Online Appendix:
"Are Low Interest Rates Deflationary? A Paradox of Perfect-Foresight Analysis"

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A Summary of Notation Used in the Paper

Var./Param.	Explanation
One dimensional parameters	
n	Degree of reflection, introduced in equation (3).
π^*	Inflation target, introduced in equation (4).
β	Discount factor when the rate of time preference is $\bar{\rho}$, introduced in equation (4).
σ	Household intertemporal elasticity of substitution, introduced in equation (4).
α	Firm's j probability of not optimizing its price, introduced in equation (8).
ξ	Elasticity of a firm's optimal relative price with respect to aggregate demand, introduced in equation (8).
κ	Parameter of aggregate supply, introduced in equation (9) and defined below.
ϕ_π	Taylor rule coefficient for inflation, introduced in equation (10).
ϕ_y	Taylor rule coefficient for output, introduced in equation (10).
δ_i	Definition of parameters for $i = \{1, 2\}$ introduced in equation (12) and defined below.
ϕ	Taylor rule coefficient for inflation in Simple Illustration, introduced in equation (17).
η_y	Coefficient of output expectation in static relation defined after equation (18).
η_π	Coefficient of inflation expectation in static relation defined after equation (18).
η_i	Coefficient of interest rate expectation in static relation defined after equation (18).
z_∞	Finite collection of real coefficients defined in equation (26).
u_k	Finite collection of real coefficients defined in equation (26).

Var./Param.	Explanation
λ_k	Real numbers defined in equation (26).
Matrix/vector parameters	
$\mathbf{c}$	2x2 Matrix relating the exogenous vector $\boldsymbol{\omega}_t$ to the endogenous vector $\mathbf{x}_t$, introduced in (11).
$\mathbf{C}$	2x2 Matrix relating the expectation vector $\mathbf{e}_t$ to the endogenous vector $\mathbf{x}_t$, introduced in (11).
$\mathbf{m}$	2x2 Matrix relating the exogenous vector $\boldsymbol{\omega}_t$ to the vector $\mathbf{a}_t$, introduced in (13).
$\mathbf{M}$	2x2 Matrix relating the expectation vector $\mathbf{e}_t$ to the vector $\mathbf{a}_t$, introduced in (13).
$\boldsymbol{\zeta}_j$	2x2 Matrix j of the FS-PFE, introduced in equation (25).
$\mathbf{m}_2$	2x1 Matrix, that is the second column of $\mathbf{m}$, introduced in (27).
Variables	
c_t^i	Consumption of household i, introduced in equation (4).
$\hat{b}_t^i$	Net real financial wealth of household i, introduced in equation (4).
y_t	Output, introduced in equation (4).
i_t	Interest rate, introduced in equation (4).
π_t	Inflation, introduced in equation (4).
ρ_t	Household's rate of time preference, introduced in equation (4).
g_t	Weighted sum of household's rate of time preference, introduced in equation (5) and defined below.
v_t^i	Expectational variable of household i, introduced in equation (5) and defined below.
e_{1t}	Average expectation of v_{t+1}^i, introduced in equation (7) and defined below.
p_t^{*j}	Optimal price in t of firm j in excess of the average prices that are not reconsidered, introduced in equation (8).
p_t	Price level in t, introduced in equation (8).
e_{2t}	Average expectation of p_{t+1}^{*j}, introduced in equation (9) and defined below.
$\bar{\imath}_t$	Intercept Taylor rule, introduced in equation (10).
e_{it}^*	Correct value for subjective expectation e_{it} for $i = \{1, 2\}$, defined in equation (12).
a_{it}	Variable to calculate e_{it}^* for $i = \{1, 2\}$, introduced in equation (12) and defined below.
π^e	Expectation of future inflation in Simple Illustration.
y^e	Expectation of future output in Simple Illustration.
i^e	Expectation of future interest rate in Simple Illustration.

Var./Param.	Explanation
Variables	
η	Expectational variable of Simple Illustration, introduced in equation (18).
η^*	Correct implied value of η given beliefs η used in Simple Illustration.
$\dot{\eta}$	Derivative of η with respect to n, introduced in equation (21).
z_t	General variable defined in equation (26).
$\bar{\imath}_{SR}$	Value of intercept in the interest rate equation or the short run introduced in section 2.3.
$\bar{\imath}_{LR}$	Value of intercept in the interest rate equation or the long run introduced in section 2.3.
Vectors	
$\mathbf{x}_t$	Vector containing endogenous variables. Defined generically in equation (1) and as the vector containing y_t and π_t in equation (11).
$\mathbf{e}_t$	Infinite-dimensional vector of average expectations defined in equation (1).
$\mathbf{e}_t^*$	Vector containing correct beliefs $\{\mathbf{e}_{1t}^*, \mathbf{e}_{2t}^*\}$ defined in equation (12).
$\mathbf{e}$	Vector that collects all expectational variables $\mathbf{e}_t$.
$\mathbf{e}^*$	Vector that collects all correct beliefs variables $\mathbf{e}_t^*$.
$\mathbf{e}(n)$	Same as $\mathbf{e}$, but making explicit its dependence on n.
$\dot{\mathbf{e}}(n)$	Derivative of $\mathbf{e}(n)$ with respect to n.
$\bar{\mathbf{e}}$	Rest point of system (12).
$\boldsymbol{\omega}_t$	Vector containing exogenous variables g_t and $\bar{\imath}_t$, introduced in equation (11).
$\mathbf{a}_t$	Vector containing variables a_{1t} and a_{2t}, introduced in equation (13).
$\mathbf{e}_t(n)$	Same as $\mathbf{e}_t$, making explicit that it depends on the degree of reflection n, introduced in equation (14).
$\mathbf{e}_t^*(n)$	Vector containing e_{1t}^* and e_{2t}^*, making explicit that it depends on the degree of reflection n, introduced in equation (14).
$\dot{\mathbf{e}}_t(n)$	Derivative of $\mathbf{e}_t(n)$ with respect to n, introduced in equation (14).
$\dot{\mathbf{e}}$	Derivative of $\mathbf{e}$ with respect to n introduced in equation (20).
$\mathbf{e}_t^{PF}$	Beliefs associated with the FS-PFE in t introduced in Proposition 1.
$\mathbf{e}_{LR}(n)$	Value of expectations of the reflective equilibrium n in the long run introduced in section 2.3.
$\bar{\mathbf{e}}_{LR}$	Value of expectations in the long run of the FS-PFE introduced in section 2.3.
$\mathbf{e}_{SR}(n)$	Value of expectations of the reflective equilibrium n in the short run introduced in section 2.3 when $T \to \infty$.
$\bar{\mathbf{e}}_{SR}$	Value of expectations of the reflective equilibrium n in the short run introduced in section 2.3 when $n \to \infty$ and $T \to \infty$.

B Mathematical Derivations

B.1 Derivation of equations (4)-(7)

The economy is made up of a continuum of identical infinite-lived households indexed by $i \in [0,1]$. Each household maximizes its estimate of its discounted utility:

$$\hat{E}_t^i \sum_{\tau=t}^{\infty} \exp[-\sum_{s=t}^{\tau-1} \hat{\rho}_s][u(C_\tau^i) - v(H_\tau^i)]$$

C_t^i is a Dixit-Stiglitz aggregate of the households' purchases of differentiated consumer goods, H_t^i is hours worked by the household in t, $\hat{\rho}_t$ is a possibly time-varying discount rate. It is assumed that the households supply the hours of work demanded by firms at a wage fixed by a union that bargains on behalf of households. This implies that its non-financial income (sum of wage income and share of profits) are outside its control. It is further assumed that the hours supplied by each household and its shares of the firms' profits is distributed equally among the household. Then, we can write the budget constraint of the household as:

$$B_{\tau+1}^i = (1+\tilde{i}_\tau)\left[B_\tau^i + W_\tau H_\tau^i + \int_{j=0}^{1} \Pi_\tau(j)dj - P_\tau C_\tau^i\right]$$

with B_t^i bond holdings by household i at t, $\tilde{i}_t$ the interest rate of the bond holdings, W_t the wage, $\Pi_t(j)$ profits of firm j, P_t the price of the consumption basket. The problem of each household can be solved with the lagrangian:

$$\begin{aligned} \mathcal{L} &= \hat{E}_t^i \sum_{\tau=t}^{\infty} \exp[-\sum_{s=t}^{\tau-1} \hat{\rho}_s]\Bigg\{ u(C_\tau^i) - v(H_\tau^i) + \\ &\lambda_\tau \left((1+\tilde{i}_\tau)\left[B_\tau^i + W_\tau H_\tau^i + \int_{j=0}^{1} \Pi_\tau(j)dj - P_\tau C_\tau^i\right] - B_{\tau+1}^i \right) \Bigg\} \end{aligned}$$

The FOCs can be written as:

$$\begin{aligned} [C_t^i] &\quad U'(C_t^i) - (1+\tilde{i}_t)P_t\lambda_t = 0 \\ [B_{t+1}^i] &\quad -\lambda_t + \exp\{-\hat{\rho}_t\}\hat{E}_t^i(1+\tilde{i}_{t+1})\lambda_{t+1} = 0 \end{aligned}$$

Which implies the Euler equation:

$$U'(C_t^i) = \exp\{-\hat{\rho}_t\}(1+\tilde{i}_t)\hat{E}_t^i \frac{U'(C_{t+1}^i)}{\Pi_{t+1}}$$

with $\Pi_t = P_t/P_{t-1}$. By replacing the equations for the profits, using the market clearing of the labor market, and dividing by P_{t-1}, we get that the budget constraint can be written as:

$$b_{t+1}^i = (1+\tilde{i}_t)\left[\frac{b_t^i}{\Pi_t} + Y_t - C_t^i\right]$$

where $Y_t = \int_{j=0}^1 Y_t(j)dj$, $b_t^i = B_t^i/P_{t-1}$.

The steady state in which these equations will be log-linearized is one with positive inflation, $\pi > 1$. The approximations are given by:

$$\hat{b}_{t+1}^i \approx \frac{1}{\beta}\hat{b}_t^i + \frac{\pi}{\beta}\left(y_t - c_t^i\right)$$

With $\hat{b}_t^i = b_t^i - b$, $y_t = \log(Y_t/Y)$, $c_t^i = \log(C_t^i/C)$ and all the variables without time subscript are steady state values. This equation uses the fact that in steady state $b^i = b = 0$, $Y = C$ and $(1+\tilde{i}) = \pi/\beta$. This implies:

$$\hat{b}_t^i = -\pi \sum_{\tau=t}^{\infty} \beta^{\tau-t}\hat{E}_t^i(y_\tau - c_\tau^i)$$

The approximation of the Euler equation:

$$\hat{E}_t^i c_{t+1}^i = c_t^i + \sigma(i_t - \rho_t - \hat{E}_t^i \pi_{t+1})$$

with $\pi_t = \log(\Pi_t/\pi)$ and $\rho_t = \hat{\rho}_t - \hat{\rho}$, $i_t = \log(1+\tilde{i}_t) - \log(1+\tilde{i})$. This implies:

$$\hat{E}_t^i c_\tau^i = c_t^i + \sigma \sum_{s=t}^{\tau-1} \hat{E}_t^i(i_s - \rho_s - \pi_{s+1})$$

And writing everything together:

$$
\begin{aligned}
\hat{b}_t^i &= -\pi \sum_{\tau=t}^{\infty} \beta^{\tau-t} \hat{E}_t^i y_\tau + \pi \left(\sum_{\tau=t}^{\infty} \beta^{\tau-t} \left(c_t^i + \sigma \sum_{s=t}^{\tau-1} \hat{E}_t^i (i_s - \rho_s - \pi_{s+1}) \right) \right) \\
&= -\pi \sum_{\tau=t}^{\infty} \beta^{\tau-t} \hat{E}_t^i y_\tau + \frac{\pi c_t^i}{1-\beta} + \frac{\sigma \pi \beta}{1-\beta} \sum_{\tau=t}^{\infty} \beta^{\tau-t} \hat{E}_t^i (i_\tau - \rho_\tau - \pi_{\tau+1}) \\
c_t^i &= \frac{1-\beta}{\pi} \hat{b}_t^i + \sum_{\tau=t}^{\infty} \beta^{\tau-t} \hat{E}_t^i \left((1-\beta) y_\tau - \beta\sigma(i_\tau - \rho_\tau - \pi_{\tau+1})\right)
\end{aligned}
$$

which is equation (4). Then the change to equation (5) is direct and also the aggregation to (7) by realizing that $\int \hat{b}_t^i di = 0$ and $\int c_t^i di = y_t$.

B.2 Derivation of equation (8)

Consider a firm j which uses labor to produce its product,

$$Y_t(j) = f(H_t(j))$$

where $Y_t(j)$ is firm j's product, $f(.)$ is its production technology and $H_t(j)$ is the labor used by the firm. Consider also that this firm faces a downward sloping demand because it produces a differentiated product:

$$Y_t(j) = Y_t \left(\frac{P_t(j)}{P_t} \right)^{-\theta}$$

where Y_t is aggregate demand, $P_t(j)$ is firm j's price. We can then write the profit of firm j as:

$$\Pi_t(j) = \Pi\left(P_t(j), P_t, Y_t, W_t\right)$$

The problem of choosing the price optimally has to take into account that prices, when not chosen are revised by the inflation target, so we can write the maximization objective as:

$$\max \hat{E}_t^j \sum_{\tau=t}^{\infty} \alpha^{\tau-t} Q_{t,\tau} \Pi\left(P_t(j)(\Pi^*)^{\tau-t}, P_\tau, Y_\tau, W_\tau\right)$$

where $Q_{t,\tau}$ is the household's stochastic discount factor. Using the homogeneity of degree zero in prices of the derivative of $\Pi(.)$ with respect to its first argument, $\Pi_1(.)$, the log-linearized version of the optimal condition of labor from the household and market clearing, we get the log-linearized FOC of this function[52]:

$$\hat{E}_t^j \sum_{\tau=t}^{\infty} (\alpha\beta)^{\tau-t} \left(\log P_t^*(j) + (\tau - t)\log \Pi^* - \log P_\tau - \xi \log Y_\tau / Y\right) = 0$$

This gives you equation (8) noting that $p_t = \log P_t$, $p_t^{*j} = \log P_t^j(j) - p_{t-1} - \pi^*$, $\pi^* = \log(\Pi^*)$.

B.3 Derivation of equation (9)

First note that the price index evolves according to:

$$\begin{aligned} p_t &= \int_{j=0}^{\alpha} (p_{t-1} + \pi^*) dj + \int_{j=\alpha}^{1} (p_t^{*j} + p_{t-1} + \pi^*) dj \\ p_t - p_{t-1} - \pi^* &= \int_{j=\alpha}^{1} p_t^{*j} dj \end{aligned}$$

so:

$$\pi_t = (1-\alpha) \int_{j=0}^{1} p_t^{*j} dj \tag{B.31}$$

since $\pi_t = p_t - p_{t-1} - \pi^*$. Starting from (8), we have:

$$\begin{aligned} p_t^{*j} &= (1-\alpha\beta) \sum_{\tau=t}^{\infty} (\alpha\beta)^{\tau-t} \hat{E}_t^j [p_\tau + \xi y_\tau - \pi^*(\tau - t)] - (p_{t-1} + \pi^*) \\ &= (1-\alpha\beta) \sum_{\tau=t}^{\infty} (\alpha\beta)^{\tau-t} \hat{E}_t^j \xi y_\tau + (1-\alpha\beta) \sum_{\tau=t}^{\infty} (\alpha\beta)^{\tau-t} \hat{E}_t^j [p_\tau - \pi^*(\tau - t) - p_{t-1} - \pi^*] \\ &= (1-\alpha\beta) \sum_{\tau=t}^{\infty} (\alpha\beta)^{\tau-t} \hat{E}_t^j \xi y_\tau + (1-\alpha\beta) \hat{E}_t^j \Big[p_t - p_{t-1} - \pi^* + \alpha\beta(p_{t+1} - p_{t-1} - 2\pi^*) + \\ &\quad (\alpha\beta)^2 (p_{t+2} - p_{t-1} - 3\pi^*) ... \Big] \\ &= (1-\alpha\beta) \sum_{\tau=t}^{\infty} (\alpha\beta)^{\tau-t} \hat{E}_t^j \xi y_\tau + (1-\alpha\beta) \hat{E}_t^j \left[\pi_t + \alpha\beta(\pi_{t+1} + \pi_t) + (\alpha\beta)^2 (\pi_{t+2} + \pi_{t+1} + \pi_t) ... \right] \end{aligned}$$

[52] To see details of this derivation, refer to Woodford 2003, chap. 3.

which we can write as:

$$p_t^{*j} = \sum_{\tau=t}^{\infty} (\alpha\beta)^{\tau-t} \hat{E}_t^j \left[\pi_\tau + (1-\alpha\beta)\xi y_\tau\right] \tag{B.32}$$

This implies:

$$p_t^{*j} = \pi_t + (1-\alpha\beta)\xi y_t + \alpha\beta \hat{E}_t^j p_{t+1}^{*j}$$

Integrating over firms we have:

$$\int_{j=0}^{1} p_t^{*j} dj = \frac{\pi_t}{1-\alpha} = \pi_t + (1-\alpha\beta)\xi y_t + \alpha\beta \int_{j=0}^{1} \hat{E}_t^j p_{t+1}^{*j} dj$$

Which by multiplying by $(1-\alpha)$, defining κ and rearranging terms gives you equation (9). Equation (12) is obtained directly by the definition of v_t^i in the text and (B.32).

B.4 Derivation of matrices and equation (11)

Replacing (10) in (7) and (9), we get the system:

$$\mathbf{C}_1 \mathbf{x}_t = \mathbf{C}_2 \mathbf{e}_t + \mathbf{C}_3 \boldsymbol{\omega}_t$$

with

$$\mathbf{x}_t = \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} \qquad \mathbf{e}_t = \begin{bmatrix} e_{1,t} \\ e_{2,t} \end{bmatrix} \qquad \boldsymbol{\omega}_t = \begin{bmatrix} g_t \\ \bar{\imath}_t \end{bmatrix}$$

and

$$\mathbf{C}_1 = \begin{bmatrix} 1+\sigma\phi_y & \sigma\phi_\pi \\ -\kappa & 1 \end{bmatrix} \qquad \mathbf{C}_2 = \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha)\beta \end{bmatrix} \qquad \mathbf{C}_3 = \begin{bmatrix} 1 & -\sigma \\ 0 & 0 \end{bmatrix}$$

Which, by inverting and pre-multiplying $\mathbf{C}_1$, gives you (11), with the matrices:

$$\mathbf{C} = \tfrac{1}{\Delta} \begin{bmatrix} 1 & -\sigma\phi_\pi(1-\alpha)\beta \\ \kappa & (1+\sigma\phi_y)(1-\alpha)\beta \end{bmatrix}, \qquad \mathbf{c} = \tfrac{1}{\Delta} \begin{bmatrix} 1 & -\sigma \\ \kappa & -\kappa\sigma \end{bmatrix},$$

and use the shorthand notation $\Delta \equiv 1+\sigma\phi_y+\sigma\kappa\phi_\pi \geq 1$. (This last inequality, that allows us to divide by Δ, holds under the sign restrictions maintained in the text.) Given this solution for $\mathbf{x}_t$, the solution for the nominal interest rate is obtained by substituting the solutions for inflation and output into the reaction function (10). You can check that $\mathbf{C} = \mathbf{C}_1^{-1}\mathbf{C}_2$, $\mathbf{c} = \mathbf{C}_1^{-1}\mathbf{C}_3$.

B.5 Derivation of equation (12)

The definition of $e_{1,t}$ is given by:

$$\begin{aligned} e_{1t} &= \int \hat{E}_t^i v_{t+1}^i di \\ v_t^i &= \sum_{\tau=t}^{\infty} \beta^{\tau-t} \hat{E}_t^i \{(1-\beta) y_\tau - \sigma(\beta i_\tau - \pi_\tau)\} \end{aligned}$$

Lets call e_{1t}^*, the implied value of e_{1t} when we actually replace the values of $\{y_t, \pi_t, i_t\}$ that are calculated using beliefs $\{e_{1t}, e_{2t}\}$

$$\begin{aligned} e_{1t}^* &= \int \hat{E}_t^i \sum_{\tau=t+1}^{\infty} \beta^{\tau-t-1} \hat{E}_{t+1}^i \{(1-\beta) y_\tau - \sigma(\beta i_\tau - \pi_\tau)\} di \\ &= (1-\beta) \sum_{\tau=t+1}^{\infty} \beta^{\tau-t-1} \int \hat{E}_t^i \{y_\tau - \frac{\sigma}{1-\beta}(\beta i_\tau - \pi_\tau)\} di \\ &= (1-\delta_1) \sum_{\tau=t+1}^{\infty} \delta_1^{\tau-t-1} \bar{E}_t \left\{ y_\tau - \frac{\sigma}{1-\beta}(\beta i_\tau - \pi_\tau) \right\} \end{aligned}$$

where $\delta_1 = \beta$ and $\bar{E}_t$ is the average expectation. For the second expectational value we follow the same steps, given the definitions provided in the text, but using the equation for the optimal price in (B.32), we have:

$$\begin{aligned} e_{2t} &= \int \hat{E}_t^j p_{t+1}^{*j} dj \\ p_t^{*j} &= \sum_{\tau=t}^{\infty} (\alpha\beta)^{\tau-t} \hat{E}_t^j \{\pi_\tau + (1-\alpha\beta)\xi y_\tau)\} \end{aligned}$$

we call e^*_{2t}, the implied value of e_{2t} when we actually replace the values of $\{y_t, \pi_t, i_t\}$ that are calculated using beliefs $\{e_{1t}, e_{2t}\}$

$$\begin{aligned}
e^*_{2t} &= \int \hat{E}^j_t \sum_{\tau=t+1}^{\infty} (\alpha\beta)^{\tau-t-1} \hat{E}^j_{t+1}\{\pi_\tau + (1-\alpha\beta)\xi y_\tau)\} dj \\
&= (1-\alpha\beta) \sum_{\tau=t+1}^{\infty} (\alpha\beta)^{\tau-t-1} \int \hat{E}^j_t \left\{ \frac{1}{1-\alpha\beta}\pi_t + \xi y_\tau \right\} dj \\
&= (1-\delta_2) \sum_{\tau=t+1}^{\infty} (\alpha\beta)^{\tau-t-1} \bar{E}_t \left\{ \frac{1}{1-\alpha\beta}\pi_t + \xi y_\tau \right\}
\end{aligned}$$

where $\delta_2 = \alpha\beta$ and $\bar{E}_t$ is the average expectation. It is assumed that the average expectation of households and firms are the same.

B.6 Derivation of equation (13): matrices m and M

Starting from the definitions of a_{1t} and a_{2t}, and replacing (10), we can write the system as:

$$\mathbf{a}_t = \mathbf{M}_1 \mathbf{x}_t + \mathbf{m}_1 \boldsymbol{\omega}_t$$

with

$$\mathbf{M}_1 = \begin{bmatrix} 1 - \frac{\beta\sigma\phi_y}{1-\beta} & \frac{\sigma}{1-\beta}(1-\sigma\phi_\pi) \\ \xi & \frac{1}{1-\alpha\beta} \end{bmatrix} \qquad \mathbf{m}_1 = \begin{bmatrix} 0 & -\frac{\beta\sigma}{1-\beta} \\ 0 & 0 \end{bmatrix}$$

we can replace $\mathbf{x}_t$ by (11) to get (13) with:

$$\mathbf{M} = \frac{1}{\Delta} \begin{bmatrix} \frac{1+\sigma\kappa-\beta\Delta}{1-\beta} & \frac{\sigma\beta(1-\alpha)(1+\sigma\phi_y-\phi_\pi)}{1-\beta} \\ \frac{\kappa}{(1-\alpha)(1-\alpha\beta)} & \frac{\beta(1+\sigma\phi_y-\alpha\Delta)}{1-\alpha\beta} \end{bmatrix}, \qquad \mathbf{m} = \frac{1}{\Delta} \begin{bmatrix} \frac{1+\sigma\kappa-\beta\Delta}{1-\beta} & -\frac{\sigma(1+\sigma\kappa)}{1-\beta} \\ \frac{\kappa}{(1-\alpha)(1-\alpha\beta)} & -\frac{\sigma\kappa}{(1-\alpha)(1-\alpha\beta)} \end{bmatrix}.$$

where you can check that $\mathbf{M} = \mathbf{M}_1\mathbf{C}$ and $\mathbf{m} = \mathbf{M}_1\mathbf{c} + \mathbf{m}_1$.

Putting together the equations (12) and (13) we can write the equation for $\mathbf{e}^*_t$ as follows:

$$\mathbf{e}^*_t = (\mathbf{I} - \boldsymbol{\Lambda}) \sum_{j=1}^{\infty} \boldsymbol{\Lambda}^{j-1} \left[\mathbf{M}\mathbf{e}_{t+j} + \mathbf{m}\boldsymbol{\omega}_{t+j}\right] \tag{B.33}$$

for all $t \geq 0$, where

$$\boldsymbol{\Lambda} \equiv \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$$

B.7 Derivations of the Simple Illustration

As given in the text, we have that the temporary equilibrium relations are given by:

$$\begin{aligned} y &= -\sigma i + e_1 \\ \pi &= \kappa y + (1-\alpha)\beta e_2 \\ i &= \bar{\imath} + \phi \pi \end{aligned}$$

Where (e_1, e_2) are given by their definitions, which in this case becomes:

$$\begin{aligned} e_1 &= \int \sum_{\tau=t+1}^{\infty} \beta^{\tau-t-1} \hat{E}_t^i \{(1-\beta) y_\tau - \sigma(\beta i_\tau - \pi_\tau)\} di = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t-1} \{(1-\beta) y^e - \sigma(\beta i^e - \pi^e)\} \\ &= y^e - \frac{\sigma}{1-\beta}(\beta i^e - \pi^e) \end{aligned}$$

and

$$\begin{aligned} e_2 &= \int \sum_{\tau=t+1}^{\infty} (\alpha\beta)^{\tau-t-1} \hat{E}_t^j \{\pi_\tau + (1-\alpha\beta)\xi y_\tau\} dj = \sum_{\tau=t+1}^{\infty} (\alpha\beta)^{\tau-t-1} \{\pi^e + (1-\alpha\beta)\xi y^e\} \\ &= \frac{\pi^e}{1-\alpha\beta} + \xi y^e \end{aligned}$$

To follow the notation given in the rest of the paper, lets replace the monetary policy in the other two equations to get:

$$\mathbf{C}_1 \mathbf{x} = \mathbf{C}_2 \mathbf{e} + \mathbf{c}_3' \bar{\imath}$$

with

$$\mathbf{x} \equiv \begin{bmatrix} y \\ \pi \end{bmatrix}, \qquad \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

and $\mathbf{C}_1$, $\mathbf{C}_2$ are the same as before just replacing $\phi_y = 0$ and $\phi_\pi = \phi$ and $\mathbf{c}_3'$ is the second column of $\mathbf{c}_3$. By inverting the first matrix, we get the equivalent to (11)

$$\mathbf{x} = \mathbf{C}\mathbf{e} + \mathbf{c}_2 \bar{\imath}$$

with $\mathbf{c}_2$ the second column of matrix $\mathbf{c}$ and again the coefficients are replaced so that $\phi_y = 0$ and $\phi_\pi = \phi$. Given $\mathbf{e}$, we have that the values of the endogenous variables is given by the above system and (10). To go to the next step and update the beliefs, we need the values for (e_1^*, e_2^*). Note in (12) that in this case $e_i^* = a_i$, since the expectation of all future variables is the same.

By their definitions, we have then:

$$\begin{aligned} e_1^* &= y - \frac{\sigma}{1-\beta}(\beta i - \pi) \\ e_2^* &= \frac{\pi}{1-\alpha\beta} + \xi y \end{aligned}$$

Note that this is the same as the equations for (e_1, e_2) given previously, just replacing the actual values by the expected values. By replacing the equation for the interest rate, we can write this as:

$$\mathbf{e}^* = \mathbf{M}_1\mathbf{x} + \mathbf{m}_1'\bar{\imath}$$

with

$$\mathbf{e}^* = \begin{bmatrix} e_1^* \\ e_2^* \end{bmatrix}$$

where $\mathbf{M}_1$ is the same as the one defined in a previous subsection of this Appendix, just replacing $\phi_y = 0$ and $\phi_\pi = \phi$ and $\mathbf{m}_1'$ is the second column of matrix $\mathbf{m}_1$. By replacing the TE relations can be written as:

$$\mathbf{e}^* = \mathbf{M}\mathbf{e} + \mathbf{m}_2\bar{\imath}$$

with $\mathbf{M}$, the same as before, just replacing $\phi_y = 0$ and $\phi_\pi = \phi$, and $\mathbf{m}_2$ is the second column of $\mathbf{m}$ replacing the same parameters as in $\mathbf{M}$. Replacing $\mathbf{e}^*$ in (14) gives you

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{e}^* - \mathbf{e} = \mathbf{M}\mathbf{e} + \mathbf{m}_2\bar{\imath} - \mathbf{e} \\ &= (\mathbf{M} - \mathbf{I})(\mathbf{e} - (\mathbf{I} - \mathbf{M})^{-1}\mathbf{m}_2\bar{\imath}) \end{aligned}$$

which becomes (20) since $\bar{\mathbf{e}}$ is the solution of the previous equation by setting $\dot{\mathbf{e}}$ to zero:

$$\begin{aligned} (\mathbf{I} - \mathbf{M})\bar{\mathbf{e}} &= \mathbf{m}_2\bar{\imath} \\ \bar{\mathbf{e}} &= (\mathbf{I} - \mathbf{M})^{-1}\mathbf{m}_2\bar{\imath} \end{aligned}$$

which is the rest point of the previous system as long as $(\mathbf{I} - \mathbf{M})$ is invertible.

The properties of the eigenvalues of $\mathbf{M} - \mathbf{I}$ are discussed in section C. It is shown that the real parts of $\mathbf{M} - \mathbf{I}$ are negative as long as the Taylor principle is satisfied, which in this case is $\phi > 1$. When the condition $\phi > 1$ is not met, one of the eigenvalues is positive.

B.8 Derivation of PFE equations

B.8.1 Neo-Keynesian IS curve: equation (23)

Starting from (4) we aggregate over households and we get:

$$\begin{aligned} y_t &= \sum_{\tau=t}^{\infty} \beta^{\tau-t} E_t\{(1-\beta)y_\tau - \beta\sigma(i_\tau - \pi_{\tau+1} - \rho_\tau)\} \\ &= (1-\beta)y_t - \beta\sigma(i_t - \pi_{t+1} - \rho_t) + \beta E_t y_{t+1} \end{aligned}$$

Which simplifying and rearranging gives you equation (23). To get to this same equation from (7) takes a little longer and you need to rearrange more terms:

$$\begin{aligned} y_t &= \sigma \sum_{\tau=t}^{\infty} \beta^{\tau-t}\rho_\tau - \sigma i_t + \sum_{\tau=t+1}^{\infty} \beta^{\tau-t-1} E_t((1-\beta)y_\tau - \sigma(\beta i_\tau - \pi_\tau)) \\ y_t &= (1-\beta)y_t + \beta y_t \\ &= (1-\beta)y_t + \beta\left(\sigma \sum_{\tau=t}^{\infty} \beta^{\tau-t}\rho_\tau - \sigma i_t + \sum_{\tau=t+1}^{\infty} \beta^{\tau-(t+1)} E_t((1-\beta)y_\tau - \sigma(\beta i_\tau - \pi_\tau))\right) \\ &= (1-\beta)y_t - \beta\sigma(i_t - \rho_t - E_t\pi_{t+1}) + \beta\left(\sum_{\tau=t+1}^{\infty} \beta^{\tau-(t+1)} E_t\left[(1-\beta)y_\tau - \beta\sigma(i_\tau - \rho_\tau - \pi_{\tau+1})\right]\right) \\ &= (1-\beta)y_t - \beta\sigma(i_t - \rho_t - E_t\pi_{t+1}) + \beta E_t y_{t+1} \end{aligned}$$

And rearranging and dividing by β gives you equation (23).

B.8.2 Neo-Keynesian Phillips curve: equation (24)

First note that equation (B.31) is:

$$\pi_t = (1-\alpha)p_t^*$$

Since p_t^{*j} is the same for all j. Using this and replacing in equation (B.32), we get:

$$\frac{\pi_t}{1-\alpha} = \pi_t + (1-\alpha\beta)\xi y_t + \alpha\beta\frac{E_t\pi_{t+1}}{1-\alpha}$$

which, rearranging terms and defining κ gives you equation (24).

B.8.3 Derivation of the 2x2 system of the PFE and and equation (25)

By replacing equation (10) in equations (23) and (24), we get the system:

$$\mathbf{C}_1\mathbf{x}_t = \mathbf{A}_2\mathbf{x}_{t+1} + \mathbf{a}(\rho_t - \bar{\imath}_t)$$

with

$$\mathbf{A}_2 = \begin{bmatrix} 1 & \sigma \\ 0 & \beta \end{bmatrix} \qquad \mathbf{a} = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}$$

By inverting and pre-multiplying $\mathbf{A}_2$, you can write this system as:

$$\mathbf{x}_t \;=\; \mathbf{B}\,\mathbf{x}_{t+1} + \mathbf{b}\,(\rho_t - \bar{\imath}_t) \tag{B.34}$$

where we define

$$\mathbf{B} = \tfrac{1}{\Delta}\begin{bmatrix} 1 & \sigma(1-\beta\phi_\pi) \\ \kappa & \sigma\kappa + \beta(1+\sigma\phi_y) \end{bmatrix}, \qquad \mathbf{b} = \tfrac{1}{\Delta}\begin{bmatrix} \sigma \\ \sigma\kappa \end{bmatrix}.$$

As shown in section C, when (22) is satisfied, this system has a unique bounded solution, since both eigenvalues of matrix $\mathbf{B}$ have modulus less than 1. This solution is given by (25) with

$$\boldsymbol{\zeta}_j = \mathbf{B}^j\mathbf{b}$$

To obtain the same 2x2 system from the equations defining the Temporary equilibrium, we need to impose $\mathbf{e}_t$ must equal $\mathbf{e}_t^*$ for all t. From (B.33) it follows that a sequence of vectors of expectations $\{\mathbf{e}_t\}$ constitute PFE expectations if and only if

$$\begin{aligned} \mathbf{e}_t \quad &= \quad \mathbf{e}_t^* \;=\; (\mathbf{I}-\boldsymbol{\Lambda})\sum_{j=1}^{\infty}\boldsymbol{\Lambda}^{j-1}\left[\mathbf{M}\mathbf{e}_{t+j} + \mathbf{m}\boldsymbol{\omega}_{t+j}\right] \\ &= \quad (\mathbf{I}-\boldsymbol{\Lambda})(\mathbf{M}\mathbf{e}_{t+1} + \mathbf{m}\boldsymbol{\omega}_{t+1}) \;+\; \boldsymbol{\Lambda}\,\mathbf{e}_{t+1} \\ &= \quad \left[(\mathbf{I}-\boldsymbol{\Lambda})\mathbf{M} \;+\; \boldsymbol{\Lambda}\right]\mathbf{e}_{t+1} \;+\; (\mathbf{I}-\boldsymbol{\Lambda})\mathbf{m}\,\boldsymbol{\omega}_{t+1} \end{aligned} \tag{B.35}$$

for all $t \geq 0$.

The dynamics implied by (B.35) are in fact equivalent to those implied by (B.34). Using (11) together with (B.35) implies that the PFE dynamics of output and inflation must satisfy

$$\begin{aligned} \mathbf{x}_t \quad &= \quad \mathbf{C}\left[(\mathbf{I}-\boldsymbol{\Lambda})\mathbf{M} \;+\; \boldsymbol{\Lambda}\right]\mathbf{e}_{t+1} \;+\; \mathbf{C}(\mathbf{I}-\boldsymbol{\Lambda})\mathbf{m}\,\boldsymbol{\omega}_{t+1} \;+\mathbf{c}\,\boldsymbol{\omega}_t \\ &= \quad \mathbf{C}\left[(\mathbf{I}-\boldsymbol{\Lambda})\mathbf{M} \;+\; \boldsymbol{\Lambda}\right]\mathbf{C}^{-1}\left[\mathbf{x}_{t+1} \;-\; \mathbf{c}\boldsymbol{\omega}_{t+1}\right] \;+\; \mathbf{C}(\mathbf{I}-\boldsymbol{\Lambda})\mathbf{m}\,\boldsymbol{\omega}_{t+1} \;+\mathbf{c}\,\boldsymbol{\omega}_t. \end{aligned}$$

But this relation is in fact equivalent to (B.34), given that our definitions above imply that

$$\begin{aligned}
\mathbf{C}\left[(\mathbf{I}-\mathbf{\Lambda})\mathbf{M} + \mathbf{\Lambda}\right]\mathbf{C}^{-1} &= \mathbf{B}, \qquad \text{(B.36)}\\
\mathbf{C}\,(\mathbf{I}-\mathbf{\Lambda})\mathbf{m} &= \mathbf{Bc} + \mathbf{b}\cdot[-\beta\sigma^{-1} \;\; 0],\\
\mathbf{c} &= \mathbf{b}\cdot[\sigma^{-1} \;\; -1].
\end{aligned}$$

B.9 Derivation of equations (27)-(30)

Since from $t \geq T$ $\bar{\imath}_t = \bar{\imath}_{LR}$ and by assumption $g_t = 0$ for all t, equation (14) can be written as:

$$\dot{\mathbf{e}}_{LR} = [\mathbf{M}-\mathbf{I}]\mathbf{e}_{LR} + \mathbf{m}_2\bar{\imath}_{LR}$$

Since, by equation (12) $e_i^* = a_i$, where the equation for $\mathbf{a}$ is given by (13) replacing $\mathbf{m}_2\bar{\imath}_{LR}$ instead of $\mathbf{m}\boldsymbol{\omega}$ since the first term in $\boldsymbol{\omega}$ is 0. If $\mathbf{M}-\mathbf{I}$ is invertible, the unique rest point of this system is given by (27), which is calculated using the previous equation setting $\dot{\mathbf{e}}_{LR} = 0$.

Given that the beliefs are started by $\mathbf{e}_{LR}(0) = 0$, which are the ones consistent with the steady state in which the inflation target π^* is achieved at all times, and assuming that $\mathbf{M}-\mathbf{I}$ is not singular, we can write the solution for general n as[53]

$$\mathbf{e}_{LR}(n) = [\mathbf{I} - \exp[n(\mathbf{M}-\mathbf{I})]]\bar{\mathbf{e}}_{LR} \qquad \text{(B.37)}$$

for all $n \geq 0$. As shown in section C, when the Taylor Principle (22) is satisfied, both eigenvalues of $\mathbf{M}-\mathbf{I}$ have negative real parts, and

$$\lim_{n\to\infty} \exp[n(\mathbf{M}-\mathbf{I})] = 0 \qquad \text{(B.38)}$$

It then follows (28).

For the periods before T, as stated in the text, we can calculate backwards the solution for any $t < T$, which depends on $\tau = T - t$. To do that, use (B.33) to get

$$\begin{aligned}
\mathbf{e}_t^* &= (\mathbf{I}-\mathbf{\Lambda})\sum_{j=1}^{\infty}\mathbf{\Lambda}^{j-1}\left[\mathbf{M}\mathbf{e}_{t+j} + \mathbf{m}\boldsymbol{\omega}_{t+j}\right]\\
&= (\mathbf{I}-\mathbf{\Lambda})\sum_{j=t+1}^{T-1}\mathbf{\Lambda}^{j-t-1}\left[\mathbf{M}\mathbf{e}_{t+j} + \mathbf{m}_2\bar{\imath}_{SR}\right] + \sum_{j=T}^{\infty}\mathbf{\Lambda}^{j-T-1}\left[\mathbf{M}\mathbf{e}_{LR} + \mathbf{m}_2\bar{\imath}_{LR}\right]
\end{aligned}$$

[53] See Hirsch and Smale (1974, p. 90).

We can also write this equivalently for $\mathbf{e}_\tau$ for $\tau \geq 1$, where $\tau = T - t$ as:

$$\mathbf{e}_\tau^* = (\mathbf{I} - \mathbf{\Lambda}) \sum_{j=1}^{\tau-1} \mathbf{\Lambda}^{j-1} \left[\mathbf{M}\mathbf{e}_{\tau-j} + \mathbf{m}_2 \bar{\imath}_{SR} \right] + \mathbf{\Lambda}^{\tau-1} \left[\mathbf{M}\mathbf{e}_{LR} + \mathbf{m}_2 \bar{\imath}_{LR} \right]$$

Using this, now we can write the differential equation (14) as

$$\mathbf{e}_\tau^*(n) = -\mathbf{e}_\tau(n) + (\mathbf{I} - \mathbf{\Lambda}) \sum_{j=1}^{\tau-1} \mathbf{\Lambda}^{j-1} \left[\mathbf{M}\mathbf{e}_{\tau-j} + \mathbf{m}_2 \bar{\imath}_{SR} \right] + \mathbf{\Lambda}^{\tau-1} \left[\mathbf{M}\mathbf{e}_{LR} + \mathbf{m}_2 \bar{\imath}_{LR} \right] \tag{B.39}$$

and integrate forward from $\mathbf{e}_\tau = 0$ for all $\tau \geq 1$ using the above solution for $\mathbf{e}_{LR}(n)$. This is done by first solving for $\tau = 1$ uniquely given $\mathbf{e}_{LR}(n)$, then for $\tau = 2$ and so on.

Equations (29) and (30) are the same as (B.37) and (27) just replacing LR by SR, since these equations are the behavior when the short run policy becomes permanent.

C Properties of Matrices

C.1 Properties of the Matrix M

A number of results turn on the eigenvalues of the matrix

$$\mathbf{M} - \mathbf{I} \;=\; \frac{1}{\Delta} \begin{bmatrix} -\frac{\sigma\phi_y + \sigma\kappa\phi_\pi - \sigma\kappa}{1-\beta} & \frac{(1-\alpha)\sigma\beta(1+\sigma\phi_y - \phi_\pi)}{1-\beta} \\ \frac{\kappa}{(1-\alpha)(1-\alpha\beta)} & \frac{\beta(1+\sigma\phi_y) - \Delta}{1-\alpha\beta} \end{bmatrix}.$$

We first note that the determinant of the matrix is given by

$$\mathrm{Det}\left[\mathbf{M} - \mathbf{I}\right] \;=\; \frac{\sigma\kappa}{\Delta(1-\beta)(1-\alpha\beta)} \left(\phi_\pi + \frac{(1-\beta)}{\kappa}\phi_y - 1 \right).$$

Under our sign assumptions, the factor pre-multiplying the factor in parentheses is necessarily positive. Hence the determinant is non-zero (and the matrix is non-singular) if

$$\phi_\pi \;+\; \frac{(1-\beta)}{\kappa}\phi_y \;-\; 1 \;\neq\; 0. \tag{C.40}$$

(In this case the steady-state vector of expectations (27) is well-defined, as asserted in the text.)

For any 2×2 real matrix $\mathbf{A}$, both eigenvalues have negative real part if and only if $\mathrm{Det}[\mathbf{A}] > 0$ and $\mathrm{Tr}[\mathbf{A}] < 0$.[54] From the result above, the first of these conditions is satisfied if the left-hand side of (C.40) is positive, which is to say, if the Taylor Principle (22) is satisfied. The trace of $\mathbf{M} - \mathbf{I}$ is given by

$$\mathrm{Tr}\,[\mathbf{M} - \mathbf{I}] \;=\; -\frac{1}{\Delta}\left(\frac{\sigma(\phi_y + \kappa\phi_\pi - \kappa)}{1-\beta} + \frac{\sigma\kappa\phi_\pi + (1-\beta)(1+\sigma\phi_y)}{1-\alpha\beta}\right).$$

The second term inside the parentheses is necessarily positive under our sign assumptions, and the first term is positive as well if the Taylor Principle is satisfied, since

$$\phi_y + \kappa\phi_\pi - \kappa \;=\; \kappa\left(\phi_\pi + \frac{\phi_y}{\kappa} - 1\right) \;>\; \kappa\left(\phi_\pi + \frac{\phi_y(1-\beta)}{\kappa} - 1\right) \;>\; 0. \tag{C.41}$$

Hence the Taylor Principle is a sufficient condition for $\mathrm{Tr}[\mathbf{M} - \mathbf{I}] < 0$. It follows that (given our other sign assumptions) the Taylor Principle is both necessary and sufficient for both eigenvalues of $\mathbf{M} - \mathbf{I}$ to have negative real part.

If instead the left-hand side of (C.40) is negative, $\mathrm{Det}[\mathbf{M} - \mathbf{I}] < 0$, and as a consequence the matrix must have two real eigenvalues of opposite sign.[55] Thus one eigenvalue is positive in this case, as asserted in the text. Note that this is the case that obtains if $\phi_\pi = \phi_y = 0$. Note also that in the case that $\phi_y = 0$, the condition becomes $\phi_\pi > 1$, which is the assumption in the Simple Illustration.

C.2 A Further Implication of the Taylor Principle

We are also interested in the eigenvalues of the related matrix $\mathbf{A}(\lambda)\mathbf{M} - \mathbf{I}$, where for an arbitrary real number $-1 \leq \lambda \leq 1$, we define

$$\mathbf{A}(\lambda) \;\equiv\; \begin{pmatrix} \frac{\lambda(1-\delta_1)}{1-\lambda\delta_1} & 0 \\ 0 & \frac{\lambda(1-\delta_2)}{1-\lambda\delta_2} \end{pmatrix}.$$

[54] See, for example, Hirsch and Smale (1974, p. 96).
[55] Again see Hirsch and Smale (1974, p. 96).

(Note that in the limiting case $\lambda = 1$, this reduces to the matrix $\mathbf{M} - \mathbf{I}$, just discussed.) In the case that the Taylor principle (22) is satisfied, we can show that for any $-1 \leq \lambda \leq 1$, both eigenvalues of $\mathbf{A}(\lambda)\mathbf{M} - \mathbf{I}$ have negative real part. This follows again from a consideration of the determinant and trace of the matrix (generalizing the above discussion).

Since

$$\mathbf{A}(\lambda)\mathbf{M} - \mathbf{I} \;=\; \frac{1}{\Delta}\begin{bmatrix} -\frac{\Delta-\lambda(1+\sigma\kappa)}{1-\beta\lambda} & -\frac{\sigma(1-\alpha)\beta(\phi_\pi-1-\sigma\phi_y)\lambda}{1-\beta\lambda} \\ \frac{\kappa\lambda}{(1-\alpha)(1-\alpha\beta\lambda)} & -\frac{\Delta-\beta\lambda(1+\sigma\phi_y)}{1-\alpha\beta\lambda} \end{bmatrix},$$

we have

$$\mathrm{Det}\left[\mathbf{A}(\lambda)\mathbf{M} - \mathbf{I}\right] \;=\; \frac{\Delta - \lambda(\beta(1+\sigma\phi_y) + 1 + \sigma\kappa) + \beta\lambda^2}{\Delta(1-\beta\lambda)(1-\alpha\beta\lambda)}.$$

Note that under our sign assumptions, the denominator is necessarily positive. The numerator defines a function $g(\lambda)$, a convex function (a parabola) with the properties

$$g'(1) \;=\; (\beta - 1) - \beta\sigma\phi_y - \kappa\sigma \;<\; 0$$

and

$$g(1) \;=\; \kappa\sigma\left(\phi_\pi + \frac{(1-\beta)}{\kappa}\phi_y - 1\right),$$

so that $g(1) > 0$ if and only if the Taylor Principle is satisfied. Hence the function $g(\lambda) > 0$ for all $\lambda \leq 1$, with the consequence that $\mathrm{Det}[\mathbf{A}(\lambda)\mathbf{M} - \mathbf{I}] > 0$ for all $|\lambda| \leq 1$, if and only if the Taylor Principle is satisfied.

The trace of the matrix is given by

$$\mathrm{Tr}\left[\mathbf{A}(\lambda)\mathbf{M} - \mathbf{I}\right] \;=\; -\frac{1}{\Delta}\left(\frac{\Delta - \lambda(1+\sigma\kappa)}{1-\beta\lambda} + \frac{\Delta - \beta\lambda(1+\sigma\phi_y)}{1-\alpha\beta\lambda}\right).$$

The denominators of both terms inside the parentheses are positive for all $|\lambda| \leq 1$, and we necessarily have $\Delta > 0$ under our sign assumptions as well. The numerator of the first term inside the parentheses is also positive, since

$$\Delta - \lambda(1+\sigma\kappa) \;=\; \sigma\left[\kappa\phi_\pi + \phi_y - \kappa\right] \;+\; (1-\lambda)(1+\sigma\kappa) \;\geq \sigma\left[\kappa\phi_\pi + \phi_y - \kappa\right] \;>\; 0$$

if the Taylor Principle is satisfied, again using (C.41). And the numerator of the second term inside the parentheses is positive as well, since

$$\Delta - \beta\lambda(1 + \sigma\phi_y) \ = \ (1 - \beta\lambda)(1 + \sigma\phi_y) \ + \ \kappa\sigma\phi_\pi \ > \ 0$$

under our sign assumptions. Thus the Taylor Principle is also a sufficient condition for $\mathrm{Tr}[\mathbf{A}(\lambda)\mathbf{M} - \mathbf{I}] \ < \ 0$ for all $|\lambda| \leq 1$.

It then follows that the Taylor Principle is necessary and sufficient for both eigenvalues of the matrix $\mathbf{A}(\lambda)\mathbf{M} - \mathbf{I}$ to have negative real part, in the case of any $|\lambda| < 1$. We use this result in the proof of Proposition 1.

C.3 Properties of the Matrix B

Necessary and sufficient conditions for both eigenvalues of a 2×2 matrix $\mathbf{B}$ to have modulus less than 1 are that (i) $\mathrm{Det}[\mathbf{B}] \ < \ \mathbf{1}$; (ii) $\mathrm{Det}[\mathbf{B}] + \mathrm{Tr}[\mathbf{B}] \ > -\mathbf{1}$; and (iii) $\mathrm{Det}[\mathbf{B}] \ - \ \mathrm{Tr}[\mathbf{B}] \ > -\mathbf{1}$. In the case of the matrix $\mathbf{B}$ defined above, we observe that

$$\Delta\, \mathrm{Det}\,[\mathbf{B}] \ = \ \beta, \tag{C.42}$$

$$\Delta\, \mathrm{Tr}\,[\mathbf{B}] \ = \ 1 \ + \ \kappa\sigma \ + \ \beta(1 + \sigma\phi_y).$$

From these facts we observe that our general sign assumptions imply that

$$\Delta\, \mathrm{Det}\,[\mathbf{B}] \ < \ \Delta,$$

$$\Delta \ (\mathrm{Det}\,[\mathbf{B}] \ + \ \mathrm{Tr}\,[\mathbf{B}] \ + 1)) \ > \ 0.$$

Thus (since Δ is positive) conditions (i) and (ii) from the previous paragraph necessarily hold. We also find that

$$\Delta\, (\mathrm{Det}\,[\mathbf{B}] \ - \ \mathrm{Tr}\,[\mathbf{B}] \ + 1) \ = \ \kappa\sigma \ \left[\phi_\pi \ + \ \left(\frac{1 - \beta}{\kappa}\right) \phi_y \ - \ 1\right],$$

from which it follows that condition (iii) is also satisfied if and only if the quantity in the square brackets is positive. Thus we conclude that both eigenvalues of $\mathbf{B}$ have modulus less than 1 if and only if the Taylor Principle (22) is satisfied.

In the case that the Taylor Principle is violated (as in the case of a fixed interest rate, in which case $\phi_\pi = \phi_y = 0$), since $\mathrm{Det}[\mathbf{B}] = \boldsymbol{\mu_1 \mu_2}$ and $\mathrm{Tr}[\mathbf{B}] = \boldsymbol{\mu_1} + \boldsymbol{\mu_2}$, where (μ_1, μ_2) are the two eigenvalues of B, the fact that condition (iii) fails to hold implies that

$$(\mu_1 - 1)(\mu_2 - 1) \ < \ 0. \tag{C.43}$$

This condition is inconsistent with the eigenvalues being a pair of complex conjugates, so in this case there must be two real eigenvalues. Condition (C.43) further implies that one must be greater than 1, while the other is less than 1. Condition (C.42) implies that Det$[\mathbf{B}] > \mathbf{0}$, which requires that the two real eigenvalues both be non-zero and of the same sign; hence both must be positive. Thus when the Taylor Principle is violated (i.e., the quantity in (C.40) is negative), there are two real eigenvalues satisfying

$$0 < \mu_1 < 1 < \mu_2,$$

as asserted in section 2.2.

We further note that in this case, $\mathbf{v}_2'$, the (real) left eigenvector associated with eigenvalue μ_2, must be such that $\mathbf{v}_2'\mathbf{b} \neq 0$ (a result that is relied upon in section 4.2). The vector $\mathbf{v}_2' \neq 0$ must satisfy

$$\mathbf{v}_2' [\mathbf{B} - \mu_2 \mathbf{I}] = 0$$

to be a left eigenvector. The first column of this relation implies that $(1 - \mu_2)\mathbf{v}_{2,1} + \kappa \mathbf{v}_{2,2} = 0$, where we use the notation $\mathbf{v}_{2,j}$ for the jth element of eigenvector $\mathbf{v}_2'$. Since $\kappa > 0$ and $\mu_2 > 1$, this requires that $\mathbf{v}_{2,1}$ and $\mathbf{v}_{2,2}$ must both be non-zero and have the same sign. But since both elements of $\mathbf{b}$ have the same sign, this implies that $\mathbf{v}_2'\mathbf{b} \neq 0$.

Finally, we note that whenever (C.40) holds, regardless of the sign, the eigenvalues must satisfy

$$(\mu_1 - 1)(\mu_2 - 1) \neq 0,$$

so that $\mathbf{B}$ has no eigenvalue equal exactly to 1. This means that the matrix $\mathbf{B} - \mathbf{I}$ must be non-singular, which is the condition needed for existence of unique steady-state levels of output and inflation consistent with a PFE. In the case of constant fundamentals $\boldsymbol{\omega}_t = \bar{\boldsymbol{\omega}}$ for all t, the unique steady-state solution to (B.34) is then given by $\mathbf{x}_t = \bar{\mathbf{x}}$ for all t, where

$$\bar{\mathbf{x}} \equiv (\mathbf{I} - \mathbf{B})^{-1} \mathbf{b} [(1 - \beta)\sigma^{-1}\bar{g} - \bar{\imath}]. \tag{C.44}$$

Note that condition (C.40) is also the condition under which $\mathbf{M} - \mathbf{I}$ is non-singular, as shown above. Moreover, since $\mathbf{I} - \boldsymbol{\Lambda}$ is non-singular, $\mathbf{M} - \mathbf{I}$ is non-singular if and only if $(\mathbf{I} - \boldsymbol{\Lambda})(\mathbf{M} - \mathbf{I}) = [(\mathbf{I} - \boldsymbol{\Lambda})\mathbf{M} + \boldsymbol{\Lambda}] - \mathbf{I}$ is non-singular. This is the condition under which equation (B.35) has a unique steady-state solution, in which $\mathbf{e}_t = \bar{\mathbf{e}}$ for all t, with

$$\bar{\mathbf{e}} \equiv (\mathbf{I} - \mathbf{M})^{-1} \mathbf{m} \bar{\boldsymbol{\omega}}.$$

This solution for steady-state PFE expectations is consistent with (C.44) because of the identities linking the **M** and **B** matrices noted above.

D Proofs of Propositions

D.1 Proof of Proposition 1

Under the hypotheses of the proposition, there must exist a date $\bar{T}$ such that the fundamental disturbances $\{\boldsymbol{\omega}_t\}$ can be written in the form

$$\boldsymbol{\omega}_t \;=\; \boldsymbol{\omega}_\infty + \sum_{k=1}^{K} \mathbf{a}_{\omega,k} \lambda_k^{t-\bar{T}}$$

for all $t \geq \bar{T}$, and the initial conjecture can also be written in the form

$$\mathbf{e}_t(0) \;=\; \mathbf{e}_\infty(0) + \sum_{k=1}^{K} \mathbf{a}_{e,k}(0) \lambda_k^{t-\bar{T}}$$

for all $t \geq \bar{T}$, where $|\lambda_k| < 1$ for all $k = 1, \ldots, K$. (There is no loss of generality in using the same date $\bar{T}$ and the same finite set of convergence rates $\{\lambda_k\}$ in both expressions.) With a driving process and initial condition of this special form, the solution to the system of differential equations (14) will be of the form

$$\mathbf{e}_t(n) \;=\; \mathbf{e}_\infty(n) + \sum_{k=1}^{K} \mathbf{a}_{e,k}(n) \lambda_k^{t-\bar{T}}$$

for all $t \geq \bar{T}$, for each $n \geq 0$. We then need simply determine the evolution as n increases of the finite set of values $\mathbf{e}_t(n)$ for $0 \leq t \leq T-1$, together with the finite set of coefficients $\mathbf{e}_\infty(n)$ and $\mathbf{a}_{e,k}(n)$. This is a set of $2(\bar{T} + K + 1)$ functions of n, which we write as the vector-valued function $\mathbf{e}(n)$ in the text.

In the case of any belief sequences and disturbances of the form assumed in the above paragraph, it follows from (B.33) that the implied correct beliefs will be of the form

$$\mathbf{e}_t^*(n) \;=\; \mathbf{e}_\infty^*(n) + \sum_{k=1}^{K} \mathbf{a}_{e,k}^*(n) \lambda_k^{t-\bar{T}}$$

for all $t \geq \bar{T}$, where

$$\mathbf{e}^*_\infty(n) \;=\; \mathbf{M}\,\mathbf{e}_\infty(n) \;+\; \mathbf{m}\,\boldsymbol{\omega}_\infty,$$

and

$$\mathbf{a}^*_{e,k}(n) \;=\; \mathbf{A}(\lambda_k)\,[\mathbf{M}\,\mathbf{a}_{e,k}(n) \;+\; \mathbf{m}\,\mathbf{a}_{\omega,k}]$$

for each $k = 1, \ldots, K$. We further observe that for any $t < \bar{T}$,

$$\begin{aligned}
\mathbf{e}^*_t(n) \;&=\; (\mathbf{I} - \boldsymbol{\Lambda}) \sum_{j=1}^{\bar{T}-t-1} \boldsymbol{\Lambda}^j [\mathbf{M}\mathbf{e}_{t+j}(n) + \mathbf{m}\boldsymbol{\omega}_{t+j}] + \boldsymbol{\Lambda}^{\bar{T}-t-1}\,[\mathbf{M}\mathbf{e}_\infty(n) \;+\; \mathbf{m}\boldsymbol{\omega}_\infty] \\
&\quad + \;\sum_{k=1}^{K} \lambda_k^{-1} \boldsymbol{\Lambda}^{\bar{T}-t-1}\,\mathbf{A}(\lambda_k)\,[\mathbf{M}\mathbf{a}_{e,k}(n) \;+\; \mathbf{m}\mathbf{a}_{\omega,k}].
\end{aligned}$$

Thus the sequence $\{\mathbf{e}^*_t(n)\}$ can also be summarized by a set of $2(\bar{T} + K + 1)$ functions of n, and each of these is a linear function of the elements of the vectors $\mathbf{e}(n)$ and $\boldsymbol{\omega}$.

It then follows that the dynamics (14) can be written in the more compact form

$$\dot{\mathbf{e}}(n) \;=\; \mathbf{V}\,\mathbf{e}(n) \;+\mathbf{W}\,\boldsymbol{\omega}, \tag{D.45}$$

where the elements of the matrices $\mathbf{V}$ and $\mathbf{W}$ are given by the coefficients of the equations in the previous paragraph. Suppose that we order the elements of $\mathbf{e}(n)$ as follows: the first two elements are the elements of $\mathbf{e}_0$, the next two elements are the elements of $\mathbf{e}_1$, and so on, through the elements of $\mathbf{e}_{\bar{T}-1}$; the next two elements are the elements of $\mathbf{a}_{e,1}$, the two elements after that are the elements of $\mathbf{a}_{e,2}$, and so on, through the elements of $\mathbf{a}_{e,K}$; and the final two elements are the elements of $\mathbf{e}_\infty$. Then we observe that the matrix $\mathbf{V}$ is of the form

$$\mathbf{V} \;=\; \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{0} & \mathbf{V}_{22} \end{bmatrix}, \tag{D.46}$$

where the first $2\bar{T}$ rows are partitioned from the last $2(K + 1)$ rows, and the columns are similarly partitioned.

Moreover, the block $\mathbf{V}_{11}$ of the matrix is of the block upper-triangular form

$$\mathbf{V}_{11} \;=\; \begin{bmatrix} -\mathbf{I} & \mathbf{v}_{12} & \cdots & \mathbf{v}_{1,\bar{T}-1} & \mathbf{v}_{1,\bar{T}} \\ \mathbf{0} & -\mathbf{I} & \cdots & \mathbf{v}_{2,\bar{T}-1} & \mathbf{v}_{2,\bar{T}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{I} & \mathbf{v}_{\bar{T}-1,\bar{T}} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{I} \end{bmatrix}, \tag{D.47}$$

where now each block of the matrix is 2×2. Furthermore, when $\mathbf{V}_{22}$ is similarly partitioned into 2×2 blocks, it takes the block-diagonal form

$$\mathbf{V}_{22} = \begin{bmatrix} \mathbf{A}(\lambda_1)\mathbf{M} - \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{A}(\lambda_K)\mathbf{M} - \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{M} - \mathbf{I} \end{bmatrix}. \tag{D.48}$$

These results allow us to determine the eigenvalues of $\mathbf{V}$. The block-triangular form (D.46) implies that the eigenvalues of $\mathbf{V}$ consist of the $2\bar{T}$ eigenvalues of $\mathbf{V}_{11}$ and the $2(K+1)$ eigenvalues of $\mathbf{V}_{22}$ (the two diagonal blocks). Similarly, the block-triangular form (D.47) implies that the eigenvalues of $\mathbf{V}_{11}$ consist of the eigenvalues of the diagonal blocks (each of which is $-\mathbf{I}$), which means that the eigenvalue -1 is repeated $2\bar{T}$ times. Finally, the block-diagonal form (D.48) implies that the eigenvalues of $\mathbf{V}_{22}$ consist of the eigenvalues of the diagonal blocks: the two eigenvalues of $\mathbf{A}(\lambda_k)\mathbf{M} - \mathbf{I}$, for each $k = 1, \ldots, K$, and the two eigenvalues of $\mathbf{M} - \mathbf{I}$.

Using the results in section C.1, it follows from the hypothesis that the reaction function coefficients satisfy (22) and the hypothesis that $|\lambda_k| < 1$ for each k that all of the eigenvalues of $\mathbf{M} - \mathbf{I}$ and of each of the matrices $\mathbf{A}(\lambda_k)\mathbf{M} - \mathbf{I}$ have negative real part. Since all of the other eigenvalues of $\mathbf{V}$ are equal to -1, all $2(\bar{T} + K + 1)$ eigenvalues of $\mathbf{V}$ have negative real part. This implies that $\mathbf{V}$ is non-singular, so that there is a unique rest point for the dynamics (D.45), defined by:

$$\mathbf{e}^{PF} \equiv -\mathbf{V}^{-1}\mathbf{W}\,\boldsymbol{\omega}.$$

It also implies that the dynamics (D.45) converge asymptotically to that rest point as n goes to infinity, for any initial condition $\mathbf{e}(0)$ (Hirsch and Smale, 1974, pp. 90-95).[56]

The rest point to which $\mathbf{e}(n)$ converges is easily seen to correspond to the unique PFE that belongs to the same linear space L^2. Beliefs in L^2 constitute a PFE if and only if $\mathbf{e}^* = \mathbf{e}$. From our characterization above of $\mathbf{e}^*$, this is equivalent to the requirement that $\mathbf{V}\,\mathbf{e} + \mathbf{W} = \mathbf{0}$, which holds if and only

[56] Of course, it is important to recognize that this result only establishes convergence for initial conjectures that belong to the linear space L^2. The result also only establishes convergence under the assumption that the linear dynamics (D.45) apply at all times; this depends on assuming that the reaction function (10) can be implemented at all times, which requires that the ZLB never binds. Thus we only established convergence for all those initial conjectures such that the dynamics implied by (14) never cause the ZLB to bind. There is however a large set of initial conditions for which this is true, given that the unconstrained dynamics are asymptotically convergent.

if $\mathbf{e} = \mathbf{e}^{PF}$, the unique rest point of the system (D.45).

Finally, the paths of output and inflation in any reflective equilibrium are given by (11), given the solution for $\{\mathbf{e}_t(n)\}$. Using (10), one obtains a similar linear equation for the nominal interest rate each period. It then follows that for any t, the reflective equilibrium values for y_t, π_t, and i_t converge to the FS-PFE values as n is made large. Furthermore, the complete sequences of values for these three variables for any value of n depend on only the finite number of elements of the vector $\mathbf{e}(n)$, in such a way that for any $\epsilon > 0$, there exists an $\tilde{\epsilon} > 0$ such that it is guaranteed that each of the variables y_t, π_t, and i_t are within distance ϵ of their FS-PFE values for all t as long as $|\mathbf{e}(n) - \mathbf{e}^{PF}| < \tilde{\epsilon}$. The convergence of $\mathbf{e}(n)$ to $\mathbf{e}^{PF}$ then implies the existence of a finite $n(\epsilon)$ for which the latter condition is satisfied, regardless of how small $\tilde{\epsilon}$ needs to be. This proves the proposition.

D.2 Proof of Proposition 2

It has already been shown in the text that under the assumptions of the proposition, we have $\mathbf{e}_t(n) = \mathbf{e}_{LR}(n)$ for all $t \geq T$, where $\mathbf{e}_{LR}(n)$ is given by (B.37). It has also been shown that for any $\tau \geq 1$, the solution for $\mathbf{e}_\tau(n)$, where $\tau \equiv T - t$ is the number of periods remaining until the regime change, is independent of T. The functions $\{\mathbf{e}_\tau(n)\}$ further satisfy the system of differential equations

$$\begin{aligned}\dot{\mathbf{e}}_\tau(n) \;=\; -\mathbf{e}_\tau(n) \;&+\; (\mathbf{I} - \mathbf{\Lambda}) \sum_{j=1}^{\tau-1} \mathbf{\Lambda}^{j-1} \left[\mathbf{M}\mathbf{e}_{\tau-j}(n) \,+\, \mathbf{m}_2 \bar{\imath}_{SR}\right] \\ &+\; \mathbf{\Lambda}^{\tau-1} \left[\mathbf{M}\mathbf{e}_{LR}(n) \,+\, \mathbf{m}_2 \bar{\imath}_{LR}\right] \qquad \text{(D.49)}\end{aligned}$$

together with the initial conditions $\mathbf{e}_\tau(0) = 0$ for all $\tau \geq 1$. (Equation (D.49) repeats equation (B.39).)

We wish to calculate the behavior of the solution to this system as $\tau \to \infty$ for an arbitrary value of n. It is convenient to use the method of z-transforms (Jury, 1964). For any n, let the z-transform of the sequence $\{\mathbf{e}_\tau(n)\}$ for $\tau \geq 1$ be defined as the function

$$\mathbf{X}_n(z) \;\equiv\; \sum_{\tau=1}^{\infty} \mathbf{e}_\tau(n)\, z^{1-\tau}. \qquad \text{(D.50)}$$

Here $\mathbf{X}_n(z)$ is a vector-valued function; each element is a function of the complex number z, defined for complex numbers $|z| > 1/\rho$, where ρ is the minimum of the radii of the convergence of the two series.

Differentiating (D.50) with respect to n, and substituting (D.49) for $\dot{\mathbf{e}}_\tau(n)$ in the resulting equation, we obtain an evolution equation for the z-transform:

$$\begin{aligned}
\dot{\mathbf{X}}_n(z) \;\; &= \;\; -\sum_{\tau=1}^{\infty} \mathbf{e}_\tau(n) z^{1-\tau} \;+\; (\mathbf{I}-\mathbf{\Lambda}) \sum_{j=0}^{\infty} \mathbf{\Lambda}^j z^{-j} \left[\mathbf{M} \sum_{\tau=1}^{\infty} \mathbf{e}_\tau(n) z^{-\tau} \;+\; \mathbf{m}_2 \bar{\imath}_{SR} \sum_{\tau=1}^{\infty} z^{-\tau} \right] \\
&\qquad\qquad + \sum_{j=0}^{\infty} \mathbf{\Lambda}^j z^{-j} \left[\mathbf{M}\, \mathbf{e}_{LR}(n) \;+\; \mathbf{m}_2 \bar{\imath}_{LR} \right] \\
&= \;\; -\mathbf{X}_n(z) \;+\; (\mathbf{I}-\mathbf{\Lambda})(\mathbf{I}-\mathbf{\Lambda} z^{-1})^{-1} \left[z^{-1} \mathbf{M}\, \mathbf{X}_n(z) \;+\; (z-1)^{-1}\, \mathbf{m}_2 \bar{\imath}_{SR} \right] \\
&\qquad + (\mathbf{I}-\mathbf{\Lambda} z^{-1})^{-1} \left[\mathbf{M} \mathbf{e}_{LR}(n) \;+\; \mathbf{m}_2 \bar{\imath}_{LR} \right], \qquad \text{(D.51)}
\end{aligned}$$

which holds for any $n > 0$ and any z in the region of convergence. We note that the right-hand side of (D.51) is well-defined for all $|z| > 1$.

The z-transform of the initial condition is simply $\mathbf{X}_0(z) = \mathbf{0}$ for all z. Thus we wish to find functions $\{\mathbf{X}_n(z)\}$ for all $n \geq 0$, each defined on the region $|z| > 1$, that satisfy (D.51) for all n and all $|z| > 1$, together with the initial condition $\mathbf{X}_0(z) = \mathbf{0}$ for all z. If we can find such a solution, then for any n we can find the implied sequence $\{\mathbf{e}_t(n)\}$ by inverse z-transformation of the function $\mathbf{X}_n(z)$.

We note that the dynamics of $\mathbf{X}_n(z)$ implied by (D.51) is independent for each value of z. (This is the advantage of z-transformation of the original system of equations (D.49).) Thus for each value of z such that $|z| > 1$, we have an independent first-order ordinary differential equation to solve for $\mathbf{X}_n(z)$, with the single initial condition $\mathbf{X}_0(z) = \mathbf{0}$. This equation has a closed-form solution for each z, given by

$$\begin{aligned}
\mathbf{X}_n(z) \;\; &= \;\; (1-z^{-1})^{-1} \left[\mathbf{I} \;-\; \exp(n(\mathbf{M}-\mathbf{I})) \right] (\mathbf{I}-\mathbf{M})^{-1} \cdot \mathbf{m}_2 \bar{\imath}_{LR} \\
&\quad + (z-1)^{-1} \left[\mathbf{I} \;-\; \exp(-n \mathbf{\Phi}(z)) \right] \mathbf{\Phi}(z)^{-1} (\mathbf{I}-\mathbf{\Lambda})(\mathbf{I}-\mathbf{\Lambda} z^{-1})^{-1} \\
&\qquad \cdot \mathbf{m}_2 (\bar{\imath}_{SR} - \bar{\imath}_{LR}) \qquad \text{(D.52)}
\end{aligned}$$

for all $n \geq 0$, where

$$\mathbf{\Phi}(z) \;\equiv\; \mathbf{I} \;-\; (\mathbf{I}-\mathbf{\Lambda})(\mathbf{I}-\mathbf{\Lambda} z^{-1})^{-1} z^{-1}\, \mathbf{M}.$$

Note also that the expression on the right-hand side of (D.52) is an analytic function of z everywhere in the complex plane outside the unit circle, and can be expressed as a sum of powers of z^{-1} that converges everywhere in that region. Such a series expansion of $\mathbf{X}_n(z)$ for any n allows us to recover the series of coefficients $\{\mathbf{e}_\tau(n)\}$ associated with the reflective equilibrium with degree of reflection n.

For any value of $n \geq 0$, we are interested in computing

$$\mathbf{e}_{SR}(n) \;\equiv\; \lim_{T\to\infty} \mathbf{e}_t(n) \;=\; \lim_{\tau\to\infty} \mathbf{e}_\tau(n).$$

The final value theorem for z-transforms[57] implies that

$$\lim_{\tau\to\infty} \mathbf{e}_\tau(n) \;=\; \lim_{z\to 1}(z-1)\mathbf{X}_n(z)$$

if the limit on the right-hand side exists. In the case of the solution (D.52), we observe that the limit is well-defined, and equal to

$$\lim_{z\to 1}(z-1)\mathbf{X}_n(z) \;=\; [\mathbf{I} - \exp(n(\mathbf{M}-\mathbf{I}))]\; (\mathbf{I}-\mathbf{M})^{-1}\,\mathbf{m}_2\bar{\imath}_{SR}.$$

Hence for any t and any n, $\mathbf{e}_t(n)$ converges to a well-defined (finite) limit as T is made large, and the limit is the one given in the statement of the proposition.

D.3 Proof of Proposition 3

The result that

$$\lim_{T\to\infty} \mathbf{e}_t(n) \;=\; \mathbf{e}_{SR}(n)$$

for all t and n follows from Proposition 2. If in addition, the Taylor Principle (22) is satisfied, then as shown in section C.1 above, both eigenvalues of $\mathbf{M}-\mathbf{I}$ have negative real part. From this (B.38) follows; substituting of this into (29) yields

$$\lim_{n\to\infty} \mathbf{e}_{SR}(n) \;=\; \bar{\mathbf{e}}_{SR}^{PF},$$

where $\bar{\mathbf{e}}_{SR}^{PF}$ is defined in (30). This establishes the first double limit in the statement of the proposition.

The result that

$$\lim_{n\to\infty} \mathbf{e}_t(n) \;=\; \mathbf{e}_t^{PF}$$

for all t follows from Proposition 1. Establishing the second double limit thus requires us to consider how $\mathbf{e}_t^{PF}$ changes as T is made large.

As discussed in section B.8 above, the FS-PFE dynamics $\{\mathbf{e}_t^{PF}\}$ satisfy equation (B.35) for all t. Under the kind of regime assumed in this proposition (with $\boldsymbol{\omega}_t$ equal to a constant vector $\bar{\boldsymbol{\omega}}$ for all $t \geq T$), the FS-PFE (obtained by "solving forward" the difference equation) involves a constant vector of

[57] See, for example, Jury (1964, p. 6).

expectations, $\mathbf{e}_t^{PF} = \bar{\mathbf{e}}_{LR}^{PF}$ for all $t \geq T-1$, where

$$\bar{\mathbf{e}}_{LR}^{PF} \equiv [\mathbf{I} - \mathbf{M}]^{-1}\, \mathbf{m}_2 \bar{\imath}_{LR}$$

is the same as the vector defined in (27).

For periods $t < T-1$, one must instead solve the difference equation backward from the terminal condition $\mathbf{e}_{T-1}^{PF} = \bar{\mathbf{e}}_{LR}^{PF}$. We thus obtain a difference equation of the form

$$\mathbf{e}_\tau \;=\; [(\mathbf{I} - \mathbf{\Lambda})\mathbf{M} \,+\, \mathbf{\Lambda}]\, \mathbf{e}_{\tau-1} \;+\; (\mathbf{I} - \mathbf{\Lambda})\, \mathbf{m}_2 \bar{\imath}_{SR} \tag{D.53}$$

for all $\tau \geq 2$, with initial condition $\mathbf{e}_1 = \bar{\mathbf{e}}_{LR}^{PF}$. The asymptotic behavior of these dynamics as τ is made large depends on the eigenvalues of the matrix

$$(\mathbf{I} - \mathbf{\Lambda})\mathbf{M} \,+\, \mathbf{\Lambda} \;=\; \mathbf{C}^{-1}\mathbf{B}\mathbf{C}, \tag{D.54}$$

which must be the same as the eigenvalues of $\mathbf{B}$. (Note that (D.54) follows from (B.36).)

Under the hypothesis that the response coefficients satisfy the Taylor Principle (22), both eigenvalues of $\mathbf{B}$ are inside the unit circle. It then follows that the dynamics (D.53) converge as $\tau \to \infty$ to the steady-state vector of expectations $\bar{\mathbf{e}}_{SR}^{PF}$ defined in (30). We thus conclude that

$$\lim_{T\to\infty} \mathbf{e}_t^{PF} \;=\; \bar{\mathbf{e}}_{SR}^{PF}$$

for any t. This establishes the second double limit.

D.4 Proof of Proposition 4

The proof of this proposition follows exactly the same lines as the proof of Proposition 1. While the definition of the matrices of coefficients $\mathbf{V}$ and $\mathbf{W}$ must be modified, it continues to be possible to write the belief revision dynamics in the compact form (D.45), for an appropriate definition of these matrices. (This depends on the fact that we have chosen $\bar{T} \geq T$, so that the coefficients of the monetary policy reaction function do not change over time during periods $t \geq \bar{T}$. Variation over time in the reaction function coefficients does not prevent us from writing the dynamics in the compact form, as long as it occurs only prior to date $\bar{T}$; and our method of analysis requires only that $\bar{T}$ be finite.)

Moreover, it continues to be the case that $\mathbf{V}$ will have the block-triangular form indicated in equations (D.46)–(D.48). In equation (D.48), the matrix $\mathbf{M}$ is defined using the coefficients (ϕ_π, ϕ_y) that apply after date T, and thus that satisfy the Taylor Principle (22), according to the hypotheses of the proposition. The eigenvalues of $\mathbf{V}$ again consist of -1 (repeated $2\bar{T}$ times); the eigenvalues of $\mathbf{A}(\lambda_k)\mathbf{M}$, for $k = 1, \ldots, K$, and the eigenvalues of $\mathbf{M}$. Because $\mathbf{M}$ is defined using coefficients that satisfy the Taylor Principle, we again find that all of the eigenvalues of $\mathbf{M}$ and of $\mathbf{A}(\lambda_k)\mathbf{M}$ have negative real part. Hence all of the eigenvalues of $\mathbf{V}$ have negative real part. This again implies that the dynamics (D.45) are asymptotically stable, and the fixed point to which they converge again corresponds to the FS-PFE expectations. This establishes the proposition.

Note that this result depends on the hypothesis that from date T onward, monetary policy is determined by a reaction function with coefficients that satisfy the Taylor Principle. If we assumed instead (as in the case emphasized in Cochrane, 2017) that after date T, policy again consists of a fixed interest rate, but one that is consistent with the long-run inflation target (i.e., $\bar{\imath}_{LR} = 0$), the belief-revision dynamics would *not* converge.

If the interest rate is also fixed after date T (albeit at some level $\bar{\imath}_{LR} \neq \bar{\imath}_{SR}$), the belief-revision dynamics can again be written in the compact form (D.45), and the matrix $\mathbf{V}$ will again have the form (D.46)–(D.48). But in this case, the matrix $\mathbf{M}$ in (D.48) would be defined using the response coefficients $\phi_\pi = \phi_y = 0$, so that the Taylor Principle is violated. It then follows from our results above that $\mathbf{M}$ will have a positive real eigenvalue. (By continuity, one can show that $\mathbf{A}(\lambda_k)\mathbf{M}$ will also have a positive real eigenvalue for all values of λ_k near enough to 1.) Hence $\mathbf{V}$ will have at least one (and possibly several) eigenvalues with positive real part, and the belief-revision dynamics (D.45) will be explosive in the case of almost all initial conjectures (even restricting our attention to conjectures within the specified finite-dimensional family).

D.5 Proof of Proposition 5

The proof of this proposition follows similar lines as the proof of Proposition 2. In general, the characterization of reflective equilibrium is more complex when the monetary policy response coefficients are not time-invariant, as in the situation considered here. However, in the case hypothesized in the proposition, $g_t = 0$ and from period T onward, monetary policy is consistent with constant inflation at the rate π^*. Under these circumstances, and initial conjecture under which $\mathbf{e}_t = 0$ for all $t \geq T$ implies correct beliefs $\mathbf{e}^*_t = 0$ for all $t \geq T$ as well. Hence under the belief-revision dynamics, the conjectured beliefs are never revised, and $\mathbf{e}_t(n) = 0$ for all degrees of reflection $n \geq 0$, and any $t \geq T$. This result would be *the same* if we were to assume a fixed interest rate for all $t \geq T$ (that is, if we were to assume response coefficients

$\phi_\pi = \phi_y = 0$ after date T, just like we do for dates prior to T), but a fixed interest rate $\bar{\imath}_t = 0$ for all $t \geq T$ (that is, the fixed interest rate consistent with the steady state with inflation rate π^*).

Thus the reflective equilibrium is the same (in this very special case) as if we assumed a fixed interest rate in all periods (and thus the same response coefficients in all periods), but $\bar{\imath}_t = \bar{\imath}_{SR}$ for $t < T$ while $\bar{\imath}_t = 0$ for $t \geq T$.[58] And the latter is a case to which Proposition 2 applies. (Note that Proposition 2 requires no assumptions about the response coefficients except that they are constant over time, and that they satisfy (C.40). Hence the case in which $\phi_\pi = \phi_y = 0$ in all periods is consistent with the hypotheses of that proposition.)

Proposition 2 can then be used to show that the reflective equilibrium beliefs $\{\mathbf{e}_t(n)\}$ for any degree of reflection n converge to a well-defined limiting value $\mathbf{e}_{SR}(n)$, which is given by (29)–(30). This establishes the proposition.

D.6 Proof of Proposition 6

Let $\{\mathbf{e}_t^1\}$ be the sequence of expectations in a reflective equilibrium when the date of the regime change is T, and $\{\mathbf{e}_t^2\}$ be the expectations in the equilibrium corresponding to the same degree of reflection n when the date of the regime change is $T' > T$. Similarly, let $\{\mathbf{a}_t^1\}$ and $\{\mathbf{a}_t^2\}$ be the evolution of the vectors of summary variables that decisionmakers need to forecast in the two equilibria, and $\{\mathbf{e}_t^{*1}\}$ and $\{\mathbf{e}_t^{*2}\}$ the implied sequences of correct forecasts in the two equilibria. We similarly use the notation $\mathbf{M}^{(i)}, \mathbf{m}^{(i)}, \mathbf{C}^{(i)}, \mathbf{c}^{(i)}$ to refer to the matrices $\mathbf{M}, \mathbf{m}, \mathbf{C}$, and $\mathbf{c}$ respectively, defined using the monetary policy response coefficients associated with regime i (for $i = 1, 2$).[59]

Let us first consider the predictions regarding reflective equilibrium in periods $t \geq T'$. Under both of the assumptions about policy, policy is expected to be the same at all dates $t \geq T'$. Since it is assumed that we start from the same initial conjecture $\{\mathbf{e}_t(0)\}$ in both cases, and the model is purely forward-looking, it follows that the belief-revision dynamics will also be the same for all $t \geq T'$ in both cases. Hence we obtain the same sequences $\{\mathbf{e}_t(n)\}$ in both cases, for all $t \geq T'$; and since the outcomes for output and inflation are then given by (11), these are the same for all $t \geq T'$ as well.

[58] Note that these two different specifications of monetary policy would *not* lead to the same reflective equilibrium expectations, under most assumptions about the real shocks or about the initial conjecture; see the discussion at the end of the proof of Proposition 4. Here we get the same result *only* because we assume $g_t = 0$ (exactly) for all $t \geq T$ and an initial conjecture under which $\mathbf{e}_t(0) = 0$ (exactly) for all $t \geq T$.

[59] By "regime 1" we mean the Taylor rule (the regime in place in periods $T \leq t < T'$ under policy 1); by "regime 2" we mean the interest-rate peg at $\bar{\imath}_{SR}$.

Moreover, it is easily shown that under our assumptions, the common solution is one in which $\mathbf{e}_t(n) = \mathbf{0}$ for all $t \geq T'$, and correspondingly $\mathbf{x}_t(n) = \mathbf{0}$ for all $t \geq T'$.

Moreover, since outcomes for output and inflation are the same for all $t \geq T'$ in the two cases, it follows that the sequences of correct forecasts $\{\mathbf{e}_t^*\}$ are the same in both cases for all $t \geq T'-1$. (Note that the correct forecasts in period $T'-1$ depend only on the equilibrium outcomes in period T' and later.) Hence the belief-revision dynamics for period $T'-1$ will also be the same in both cases, and we obtain the same vector $\mathbf{e}_{T'-1}(n)$ for all n; and again the common beliefs are $\mathbf{e}_{T'-1}(n) = \mathbf{0}$.

Let us next consider reflective equilibrium in periods $T \leq t \leq T'-1$. Suppose that for such t and some n, $\mathbf{e}_t^2 \geq \mathbf{e}_t^1 \geq \mathbf{0}$ (in both components). Then

$$\mathbf{a}_t^2 - \mathbf{a}_t^1 \;=\; \mathbf{M}^{(2)}\,(\mathbf{e}^2 - \mathbf{e}_t^1) \;+\; [\mathbf{M}^{(2)} - \mathbf{M}^{(1)}]\,\mathbf{e}_t^1 \mid + \;\mathbf{m}_2^{(2)}\,\bar{\imath}_{SR}.$$

Moreover, we observe from the above definitions of $\mathbf{M}$ and $\mathbf{m}$ that $\mathbf{M}^{(2)}$ is positive in all elements; $\mathbf{M}^{(2)} - \mathbf{M}^{(1)}$ is positive in all elements; and $\mathbf{m}_2^{(2)}$ is negative in both elements. Under the hypotheses that $\mathbf{e}_t^2 \geq \mathbf{e}_t^1 \geq \mathbf{0}$ and $\bar{\imath}_{SR} < 0$, it follows that $\mathbf{a}_t^2 - \mathbf{a}_t^1 \;>>\; \mathbf{0}$, where we use the symbol $>>$ to indicate that the first vector is greater in both elements.

Now suppose that for some n, $\mathbf{e}_t^2 \geq \mathbf{e}_t^1 \geq \mathbf{0}$ for all $T \leq t \leq T'-1$. It follows from our conclusions above that these inequalities then must hold for all $t \geq T$. It also follows from the argument in the paragraph above that we must have $\mathbf{a}_t^2 >> \mathbf{a}_t^1$ for all $T \leq t \leq T'-1$, along with $\mathbf{a}_t^2 = \mathbf{a}_t^1$ for all $t \geq T'$. This implies that $\mathbf{e}_t^{*2}(n) >> \mathbf{e}_t^{*1}(n)$ for all $T \leq t < T'-1$, while $\mathbf{e}_t^{*2}(n) = \mathbf{e}_t^{*1}(n)$ for $t = T'-1$.

The fact that $\mathbf{e}_t^{*2}(n) = \mathbf{e}_t^{*1}(n)$ for $t = T'-1$ means that the belief-revision dynamics for period $T'-1$ will again be the same in both cases, and we obtain the same vector $\mathbf{e}_{T'-1}(n)$ for all n; and again the common beliefs are $\mathbf{e}_{T'-1}(n) = 0$. For periods $T \leq t < T'-1$, we continue to have $\mathbf{e}_t^{*1}(n) = 0$ for all n, for the same reason as in the case of periods $t \geq T'$. But now the fact that we start from the common initial conjecture $\mathbf{e}_t^2(0) = \mathbf{e}_t^1(0) = \mathbf{0}$ implies that $\mathbf{e}_t^{*2}(0) >> \mathbf{e}_t^{*1}(0) = \mathbf{0}$ and hence $\dot{\mathbf{e}}_t^2(0) >> \dot{\mathbf{e}}_t^1(0) = \mathbf{0}$. This implies that for small enough $n > 0$, we will have $\mathbf{e}_t^2(n) >> \mathbf{e}_t^1(n) = \mathbf{0}$ for all $T \leq t < T'-1$.

Moreover, for any n, as long as we continue to have $\mathbf{e}_t^2(n) \geq \mathbf{e}_t^1(n) = \mathbf{0}$ for all $t \geq T$, we will continue to have $\mathbf{e}_t^{*2}(n) >> \mathbf{e}_t^{*1}(n) = \mathbf{0}$ for all $T \leq t < T'-1$. Since the belief-revision dynamics (14) imply that for any $n > 0$, $\mathbf{e}_t(n)$ is an average of $\mathbf{e}_t(0)$ and the vectors $\mathbf{e}_t^*(\tilde{n})$ for values $0 \leq \tilde{n} < n$, as long as we have had $\mathbf{e}_t^{*2}(\tilde{n}) >> \mathbf{0}$ for all $0 \leq \tilde{n} < n$, we will necessarily have $\mathbf{e}_t^2(n) >> 0$.

Thus we conclude by induction that $\mathbf{e}_t^2(n) >> \mathbf{e}_t^1(n) = \mathbf{0}$ for all $n > 0$, and any $T \leq t < T' - 1$.

The associated reflective equilibrium outcomes are given by (11) in each case. This implies that

$$\mathbf{x}_t^2 - \mathbf{x}_t^1 \ = \ \mathbf{C}^{(2)}\,(\mathbf{e}^2 - \mathbf{e}_t^1) \ + \ [\mathbf{C}^{(2)} - \mathbf{C}^{(1)}]\,\mathbf{e}_t^1 \mid + \ \mathbf{c}_2^{(2)}\,\bar{\imath}_{SR}.$$

Note furthermore that all elements of $\mathbf{C}^{(2)}$ are non-negative, with at least one positive element in each row; that all elements of $\mathbf{C}^{(2)} - \mathbf{C}^{(1)}$ are positive; and that all elements of $\mathbf{c}_2^{(2)}$ are negative. Then the fact that $\mathbf{e}_t^2(n) \geq \mathbf{e}_t^1(n) = \mathbf{0}$ for all $T \leq t \leq T' - 1$ and $\bar{\imath}_{SR} < 0$ implies that $\mathbf{x}_t^2 \ >> \ \mathbf{x}_t^1$ for all $T \leq t \leq T' - 1$.

Finally, let us consider reflective equilibrium in periods $0 \leq t < T$. In these periods, the monetary policy is expected to be the same in both cases (the fixed interest rate). Suppose that for some such t and some n, $\mathbf{e}_t^2 \geq \mathbf{e}_t^1$. Then

$$\mathbf{a}_t^2 - \mathbf{a}_t^1 \ = \ \mathbf{M}^{(2)}\,(\mathbf{e}^2 - \mathbf{e}_t^1) \ \geq \ \mathbf{0},$$

because all elements of $\mathbf{M}^{(2)}$ are positive. Since we have already concluded above that $\mathbf{a}_t^2 \ >> \ \mathbf{a}_t^1$ for all $T \leq t \leq T' - 1$, and that $\mathbf{a}_t^2 = \mathbf{a}_t^1$ for all $t \geq T'$, this implies that $\mathbf{e}_t^{*2} >> \mathbf{e}_t^{*1}$ for all $0 \leq t < T$.

We can then use an inductive argument, as above, to show that $\mathbf{e}_t^2(n) >> \mathbf{e}_t^1(n)$ for any $n > 0$, and any $0 \leq t < T$. It follows from this that

$$\mathbf{x}_t^2 - \mathbf{x}_t^1 \ = \ \mathbf{C}^{(2)}\,(\mathbf{e}^2 - \mathbf{e}_t^1) \ >> \ \mathbf{0}$$

for any $n > 0$, and any $0 \leq t < T$, given that all elements of $\mathbf{C}^{(2)}$ are non-negative, with at least one positive element in each row. This establishes the proposition.

E Comparison with a Discrete Model of Belief Revision

Here we note that the convergence result in Proposition 1 would not hold with the same generality were we instead to assume a discrete model of belief revision in which, instead of the continuous model of belief revision (14), we iterate the mapping

$$\mathbf{e}_t(N + 1) \ = \ \mathbf{e}_t^*(N) \tag{E.55}$$

for $N = 0, 1, 2, \ldots,$ where for each N, $\{\mathbf{e}_t^*(N)\}$ is the sequence of correct beliefs implied by average expectations specified by the sequence $\{\mathbf{e}_t(N)\}$.

As with the continuous model, we might take as given some "naive" initial conjecture, and then consider how it evolves as a result of further iterations of the mapping. And as with the continuous model, *if* the process converges to a fixed point, such a fixed point must correspond to PFE beliefs.

However, the conditions for convergence of the discrete process, while related to the conditions under which the continuous process converges, are more stringent. Convergence need not obtain under the conditions hypothesized in Proposition 1, as the following numerical example illustrates. In Figure E.1 the intercept of the Taylor rule is expected to be lowered for 200 quarters, after which it is expected to return to the level consistent with the inflation target π^*. All model parameters are also the same as in Figure 2, and the initial conjecture is assumed to be $\mathbf{e}_t(0) = 0$ for all t. In the panel on top, the continuous belief revision process is assumed and in the panel below, the discrete model of belief revision (E.55) is assumed.

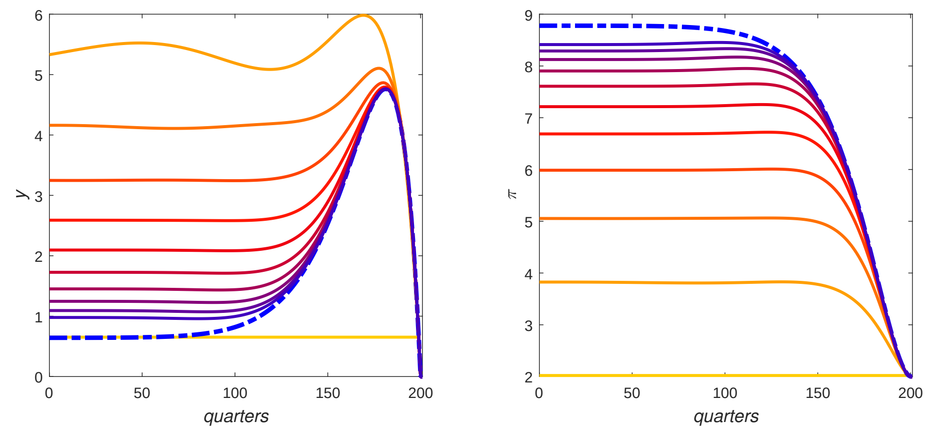
The figure plots the implied TE dynamics of output and inflation for iterations $N = 0, 1, 2, 3,$ and 4 for the discrete case. While the continuous case converges as is expected by Proposition 1, the belief-revision dynamics in the discrete case are explosive. The first revision of the initial conjecture (which takes account of the fact that if people maintain the initial beliefs, consistent with the unperturbed steady state, the temporary policy will lead to higher inflation and output) raises both output and inflation further. But anticipation of *these* effects (and the associated increase in the interest rate that they must provoke) should actually lead output and inflation to be *lower* in stage $N = 2$. Anticipation of the $N = 2$ outcomes (which imply an even deeper cut in the interest rate) then leads output and inflation to be *high* again in stage $N = 3$, and to an even greater extent than in stage $N = 1$. Anticipating of this then leads output and inflation to be *low* again in stage $N = 4$, to an even greater extent than in stage $N = 2$. The oscillations continue, growing larger and larger, as N is increased; but as the figure shows, the predicted expectations are already very extreme after only four iterations of the belief updating mapping.

It is not accidental that the unstable dynamics of belief revision in this case are oscillatory. In terms of the compact notation introduced in the proof of Proposition 1 (under the assumption of exponentially convergent fundamentals and average beliefs), the discrete model of belief revision (E.55) replaces the continuous dynamics (D.49) by the discrete process

$$\mathbf{e}(N+1) \;=\; (\mathbf{I} + \mathbf{V})\,\mathbf{e}(N) \;+ \mathbf{W}\,\omega$$

This process is unstable if not all eigenvalues of $\mathbf{I} + \mathbf{V}$ are of modulus less than 1. Since the eigenvalues of $\mathbf{I} + \mathbf{V}$ are equal to $1 + \mu_i$, where μ_i is an

(a) Continuous process



(b) Discrete process

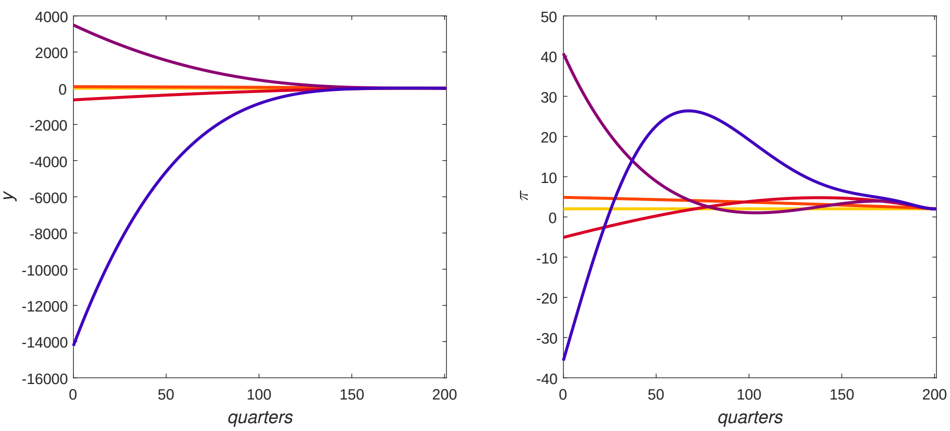


Figure E.1: Belief Revision Process using a Continuous vs. a Discrete process

Note: The graphs on top show the result for $n = 0$ through 20 (progressively darker lines) when the Taylor-rule intercept is reduced for 200 quarters. The graphs on the bottom show reflective equilibrium outcomes for $N = 0$ through 4 (progressively darker lines) when the Taylor-rule intercept is reduced for 200 quarters assuming a discrete process of iterative belief revision. See section F for details.

eigenvalue of $\mathbf{V}$, and we have shown above that all eigenvalues of $\mathbf{V}$ have negative real part, $\mathbf{I}+\mathbf{V}$ cannot have a real eigenvalue greater than 1. It can, however, have a real eigenvalue with *modulus* greater than 1, if $\mathbf{V}$ has a real eigenvalue that is less than -2. This is the case shown in Figure E.1, in which a large negative eigenvalue results in explosive oscillations.

We feel, however, that the kind of unstable process of belief revision illustrated by Figure E.1 is unrealistic, as it is requires that at each stage in the reasoning, one must conjecture that *everyone* else should reason in one precise way, even though that assumed reasoning changes dramatically from each stage in the process of reflection to the next. The continuous process of belief revision proposed in the text avoids making such an implausible assumption.

F Algorithm to Construct the Figures

The figures were constructed using the parameters listed in Table F.1.

Table F.1: Parameters used in Numerical Exercises

Parameter	Definition	Value	Source
α	Prob. not choosing price	0.784	Denes, Eggertsson and Gilbukh (2013)
β	Discount factor	0.997	
σ	Int. elast. substitution	1/1.22	
ξ	Elast. firm's optimal price wrt AD	0.125	
ϕ_y	Coef. output in Taylor rule	0.5/4	Taylor (1993)
ϕ_π	Coef. inflation in Taylor rule	1.5	
π^*	Inflation Target	$\log(1.02)^{1/4}$	

The initial steady state, that determined the initial value for $\mathbf{e}$, was assumed to be one with $\bar{\imath}=0$. The temporary policy was set to be one with $\bar{\imath}=0.0088$, which implies a zero nominal interest rate when $n=0$. To calculate and graph the exercises, the continuous updating procedure was approximated by the following discrete procedure:

$$\mathbf{e}^{N+1}=(1-\gamma)\mathbf{e}^{*,N}+\gamma\mathbf{e}^{N} \tag{F.56}$$

where $\mathbf{e}$ is the whole vector of $\mathbf{e}_t$. The N chosen for each figure depends of the desired n and the γ, since the approximation is given by:

$$n \approx N\gamma$$

The general algorithm for the figures can be described as:

1. **Calculate initial values**: The initial values of variables $\{y_t, \pi_t, i_t, e_{1t}, e_{2t}\}$ for all t are the ones corresponding to the steady state with $\bar{\imath} = 0$ and $\rho_t = 0$ for all t such that parameters are those in Table F.1. This means that the values for all variables are zero, since all variables are defined as their deviations from that steady state. Set the initial values of the expectations $e_{1t}^0 = e_{2t}^0 = 0$ for all t.

2. **Introduce the change in policy**: It is one of the two following:

   - For Figures 2 and E.1: Maintaining the values for ϕ_y and ϕ_π as in Table F.1, set $\bar{\imath} = -0.0088$ for $T = \{8, 200\}$ periods respectively, and then go back to $\bar{\imath} = 0$ forever.
   - For Figures 3 and 4: Set the values $\phi_y = 0$, $\phi_\pi = 0$ and $\bar{\imath} = -0.0088$ for $T = \{8, 2000\}$ periods respectively, and then go back to the values of ϕ_y, ϕ_π of Table F.1 and $\bar{\imath} = 0$ forever.

3. **Calculate the FS-PFE**: This is done by using (B.34) and (10).

4. **Given e_{1t}^N, e_{2t}^N, calculate y_t^N, π_t^N, i_t^N**: This is done by using (11) and (10).

5. **Given e_{1t}^N, e_{2t}^N, calculate $e_{1t}^{*,N}, e_{2t}^{*,N}$**: To do this, note that for $t \geq T$, we stay in the same steady state that we started with $y_t = \pi_t = i_t = e_{1t}^* = e_{2t}^* = 0$. Given that, calculate $\mathbf{e}_t^{*,N}$ using equation (12). You can also use a recursive formulation noting that:

$$
\begin{aligned}
e_{1t}^{*,N} &= (1-\beta)\left(y_t^N - \frac{\sigma}{1-\beta}(\beta i_t^N - \pi_t^N)\right) + \beta e_{1t+1}^{*,N} \\
e_{2t}^{*,N} &= (1-\alpha\beta)\left(\frac{1}{1-\alpha\beta}\pi_t^N + \xi y_t^N\right) + \alpha\beta e_{2t+1}^{*,N}
\end{aligned}
$$

6. **Given $e_{1t}^{*,N}, e_{2t}^{*,N}$, calculate $e_{1t}^{N+1}, e_{2t}^{N+1}$**: This is done by using the formula (F.56):

$$
\begin{aligned}
e_{1t}^{N+1} &= (1-\gamma)e_{1t}^{*,N} + \gamma e_{1t}^N \\
e_{2t}^{N+1} &= (1-\gamma)e_{2t}^{*,N} + \gamma e_{2t}^N
\end{aligned}
$$

γ is set equal to 0.02 for figures 2 and 3 and panel (a) of E.1, 0.001 for figure 4 (because the approximation was inaccurate for higher values of γ) and 0 for panel (b) of figure E.1.

7. **Repeat 4-6** N **times**.

8. **Transform the variables to be graphed**: Use the following

$$\begin{aligned}
\pi_t^{Graph} &= 100((\exp(\pi_t + \pi^*))^4 - 1) \\
y_t^{Graph} &= 100 y_t \\
i_t^{Graph} &= 100((\exp(i_t + \pi^*)/\beta).^4 - 1)
\end{aligned}$$

ADDITIONAL REFERENCES

Hirsch, Morris W. and Stephen Smale. 1974. *Differential Equations, Dynamical Systems, and Linear Algebra.* Academic Press.

Jury, Eliahu I. 1964. *Theory and Applications of the z-Transform Method.* Wiley.