)\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-1}\right].
\end{aligned}$$

Hence

$$\hat{p}_{k-1}\mu_{k-1} = \mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-2}\right] - (1-\hat{p}_{k-1})\,\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}\right].$$

Solving for $\hat{p}_{k-1}$ delivers the desired conclusion.

Observe that $(1-\hat{p}_{k-2})\cdot\hat{p}_{k-1}$ is the probability of the event $\tilde{\theta} \in [a_{k-2}, a_{k-1}]$ conditional on $\tilde{\theta} \geq a_{k-3}$, and $(1-\hat{p}_{k-2})\cdot(1-\hat{p}_{k-1})$ is the probability of the event $\tilde{\theta} \geq a_{k-1}$ conditional on $\tilde{\theta} \geq a_{k-3}$. To see this, note that $1-\hat{p}_{k-2} = \Pr\left[\tilde{\theta} \geq a_{k-2}\middle|\tilde{\theta} \geq a_{k-3}\right] = \frac{1-F(a_{k-2})}{1-F(a_{k-3})}$ and recall that $\hat{p}_{k-1} = \frac{F(a_{k-1})-F(a_{k-2})}{1-F(a_{k-2})}$.

Induction

Induction Basis:

Recall that $\mu_+ \equiv \mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq 0\right]$. Let the distribution satisfy $\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq \overline{\theta}\right] = \mu_+ + \alpha\cdot\overline{\theta}$ for all $\overline{\theta} \geq 0$ and for some constant α. Note that for the Laplace distribution, $\alpha = 1$. Finally, define

$$\hat{c} \equiv \alpha c.$$

Assume that $\hat{c} \in (0,2)$. Let

$$X_n^n\left(a_{n-1}^n\right) \equiv \hat{p}_n^n\left(\hat{c}\mu_n^n - \hat{c}\mu_+\right)^2 + (1-\hat{p}_n^n)\left(\hat{c}\mu_{n+1}^n - \hat{c}\mu_+\right)^2.$$

$X_n^n\left(a_{n-1}^n\right)$ is equal to $\hat{c}^2$ times the expected squared deviation of the truncated means from μ_+, conditional on $\tilde{\theta} \geq a_{n-1}^n$. Substituting for $\hat{p}_n^n$ from (34) , and multiplying and dividing by $\hat{c}$ for convenience, we can write

$$\begin{aligned} X_n^n\left(a_{n-1}^n\right) \quad &= \quad \frac{\hat{c}\mu_{n+1}^n - \hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{n-1}^n\right]}{\hat{c}\mu_{n+1}^n - \hat{c}\mu_n^n}\left(\hat{c}\mu_n^n - \hat{c}\mu_+\right)^2 \\ &\quad + \frac{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{n-1}^n\right] - \hat{c}\mu_n^n}{\hat{c}\mu_{n+1}^n - \hat{c}\mu_n^n}\left(\hat{c}\mu_{n+1}^n - \hat{c}\mu_+\right)^2. \end{aligned}$$

Expanding the numerators of the probabilities by $\pm\hat{c}\mu_+$, reorganizing according to common factors, and simplifying (using lengthy but straightforward computations), we can write

$$X_n^n\left(a_{n-1}^n\right) = A_n^n + B_n^n,$$

where

$$A_n^n \equiv \left(\hat{c}\mu_{n+1}^n - \hat{c}\mu_+\right)\left(\hat{c}\mu_+ - \hat{c}\mu_n^n\right)$$

and

$$B_n^n \equiv \left(\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{n-1}^n\right] - \hat{c}\mu_+\right)\left(\left(\hat{c}\mu_n^n + \hat{c}\mu_{n+1}^n\right) - 2\hat{c}\mu_+\right).$$

We can further simplify the terms A_n^n and B_n^n, using the indifference condition of the marginal type a_n^n (multiplied by α), $\hat{c}\mu_n^n + \hat{c}\mu_{n+1}^n = 2\alpha a_n^n$ and the linearity of the tail conditional expectation, $\alpha a_n^n = \mu_{n+1}^n - \mu_+$. Substituting the latter condition into the former one, and solving for μ_{n+1}^n, we obtain

$$\frac{\hat{c}\mu_n^n + 2\mu_+}{2-\hat{c}} = \mu_{n+1}^n.$$

Substituting back into A_n^n and B_n^n, and simplifying, we have shown that

$$\begin{aligned} X_n^n\left(a_{n-1}^n\right) &= \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_n^n + \hat{c}\mu_+\right)\left(\hat{c}\mu_+ - \hat{c}\mu_n^n\right) \\ &\quad +2\left(\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{n-1}^n\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_n^n + \mu_+\right) - \hat{c}\mu_+\right). \end{aligned}$$

Induction hypothesis:

$$X_k^n\left(a_{k-1}^n\right) = \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ + \hat{c}\mu_k^n\right)\left(\hat{c}\mu_+ - \hat{c}\mu_k^n\right) + 2\left(\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu_k^n\right) - \hat{c}\mu_+\right).$$

Inductive step:

By definition

$$X_{k-1}^n\left(a_{k-2}^n\right) = \hat{p}_{k-1}^n\left(\hat{c}\mu_{k-1}^n - \hat{c}\mu_+\right)^2 + \left(1 - \hat{p}_{k-1}^n\right)X_k^n\left(a_{k-1}^n\right).$$

Substituting for the probability distribution from (34) and using the inductive hypothesis, we have

$$\begin{aligned} X_{k-1}^n\left(a_{k-2}^n\right) &= \frac{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-2}^n\right]}{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}\left(\hat{c}\mu_{k-1}^n - \hat{c}\mu_+\right)^2 \\ &\quad + \frac{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-2}^n\right] - \hat{c}\mu_{k-1}^n}{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}\left(\begin{array}{c} \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ + \hat{c}\mu_k^n\right)\left(\hat{c}\mu_+ - \hat{c}\mu_k^n\right) \\ +2\left(\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\,\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu_k^n\right) - \hat{c}\mu_+\right) \end{array}\right). \end{aligned}$$

Expanding the numerators of the probabilities by $\pm\hat{c}\mu_+$ and reorganizing according to common factors, we can write

$$X_{k-1}^n\left(a_{k-2}^n\right) = A_{k-1}^n + B_{k-1}^n$$

with

$$A_{k-1}^n \equiv \frac{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}\left(\hat{c}\mu_{k-1}^n - \hat{c}\mu_+\right)^2 + \frac{\hat{c}\mu_+ - \hat{c}\mu_{k-1}^n}{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}$$
$$\cdot\left(\frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ + \hat{c}\mu_k^n\right)\left(\hat{c}\mu_+ - \hat{c}\mu_k^n\right) + 2\left(\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu_k^n\right) - \hat{c}\mu_+\right)\right)$$

and

$$B_{k-1}^n \equiv \frac{\hat{c}\mu_+ - \hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-2}^n\right]}{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}\left(\hat{c}\mu_{k-1}^n - \hat{c}\mu_+\right)^2 + \frac{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-2}^n\right] - \hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}$$
$$\cdot\left(\frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ + \hat{c}\mu_k^n\right)\left(\hat{c}\mu_+ - \hat{c}\mu_k^n\right) + 2\left(\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu_k^n\right) - \hat{c}\mu_+\right)\right).$$

We consider each term in sequence. We first show that

$$A_{k-1}^n = \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu - \hat{c}\mu_{k-1}^n\right)\left(\hat{c}\mu + \hat{c}\mu_{k-1}^n\right).$$

The indifference condition of type a_{k-1}^n,$\hat{c}\mu_k^n = 2\alpha a_{k-1}^n - \hat{c}\mu_{k-1}^n$, allows us to substitute for $\hat{c}\mu_k^n$. Hence,

$$\begin{aligned} A_{k-1}^n =& \frac{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}\left(\hat{c}\mu_{k-1}^n - \hat{c}\mu_+\right)^2 + \frac{\hat{c}\mu_+ - \hat{c}\mu_{k-1}^n}{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_{k-1}^n} \\ &\cdot\Bigg(\frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ + 2\alpha a_{k-1}^n - \hat{c}\mu_{k-1}^n\right)\left(\hat{c}\mu_+ - \left(2\alpha a_{k-1}^n - \hat{c}\mu_{k-1}^n\right)\right) \\ &\quad +2\left(\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}\right] - \hat{c}\mu_+\right)\left(\frac{1}{2-\hat{c}}\left(\hat{c}\mu + 2\alpha a_{k-1} - \hat{c}\mu_{k-1}^n\right) - \hat{c}\mu_+\right)\Bigg). \end{aligned}$$

Collecting terms with the common factor $\frac{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta}\geq a_{k-1}^n\right]-\hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta}\geq a_{k-1}^n\right]-\hat{c}\mu_{k-1}^n}\left(\hat{c}\mu_{k-1}^n - \hat{c}\mu_+\right)$ and simplifying, we get

$$\begin{aligned} A_{k-1}^n =& \frac{\hat{c}\mu_+ - \hat{c}\mu_{k-1}^n}{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}\Bigg(\frac{\hat{c}}{2-\hat{c}}\left(\left(\hat{c}\mu_+ - \hat{c}\mu_{k-1}^n\right)\left(\hat{c}\mu_+ + \hat{c}\mu_{k-1}^n\right) + \left(-4\left(\alpha a_{k-1}^n\right)^2 + 4\alpha a_{k-1}^n\hat{c}\mu_{k-1}^n\right)\right) \\ &+\left(\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}^n\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_{k-1}^n + \hat{c}\mu_+\right) + \frac{4}{2-\hat{c}}\left(\alpha a_{k-1}^n - \hat{c}\mu_{k-1}^n\right)\right)\Bigg). \end{aligned}$$

It is easy to see that

$$
\begin{aligned}
& \frac{\hat{c}\mu_+ - \hat{c}\mu^n_{k-1}}{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a^n_{k-1}\right] - \hat{c}\mu^n_{k-1}} \frac{\hat{c}}{2-\hat{c}} \left(\hat{c}\mu_+ - \hat{c}\mu^n_{k-1}\right)\left(\hat{c}\mu_+ + \hat{c}\mu^n_{k-1}\right) \\
+ & \frac{\hat{c}\mu_+ - \hat{c}\mu^n_{k-1}}{\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a^n_{k-1}\right] - \hat{c}\mu^n_{k-1}} \left(\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a^n_{k-1}\right] - \hat{c}\mu_+\right) \frac{\hat{c}}{2-\hat{c}} \left(\hat{c}\mu^n_{k-1} + \hat{c}\mu_+\right) \\
= & \frac{\hat{c}}{2-\hat{c}} \left(\hat{c}\mu_+ - \hat{c}\mu^n_{k-1}\right)\left(\hat{c}\mu_+ + \hat{c}\mu^n_{k-1}\right).
\end{aligned}
$$

Moreover, since $\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a_{k-1}\right] - \hat{c}\mu_+ = \alpha\hat{c}a^n_{k-1}$, all the terms involving a^n_{k-1} exactly cancel out. Hence, the desired conclusion follows.

The term B^n_{k-1} is simplified using the same essential steps: the indifference condition of the marginal type to substitute for $\hat{c}\mu^n_k$, collecting terms with common factors and terms that add up conveniently, and the linear tail conditional expectation. Hence we can conclude that

$$
B^n_{k-1} = 2\left(\hat{c}\mathbb{E}\left[\tilde{\theta}\middle|\tilde{\theta} \geq a^n_{k-2}\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu^n_{k-1}\right) - \hat{c}\mu_+\right).
$$

This completes the induction.

Building on the characterization, we can compute $\mathbb{E}\left[\tilde{\mu}^2\right]$ in any equilibrium.

Finite Class I: In a Class I equilibrium, $a^n_0 = 0$. Hence,

$$
X^n_1\left(a^n_0\right) = \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ - \hat{c}\mu^n_1\right)\left(\hat{c}\mu_+ + \hat{c}\mu^n_1\right).
$$

Recalling the definition of $X^n_{k-1}\left(a^n_{k-2}\right)$, we also have

$$
X^n_1\left(a^n_0\right) = \hat{c}^2 \sum_{i=1}^{n+1} \hat{p}^n_i \left(\mu^n_i - \mu_+\right)^2 = \hat{c}^2 \sum_{i=1}^{n+1} \hat{p}^n_i \left(\mu^n_i\right)^2 - \hat{c}^2\mu^2_+,
$$

where the second equality follows from the fact that $\sum_{i=1}^{n+1} \hat{p}^n_i \left(\mu^n_i - \mu_+\right) = 0$. Solving for $\sum_{i=1}^{n+1} \hat{p}^n_i \left(\mu^n_i\right)^2$ between these equations, we get

$$
\sum_{i=1}^{n+1} \hat{p}^n_i \left(\mu^n_i\right)^2 = \frac{X^n_1\left(a^n_0\right)}{\hat{c}^2} + \mu^2_+ = \frac{2}{2-\hat{c}}\mu^2_+ - \frac{\hat{c}}{2-\hat{c}}\left(\mu^n_1\right)^2.
$$

For the uni-dimensional Laplace distribution with density

$$f(\theta) = \frac{1}{2}\lambda e\left(-\lambda\left|\theta\right|\right),$$

the scale parameter λ determines all the relevant moments of the distribution. In particular, $\mu_+ = \frac{1}{\lambda}$ and $\sigma_\theta^2 = \frac{2}{\lambda^2} = 2\mu_+^2$. Moreover, $\alpha = 1$. Hence, we have $\sum_{i=1}^{n+1}\hat{p}_i\left(\mu_i^n\right)^2 = \frac{1}{2-c}\sigma_\theta^2 - \frac{c}{2-c}\left(\mu_1^n\right)^2$. By the symmetry of the distribution, $\Pr\left[\tilde{\theta} \geq 0\right] = \Pr\left[\tilde{\theta} \leq 0\right] = \frac{1}{2}$ and $\mathbb{E}\left[\tilde{\mu}^2\middle|\tilde{\theta} \geq 0\right] = \mathbb{E}\left[\tilde{\mu}^2\middle|\tilde{\theta} \leq 0\right]$, so that

$$\mathbb{E}\left[\tilde{\mu}^2\right] = \frac{\mathbb{E}\left[\tilde{\mu}^2\middle|\tilde{\theta} \geq 0\right] + \mathbb{E}\left[\tilde{\mu}^2\middle|\tilde{\theta} \leq 0\right]}{2} = \mathbb{E}\left[\tilde{\mu}^2\middle|\tilde{\theta} \geq 0\right].$$

Hence, we have shown that in a Class I equilibrium

$$\mathbb{E}\left[\tilde{\mu}^2\right] = \frac{1}{2-c}\sigma_\theta^2 - \frac{c}{2-c}\left(\mu_1^n\right)^2.$$

Finite Class II: In a Class II equilibrium, a_0 is eliminated. We have

$$\begin{aligned} X_2^n\left(a_1^n\right) &= \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ - \hat{c}\mu_2^n\right)\left(\hat{c}\mu_+ + \hat{c}\mu_2^n\right) + 2\left(\hat{c}\mathbb{E}\left[\theta\middle|\theta \geq a_1^n\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu_2^n\right) - \hat{c}\mu_+\right) \\ &= \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ - \hat{c}\mu_2^n\right)\left(\hat{c}\mu_+ + \hat{c}\mu_2^n\right) + 2\alpha\hat{c}a_1^n\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu_2^n\right) - \hat{c}\mu_+\right). \end{aligned}$$

Using the definition of X_2^n and then the fact that $\sum_{i=2}^{n+1}\hat{p}_i^n\mu_i^n = \mu_+ + \alpha a_1^n$ for a distribution with an linear tail conditional expectation, we get

$$\begin{aligned} \frac{X_2^n\left(a_1^n\right)}{\hat{c}^2} &= \sum_{i=2}^{n+1}\hat{p}_i^n\left(\mu_i^n\right)^2 - 2\mu_+\sum_{i=2}^{n+1}\hat{p}_i^n\mu_i^n + \mu_+^2 \\ &= \sum_{i=2}^{n+1}\hat{p}_i^n\left(\mu_i^n\right)^2 - \mu_+^2 - 2\alpha a_1^n\mu_+. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i=2}^{n+1}\hat{p}_i^n\left(\mu_i^n\right)^2 &= \frac{X_2^n\left(a_1^n\right)}{\hat{c}^2} + \mu_+^2 + 2\alpha a_1^n\mu_+ \\ &= \frac{2}{2-\hat{c}}\mu_+^2 - \frac{\hat{c}}{2-\hat{c}}\left(\mu_2^n\right)^2 + \frac{2\alpha a_1^n}{2-\hat{c}}\left(\mu_+ + \mu_2^n\right). \end{aligned}$$

Now, we may write

$$\begin{aligned}\mathbb{E}\left[\tilde{\mu}^2\right] &= \Pr\left[\tilde{\theta}\geq a_1^n\right]\cdot\mathbb{E}\left[\tilde{\mu}^2\middle|\,\tilde{\theta}\geq a_1^n\right]+\Pr\left[\tilde{\theta}\leq -a_1^n\right]\cdot\mathbb{E}\left[\tilde{\mu}^2\middle|\,\tilde{\theta}\leq -a_1^n\right]\\ &= 2\Pr\left[\tilde{\theta}\geq a_1^n\right]\cdot\mathbb{E}\left[\tilde{\mu}^2\middle|\,\tilde{\theta}\geq a_1^n\right]\\ &= \left(1-\Pr\left[\tilde{\theta}\in[-a_1^n,a_1^n)\right]\right)\cdot\mathbb{E}\left[\tilde{\mu}^2\middle|\,\tilde{\theta}\geq a_1^n\right].\end{aligned}$$

The first equality uses the fact that $\mu_1^n = 0$ in a Class II equilibrium, and the other two equalities use the symmetry of the distribution, which implies that $\mathbb{E}\left[\tilde{\mu}^2\middle|\,\tilde{\theta}\geq a_1^n\right] = \mathbb{E}\left[\tilde{\mu}^2\middle|\,\tilde{\theta}\leq -a_1^n\right]$ and $\Pr\left[\tilde{\theta}\geq a_1^n\right]=\Pr\left[\tilde{\theta}\leq -a_1^n\right]$. Hence, we have shown that

$$\mathbb{E}\left[\tilde{\mu}^2\right]=\left(1-\Pr\left[\tilde{\theta}\in[-a_1^n,a_1^n)\right]\right)\left[\frac{2}{2-\hat{c}}\mu_+^2-\frac{\hat{c}}{2-\hat{c}}\left(\mu_2^n\right)^2+\frac{2\alpha a_1^n}{2-\hat{c}}\left(\mu_++\mu_2^n\right)\right].$$

The indifference condition of the marginal type a_1^n requires that $c\mu_2^n-a_1^n=a_1^n$. Substituting for $2a_1^n=c\mu_2^n$, noting that $\hat{c}=\alpha c$, and simplifying, we obtain

$$\mathbb{E}\left[\tilde{\mu}^2\right]=\left(1-\Pr\left[\tilde{\theta}\in\left[-\frac{c\mu_2^n}{2},\frac{c\mu_2^n}{2}\right)\right]\right)\left[\frac{2}{2-\hat{c}}\mu_+^2+\frac{\hat{c}}{2-\hat{c}}\mu_2^n\mu_+\right],$$

which coincides with expression (9) for $\alpha=1$ and $\sigma_\theta^2=2\mu_+^2$, the Laplace case.

Limit: In a limit equilibrium resulting from the limit of a Class I equilibrium, the sequence $(\mu_1^n)_n$ satisfies $\lim_{n\to\infty}\mu_1^n=0$. In a limit equilibrium resulting from the limit of a Class II equilibrium, the sequences $(a_1^n)_n$ and $(\mu_2^n)_n$ satisfy $\lim_{n\to\infty}a_1^n=0$ and $\lim_{n\to\infty}\mu_2^n=0$. Hence, in the limit

$$\mathbb{E}\left[\tilde{\mu}^2\right]=\frac{2}{2-\hat{c}}\mu_+^2.$$

Substituting for the Laplace case, $\alpha=1$ and $\sigma_\theta^2=2\mu_+^2$, gives expression (10).

In the limit equilibrium resulting from the limit of finite Class I and Class II equilibria, if it exists, $\mathbb{E}\left[\tilde{\mu}^2\right]$ is maximized. The right-hand side of (10) exceeds the right-hand side of (8) for all finite n, since $\mu_1^n>0$ for finite n. We now show that the right-hand side of (10) also exceeds the right-hand side of (9) for all finite n. Noting that $\left(1-\Pr\left[\tilde{\theta}\in\left[-\frac{c\mu_2^n}{2},\frac{c\mu_2^n}{2}\right)\right]\right)=\exp\left(-\lambda\frac{c\mu_2^n}{2}\right)=\exp\left(-\frac{c\mu_2^n}{2\mu_+}\right)$,

$$\frac{2}{2-\hat{c}}\mu_+^2>\left(1-\Pr\left[\tilde{\theta}\in\left[-\frac{c\mu_2^n}{2},\frac{c\mu_2^n}{2}\right)\right]\right)\left[\frac{2}{2-\hat{c}}\mu_+^2+\frac{\hat{c}}{2-\hat{c}}\mu_2^n\mu_+\right]$$

is equivalent to

$$1 - \exp\left(-\frac{c\mu_2^n}{2\mu_+}\right) > \exp\left(-\frac{c\mu_2^n}{2\mu_+}\right)\frac{c\mu_2^n}{2\mu_+}.$$

This is true for all $\mu_2^n > 0$ since the function $\exp(-x)(1+x)$ satisfies $\exp(-x)(1+x) < 1$ for all $x > 0$. ■

Proof of Lemma 5. We derive here the density of the marginal distribution of $\tilde{\theta}$. Let $\hat{f}(\theta;\alpha)$ and $\hat{F}(\theta;\alpha)$ denote the density and cdf of the distribution, conditional on $\tilde{\theta} \geq 0$. After an integration by parts, (5) is equivalent to

$$\mu_+ + \alpha\theta = \theta + \frac{\int_\theta^{\overline{\theta}} \left(1 - \hat{F}(t;\alpha)\right) dt}{1 - \hat{F}(\theta;\alpha)}. \tag{35}$$

Define $q(\theta) = \int_\theta^{\overline{\theta}} \left(1 - \hat{F}(t;\alpha)\right) dt$ and note that $\dot{q} \equiv \frac{\partial q(\theta)}{\partial \theta} = -\left(1 - \hat{F}(\theta;\alpha)\right)$. In terms of these functions, we can write (35) as the ordinary differential equation

$$\frac{\dot{q}}{q} = \frac{1}{(1-\alpha)\theta - \mu_+}$$

with initial condition $q(0) = \mu_+$. The solution is

$$q(\theta) = \left(\mu_+\right)^{-\frac{\alpha}{1-\alpha}} \left(\mu_+ - \theta(1-\alpha)\right)^{\frac{1}{1-\alpha}}.$$

To satisfy $\lim_{\theta\to\overline{\theta}} \hat{F}(\theta;\alpha) = 1$, we have $\overline{\theta} = \frac{\mu_+}{1-\alpha}$ for $\alpha < 1$. For $\alpha \geq 1$, the support is $\mathbb{R}^+$. Differentiating twice, we obtain the density

$$\hat{f}(\theta;\alpha) = \alpha\left(\mu_+\right)^{-\frac{\alpha}{1-\alpha}} \left(\mu_+ - \theta(1-\alpha)\right)^{\frac{2\alpha-1}{1-\alpha}}. \tag{36}$$

For future reference, the cdf is

$$\hat{F}(\theta;\alpha) = 1 - \left(\mu_+\right)^{-\frac{\alpha}{1-\alpha}} \left(\mu_+ - \theta(1-\alpha)\right)^{\frac{\alpha}{1-\alpha}}.$$

The density is square integrable since $\alpha < 2$. Straightforward integration reveals that the variance v_+^2 of the distribution is $v_+^2 = \frac{\alpha}{2-\alpha}\mu_+^2$.

Consider now the density on the whole support. By symmetry and the variance decomposition, $\sigma_\theta^2 = v_+^2 + \mu_+^2$, so

$$\sigma_\theta^2 = \frac{2}{2-\alpha}\mu_+^2.$$

Hence, we get expression (14). ■

Proof of Proposition 4. Note that the first part is a corollary to Proposition 3. So, we only need to verify the upper bound on $\mathbb{E}\left[\tilde{\mu}^2\right]$ in any symmetric equilibrium. For Class I equilibria this is obvious, so consider Class II equilibria. Note that $\Pr\left[\tilde{\theta} \in \left[-a_1^n, a_1^n\right)\right] = \hat{F}\left(a_1^n; \alpha\right)$. Moreover,

$$\hat{F}\left(\theta; \alpha\right) = 1 - \left(\mu_+\right)^{-\frac{\alpha}{1-\alpha}} \left(\mu_+ - \theta\left(1-\alpha\right)\right)^{\frac{\alpha}{1-\alpha}}.$$

Hence,

$$\frac{2}{2-\hat{c}}\mu_+^2 > \left(1 - \Pr\left[\tilde{\theta} \in \left[-\frac{c\mu_2^n}{2}, \frac{c\mu_2^n}{2}\right)\right]\right)\left[\frac{2}{2-\hat{c}}\mu_+^2 + \frac{\hat{c}}{2-\hat{c}}\mu_2^n\mu_+\right]$$

is equivalent to

$$\left(1 - \left(\mu_+\right)^{-\frac{\alpha}{1-\alpha}}\left(\mu_+ - \frac{c\mu_2^n}{2}\left(1-\alpha\right)\right)^{\frac{\alpha}{1-\alpha}}\right)\frac{2}{2-\hat{c}}\mu_+^2 > \left(\mu_+\right)^{-\frac{\alpha}{1-\alpha}}\left(\mu_+ - \frac{c\mu_2^n}{2}\left(1-\alpha\right)\right)^{\frac{\alpha}{1-\alpha}}\frac{\hat{c}}{2-\hat{c}}\mu_2^n\mu_+.$$

Simplifying, we obtain

$$1 > \left(1 - \frac{c\mu_2^n}{2\mu_+}\left(1-\alpha\right)\right)^{\frac{\alpha}{1-\alpha}}\left(1 + \alpha\frac{c\mu_2^n}{2\mu_+}\right).$$

To see this is always satisfied, consider the function $h\left(x\right) \equiv \left(1 - x\left(1-\alpha\right)\right)^{\frac{\alpha}{1-\alpha}}\left(1+\alpha x\right)$. Note that $h\left(0\right) = 1$. Moreover, $h'\left(x\right) < 0$ for $x > 0$. ■