EVALUATING STRATEGIC FORECASTERS
ONLINE APPENDIX

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1. A Revelation Principle for Deterministic Dynamic Mechanisms

A deterministic mechanism $\mathcal{M}$ in our environment is simply a sequence of message spaces $M_0, M_1, \ldots, M_T$ (where we will let $M^t := \times_{\tau=0}^{t} M_\tau$ denote the set of period-t sequences of messages) and a decision rule $x : M^T \times \{h, l\} \to \{0, 1\}$.

Given a mechanism $\mathcal{M} = (M^T, x)$, the agent's reporting strategy μ is a sequence of rules

$$\mu_t : \theta \times S^t \times M^{t-1} \to \Delta(M_t),$$

where we write

$$\mu_t(m_t|\theta, s_1, \ldots, s_t, m_0, m_1, \ldots, m_{t-1})$$

to denote the probability of sending message $m_t \in M_t$ when the agent's private information is (θ, s^t) and she has already sent messages m^{t-1}. (Note that, as in any sequential game, the agent's strategy must specify the messages that she sends in some period t even after sequences of messages m^{t-1} that are not in the support of her strategy.)

A mechanism is a *direct mechanism* if $M_0 = \Theta$ and $M_t = S$ for all $t = 1, \ldots, T$.

Lemma. *Consider an equilibrium μ of a game induced by a deterministic mechanism $\mathcal{M} = (M^T, x)$. Then there exists a deterministic direct mechanism $\widehat{\mathcal{M}} = (\theta \times S^T, \chi)$ that induces an equilibrium $\hat{\mu}$ with truthful revelation. Moreover, the principal's expected payoff under $\hat{\mu}$ in $\widehat{\mathcal{M}}$ is (weakly) greater than her expected payoff under μ in $\mathcal{M}$.*[1]

Proof. Consider a deterministic mechanism $\mathcal{M} = (M^T, x)$ and equilibrium reporting strategy μ.

Fix an arbitrary period $t \in \{0, 1, \ldots, T\}$, and let $\lambda_t := (\theta, s^t)$ denote the agent's period-t (private) history of type and signals. For each $\lambda_t \in \Lambda_t := \Theta \times S^t$ and each $m^{t-1} \in M^{t-1}$, define

$$M_t^{\lambda_t, m^{t-1}} := \left\{ m \in M_T | \mu_t(m_t|\lambda_t, m^{t-1}) > 0 \right\}$$

to be the set of equilibrium period-t messages sent by the agent with positive probability when her private type is λ_t and she has already reported messages m^{t-1}. Note that, by definition of equilibrium, it must therefore be the case that

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[1]This result extends Strausz's (2003) deterministic revelation principle (in terms of payoffs) for a single agent to our dynamic environment.

$$\sum_{\substack{r\in\{h,l\}\\ s_{t+1}^T\in S^{T-t}\\ m_{t+1}^T\in M_{t+1}\times\cdots\times M_T}} \Pr(r,s_{t+1}^T|\lambda_t)\begin{bmatrix}\mu_{t+1}(m_{t+1}|(\lambda_t,s_{t+1}),(m^{t-1},m_t))\\ \times\cdots\times\\ \mu_T(m_T|(\lambda_t,s_{t+1}^T),(m^{t-1},m_t,m_{t+1}^T))\end{bmatrix} x(m^{t-1},m_t,m_{t+1}^T,r)$$

$$\geq \sum_{\substack{r\in\{h,l\}\\ s_{t+1}^T\in S^{T-t}\\ m_{t+1}^T\in M_{t+1}\times\cdots\times M_T}} \Pr(r,s_{t+1}^T|\lambda_t)\begin{bmatrix}\mu_{t+1}(m_{t+1}|(\lambda_t,s_{t+1}),(m^{t-1},m_t'))\\ \times\cdots\times\\ \mu_T(m_T|(\lambda_t,s_{t+1}^T),(m^{t-1},m_t',m_{t+1}^T))\end{bmatrix} x(m^{t-1},m_t',m_{t+1}^T,r)$$

for all $m_t \in M_t^{\lambda_t,m^{t-1}}$ and $m_t' \in M_t$, and where the above holds with equality when $m_t' \in M_t^{\lambda_t,m^{t-1}}$.

So define $\widehat{M}_t^{\lambda_t,m^{t-1}}$ to be the set of all messages that yield the principal her highest payoff from type λ_t among the messages that are sent with positive probability in equilibrium; that is,

$$\widehat{M}_t^{\lambda_t,m^{t-1}} := \operatorname*{argmax}_{m_t'\in M_t^{\lambda_t,m^{t-1}}} \left\{ \begin{array}{r} \displaystyle\sum_{\substack{r\in\{h,l\}\\ s_{t+1}^T\in S^{T-t}\\ m_{t+1}^T\in M_{t+1}\times\cdots\times M_T}} \Pr(r,s_{t+1}^T|\lambda_t)\begin{bmatrix}\mu_{t+1}(m_{t+1}|(\lambda_t,s_{t+1}),(m^{t-1},m_t'))\\ \times\cdots\times\\ \mu_T(m_T|(\lambda_t,s_{t+1}^T),(m^{t-1},m_t',m_{t+1}^T))\end{bmatrix} \\ x(m^{t-1},m_t',m_{t+1}^T,r)\left[\mathbb{1}_g(\theta)-\mathbb{1}_b(\theta)\right] \end{array} \right\}.$$

With this in hand, define the mechanism $\mathcal{M}' := (M_0,\ldots,M_{t-1},\Lambda_t,M_{t+1},\ldots,M_T,x')$, where for all $m^{t-1}\in M^{t-1}$ and all $\lambda_t\in\Lambda_t$, we let

$$x'(m^{t-1},\lambda_t,m_{t+1}^T,r) := x(m^{t-1},\hat{m}_t^{\lambda_t,m^{t-1}},m_{t+1}^T,r) \text{ for an arbitrary } \hat{m}_t^{\lambda_t,m^{t-1}} \in \widehat{M}_t^{\lambda_t,m^{t-1}}.$$

Thus, the (also deterministic) mechanism $\mathcal{M}'$ is identical to $\mathcal{M}$ in all periods except period t, where the agent is asked to report her entire private history up to that point; the mechanism then "translates" the reported private history into its corresponding principal-optimal period-t message chosen by the equilibrium μ. Since μ is an equilibrium reporting strategy in mechanism $\mathcal{M}$, then the strategy μ' defined by

$$\mu_\tau'(m_\tau|\theta,s^\tau,m^{\tau-1}) := \mu_\tau(m_\tau|\theta,s^\tau,m^{\tau-1}) \text{ for all } \tau<t;$$

$$\mu_t'(\lambda_t|\theta,s^t,m^{t-1}) := \begin{cases}1 & \text{if } \lambda_t=(\theta,s^t),\\ 0 & \text{otherwise;}\end{cases} \text{ and}$$

$$\mu_\tau'(m_\tau|\theta,s^\tau,(m^{t-1},\lambda_t,m_{t+1}^\tau)) := \mu_\tau(m_\tau|\theta,s^\tau,(m^{t-1},\hat{m}_t^{\lambda_t,m^{t-1}},m_{t+1}^\tau)) \text{ for all } \tau>t,$$

is by construction an equilibrium reporting strategy in mechanism $\mathcal{M}'$. (Note that μ' is identical to μ for all period $\tau<t$; optimally reports the private history truthfully in period t, which corresponds to an optimal message from μ; and follows the equilibrium continuation play of μ after any period-t report, truthful or otherwise.) Moreover, the agent's expected payoff is unchanged, while the principal's payoff is (weakly) higher in the equilibrium μ' of the new mechanism $\mathcal{M}'$.

Note, however, that the period t that we chose above was entirely arbitrary. Therefore, we can define a new (and still deterministic) mechanism $\mathcal{M}'' := (\Lambda_0,\Lambda_1,\ldots,\Lambda_T,x'')$ by iteratively applying the procedure above $T+1$ times, starting in the final period T and working backwards until we reach period 0. Note, however, that the message spaces induced by this iterative procedure contain some redundancy, in that the agent is asked to re-report her *entire* private history each period. However, the procedure above also generates a truthful equilibrium μ'' in which the agent *truthfully* re-reports that history in each period; this implies that (in equilibrium) misreports occur with zero probability.

Thus, we define a dynamic *direct* mechanism $\widehat{\mathcal{M}} := (\Theta, S, \ldots, S, \hat{x})$ in which the agent is asked to report only her *new* private information in each period and the decision rule $\hat{x}$ is defined by

$$\hat{x}(\theta, s^T, r) := x''((\theta), (\theta, s^1), (\theta, s^2), \ldots, (\theta, s^T), r) \text{ for all } (\theta, s^T) \in \Theta \times S^T \text{ and } r \in \{h, l\}.$$

Note that since the iterative procedure above preserves the deterministic nature of the decision rule, $\widehat{\mathcal{M}}$ is also deterministic; in addition, since the set of reporting strategies under $\widehat{\mathcal{M}}$ is a subset of those in $\mathcal{M}''$ (but still contains the equilibrium strategy of truthful reporting after all possible histories), the new direct mechanism $\widehat{\mathcal{M}}$ is incentive compatible. This also implies that the agent's payoff is the same as in the original mechanism $\mathcal{M}$, while the principal's payoff under $\widehat{\mathcal{M}}$ is (weakly) greater. ■

2. Optimal Static Mechanism

PROOF OF THEOREM 8. Before proceeding, note that Lemma 1 applies immediately in this setting with a single period-T report.

CLAIM. *It is without loss of generality to consider contracts such that $x_r(s^T) = x_r(\hat{s}^T)$ for all $s^T, \hat{s}^T$ such that $\sum_t \mathbb{1}_h(s_t) = \sum_t \mathbb{1}_h(\hat{s}_t)$.*

PROOF OF CLAIM. Suppose there exists some $\tilde{s}^T, \hat{s}^T \in \{h, l\}^T$ with $\sum_t \mathbb{1}_h(\tilde{s}_t) = \sum_t \mathbb{1}_h(\hat{s}_t)$ but $x_r(\tilde{s}^T) \neq x_r(\hat{s}^T)$ for some $r \in \{h, l\}$. Since signals are conditionally i.i.d., the agent has identical posterior beliefs $q_\theta = \Pr(\omega = h | \tilde{s}^T, \theta) = \Pr(\omega = h | \hat{s}^T, \theta)$ about the underlying state of the world after observing $\tilde{s}^T$ or $\hat{s}^T$.

Since the contract must be incentive compatible for the type-g agent, he must prefer reporting $\tilde{s}^T$ truthfully to misreporting $\tilde{s}^T$ as $\hat{s}^T$, implying

$$\begin{aligned} &q_g(\gamma x_h(\tilde{s}^T) + (1-\gamma) x_l(\tilde{s}^T)) \\ &\quad + (1-q_g)(\gamma x_l(\tilde{s}^T) + (1-\gamma) x_h(\tilde{s}^T)) \end{aligned} \geq \begin{aligned} &q_g(\gamma x_h(\hat{s}^T) + (1-\gamma) x_l(\hat{s}^T)) \\ &\quad + (1-q_g)(\gamma x_l(\hat{s}^T) + (1-\gamma) x_h(\hat{s}^T)). \end{aligned}$$

The agent must also prefer reporting $\hat{s}^T$ truthfully to misreporting $\hat{s}^T$ as $\tilde{s}^T$, implying

$$\begin{aligned} &q_g(\gamma x_h(\hat{s}^T) + (1-\gamma) x_l(\hat{s}^T)) \\ &\quad + (1-q_g)(\gamma x_l(\hat{s}^T) + (1-\gamma) x_h(\hat{s}^T)) \end{aligned} \geq \begin{aligned} &q_g(\gamma x_h(\tilde{s}^T) + (1-\gamma) x_l(\tilde{s}^T)) \\ &\quad + (1-q_g)(\gamma x_l(\tilde{s}^T) + (1-\gamma) x_h(\tilde{s}^T)). \end{aligned}$$

Of course, these two inequalities jointly imply that the type-g agent with belief q_g is indifferent between reporting $\tilde{s}^T$ or $\hat{s}^T$.

So consider the alternative mechanism $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ defined by, for $r = h, l$,

$$\hat{x}_r(s^T) := \begin{cases} x_r(\tilde{s}^T) & \text{if } s^T = \hat{s}^T \\ x_r(s^T) & \text{otherwise.} \end{cases}$$

Thus, $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ simply "deletes" the option of reporting as $\hat{s}^T$ and replaces it by the report of $\tilde{s}^T$. Since the original mechanism $\{x_h(\cdot), x_l(\cdot)\}$ was incentive compatible for the type-g agent and the type-g agent who observed $\hat{s}^T$ was indifferent between the two reports, $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ is also incentive compatible for the type-g agent. Moreover, $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ leaves the ex ante expected payoff of the type-g agent unchanged.

Meanwhile, the type-b agent's ex ante expected payoff is (weakly) lower under $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ than under $\{x_h(\cdot), x_l(\cdot)\}$ since there is one fewer potential report available to him. Since the principal's payoff is increasing in U_g and decreasing in U_b, this implies that $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ (weakly) raises the principal's expected payoff. $\Diamond$

With this property in hand, we abuse notation somewhat and write $x_r(n)$ to denote $x_r(s^T)$, where $n = \sum_t \mathbb{1}_h(s_t)$. We also write $q_\theta(n)$ to denote the associated posterior belief $\Pr(\omega = h | s^T, \theta)$.

CLAIM. *It is without loss of generality to consider symmetric contracts in which $x_h(n) = x_l(T-n)$ for all $n = 0, 1, \ldots, T$.*

PROOF OF CLAIM. Fix any contract $\{x_h(\cdot), x_l(\cdot)\}$ that is incentive compatible for the type-g agent, and define the alternative contract $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ by

$$\hat{x}_h(n) := x_l(T-n) \text{ and } \hat{x}_l(n) := x_h(T-n) \text{ for all } n = 0, 1, \ldots, T.$$

Then the expected utility of a type-θ agent who observes s^T with $\sum_t \mathbb{1}_h(s_t) = n$ but reports n' is

$$\begin{aligned}
\widehat{U}_\theta(n'|n) &= q_\theta(n)(\gamma \hat{x}_h(n') + (1-\gamma)\hat{x}_l(n')) + (1 - q_\theta(n))(\gamma \hat{x}_l(n') + (1-\gamma)\hat{x}_h(n')) \\
&= q_\theta(n)(\gamma x_l(T-n') + (1-\gamma) x_h(T-n')) \\
&\qquad + (1 - q_\theta(n))(\gamma x_h(T-n') + (1-\gamma) x_l(T-n')) \\
&= (1 - q_\theta(T-n))(\gamma x_l(T-n') + (1-\gamma) x_h(T-n')) \\
&\qquad + q_\theta(T-n)(\gamma x_h(T-n') + (1-\gamma) x_l(T-n')) \\
&= U_\theta(T-n' | T-n).
\end{aligned}$$

Letting $\sigma^\theta(\cdot)$ denote type-θ's optimal strategy under the original mechanism $\{x_h(\cdot), x_l(\cdot)\}$, this implies that type-θ's optimal reporting strategy $\hat{\sigma}^\theta(\cdot)$ under the new contract $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ is

$$\hat{\sigma}^\theta(n) = T - \sigma^\theta(T-n).$$

In particular, the type-g incentive compatibility of the original mechanism (that is, $\sigma^g(n) = n$ for all n) implies that $\hat{\sigma}^g(n) = n$ for all n. Moreover, the symmetry of the signal distributions implies that the agent's expected utility (conditional on quality) is the same across both mechanisms (that is, $\widehat{U}_g = U_g$ and $\widehat{U}_b = U_b$), so the principal's expected payoff is

$$\widehat{\Pi} := \frac{1}{2}\widehat{U}_g - \frac{1}{2}\widehat{U}_b = \frac{1}{2}U_g - \frac{1}{2}U_b.$$

Now define the (symmetric) mechanism $\{\overline{x}_h(\cdot), \overline{x}_l(\cdot)\}$ by

$$\overline{x}_r(n) := \frac{x_r(n) + \hat{x}_r(n)}{2} \text{ for all } n.$$

Then the expected utility of a type-θ agent who observes s^T with $\sum_t \mathbb{1}_h(s_t) = n$ but reports n' is

$$\begin{aligned}\overline{U}_\theta(n'|n) &= q_\theta(n)(\gamma\overline{x}_h(n') + (1-\gamma)\overline{x}_l(n')) + (1-q_\theta(n))(\gamma\overline{x}_l(n') + (1-\gamma)\overline{x}_h(n')) \\ &= \frac{1}{2}U_\theta(n'|n) + \frac{1}{2}\widehat{U}_\theta(n'|n).\end{aligned}$$

Since $U_g(n|n) \geq U_g(n'|n)$ and $\widehat{U}_g(n|n) \geq \widehat{U}_g(n'|n)$ for all $n, n' \in \{0, 1, \ldots, T\}$, it must also be the case that $\overline{U}_g(n|n) \geq \overline{U}_g(n'|n)$ for all n and n'; that is, this new symmetric mechanism is type-g incentive compatible. This also implies that the type-g expected utility is unchanged, so $\overline{U}_g = U_g$. On the other hand, note that

$$\begin{aligned}\overline{U}_b &:= \sum_n \Pr(n|\theta = b) \sup_{n'} \left\{\overline{U}_b(n'|n)\right\} \\ &= \sum_n \Pr(n|\theta = b) \sup_{n'} \left\{\frac{1}{2}U_b(n'|n) + \frac{1}{2}\widehat{U}_b(n'|n)\right\} \\ &\leq \sum_n \Pr(n|\theta = b) \sup_{n'} \left\{\frac{1}{2}U_b(n'|n)\right\} + \sum_n \Pr(n|\theta = b) \sup_{n'} \left\{\frac{1}{2}\widehat{U}_b(n'|n)\right\} \\ &= \frac{1}{2}U_b + \frac{1}{2}\widehat{U}_b = U_b.\end{aligned}$$

Thus, the new symmetric mechanism leaves the type-g agent's expected utility unchanged while decreasing that of the type-b agent, thereby increasing the principal's payoff. $\Diamond$

We now move to an equivalent posterior-space setting where, instead of focusing on the signals received by an agent, we consider the posterior beliefs induced by those signals. (Note that this is equivalent due to the two lemmas above as well as the one-to-one mapping between the number of h signals and the agent's posterior belief.) We denote the agent's posterior beliefs that the state of the world is $\omega = h$ by $q \in [0, 1]$, and let F_θ denote the distribution of type-θ's posterior beliefs.

CLAIM. *The distributions F_θ are symmetric about $\frac{1}{2}$; that is, $F_\theta(q) = 1 - F_\theta(1-q)$ for all $q \in [0, 1]$ and $\theta \in \{g, b\}$. In addition, the type-g agent puts more mass on extremal posteriors than the type-b agent, so $F_g(q) \geq F_b(q)$ for all $q \in (0, \frac{1}{2})$.*

PROOF OF CLAIM. To see that the distributions are symmetric, note that the symmetry of the signal-generating process implies that, for all $n = 0, \ldots, T$, it is equally likely for the number of h signals observed by the agent to equal n or to equal $T - n$; moreover, it is straightforward to show that $q_\theta(n) = 1 - q_\theta(T - n)$.

To see that the second property holds, note that the probability an agent with signal precision α observes signals s^T with $n \leq \sum_t \mathbb{1}_h(s_t) \leq T - n$ is

$$\begin{aligned}\pi(n, T, \alpha) &= \sum_{k=n}^{T-n} \binom{T}{k} \left[\frac{1}{2}\alpha^k(1-\alpha)^{T-k} + \frac{1}{2}\alpha^{T-k}(1-\alpha)^k\right] \\ &= \frac{1}{2}\sum_{k=n}^{T-n}\binom{T}{k}\alpha^k(1-\alpha)^{T-k} + \frac{1}{2}\sum_{k=n}^{T-n}\binom{T}{T-k}\alpha^{T-k}(1-\alpha)^k = \sum_{k=n}^{T-n}\binom{T}{k}\alpha^k(1-\alpha)^{T-k}.\end{aligned}$$

Note that

$$
\begin{aligned}
\frac{\partial \pi(n,T,\alpha)}{\partial \alpha} &= \sum_{k=n}^{T-n} \binom{T}{k} \alpha^{k-1}(1-\alpha)^{T-k-1}(k-T\alpha) \\
&= \sum_{k=n}^{T-n} \binom{T}{k} k\alpha^{k-1}(1-\alpha)^{T-k-1} - \sum_{k=n}^{T-n} \binom{T}{k} T\alpha^{k}(1-\alpha)^{T-k-1} \\
&= \sum_{k=n}^{T-n} \binom{T-1}{k-1} T\alpha^{k-1}(1-\alpha)^{T-k-1} - \sum_{k=n}^{T-n} \binom{T}{k} T\alpha^{k}(1-\alpha)^{T-k-1} \\
&= \frac{T}{1-\alpha}\left(\sum_{k=n}^{T-n} \binom{T-1}{k-1} \alpha^{k-1}(1-\alpha)^{T-k} - \sum_{k=n}^{T-n} \binom{T}{k} \alpha^{k}(1-\alpha)^{T-k} \right) \\
&= \frac{T}{1-\alpha}\left(\sum_{k=n-1}^{T-n-1} \binom{T-1}{k} \alpha^{k}(1-\alpha)^{T-k-1} - \pi(n,T,\alpha) \right).
\end{aligned}
$$

Now recall that $\pi(n,T,\alpha)$ is the probability of observing between n and $T-n$ signals equal to h. There are three possible ways in which this event can occur:

- $\sum_{t=1}^{T-1} \mathbb{1}_h(s_t) = n-1$ and $s_T = h$, occurring with probability $\alpha\binom{T-1}{n-1}\alpha^{n-1}(1-\alpha)^{T-n}$;
- $n \leq \sum_{t=1}^{T-1} \mathbb{1}_h(s_t) \leq T-n-1$, occurring with probability $\sum_{k=n}^{T-n-1} \binom{T-1}{k}\alpha^{k}(1-\alpha)^{T-k-1}$; or
- $\sum_{t=1}^{T-1} \mathbb{1}_h(s_t) = T-n$ and $s_T = l$, occurring with probability $(1-\alpha)\binom{T-1}{T-n}\alpha^{T-n}(1-\alpha)^{n-1}$.

Since $\pi(n,T,\alpha)$ is the sum of these three probabilities, we can rewrite the expression above as

$$
\begin{aligned}
\frac{\partial \pi(n,T,\alpha)}{\partial \alpha} &= \frac{T}{1-\alpha}\Bigg(\sum_{k=n-1}^{T-n-1} \binom{T-1}{k} \alpha^{k}(1-\alpha)^{T-k-1} - \alpha\binom{T-1}{n-1}\alpha^{n-1}(1-\alpha)^{T-n} \\
&\qquad\qquad - \sum_{k=n}^{T-n-1} \binom{T-1}{k} \alpha^{k}(1-\alpha)^{T-k-1} - (1-\alpha)\binom{T-1}{T-n}\alpha^{T-n}(1-\alpha)^{n-1} \Bigg) \\
&= \frac{T}{1-\alpha}\left((1-\alpha)\binom{T-1}{n-1}\alpha^{n-1}(1-\alpha)^{T-n} - (1-\alpha)\binom{T-1}{T-n}\alpha^{T-n}(1-\alpha)^{n-1} \right) \\
&= T\binom{T-1}{n-1}\left(\alpha^{n-1}(1-\alpha)^{T-n} - \alpha^{T-n}(1-\alpha)^{n-1} \right).
\end{aligned}
$$

It is easy to see that this expression is negative whenever $\alpha \geq \frac{1}{2}$ and $n \leq \frac{T}{2}$, thereby implying that the type-g agent is less likely to observe an "intermediate" number of h signals than the type-b agent; that is, since $\alpha_g > \alpha_b$, the type-g agent is more likely to observe extremal numbers of h signals than the type-b agent.

Finally, note that $q_g(n) \leq q_b(n)$ for $n \leq \frac{T}{2}$ and $q_g(n) \geq q_b(n)$ for $n \geq \frac{T}{2}$; therefore, the posteriors induced by these more extremal signals are themselves more extreme. This implies $F_g(q) \geq F_b(q)$ for all $q \in (0, \frac{1}{2})$ and $F_g(q) \leq F_b(q)$ for all $q \in (\frac{1}{2}, 1)$, as desired. ◊

So now consider the principal's problem in this setting. Applying our results above and treating the agent's posterior as his type, the principal offers a mechanism $\{x_h(q), x_l(q)\}$ that must be incentive compatible for the type-g agent.

With this in mind, let $U_\theta(q'|q)$ denote the expected payoff of an agent who is of type θ, has posterior q, and reports q':

$$U_\theta(q'|q) := (\gamma q + (1-\gamma)(1-q))x_h(q') + ((1-\gamma)q + \gamma(1-q))x_h(1-q').$$

Note that U_θ is, in fact, independent of the agent's type θ; this implies that whenever the mechanism is incentive compatible for the type-g agent, it will also be incentive compatible for the type-b agent. Combining this observation with the symmetry property derived above (which implies $x_h(q) = x_l(1-q)$ for all q), we write the agent's (type-independent) indirect utility as

$$\begin{aligned}
U(q'|q) &= (\gamma q + (1-\gamma)(1-q))x_h(q') + ((1-\gamma)q + \gamma(1-q))x_h(1-q') \\
&= ((2\gamma-1)q + (1-\gamma))x_h(q') + (\gamma - (2\gamma-1)q)x_h(1-q') \\
&= ((2\gamma-1)q - \gamma)(x_h(q') - x_h(1-q')) + x_h(q').
\end{aligned}$$

The principal's problem is then to

$$\max_{x_h} \left\{ \int_0^1 U(q|q) d[F_g(q) - F_b(q)] \right\} \text{ s.t. } U(q|q) \geq U(q'|q) \text{ for all } q, q' \in [0,1].$$

The incentive compatibility constraint implies that we must have both $U(q|q) \geq U(q'|q)$ and $U(q'|q') \geq U(q|q')$ for all $q, q' \in [0,1]$. Summing these incentive constraints yields

$$(2\gamma-1)(q-q')\left[(x_h(q) - x_h(1-q)) - \left(x_h(q') - x_h(1-q')\right)\right] \geq 0.$$

This implies that $x_h(q) - x_h(1-q)$ must be nondecreasing in q, which in addition implies that $x_h(q) - x_h(1-q) \geq 0$ for all $q \geq \frac{1}{2}$.

The standard "sandwich" arguments can be used to further characterize incentive compatible mechanisms. Letting $U^*(q) := U(q|q)$ for all q, we have

$$U^*(q) \geq U^*(q') + (2\gamma-1)(q-q')(x_h(q') - x_h(1-q')).$$

Reversing the roles of q and q' above and summing the resulting inequalities yields

$$(2\gamma-1)(q-q')(x_h(q') - x_h(1-q')) \leq U^*(q) - U^*(q') \leq (2\gamma-1)(q-q')(x_h(q) - x_h(1-q)).$$

Since $-1 \leq x_h(q) - x_h(1-q) \leq 1$, $U^*(q)$ is Lipschitz continuous. In addition, $x_h(q) - x_h(1-q)$ is monotone and therefore continuous almost everywhere, and so $U^*(q)$ is differentiable almost everywhere. Applying the Envelope Theorem,

$$\frac{dU^*(q)}{dq} = (2\gamma-1)(x_h(q) - x_h(1-q))$$

at every point of continuity of $x_h(q) - x_h(1-q)$ (which is almost everywhere).

We now integrate the principal's objective function by parts. (This is proper since U^* is absolutely continuous and the distribution functions F_θ are monotone.) Note that

$$\begin{aligned}
\int_0^1 U^*(q) d[F_g(q) - F_b(q)] &= \left[U^*(q)(F_g(q) - F_b(q))\right]_0^1 - \int_0^1 \frac{dU^*(q)}{dq}(F_g(q) - F_b(q))dq \\
&= -(2\gamma-1)\int_0^1 (x_h(q) - x_h(1-q))(F_g(q) - F_b(q))dq
\end{aligned}$$

$$= -2(2\gamma - 1) \int_{\frac{1}{2}}^{1} (x_h(q) - x_h(1-q))(F_g(q) - F_b(q))dq,$$

where the final step follows from the symmetry of the distributions about $\frac{1}{2}$.

Recall that $F_g(q) - F_b(q) \leq 0$ for all $q \geq \frac{1}{2}$; therefore, since $x_h(q) - x_h(1-q)$ is constrained by feasibility to lie within $[-1, 1]$, the objective function is easily maximized pointwise by setting $x_h(q) - x_h(1-q) = 1$ for all $q > \frac{1}{2}$, yielding the solution

$$x_h^*(q) = x_l^*(1-q) = \begin{cases} 0 & \text{if } q < \frac{1}{2}, \\ \frac{1}{2} & \text{if } q = \frac{1}{2}, \\ 1 & \text{if } q > \frac{1}{2}. \end{cases}$$

It is easy to see (by observation) that this mechanism does indeed satisfy the full set of incentive compatibility constraints, implying that it is indeed optimal. Of course, this is precisely equivalent to a period-T prediction mechanism: after observing all T signals, the agent reports to the principal whether they view state h or state l as more likely, and the agent is hired if (and only if) their prediction matches the principal's signal $r \in \{h, l\}$. ■

References

Strausz, R. (2003): "Deterministic Mechanisms and the Revelation Principle," *Economics Letters*, 79(3), 333–337.