Online Appendix for: Auctions with Limited Commitment

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B Appendix B: Omitted Proofs

B.1 Proof of Lemma 2

Proof. In the main paper we slightly abuse notation by using p_t both for the seller's (possibly mixed) strategy and the announced reserve price at a given history. This should not lead to confusion in the main part but for this proof we make a formal distinction. We denote the reserve price announced in period t by x_t. A history is therefore given by $h_t = (x_0, \dots, x_{t-\Delta})$. Furthermore we denote by $h_{t^+} = (h_t, x_t) = (x_0, \dots, x_{t-\Delta}, x_t)$ a history in which the reserve prices $x_0, \dots, x_{t-\Delta}$ have been announced in periods $t = 0, \dots, t - \Delta$ but no buyer has bid in these periods, and the seller has announced x_t in period t, but buyers have not yet decided whether they bid or not. For any two histories $h_t = (x_0, x_\Delta, \dots, x_{t-\Delta})$ and $h'_s = (x'_0, x'_\Delta, \dots, x'_{s-\Delta})$, with $s \leq t$, we define a new history

$$h_t \oplus h'_s = (x'_0, x'_\Delta, \dots, x'_{s-\Delta}, x_s, \dots, x_{t-\Delta}).$$

That is, $h_t \oplus h'_s$ is obtained by replacing the initial period s sub-history in h_t with h'_s. Finally, we can similarly define $h_{t^+} \oplus h'_s$ for $s < t$. With this notation we can state the proof of the lemma.

Consider any equilibrium $(p, b) \in \mathcal{E}(\Delta)$ in which the seller randomizes on the equilibrium path. The idea of the proof is that we can inductively replace randomization on the equilibrium path by a deterministic reserve price and at the same time weakly increase the seller's ex-ante revenue. We first construct an equilibrium $(p^0, b^0) \in \mathcal{E}(\Delta)$ in which the seller earns the same expected profit as in (p, b), but does not randomize at $t = 0$. If the seller uses a pure action at $t = 0$, we can set $(p^0, b^0) = (p, b)$. Otherwise, if the seller randomizes over several prices at $t = 0$, she must be indifferent between all prices in the support of $p_0(h_0)$. Therefore, we can define $p^0_0(h_0)$ as the distribution that puts probability one on a single price $x_0 \in \operatorname{supp} p_0(h_0)$. If we leave the seller's strategy unchanged for all other histories ($p^0_t(h_t) = p_t(h_t)$, for all $t > 0$ and all $h_t \in H_t$) and set $b^0 = b$, we have defined an equilibrium

(p^0, b^0) that gives the seller the same payoff as (p, b) and specifies a pure action for the seller at $t = 0$.

Next we proceed inductively. Suppose we have already constructed an equilibrium (p^m, b^m) in which the seller does not randomize on the equilibrium path up to $t = m\Delta$, but uses a mixed action on the equilibrium path at $(m+1)\Delta$. We want to construct an equilibrium (p^{m+1}, b^{m+1}) with a pure action for the seller on the equilibrium path at $(m+1)\Delta$. Suppose that in the equilibrium (p^m, b^m), the highest type in the posterior at $(m+1)\Delta$ is some type $\beta^0_{(m+1)\Delta} > 0$. We select a price in the support of the seller's mixed action at $(m+1)\Delta$, which we denote by $x^0_{(m+1)\Delta}$, such that the expected payoff of $\beta^0_{(m+1)\Delta}$ at $h_{t^+} = (h_t, x^0_{(m+1)\Delta})$ is weakly smaller than the expected payoff at h_t. In other words, we pick a price that is (weakly) bad news for the buyer with type $\beta^0_{(m+1)\Delta}$. This will be the equilibrium price announced in period $t = (m+1)\Delta$ in the equilibrium (p^{m+1}, b^{m+1}). The formal construction of the equilibrium is rather complicated. The rough idea is that, first we posit that after $x^0_{(m+1)\Delta}$ was announced in period $(m+1)\Delta$, (p^{m+1}, b^{m+1}) prescribes the same continuation as (p^m, b^m). Second, on the equilibrium path up to period $m\Delta$, we change the reserve prices such that the same marginal types β^0_t as before are indifferent between buying immediately and waiting in all periods $t = 0, \ldots, m\Delta$. Since we have chosen $x^0_{(m+1)\Delta}$ to be bad news, this leads to (weakly) higher prices for $t = 0, \ldots, m\Delta$, and therefore we can show that the seller's expected profit increases weakly. Finally, we have to specify what happens after a deviation from the equilibrium path by the seller in periods $t = 0, \ldots, (m+1)\Delta$. Consider the on-equilibrium history h_t in period t for (p^{m+1}, b^{m+1}). We identify a history $\hat{h}_t$ for which the posterior in the original equilibrium (p, b) is the same posterior as at h_t in the new equilibrium. If at h_t, the seller deviates from p^{m+1} by announcing the reserve price $\hat{x}_t$, then we define (p^{m+1}, b^{m+1}) after $h_{t^+} = (h_t, \hat{x}_t)$ using the strategy prescribed by (p, b) for the subgame starting at $\hat{h}_{t^+} = (\hat{h}_{t^+}, \hat{x}_t)$. We will show that with this definition, the seller does not have an incentive to deviate.

Next, we formally construct the sequence of equilibria (p^m, b^m), $m = 1, 2, \ldots$, and show that this sequence converges to an equilibrium (p^∞, b^∞) in which the seller never randomizes on the equilibrium path and achieves an expected revenue at least as high as the expected revenue in (p, b). We first identify a particular equilibrium path of (p^0, b^0) with a sequence of reserve prices $h^0_\infty = (x^0_0, x^0_\Delta, \ldots)$ and the corresponding buyer cutoffs $\beta^0 = (\beta^0_0, \beta^0_\Delta, \ldots)$ that specify the seller's posteriors along the path $h^0_\infty = (x^0_0, x^0_\Delta, \ldots)$.[1] Then we construct an equilibrium (p^m, b^m) such that the following properties hold: for $t = 0, \ldots, m\Delta$, the

[1] Note that the cutoffs β^0_t are the equilibrium cutoffs which may be different from the cutoffs that would arise if the seller used pure actions with prices $x^0_0, x^0_\Delta, \ldots$ on the equilibrium path.

equilibrium prices x_t^m chosen by the seller are weakly higher than x_t^0 and the equilibrium cutoffs β_t^m are exactly β_t^0; for $t > m\Delta$, or off the equilibrium path, the strategies coincide with what (p^0, b^0) prescribes at some properly identified histories, so that the two strategy profiles prescribe the same continuation payoffs at their respective histories.

In order to determine $h_\infty^0 = (x_0^0, x_\Delta^0, \ldots)$ and $\beta^0 = (\beta_0^0, \beta_\Delta^0, \ldots)$ we start at $t = 0$ and define x_0^0 as the seller's pure action in period zero in the equilibrium (p^0, b^0) and set $\beta_0^0 = 1$. Next we proceed inductively. Suppose we have fixed x_t^0 and β_t^0 for $t = 0, \Delta, \ldots$. To define $x_{t+\Delta}^0$, we select a price in the support of the seller's mixed action at history $h_{t+\Delta}^0 = (x_0^0, \ldots, x_t^0)$ in the equilibrium (p^0, b^0) such that the expected payoff of the cutoff buyer type β_t^0, conditional on $x_{t+\Delta}^0$ is announced, is no larger than this type's expected payoff at the beginning of period $t + \Delta$ before a reserve price is announced.[2] We then pick $\beta_{t+\Delta}^0$ as the cutoff buyer type following history $\left(x_0^0, \ldots, x_t^0, x_{t+\Delta}^0\right)$.

(p^0, b^0) was already defined. We proceed inductively and construct equilibrium (p^{m+1}, b^{m+1}) for $m = 0, 1, \ldots$ as follows.

(1) On the equilibrium path at $t = (m+1)\Delta$, the seller plays a pure action and announces the reserve price $x_{(m+1)\Delta}^{m+1} := x_{(m+1)\Delta}^0$.

(2) On the equilibrium path at $t = 0, \Delta, \ldots, m\Delta$, the seller's pure action x_t^{m+1} is chosen such that the buyers' on-path cutoff types in periods $t = \Delta, \ldots, (m+1)\Delta$ is $\beta_t^{m+1} = \beta_t^0$, where β_t^0 was defined above.

(3) On the equilibrium path at the history $h_{t^+} = (x_0, \ldots, x_t)$ for $t = 0, \Delta, \ldots, (m+1)\Delta$, each buyer bids if and only if $v^i \geq \beta_t^{m+1} = \beta_t^0$.

(4) at $t > (m+1)\Delta$: for any history $h_t = (x_0, \ldots, x_{t-\Delta})$ in which no deviation has occurred at or before $(m+1)\Delta$, the seller's (mixed) action is $p^{m+1}(h_t) := p^0\left(h_t \oplus \left(x_0^0, \ldots, x_{(m+1)\Delta}^0\right)\right)$. For any history $h_{t^+} = (x_0, \ldots, x_{t-\Delta}, x_t)$ in which no deviation has occurred at or before $(m+1)\Delta$, the buyer's strategy is defined by $b^{m+1}(h_{t^+}) := b^0\left(h_{t^+} \oplus \left(x_0^0, \ldots, x_{(m+1)\Delta}^0\right)\right)$.

(5) For any off-path history $h_t = (x_0, \ldots, x_{t-\Delta})$ in which the seller's first deviation from the equilibrium path occurs at $s \leq (m+1)\Delta$, the seller's (mixed) action is prescribed by $p^{m+1}(h_t) := p^0\left(h_t \oplus \left(x_0^0, \ldots, x_{s-\Delta}^0\right)\right)$. For any off-path history $h_{t^+} = (x_0, \ldots, x_{t-\Delta}, x_t)$ in which the seller's first deviation from the equilibrium path occurs in period $s \leq (m+1)\Delta$, the buyer's strategy is $b^{m+1}(h_{t^+}) := b^0\left(h_{t^+} \oplus \left(x_0^0, \ldots, x_{s-\Delta}^0\right)\right)$.

[2] If the seller plays a pure action at $h_{t+\Delta}^0$, then $x_{t+\Delta}^0$ the price prescribed with probability one by the pure action. If the seller randomizes at $h_{t+\Delta}^0$, there must be one realization, which, together with the continuation following it, gives the buyer a payoff weakly smaller than the average.

In this definition, (1) and (2) define the seller's pure actions on the equilibrium path up to $(m+1)\,\Delta$. The prices defined in (1) and (2) are chosen such that bidding according to the cutoffs β_t^{m+1} is optimal for the buyers. Part (4) defines the equilibrium strategies for all remaining on-path histories and after deviations that occur in periods after $(m+1)\Delta$, that is, in periods where the seller can still mix on the equilibrium path. The equilibrium proceeds as in (p^0, b^0) at the history where the seller used the prices $x_0^0, \ldots, x_{(m+1)\Delta}^0$ in the first $m+1$ periods. This ensures that the continuation strategy profile is taken from the continuation of an on-path history of the equilibrium (p^0, b^0), where the seller's posterior in period $(m+1)\Delta$ is the same as in the equilibrium (p^{m+1}, b^{m+1}). Finally, (5) defines the continuation after a deviation by the seller at a period in which we have already defined a pure action. If the seller deviates at a history $h_t = \left(x_0^m, \ldots, x_{s-\Delta}^m\right)$, then we use the continuation strategy of (p^0, b^0), at the history $\left(x_0^0, \ldots, x_{s-\Delta}^0\right)$.

We proceed by proving a series of claims showing that we have indeed constructed an equilibrium.

Claim 1. *The expected payoff of the cutoff buyer* $\beta_{(m+1)\Delta}^m = \beta_{(m+1)\Delta}^0$ *at the on-path history* $h_{(m+1)\Delta}^m = (x_0^m, \ldots, x_{m\Delta}^m)$ *in the candidate equilibrium* (p^m, b^m) *is the same as its payoff at the on-path history* $h_{(m+1)\Delta}^0 = (x_0^0, \ldots, x_{m\Delta}^0)$ *in the candidate equilibrium* (p^0, b^0).

Proof. This follows immediately from (1)–(3) above. □

Claim 2. *The expected payoff of the cutoff buyer* $\beta_{(m+1)\Delta}^{m+1} = \beta_{(m+1)\Delta}^0$ *at the on-path history* $h_{((m+1)\Delta)^+}^{m+1} = \left(x_0^{m+1}, \ldots, x_{m\Delta}^{m+1}, x_{(m+1)\Delta}^{m+1}\right)$ *in the candidate equilibrium* (x^{m+1}, x^{m+1}) *is the same as this cutoff type's expected payoff at the on-path history* $h_{((m+1)\Delta)^+}^0 = \left(x_0^0, \ldots, x_{m\Delta}^0, x_{(m+1)\Delta}^0\right)$ *in the candidate equilibrium* (p^0, b^0).

Proof. By construction, $x_{(m+1)\Delta}^{m+1} = x_{(m+1)\Delta}^0$. It follows from part (4) that (p^{m+1}, b^{m+1}) and (p^0, b^0) are identical on the equilibrium path from period $(m+2)\,\Delta$ onwards. The claim follows. □

Claim 3. *The expected payoff of the cutoff buyer* $\beta_{(m+1)\Delta}^{m+1} = \beta_{(m+1)\Delta}^0$ *at the on-path history* $h_{(m+1)\Delta}^{m+1} = \left(x_0^{m+1}, \ldots, x_{m\Delta}^{m+1}\right)$ *in the candidate equilibrium* (p^{m+1}, b^{m+1}) *is weakly lower than this cutoff type's expected payoff at the on-path history* $h_{(m+1)\Delta}^0 = (x_0^0, \ldots, x_{m\Delta}^0)$ *in the equilibrium* (p^0, b^0).

Proof. In the candidate equilibrium (p^{m+1}, b^{m+1}), the cutoff type's payoffs at histories $h_{(m+1)\Delta}^{m+1}$ and $h_{((m+1)\Delta)^+}^{m+1}$ are the same because the seller plays a pure action in period $(m+1)\,\Delta$. In the equilibrium (p^0, b^0), the cutoff type's payoff at history $h^{m+1}_{((m+1)\Delta)^+}$ is weakly lower than his payoff at history $h^0_{(m+1)\Delta}$ because of the definition of $x^0_{(m+1)\Delta}$ (which chosen to give the cutoff type a lower expected payoff than the expected payoff at $h^0_{(m+1)\Delta}$). The claim then follows from Claim 2. $\square$

Claim 4. *The expected payoff of the cutoff buyer $\beta^{m+1}_{(m+1)\Delta} = \beta^0_{(m+1)\Delta}$ at the on-path history $h^{m+1}_{(m+1)\Delta} = \left(x^{m+1}_0, \ldots, x^{m+1}_{m\Delta}\right)$ in the candidate equilibrium (p^{m+1}, b^{m+1}) is weakly lower than this cutoff type's expected payoff at the on-path history $h^m_{(m+1)\Delta} = (x^m_0, \ldots, x^m_{m\Delta})$ in the candidate equilibrium (p^m, b^m).*

Proof. By Claim 1, the cutoff type's expected payoff at the on-path history $h^m_{(m+1)\Delta} = (x^m_0, \ldots, x^m_{m\Delta})$ in the candidate equilibrium (p^m, b^m) is the same as its payoff at the on-path history $h^0_{(m+1)\Delta} = (x^0_0, \ldots, x^0_{m\Delta})$ in the candidate equilibrium (p^0, b^0). The claim then follows from Claim 3. $\square$

Claim 5. *For each $m = 0, 1, \ldots$ and $t = 0, 1, \ldots, m\Delta$, we have $x^{m+1}_t \geq x^m_t$.*

Proof. By Claim 4, the cutoff type $\beta^{m+1}_{(m+1)\Delta} = \beta^m_{(m+1)\Delta} = \beta^0_{(m+1)\Delta}$ in period $(m+1)\Delta$ on the equilibrium path in the candidate equilibrium (p^{m+1}, b^{m+1}) has a weakly lower payoff than its expected payoff in the candidate equilibrium (p^m, b^m). To keep this cutoff indifferent in period $m\Delta$ in both candidate equilibria, we must have $x^{m+1}_{m\Delta} \geq x^m_{m\Delta}$. Then to keep the cutoff type $\beta^{m+1}_{m\Delta} = \beta^m_{m\Delta} = \beta^0_{m\Delta}$ indifferent in period $(m-1)\Delta$, we must have $x^{m+1}_{(m-1)\Delta} \geq x^m_{(m-1)\Delta}$. The proof is then completed by induction. $\square$

Claim 6. *The seller's (time 0) expected payoff in the candidate equilibrium (p^{m+1}, b^{m+1}) is weakly higher than the seller's expected payoff in the equilibrium (p^0, b^0).*

Proof. By parts (1)–(3) of the construction, at $t = 0, \ldots, m\Delta$, (p^{m+1}, b^{m+1}) and (p^m, b^m) have the same buyer cutoffs on the equilibrium path. At $t = (m+1)\Delta$, the seller in (p^{m+1}, b^{m+1}) chooses $x^{m+1}_{(m+1)\Delta}$ that is in the support of the seller's strategy in (p^m, b^m) in that period (note that even though we haven't show that (p^m, b^m) is an equilibrium, the seller is indeed indifferent in (p^m, b^m) at $(m+1)\Delta$ because play switches to (p^0, b^0) with identical continuation payoffs by Part (4) of the construction). It then follows from Claim 5 that the seller's (time 0) expected payoff in (p^{m+1}, b^{m+1}) is weakly higher than the seller's (time 0) expected payoff in (p^m, b^m). The claim is proved by repeating this argument. $\square$

Claim 7. *For $t = \Delta, \ldots, (m+1)\Delta$, the seller's expected payoff at the on-path history $\left(x^{m+1}_0, \ldots, x^{m+1}_{t-\Delta}\right)$, in the candidate equilibrium (p^{m+1}, b^{m+1}) is weakly higher than the seller's expected at the history $\left(x^0_0, \ldots, x^0_{t-\Delta}\right)$ in equilibrium (p^0, b^0).*

Proof. Denote $m_t = t/\Delta$ so that $t = m_t\Delta$ and consider (p^{m_t}, b^{m_t}). By parts (1)–(3) of the construction, the buyer's cutoff type at $\left(x_0^{m_t}, \ldots, x_{t-\Delta}^{m_t}\right)$ in this equilibrium is the same as the buyer's cutoff type at $\left(x_0^0, \ldots, x_{t-\Delta}^0\right)$ in equilibrium (p^0, b^0). By part (4) of the construction, the seller's payoff at history $\left(x_0^{m_t}, \ldots, x_{t-\Delta}^{m_t}\right)$ in (p^{m_t}, b^{m_t}) coincides with the seller's payoff at history $\left(x_0^0, \ldots, x_{t-\Delta}^0\right)$ in equilibrium (p^0, b^0). Now consider the candidate equilibrium (p^{m_t+1}, b^{m_t+1}) and the history $\left(x_0^{m_t+1}, \ldots, x_{t-\Delta}^{m_t+1}\right)$. By claim 5, $\left(x_0^{m_t+1}, \ldots, x_{t-\Delta}^{m_t+1}\right) \geq \left(x_0^{m_t}, \ldots, x_{t-\Delta}^{m_t}\right)$. Note that the candidate equilibrium (p^{m_t+1}, b^{m_t+1}) further differs from the equilibrium (p^{m_t}, b^{m_t}) on the equilibrium path in period $t + \Delta$. But $x_t^{m_t+1}$ is in the support of the seller's randomization in (p^{m_t}, b^{m_t}) (which makes the seller indifferent by part (4) of the equilibrium construction—see the proof in Claim 6). Therefore, the seller's payoff at $\left(x_0^{m_t+1}, \ldots, x_{t-\Delta}^{m_t+1}\right)$ in the equilibrium (p^{m_t+1}, b^{m_t+1}) is weakly greater than at $\left(x_0^{m_t}, \ldots, x_{t-\Delta}^{m_t}\right)$ in the equilibrium (p^{m_t+1}, b^{m_t+1}). This completes the proof of the claim. $\square$

Claim 8. *For each $m = 0, 1, \ldots, (p^{m+1}, b^{m+1})$ such constructed is indeed an equilibrium.*

Proof. The buyer's optimality condition follows immediately from the construction. Now consider the seller. By part (5) of the construction, for any off-path history $h_t = (x_0, \ldots, x_{t-\Delta})$ in which the seller's first deviation from the equilibrium path occurs at $s \leq (m+1)\Delta$, the continuation strategy profile prescribed by (p^{m+1}, b^{m+1}) is exactly that prescribed by (p^0, b^0) at a corresponding history $h_t \oplus \left(x_0^0, \ldots, x_{s-\Delta}^0\right)$ with exactly the same expected payoff (the payoff is the same due to the fact that the seller's strategies coincide and the fact that the buyer's cutoff at h_t in (p^{m+1}, b^{m+1}) is the same as that at $h_t \oplus \left(x_0^0, \ldots, x_{s-\Delta}^0\right)$ in (p^0, b^0)). Hence there is no profitable deviation at h_t in (p^{m+1}, b^{m+1}) just as there is no profitable deviation at $h_t \oplus \left(x_0^0, \ldots, x_{s-\Delta}^0\right)$ in (p^0, b^0).

By part (4) of the construction, at $t > (m+1)\Delta$, for any history $h_t = (x_0, \ldots, x_{t-\Delta})$ in which no deviation has occurred at or before $(m+1)\Delta$, the seller's strategy at h_t in (p^{m+1}, b^{m+1}) coincides with the seller's strategy at $h_t \oplus \left(x_0^m, \ldots, x_{(m+1)\Delta}^m\right)$, with exactly the same continuation payoffs (see the previous paragraph). Hence there is no profitable deviation at h_t in (p^{m+1}, b^{m+1}).

Now consider parts (1)–(3) of the construction, for $t = 0, \ldots, (m+1)\Delta$. By Claim 6 and 7, staying on the equilibrium path gives the seller a weakly higher payoff than that from the equilibrium (p^0, b^0) at the corresponding history. But deviation from the equilibrium path triggers a switch to (p^0, b^0) at a corresponding history. Since there is no deviation in (p^0, b^0), deviation becomes even less desirable in (p^{m+1}, b^{m+1}). This completes the proof of the claim. $\square$

So far, we have obtained a sequence of equilibria $\{(p^m, b^m)\}_{m=0}^{\infty}$. We construct a limit equilibrium candidate (p^∞, b^∞) as follows. First, note that for each t the on-path equilibrium actions x_t^m of the seller are monotonically increasing as $m \to \infty$. Therefore we obtain a well-defined limit path x_t^∞. For any history on the limit path, let (p^∞, b^∞) prescribe to follow the limit path. Next consider a history h_t off the limit path x_t^∞. Let $s < t$ be the time of the first deviation. Then, by monotonicity of the on-path actions of the seller, there exists $M > t$ such that for all $m \geq M$, h_t is an off-path history for (p^m, b^m) and the first deviation is at time s. Hence by definition, behavior converges at this history and we can define $p^\infty(h_t) = p^M(h^t)$. We use an analogous construction for histories h_{t+} off the limit path.

It remains to show that (p^∞, b^∞) is an equilibrium. It is clear that buyers do not have an incentive to deviate. For the seller, suppose the seller has a profitable deviation at some history $h_{m\Delta}$. By the definition of (p^∞, b^∞) and the construction of the sequence $\{(p^m, b^m)\}_{m=0}^{\infty}$, the continuation play at h_t in the candidate equilibrium (p^∞, b^∞), where h_t is a history with $h_{m\Delta}$ as its sub-history, will coincide with continuation play at h_t prescribed by equilibrium $\left(p^{m'}, b^{m'}\right)$ for any $m' \geq m$, which is in turn described by $p^0\left(h_t \oplus \left(x_0^0, \ldots, x_{(m-1)\Delta}^0\right)\right)$ and $b^0\left(h_{t+} \oplus \left(x_0^0, \ldots, x_{(m-1)\Delta}^0\right)\right)$ by part (5) of the equilibrium construction. Since $\left(p^{m'}, b^{m'}\right)$ is an equilibrium, this particular deviation is not profitable in the equilibrium $\left(p^{m'}, b^{m'}\right)$ for any $m' \geq m$. But the on-path payoff of $\left(p^{m'}, b^{m'}\right)$ converges to that of (p^∞, b^∞), and we have just argued that the payoff after this particular deviation is the same for both $\left(p^{m'}, b^{m'}\right)$ and (p^∞, b^∞). This contradicts the assumption of profitable deviation. $\square$

B.2 Proof of Lemma 3

Proof. Fix a history h_t. Note that if all buyers bid, then by the standard argument, it is optimal for each buyer to bid their true values. Therefore, it is sufficient to show that each buyer will submit a bid. By the skimming property (Lemma 1), we only need to show $\beta_t(h_t, p_t) = 0$. Suppose by contradiction that $\beta_t(h_t, p_t) > 0$. Consider a positive type $\beta_t(h_t, p_t) - \varepsilon$, where $\varepsilon > 0$. By Lemma 1, if this type follows the equilibrium strategy and waits, he wins only if his opponents all have types lower than $\beta_t(h_t, p_t) - \varepsilon$, and he can only win in period $t + \Delta$ or later at a price no smaller than 0. If he deviates and bids his true value in period t, it follows from Lemma 1 that he wins in period t at a price 0 if all of his opponents have types lower than $\beta_t(h_t, p_t)$. Therefore, the deviation is strictly profitable for type $\beta_t(h_t, p_t) - \varepsilon$, contradicting the definition of $\beta_t(h_t, p_t)$. $\square$

B.3 Proof of Proposition 1

Proof. Let $\delta(v) := e^{-rT(v)}$ denote the discount factor for type v who trades at time $T(v)$. We can rewrite the auxiliary problem as a maximization problem with $\delta(v)$ as the choice variable:

$$\sup_{\delta} \int_0^1 \delta(v)\, J(v)\, f^{(n)}(v)\, dv$$
$$\text{s.t. } \delta(v) \in [0,1], \text{ and non-decreasing,}$$
$$\forall v \in [0,1]: \int_0^v \delta(s)\, J(s|s \leq v)\, f^{(n)}(s)\, ds \geq \delta(v) \int_0^v J(s|s \leq v)\, f^{(n)}(s)\, ds. \tag{1}$$

We show that (PF) is equivalent to (1). First suppose that (PF) holds. If v' is not part of an atom, i.e., $T^{-1}(T(v')) = \{v'\}$, then (PF) at $t' = T(v')$ is equivalent to (1) at v'. If v' is part of an atom, Lemma 4 (slack PF before atom), implies that if (PF) holds for all $t > T(v')$ in a neighborhood of $T(v')$, then (1) must hold for v'.

Conversely, suppose that (1) holds for all $v \in [0,1]$. If $t \in T([0,1])$ then the (1) for v_t implies that (PF) holds at t. Next, suppose that t is in a "quiet period," i.e., $t \notin T([0,1])$. Let t' be the start of the quiet period, i.e., $t' = \sup\{s|v_s > v_t\}$. Let $v^m \searrow v_t$ be a sequence of valuations such that $T(v^m) \to t'$ and hence $\delta(v^m) \to \delta^+(v_t)$. Since (1) holds for all v^m, we have

$$\int_0^{v_t} \delta(s)\, J(s|s \leq v_t)\, f^{(n)}(s)\, ds \geq \delta^+(v_t) \int_0^{v_t} J(s|s \leq v_t)\, f^{(n)}(s)\, ds.$$

But this is equivalent to (PF) for t'^+. Since the RHS of (PF) is constant and the LHS is increasing in the quiet period $(t', t]$, this implies that (PF) is satisfied for t.

To summarize, we have shown that the constraint set of the above problem is isomorphic to the auxiliary problem (with $\delta(v) = e^{-rT(v)}$). This shows that existence of an optimal function δ in the above problem implies existence of an optimal solution to the payoff floor constraint which proves the Proposition.

Let $\bar{\pi}$ be the supremum of this maximization problem and let (δ_k) be a sequence of feasible solutions of this problem such that

$$\lim_{k\to\infty} \int_0^1 \delta_k(v)\, J(v)\, f^{(n)}(v)\, dv = \bar{\pi}. \tag{2}$$

By Helly's selection theorem, there exists a subsequence (δ_{k_ℓ}), and a non-decreasing function $\bar{\delta} : [0,1] \to [0,1]$ such that $\delta_{k_\ell}(v) \to \bar{\delta}(v)$ for all points of continuity of $\bar{\delta}$. Hence (after selecting a subsequence), we can take (δ_k) to be almost everywhere convergent with a.e.-limit

$\bar{\delta}$ (taking the subsequence does not change the limit in (2)). By Lebesgue's dominated convergence theorem, we also have convergence w.r.t. the L^2-norm and hence weak convergence in L^2. Therefore

$$\int_0^1 \bar{\delta}(v)\, J(v)\, f^{(n)}(v)\, dv = \lim_{k\to\infty} \int_0^1 \delta_k(v)\, J(v)\, f^{(n)}(v)\, dv = \bar{\pi}.$$

It remains to show that $\bar{\delta}$ satisfies the payoff floor constraint. Suppose not. Then there exists $\hat{v} \in [0, 1)$ such that

$$\int_0^{\hat{v}} \bar{\delta}(s)\, J(s|s \le \hat{v})\, f^{(n)}(s)\, ds < \bar{\delta}(\hat{v}) \int_0^{\hat{v}} J(s|s \le \hat{v})\, f^{(n)}(s)\, ds.$$

Then there also exists $v \ge \hat{v}$ such that $\bar{\delta}$ is continuous at v, and

$$\int_0^{v} \bar{\delta}(s)\, J(s|s \le v)\, f^{(n)}(s)\, ds < \bar{\delta}(v) \int_0^{v} J(s|s \le v)\, f^{(n)}(s)\, ds.$$

Define

$$S := \bar{\delta}(v) \int_0^{v} J(s|s \le v)\, f^{(n)}(s)\, ds - \int_0^{v} \bar{\delta}(s)\, J(s|s \le v)\, f^{(n)}(s)\, ds.$$

Since v is a point of continuity we have $\bar{\delta}(v) = \lim_{k\to\infty} \delta_k(v)$. Therefore, there exists k_v such that for all $k > k_v$,

$$\left| \bar{\delta}(v) \int_0^{v} J(s|s \le v)\, f^{(n)}(s)\, ds - \delta_k(v) \int_0^{v} J(s|s \le v)\, f^{(n)}(s)\, ds \right| < \frac{S}{2},$$

and furthermore, since $\delta_k \to \bar{\delta}$ weakly in L^2, we can choose k_v such for all $k > k_v$ also

$$\left| \int_0^{v} \bar{\delta}(s)\, J(s|s \le v)\, f^{(n)}(s)\, ds - \int_0^{v} \delta_k(s)\, J(s|s \le v)\, f^{(n)}(s)\, ds \right| < \frac{S}{2}.$$

Together, this implies that for all $k > k_v$,

$$\int_0^{v} \delta_k(s)\, J(s|s \le v)\, f^{(n)}(s)\, ds < \delta_k(v) \int_0^{v} J(s|s \le v)\, f^{(n)}(s)\, ds,$$

which contradicts the assumption that δ_k is a feasible solution of the reformulated auxiliary problem defined above. $\square$

B.4 Proof of Lemma 4

Proof. Fix $v \in (v_t^+, v_t]$. We obtain a lower bound for the LHS of (A2) as follows:

$$\begin{aligned}
&\int_0^v e^{-r(T(x)-t)} J(x|x \le v) dF^{(n)}(x) \\
=&\int_{v_t^+}^v J(x|x \le v) dF^{(n)}(x) + \int_0^{v_t^+} e^{-r(T(x)-t)} J(x|x \le v_t^+) dF^{(n)}(x) \\
&- \int_0^{v_t^+} e^{-r(T(x)-t)} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x) \\
\ge &\int_{v_t^+}^v J(x|x \le v) dF^{(n)}(x) + \int_0^{v_t^+} J(x|x \le v_t^+) dF^{(n)}(x) \\
&- \int_0^{v_t^+} e^{-r(T(x)-t)} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x).
\end{aligned}$$

The equality follows because all types in $(v_t^+, v]$ trade at time t, and the inequality follows from (A1). To prove (A2), it is sufficient to show that the RHS of (A2) is smaller than the above lower bound. The RHS can be written as

$$\int_{v_t^+}^v J(x|x \le v) dF^{(n)}(x) + \int_0^{v_t^+} J(x|x \le v_t^+) dF^{(n)}(x) - \int_0^{v_t^+} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x).$$

Comparing this with the above lower bound, we only need to show:

$$-\int_0^{v_t^+} e^{-r(T(x)-t)} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x) > -\int_0^{v_t^+} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x),$$

or equivalently

$$\int_0^{v_t^+} (1 - e^{-r(T(x)-t)}) \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x) > 0.$$

Since $T(x) > t$ for $x < v_t^+$ and $F(v) - F(v_t^+) > 0$ for $v > v_t^+$, the last inequality holds and the proof is complete. □

B.5 Proof of Lemma 5

Proof. Suppose by contradiction that for some t with $v_t > 0$, we have $T(v) = t$ for all $v \in [0, v_t]$. Then for all $\varepsilon > 0$ the payoff floor constraint at $t - \varepsilon$ is

$$\int_0^{v_t} e^{-r\varepsilon} J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v) + \int_{v_t}^{v_{t-\varepsilon}} e^{-r(T(v)-(t-\varepsilon))} J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v) \ge \int_0^{v_{t-\varepsilon}} J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v).$$

Rearranging this we get

$$\int_{v_t}^{v_{t-\varepsilon}} \left(e^{-r(T(v)-(t-\varepsilon))} - 1\right) J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v) \geq \left(1 - e^{-r\varepsilon}\right) \int_0^{v_t} J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v).$$

The RHS is strictly positive for $\varepsilon > 0$ but sufficiently small because, by the left-continuity of v_t and continuity of $J_t(v)$ in t, we have

$$\lim_{\varepsilon \to 0} \int_0^{v_t} J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v) = \int_0^{v_t} J_t(v) dF_t^{(n)}(v) > 0.$$

On the other hand, since $J_t(v_t) = v_t > 0$, we have $J_{t-\varepsilon}(v) > 0$ for $v \in (v_t, v_{t-\varepsilon})$ with $\varepsilon > 0$ but sufficiently small. Note that

$$T(v) \geq t - \varepsilon \text{ for all } v \in (v_t, v_{t-\varepsilon})$$

Therefore, $e^{-r(T(v)-(t-\varepsilon))} \leq 1$ for all $v \in (v_t, v_{t-\varepsilon})$, and thus the LHS is non-positive. A contradiction. □

B.6 Proof of Lemma 6

Proof. For $t \in (a, b]$, the right-hand side of (PF) is independent of t since v_t is constant. The left-hand side is increasing in t, since t enters the discount factor. Feasibility of T implies that (PF) is satisfied at a^+ and therefore it must be strictly slack for $t \in (a, b]$. □

B.7 Proof of Lemma 7

Proof. Suppose by contradiction that T is feasible but $T(v) = \infty$ for some $v > 0$. Since T is non-increasing, there exists $w \in (0, 1)$ such that $T(v) = \infty$ for all $v \in [0, w)$ and $T(v) < \infty$ for all $v \in (w, 1]$. The left-hand side of the payoff floor constraint can be rewritten as, for all $t < \infty$,

$$\int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x) = \int_w^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x).$$

Since $T(v) < \infty$ for all $v \in (w, 1]$, we have $v_t \to w$ as $t \to \infty$. Hence, as $t \to \infty$, the limit of the left-hand side is zero:

$$\lim_{t \to \infty} \int_w^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x) = 0.$$

The limit of right-hand side of the payoff floor constraint as $t \to \infty$, however, is strictly positive:

$$\lim_{t\to\infty} \int_0^{v_t} J_t(x) dF^{(n)}(x) = \int_0^{w} J(x|x \le w) dF^{(n)}(x) > 0.$$

Therefore, the payoff floor constraint must be violated for sufficiently large t, which contradicts the feasibility of T. $\square$

B.8 Proof of Lemma 8

Proof. For $t > b$, $\hat{T}$ satisfies the (PF) because $\hat{T}(v) = T(v)$ for all $v < v_b$. If $\hat{v}_b = v_b$, the same argument extends to $t = b$. If $\hat{v}_b > v_b$, $\hat{T}$ satisfies (PF) for all $t > b$. Therefore, Lemma 4 (slack PF before atom) implies that $\hat{T}$ satisfies (PF) at $t = b$.

To show that $\hat{T}$ satisfies the (PF) for $t \le a$, we define $\psi_t(v) := J_t(v)f(v)$. For any $t > 0$, (PF) can be written as

$$ne^{rt} \int_0^{v_t} Q(v)\psi_t(v)dv \ge F^{(n)}(v_t)\Pi^E(v_t).$$

For $t \le a$, $\hat{v}_t = v_t$. Therefore the right-hand side does not change if we replace Q by $\hat{Q}$. Therefore, it suffices to show that

$$ne^{rt} \int_0^{v_t} \hat{Q}(v)\psi_t(v)dv \ge ne^{rt} \int_0^{v_t} Q(v)\psi_t(v)dv$$

for all $t \le a$. Defining $\Psi_t(v) := \int_0^v \psi_t(x)dx$, this inequality is equivalent to

$$\begin{aligned}
& \int_0^{v_t} \hat{Q}(v)\psi_t(v)dv \ge \int_0^{v_t} Q(v)\psi_t(v)dv \\
\iff\ & \hat{Q}(b)\Psi_t(b) - \hat{Q}(a)\Psi_t(a) - \int_0^{v_t} \Psi_t(v)d\hat{Q}(v) \ge Q(b)\Psi_t(b) - Q(a)\Psi_t(a) \\
& \qquad - \int_0^{v_t} \Psi_t(v)dQ(v) \\
\iff\ & \int_0^{v_t} \Psi_t(v)d\hat{Q}(v) \le \int_0^{v_t} \Psi_t(v)dQ(v).
\end{aligned}$$

To obtain the first and third lines we have used that $\hat{Q}(v) = Q(v)$ for $v \notin (a, b)$.

To establish the last line note first that both Q and $\hat{Q}$ are increasing, and hence up to an affine transformation, they are distribution functions on $[v_b, v_a]$. It follows from (A3) that Q is a mean-preserving spread of $\hat{Q}$. Second, note that $\psi_t(v) = \psi_0(v) + (1 - F(v_t))$. Since

$\psi_0(v) = J(v)f(v)$ is strictly increasing by assumption, ϕ_s is strictly increasing and Ψ_t is strictly convex. Convexity of Ψ together with the mean-preserving spread implies that the last line holds.

If (A3) is a strict inequality for a set with strictly positive measure, then all inequalities are strict which implies that (PF) becomes a strict inequality for $t \leq a$, and the ex-ante revenue is strictly increased by replacing T with $\hat{T}$. □

B.9 Proof of Proposition 3

Proof of Proposition 3. Let T be an optimal solution to the auxiliary problem with associated cutoffs v_t, and suppose by contradiction that there exists $s > 0$ such that $v_s \in (0, \overline{v})$ and the payoff floor constraint is slack at s. Define

$$\begin{aligned} s' &:= \inf \{\sigma \in (T(\overline{v}), s] \,|\, \text{(PF) is a strict inequality for all } t \in [\sigma, s]\} \\ s'' &:= \sup \{\sigma \geq s \,|\, \text{(PF) is a strict inequality for all } t \in [s, \sigma]\} . \end{aligned}$$

Since v_t is left-continuous everywhere, $s' < s$ and hence $s' < s''$. In the following, we consider two cases:

Case 1: $v_{s'}^+ > v_{s''}$

In this case, there exists an interval $(a, b) \subset [s', s'']$ such that $v_a > v_b$, and for a positive measure of types $v \in (v_b, v_a)$, $T(v) \in (a, b)$. In other words, (a, b) is not a "quiet period."

We construct an alternative solution $\hat{T}$ that satisfies the conditions of the MPS-Lemma 8 as follows:

$$\hat{T}(v) := \begin{cases} T(v), & \text{if } v \notin (v_b, v_a), \\ a, & \text{if } v \in (w, v_a), \\ b, & \text{if } v \in (v_b, w]. \end{cases}$$

We choose w such that

$$\begin{aligned} &\int_{v_b}^{v_a} \left(e^{-r\hat{T}(v)} - e^{-rT(v)}\right) (F(v))^{n-1} dv \\ = &\int_{w}^{v_a} \left(e^{-ra} - e^{-rT(v)}\right) (F(v))^{n-1} dv + \int_{v_b}^{w} \left(e^{-rb} - e^{-rT(v)}\right) (F(v))^{n-1} dv = 0. \end{aligned} \tag{3}$$

The existence of such w follows from the intermediate value theorem: The second line is continuous in w. For $w = v_a$ the first integral in the second line vanishes and the second is negative. Conversely, for $w = v_b$ the second integral in the second line vanishes and the first is positive. Hence there exists $w \in (v_b, v_a)$ for which the second line is equal to zero.

Next, note that

$$\int_{v_b}^{x} \left(e^{-r\hat{T}(v)} - e^{-rT(v)}\right) (F(v))^{n-1} dv$$

is decreasing in x for $x < w$ and increasing for $x > w$. This together with (3) implies that $\hat{T}$ satisfies the conditions of Lemma 8. There is a positive measure of types $v \in (v_b, v_a)$ for which $T(v) \neq \hat{T}(v)$. $\hat{T}$ therefore satisfies the payoff floor constraint for $t \notin (a, b)$, and yields strictly higher ex-ante profit than T.

For the contradiction, it remains to show that $\hat{T}$ satisfies the payoff floor constraint for $t \in (a, b)$. Since $(a, b) \in [s', s'']$, the payoff floor constraint with T is a strict inequality for all $t \in (a, b)$. By choosing the interval (a, b) sufficiently small, we can ensure that replacing T by $\hat{T}$ does not violate the payoff floor constraint on (a, b). This concludes the proof for Case 1.

Case 2: $v_t = v_{s'}$ *for all* $t \in (s', s'']$.

In this case, the interval where the payoff floor constraint is slack is a "quiet period" without trade. This implies that v_t is discontinuous at s''. Otherwise the payoff floor constraint would be continuous in t at s'' which would require that it is binding at s''. However, if the payoff floor is binding at the endpoint of the "quiet period," it must be violated for $t \in (s', s'')$.[3] Therefore v_t must be discontinuous at s''—i.e., $v_{s''} > v^+_{s''}$.

Similar to Case 1, we construct an alternative solution $\hat{T}$ that satisfies the conditions of the MPS-Lemma 8. The alternative solution is parametrized by two trading times $a < s'' < b$ and a cutoff valuation w which we set to $w = (v_{s''} + v^+_{s''})/2$.

$$\hat{T}(v) := \begin{cases} T(v), & \text{if } v \notin (v_b, v_{s''}), \\ a, & \text{if } v \in (w, v_{s''}), \\ b, & \text{if } v \in (v_b, w). \end{cases}$$

In words, we "split the atom" at w. For the higher types in the atom we set an earlier trading time a and for the low types we delay the trading time to b. To preserve monotonicity we also delay the trading times of all $v \in (v_b, v^+_{s''})$ to b.

If we fix $b > s''$ we need to select a such that we preserve the mean preserving spread of Q:

[3]For $t \in (s', s'')$, the right-hand side of (PF) is independent of t whereas the left-hand side is increasing in t.

$$\int_{w}^{v_a} \left(e^{-ra} - e^{-rT(v)}\right) (F(v))^{n-1} dv + \int_{v_b}^{w} \left(e^{-rb} - e^{-rT(v)}\right) (F(v))^{n-1} dv = 0. \tag{4}$$

The second integral is negative and decreasing in b and the first is positive and decreasing in a. Therefore, for b sufficiently close to s'' there is a unique $a \in (s', s'')$ so that the equation is satisfied. $a \in (s', s'')$ implies that $v_a = v_{s''}$ so that $\hat{T}$ is monotone. We have constructed $\hat{T}$ such that (A3) holds with equality for $x = v_a$ and by a similar argument as in case 1 it is satisfied for all $x \in [v_a, v_b]$. Therefore, by the MPS-Lemma 8, $\hat{T}$ yields higher ex-ante revenue than T and satisfies (PF) for all $t \notin (a, b)$. It remains to show that we can choose b such that (PF) is satisfied for all $t \in (a, b)$.

T satisfies (PF) for all t, and v_t is discontinuous at s''. Therefore, Lemma 5 (no final atom) implies that $v^+_{s''} > 0$ and we can apply Lemma 4 (slack PF before atom). This yields

$$\int_0^w e^{-r(T(v)-s'')} J(v|v \le w) dF^{(n)}(v) > \int_0^w J(v|v \le w) dF^{(n)}.$$

If we choose b sufficiently close to s'' this inequality also holds for $\hat{T}$. Moreover, a is decreasing in b and $a \to s''$ for $b \to s''$, therefore we have

$$\int_0^w e^{-r(T(v)-a)} J(v|v \le w) dF^{(n)} > \int_0^w J(v|v \le w) dF^{(n)}.$$

This shows that $\hat{T}$ satisfies (PF) for $t = a^+$. Since the cutoff $\hat{v}_t$ defined by $\hat{T}$ is constant on (a, b), this implies that the payoff floor constraint is satisfied for all $t \in (a, b)$ (see footnote 3). This completes the proof for Case 2. □

B.10 Proof of Lemma 9

Proof. We first show that v_t is continuously differentiable for all $t \in (a, b)$ where $v_t > 0$. To show this we establish several claims.

Claim 1. *$v^+_a > v_b$ and T is continuous on $v \in (v_b, v^+_a)$.*

Proof. $v^+_a = v_b$ would imply a that (a, b) is a quiet period. By Lemma 6 (slack PF in quiet period) this would required that (PF) is a strict inequality for $t \in (a, b)$. Similarly, if T has a discontinuity at $v \in (v_b, v^+_a)$, then there is a quiet period (s, s') which contradicts that (PF) is binding for all $t \in (a, b)$ by Lemma 6. □

Claim 2. *T is strictly decreasing for $v \in (v_b, v^+_a)$.*

Proof. Suppose by contradiction, that there exists a trading time $s \in (a, b)$ such that $T^{-1}(s) = (v_s^+, v_s]$ where $v_s^+ < v_s$. Since T is a feasible solution, Lemma 5 (no final atom) implies that $v_s^+ > 0$. By Lemma 4 (slack PF before atom), this implies that (PF) is a strict inequality at s which is a contradiction. □

Claim 3. *T is continuously differentiable with $T'(v) < 0$ for all $v \in (v_b, v_a^+)$*

Proof. Since T is continuous and strictly decreasing for $v \in (v_b, v_a^+)$, a binding payoff floor constraint for all $t \in (a, b)$ is equivalent to the condition that, for all $v \in (v_b, v_a^+)$,

$$\int_0^v e^{-rT(x)} J(x|x \leq v)\, dF^{(n)}(x) = e^{-rT(v)} \int_0^v J(x|x \leq v) dF^{(n)}(x),$$

which can be rearranged into

$$e^{-rT(v)} = \frac{\int_0^v e^{-rT(x)} J(x|x \leq v) dF^{(n)}(x)}{\int_0^v J(x|x \leq v)\, dF^{(n)}(x)}.$$

Continuity of T and continuous differentiability of F imply that the right-hand side of this expression is continuously differentiable, and thus T is also continuously differentiable.

Differentiating with respect to v and solving for $T'(v)$ yields

$$\begin{aligned}
T'(v) &= \frac{1}{r} \frac{\left[f^{(n)}(v)v - \int_0^v \frac{f(v)}{f(x)}\, dF^{(n)}(x)\right] \int_0^v e^{-r(T(x)-T(v))} J(x|x \leq v)\, dF^{(n)}(x)}{\left(\int_0^v J(x|x \leq v)\, dF^{(n)}(x)\right)^2} \\
&\quad - \frac{1}{r} \frac{\left[f^{(n)}(v)v - \int_0^v e^{-r(T(x)-T(v))} \frac{f(v)}{f(x)}\, dF^{(n)}(x)\right]}{\int_0^v J(x|x \leq v)\, dF^{(n)}(x)} \\
&= \frac{1}{r} \frac{\left[f^{(n)}(v)v - \int_0^v \frac{f(v)}{f(x)}\, dF^{(n)}(x)\right] \int_0^v J(x|x \leq v)\, dF^{(n)}(x)}{\left(\int_0^v J(x|x \leq v)\, dF^{(n)}(x)\right)^2} \\
&\quad - \frac{1}{r} \frac{\left[f^{(n)}(v)v - \int_0^v e^{-r(T(x)-T(v))} \frac{f(v)}{f(x)}\, dF^{(n)}(x)\right]}{\int_0^v J(x|x \leq v)\, dF^{(n)}(x)} \\
&= \frac{f(v)}{r} \frac{\int_0^v \left(e^{-r(T(x)-T(v))} - 1\right) \frac{1}{f(x)}\, dF^{(n)}(x)}{\int_0^v J(x|x \leq v)\, dF^{(n)}(x)}.
\end{aligned}$$

where the second equality follows from the binding payoff floor constraint. In the last line, the numerator is strictly negative and the denominator is positive. Therefore $T'(v) < 0$. □

Together Claims 1–3 imply that v_t is continuously differentiable for $t \in (a, b)$ where $v_t > 0$.

Next we derive a differential equation for v_t from the binding payoff floor constraint. In the process we also show that v_t is twice continuously differentiable.

Since v_t is continuously differentiable on (a,b), we can differentiate (PF) on both sides to obtain

$$
\begin{aligned}
&e^{-rt}v_t f^{(n)}(v_t)\dot{v}_t - \int_0^{v_t} e^{-rT(x)}\frac{f(v_t)\dot{v}_t}{f(x)}dF^{(n)}(x) \\
=&-re^{-rt}\int_0^{v_t} J_t(x)dF^{(n)}(x) + e^{-rt}v_t f^{(n)}(v_t)\dot{v}_t - e^{-rt}\int_0^{v_t}\frac{f(v_t)\dot{v}_t}{f(x)}dF^{(n)}(x),
\end{aligned}
$$

where we have used $\frac{\partial J_t(x)}{\partial t} = -\frac{f(v_t)\dot{v}_t}{f(x)}$, and $T(v_t)=t$ which follows from continuity of $T(v)$. This equation can be further simplified

$$
\begin{aligned}
&-\int_0^{v_t} e^{-rT(x)}\frac{f(v_t)\dot{v}_t}{f(x)}dF^{(n)}(x) \\
&\qquad\qquad = -re^{-rt}\int_0^{v_t} J_t(x)dF^{(n)}(x) - e^{-rt}\int_0^{v_t}\frac{f(v_t)\dot{v}_t}{f(x)}dF^{(n)}(x).
\end{aligned}
$$

Since T is continuously differentiable for all $v\in(v_b,v_a^+)$ by Claim 3, $\dot{v}_t<0$ for $t\in(a,b)$. By assumption, $f(v_t)>0$, so we can divide the previous equation by $-f(v_t)\dot{v}_t$ to obtain

$$
\int_0^{v_t} e^{-rT(x)}\frac{1}{f(x)}dF^{(n)}(x) = \frac{re^{-rt}}{f(v_t)\dot{v}_t}\int_0^{v_t} J_t(x)dF^{(n)}(x) + e^{-rt}\int_0^{v_t}\frac{1}{f(x)}dF^{(n)}(x). \tag{5}
$$

This equation, together with our assumption that $f(v)$ is continuously differentiable, implies that v_t is twice continuously differentiable. Differentiating (5) on both sides yields

$$
\begin{aligned}
&e^{-rt}nF^{n-1}(v_t)\dot{v}_t \\
=&re^{-rt}\left(\frac{v_t f^{(n)}(v_t)\dot{v}_t - f(v_t)\int_0^{v_t}\frac{f^{(n)}(x)}{f(x)}dx\,\dot{v}_t}{f(v_t)\dot{v}_t}\right) \\
&-re^{-rt}\left(\frac{\left(\dot{v}_t\frac{f'(v_t)}{f(v_t)}+\frac{\ddot{v}_t}{\dot{v}_t}+r\right)\int_0^{v_t} J_t(x)f^{(n)}(x)dx}{f(v_t)\dot{v}_t}\right) \\
&-re^{-rt}\int_0^{v_t}\frac{1}{f(x)}dF^{(n)}(x) + e^{-rt}nF^{n-1}(v_t)\dot{v}_t.
\end{aligned}
$$

Multiplying both side by $f(v_t)\,\dot{v}_t$, and rearranging we get

$$\frac{\ddot{v}_t}{\dot{v}_t}+\underbrace{\left(\frac{f'(v_t)}{f(v_t)}-\frac{f^{(n)}(v_t)v_t-2f(v_t)\,n\int_0^{v_t}F^{n-1}(x)dx}{\int_0^{v_t}J_t(x)f^{(n)}(x)dx}\right)}_{=:g(v_t)}\dot{v}_t+r=0.$$

Some further algebra yields

$$\int_0^{v_t}J_t(x)f^{(n)}(x)dx=(n-1)n\int_0^{v_t}\left(F(v_t)-F(x)\right)F^{n-2}(x)f(x)xdx,$$

which implies

$$g(v_t)=\frac{f'(v_t)}{f(v_t)}-\frac{\left\{v_tF^{n-1}(v_t)-2\int_0^{v_t}F^{n-1}(v)\,dv\right\}f(v_t)}{(n-1)\int_0^{v_t}\left[F(v_t)-F(v)\right]F^{n-2}(v)\,f(v)\,vdv}.$$

□

B.11 Proof of Lemma 10

Proof. Since $f(v)$ is continuously differentiable $\lim_{v\to 0}vf'(v)$ exists. We first show that $\lim_{v\to 0}vf'(v)=0$. Suppose by contradiction that $\lim_{v\to 0}vf'(v)=z\neq 0$. If $z>0$, we must have $f'(v)\geq z/(2v)$ for a neighborhood $(0,\varepsilon)$, which implies that $f(\varepsilon)=f(0)+\int_0^{\varepsilon}f'(v)dv\geq f(0)+\int_0^{\varepsilon}(z/(2v))dv=\infty$ which contradicts the assumption of a finite density. If $z<0$, we have $f'(v)\leq z/(2v)$ for a neighborhood $(0,\varepsilon)$, which implies that $f(\varepsilon)=f(0)+\int_0^{\varepsilon}f'(v)dv\leq f(0)+\int_0^{\varepsilon}(z/(2v))dv=-\infty$ which contradicts $f(v)>0$. Since $f(0)>0$ and $\lim_{v\to 0}vf'(v)=0$ together imply $\phi=\lim_{v\to 0}\frac{vf'(v)}{f(v)}=0$, we have $\overline{N}(F):=1+\frac{\sqrt{2+\phi}}{1+\phi}=1+\sqrt{2}\in(2,3)$.

If $f(0)=0$, we use a Taylor expansion of $f(v)$ at zero to obtain

$$\phi=\lim_{v\to 0}\frac{f'(v)v}{f(v)}=\lim_{v\to 0}\frac{f'(v)v}{f'(0)v}=1.$$

This implies $\overline{N}(F)=1+\sqrt{3}/2<2$. □

B.12 Proof of Lemma 11

Proof. If Assumption 2 is satisfied, we can repeatedly use l'Hospital's rule, and

$$\lim_{v\to 0}\frac{vf(v)}{F(v)}=\lim_{v\to 0}\frac{f'(v)v+f(v)}{f(v)}=1+\phi\quad\text{and}\quad\lim_{v\to 0}\frac{F(v)}{vf(v)}=\frac{1}{1+\phi},$$

to get

$$\kappa := \lim_{v\to 0} g(v)v = \phi - \frac{((n-1)\,\phi + n - 2)\,(n\phi + n + 1)}{(n-1)\,(1+\phi)}.$$

Simple algebra shows that if $\phi > -1$,

$$\kappa > -1 \qquad \Longleftrightarrow \qquad n < \overline{N}(F).$$

Next, we transform the ODE (16) using the change of variables $y = \dot{v}_t$. This yields

$$y'(v) + g(v)y(v) + r = 0.$$

The general solution is given by

$$y\,(v) = e^{-\int_m^v g(x)dx} \left(C - \int_m^v re^{\int_m^w g(x)dx} dw \right), \tag{6}$$

where $m > 0$.[4] Feasibility requires that $y(v) \leq 0$ for all $v \in (0, v_0^+)$. This implies that

$$\forall v \in (0, v_0^+): \quad C \leq \int_m^v re^{\int_m^w g(x)dx} dw,$$

Since the right-hand side is increasing in v this implies

$$C \leq \overline{C} := -\int_0^m re^{\int_m^w g(x)dx} dw$$

and

$$\overline{C} = \lim_{v\to 0} \int_m^v re^{\int_m^w g(x)dx} dw > -\infty. \tag{7}$$

(i) Suppose $\kappa < -1$. Since $\kappa = \lim_{v\to 0} g(v)v$, there must exist $\gamma > 0$ such that $g(v) \leq -\frac{1}{v}$ for all $v \in (0, \gamma]$. We may assume that $0 < m < \gamma$. In this case, the limit in (7) can be computed as follows:

$$\begin{aligned} \lim_{v\to 0} \int_m^v re^{\int_m^w g(x)dx} dw &= \lim_{v\to 0} - \int_v^m re^{-\int_w^m g(x)dx} dw \\ &\leq \lim_{v\to 0} - \int_v^m re^{\int_w^m \frac{1}{x}dx} dw = \lim_{v\to 0} - \int_v^m r\frac{m}{w} dw = -\infty. \end{aligned}$$

[4] For $m = 0$, the solution candidate is not well defined for all κ because $e^{-\int_m^v g(x)dx} = \infty$.

Given that $C \leq \overline{C}$ there exits no finite C such that the general solution in (6) satisfies $y(v) \leq 0$ for all $v \in (0, v_0^+)$. This shows part (i).

To prove part (ii), we set $C = \overline{C}$. We show that the resulting solution

$$y(v) = -e^{-\int_m^v g(x)dx} \int_0^v re^{\int_m^w g(x)dx} dw = -\int_0^v re^{-\int_w^v g(x)dx} dw, \tag{8}$$

is negative and finite for all v. It is clear that $y(v) < 0$, so it suffices to rule out $y(v) = -\infty$. Since $\kappa = \lim_{v \to 0} g(v)v > -1$, there exist $\hat{\kappa} > -1$ and $\gamma > 0$ such that $g(v) \geq \frac{\hat{\kappa}}{v}$ for all $v \in (0, \gamma]$. Hence the limit in (7) can be computed as (where we may again assume that $0 < m < \gamma$):

$$\begin{aligned}
\lim_{v_t \to 0} \int_m^{v_t} re^{\int_m^v g(x)dx} dv &= \lim_{v_t \to 0} -\int_{v_t}^m re^{-\int_v^m g(x)dx} dv \\
&\geq \lim_{v_t \to 0} -\int_{v_t}^m re^{-\hat{\kappa} \ln \frac{m}{v}} dv = \lim_{v_t \to 0} -\int_{v_t}^m r\left(\frac{v}{m}\right)^{\hat{\kappa}} dv \\
&= -rm^{-\hat{\kappa}} \frac{1}{\hat{\kappa}+1} \lim_{v_t \to 0} \left(m^{\hat{\kappa}+1} - v_t^{\hat{\kappa}+1}\right) > -\infty.
\end{aligned}$$

Therefore, $y(v)$ is finite and $y(v) < 0$ for all v. Next we have to show that (8) can be integrated to obtain a feasible solution of the auxiliary problem. It suffices to verify that the following boundary condition from Lemma 7 (cutoffs converge to zero):

$$\lim_{t \to \infty} v_t = 0, \tag{9}$$

is satisfied. Recall that $\dot{v}_t = y(v_t)$. Therefore, we have

$$\dot{v}_t = -e^{-\int_m^{v_t} g(v)dv} \left(\int_0^{v_t} re^{\int_m^v g(x)dx} dv \right).$$

We first show that, for any $v_0^+ \in [0, 1]$, the solution to this differential equation satisfies (9). Since the term in the parentheses is strictly positive we have

$$\frac{e^{\int_m^{v_t} g(v)dv} \dot{v}_t}{\int_0^{v_t} e^{\int_m^v g(x)dx} dv} = -r.$$

Integrating both sides with respect to t, we get

$$\ln \int_0^{v_t} e^{\int_m^v g(x)dx} dv - \ln \int_0^{v_0^+} e^{\int_m^v g(x)dx} dv = -rt.$$

Now take $t \to \infty$. The RHS diverges to $-\infty$ and the second term on the LHS is constant, so we must have

$$\lim_{t\to\infty} \ln \int_0^{v_t} e^{\int_m^v g(x)dx} dv = -\infty$$

which holds if and only if $\lim_{t\to\infty} v_t = 0$. Therefore, we have found a solution that satisfies the boundary condition and is decreasing for all starting values v_0^+. This completes the proof of part (ii).

(iii) Let $\hat{v}_t$ be a decreasing solution to the binding payoff floor constraint that does not satisfy (7). Then $z(v) = \dot{\hat{v}}_t$ must be given by (6) for some $C \leq \overline{C}$. The solution v_t^x satisfies (7). If we define $y^x(v) = \dot{v}_t^x$, $y^x(v)$ satisfies (8). Therefore we have

$$z(v) = y(v) - \left(\overline{C} - C\right) e^{-\int_m^v g(x)dx} < y(v).$$

This implies that $\hat{v}_t = v_t^x$ implies $\dot{\hat{v}}_t < \dot{v}_t^x$. We have established a single crossing property: For any $x \in [0, 1]$, $\hat{v}_t$ crosses v_t^x at most once, and from above.

Now we pick x so that we can apply the MPS-Lemma 8. Let $\hat{Q}(v)$ be the expected discounted winning probability times associated with the cutoff path $\hat{v}_t$ and $Q^x(v)$ the one associated with v_t^x. Define

$$D(x) = \int_0^1 Q^x(v) - \hat{Q}(v)(v)dv.$$

Clearly $D(x)$ is continuous in x. $x = 0$ implies that $Q^x(v) = (F(v))^{(n-1)} > (F(v))^{(n-1)} e^{-r\widehat{T}(v)} = \hat{Q}(v)$ for all $v < \hat{v}_t^+$ Therefore $D(0) > 0$. If we set $x = \hat{v}_t^+$, then v_t^x and $\hat{v}_t$ intersect at $t = 0$ and the crossing property implies that $v_t^x > \hat{v}_t$ for all $t > 0$. This implies $Q^x(v) < \hat{Q}(v)$ for all $v < \hat{v}_t^+$ and thus $D(v_0^+) < 0$. Hence, the intermediate value theorem implies that there exists $x^* \in (0, v_0^x)$ such that $D(x^*) = 0$. Moreover, $\hat{v}_t$ crosses $v_t^{x^*}$ exactly once and from above. This implies that $\hat{Q}(v)$ crosses $Q^x(v)$ once and from below. Therefore we must have

$$\int_0^z Q^{x^*}(v) - \hat{Q}(v)dv \leq 0, \forall z \in [0, 1].$$

Hence Lemma 8 implies that v_t^x yields strictly higher profit than $\hat{v}_t$. □

B.13 Proof of Proposition 5

Proof. The lower bound follows directly from Lemma 3. For the upper bound, by Lemma 2, we can restrict attention to equilibria (p_m, b_m) in which the seller does not randomize on the equilibrium path.

We first define an ε-relaxed continuous-time auxiliary problem. We replace the payoff floor constraint by

$$\int_0^{v_t} e^{-r\,(T(x)-t)} J_t(v) dF^{(n)}(v) \geq (1-\varepsilon) F^{(n)}(v_t) \Pi^E(v_t).$$

By similar arguments as in the proof of Proposition 1, there exists an optimal solution to the ε-relaxed continuous-time auxiliary problem for every $\varepsilon \geq 0$. Denote the value of this problem by V_ε. Clearly V_ε is increasing in ε so that the limit $\lim_{\varepsilon\to 0} V_\varepsilon$ exists. Moreover, by similar arguments as in the proof of Proposition 1, there exists a sequence $\varepsilon^m \searrow 0$ such that the corresponding optimal solutions converge to a feasible solution of the auxiliary problem for $\varepsilon = 0$. Therefore we have

$$\lim_{\varepsilon\to 0} V_\varepsilon = \lim_{m\to\infty} V_{\varepsilon^m} = V. \tag{10}$$

The first equality follows from the existence of the first limit and the second follows because the objective function is continuous.[5]

Next, we formulate a discrete version of the auxiliary problem. For given Δ, the feasible set of this problem is given by

$$\begin{aligned} & T : [0,1] \to \{0, \Delta, 2\Delta, \ldots\} \text{ non-increasing,} \\ \text{and} \quad & \int_0^{v_{k\Delta}} e^{-r\,(T(x)-k\Delta)} J_{k\Delta}(v) dF^{(n)}(v) \geq F^{(n)}(v_{k\Delta}) \Pi^E(v_{k\Delta}) \qquad \forall k \in \mathbb{N}. \end{aligned}$$

We denote the value of this problem by $V(\Delta)$. Let $\mathcal{E}^d(\Delta) \subset \mathcal{E}(\Delta)$ denote the set of equilibria in which the seller does not randomize on the equilibrium path. The first constraint is clearly satisfied for outcomes of any equilibrium in $\mathcal{E}^d(\Delta)$. The second constraint requires that in each period, the seller's continuation profit on the equilibrium path exceeds the revenue from an efficient auction given the current posterior. Lemma 3 shows that this lower bound is a necessary condition for an equilibrium. Therefore, the seller's expected revenue in any equilibrium $(p, b) \in \mathcal{E}^d(\Delta)$ cannot exceed $V(\Delta)$. Moreover, for given ε, the feasible set of the discrete auxiliary problem is contained in the feasible set of the ε-relaxed continuous-time auxiliary problem if Δ is sufficiently small. Formally, we have:

[5] As in the proof of Proposition 1, we need to formulate the problem in terms of $\delta(v) = e^{-rT(v)}$ and then use weak convergence. We omit replicating the rigorous argument here.

Claim: Let $\varepsilon > 0$ and $\Delta_\varepsilon = -\ln(1-\varepsilon)/r$. For all $\Delta < \Delta_\varepsilon$ we have

$$\sup_{(p,b)\in\mathcal{E}^d(\Delta)} \Pi^\Delta(p,b) \leq V(\Delta) \leq V_\varepsilon.$$

Proof of the claim: The first inequality has been shown in the text above. For the second, let T^Δ be an element of the feasible set of the discrete auxiliary problem for $\Delta \leq \Delta_\varepsilon$. Let v_t^Δ be the corresponding cutoff path. Note that for $t \in (k\Delta, (k+1)\Delta]$ we have $v_t^\Delta = v_{(k+1)\Delta}^\Delta$ and hence

$$\begin{aligned}
&\int_0^{v_t^\Delta} e^{-r\,(T^\Delta(v)-t)} J_t(v) n(F(v))^{n-1} f(v)dv \\
=&e^{-r((k+1)\Delta-t)} \int_0^{v_{(k+1)\Delta}^\Delta} e^{-r\,(T^\Delta(v)-(k+1)\Delta)} J_{(k+1)\Delta}(v) n(F(v))^{n-1} f(v)dv \\
\geq&e^{-r\Delta} \int_0^{v_{(k+1)\Delta}^\Delta} e^{-r\,(T^\Delta(v)-(k+1)\Delta)} J_{(k+1)\Delta}(v) n(F(v))^{n-1} f(v)dv \\
\geq&e^{-r\Delta} F^{(n)}(v_{(k+1)\Delta}^\Delta)\Pi^E(v_{(k+1)\Delta}^\Delta) \\
=&e^{-r\Delta} F^{(n)}(v_t^\Delta)\Pi^E(v_t^\Delta) \geq (1-\varepsilon)F^{(n)}(v_t^\Delta)\Pi^E(v_t^\Delta).
\end{aligned}$$

The first inequality holds because $t \geq k\Delta$, the second inequality follows from the payoff floor constraint of the discretized auxiliary problem, and the last inequality holds because $\Delta \leq \Delta_\varepsilon$. Therefore, T^Δ is a feasible solution for the ε-relaxed continuous time auxiliary problem, and hence $V(\Delta) \leq V_\varepsilon$ if $\Delta < \Delta_\varepsilon$. Thus the claim is proved.

To complete the proof for Proposition 5, it suffices to show $\Pi^* \leq V$. We have:

$$\Pi^* = \limsup_{\Delta\to 0} \sup_{(p,b)\in\mathcal{E}^d(\Delta)} \Pi^\Delta(p,b) \leq \lim_{\varepsilon\to 0} V_\varepsilon = V.$$

The first equality follows from Lemma 2 which shows that the maximal revenue can be achieved without randomization on the equilibrium path by the seller. The previous claim implies that inequality. The second equality was shown above (see (10)). □

B.14 Proof of Lemma 12

Proof. The proof follows the same steps as in the proof of Lemma 11 but when taking the limit $\kappa = \lim_{v\to 0} g(v)\, v$, we have to take into account that $F(0) > 0$. Applying l'Hospital's rule, we can compute κ as

$$\kappa = \phi - \lim_{v \to 0} \frac{(n-1)v^2 f^2(v) F^{n-2}(v) + v^2 f'(v) F^{n-1}(v) - 2\left(f'(v)v + f(v)\right) \int_0^v F^{n-1}(s)\, ds}{(n-1) f(v) \int_0^v s\, F^{n-2}(s) f(s)\, ds}$$

Noting that $F(0) > 0$, we can again apply l'Hospital's rule to obtain

$$\lim_{v \to 0} \frac{(n-1)v^2 f^2(v) F^{n-2}(v)}{(n-1) f(v) \int_0^v s\, F^{n-2}(s) f(s)\, ds} = 2$$

and

$$\lim_{v \to 0} \frac{v^2 f'(v) F^{n-1}(v) - 2\left(f'(v)v + f(v)\right) \int_0^v F^{n-1}(s)\, ds}{(n-1) f(v) \int_0^v s\, F^{n-2}(s) f(s)\, ds} = -\infty$$

It follows that

$$\kappa = \lim_{v \to 0} v g(v) = +\infty.$$

The rest of the proof of part (i) follows from the proof of Part (ii) of Lemma 11. Part (ii) is proven by the same steps as in the proof of Part (iii) of Lemma 11. □

C Appendix C: Equilibrium Approximation of the Solution to the Binding Payoff Floor Constraint

C.1 Equilibrium Approximation (Proof of Proposition 6)

In this section we construct equilibria that approximate the solution to the binding payoff floor constraint. We proceed in three steps. First, we show that if the binding payoff floor constraint has a decreasing solution, then there exists a nearby solution for which the payoff floor constraint is strictly slack. In particular, we show that for each $K > 1$ sufficiently small, there exists a solution with a decreasing cutoff path to the following *generalized* payoff floor constraint:

$$\int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF_t^{(n)}(x) = K\, \Pi^E(v_t). \tag{11}$$

For $K = 1$, (11) reduces to the original payoff floor constraint in (PF) (divided by $F_t(v_t)$). Therefore, a decreasing solution that satisfies (11) for $K > 1$ is a feasible solution to the auxiliary problem. Moreover, the slack in the original payoff floor constraint is proportional to $\Pi^E(v_t)$.

Lemma 13. *Suppose Assumption 2 holds and $n < \overline{N}(F)$. Then there exists $\Gamma > 1$ such that* *for all $K \in [1, \Gamma]$, there exists a feasible solution T^K to the auxiliary problem that satisfies* (11). *For $K \searrow 1$, $T^K(v)$ converges to $T(v)$ for all $v \in [0, 1]$, and the seller's expected revenue converges to the value of the auxiliary problem.*

In the second step, we discretize the solution obtained in the first step so that all trades take place at times $t = 0, \Delta, 2\Delta, \ldots$. For given K and Δ, we define the discrete approximation $T^{K,\Delta}$ of T^K by delaying all trades in the time interval $(k\Delta, (k+1)\Delta]$ to $(k+1)\Delta$:

$$T^{K,\Delta}(v) := \Delta \min \left\{ k \in \mathbb{N} \,\middle|\, k\Delta \geq T^K(v) \right\}. \tag{12}$$

In other words, we round up all trading times to the next integer multiple of Δ. Clearly, for all $v \in [0, 1]$ we have,

$$\lim_{K \to 1} \lim_{\Delta \to 0} T^{K,\Delta}(v) = \lim_{\Delta \to 0} \lim_{K \to 1} T^{K,\Delta}(v) = T(v),$$

and the seller's expected revenue also converges. Therefore, if we show that the functions T^{K_m,Δ_m} for some sequence (K_m, Δ_m) describe equilibrium outcomes for a sequence of equilibria $(p^m, b^m) \in \mathcal{E}(\Delta_m)$, we have obtained the desired approximation result.

The discretization changes the continuation revenue, but we can show that the approximation loss vanishes as Δ becomes small. In particular, if Δ is sufficiently small, then the approximation loss is less than half of the slack in the payoff floor constraint at the solution T^K. More precisely, we have the following lemma.

Lemma 14. *Suppose Assumption 3 is satisfied and let $n < \overline{N}(F)$. For each $K \in [1, \Gamma]$, where Γ satisfies the condition of Lemma 13, there exists $\bar{\Delta}^1_K > 0$ such that for all $\Delta < \bar{\Delta}^1_K$, and all $t = 0, \Delta, 2\Delta, \ldots$,*

$$\int_0^{v_t^{K,\Delta}} e^{-r\left(T^{K,\Delta}(x) - t\right)} J_t(x) dF_t^{(n)}(x) \geq \frac{K+1}{2} \Pi^E(v_t^{K,\Delta}).$$

This lemma shows that if Δ is sufficiently small, at each point in time $t = 0, \Delta, 2\Delta, \ldots$, the continuation payoff of the discretized solution is at least as high as $1 + (K-1)/2$ times the profit of the efficient auction.

In the final step, we show that the discretized solution $T^{K,\Delta}$ can be implemented in an equilibrium of the discrete time game. To do this, we use weak-Markov equilibria as a threat to deter any deviation from the equilibrium path by the seller. The threat is effective because the uniform Coase conjecture (Proposition 4.(ii)) implies that the profit of a weak-Markov equilibrium is close to the profit of an efficient auction for any posterior along the equilibrium path. More precisely, let $\Pi^{\Delta}(p, b|v)$ be the continuation profit at posterior v for a given equilibrium $(p, b) \in \mathcal{E}(\Delta)$ as before.[6] Then Proposition 4.(ii) implies that for all $K \in [1, \Gamma]$, where Γ satisfies the condition of Lemma 13, there exists $\bar{\Delta}_K^2 > 0$ such that, for all $\Delta < \bar{\Delta}_K^2$, there exists an equilibrium $(p, b) \in \mathcal{E}(\Delta)$ such that, for all $v \in [0, 1]$,

$$\Pi^{\Delta}(p, b|v) \leq \frac{K+1}{2}\Pi^E(v). \tag{13}$$

Now suppose we have a sequence $K_m \searrow 1$, where $K_m \in [1, \Gamma]$ as in Lemma 13. Define $\bar{\Delta}_K := \min\left\{\bar{\Delta}_K^1, \bar{\Delta}_K^2\right\}$. We can construct a decreasing sequence $\Delta_m \searrow 0$ such that for all m, $\Delta_m < \bar{\Delta}_{K_m}$. By Lemma 14 and (13), there exists a sequence of (punishment) equilibria $(\hat{p}^m, \hat{b}^m) \in \mathcal{E}(\Delta_m)$ such that for all m and all $t = 0, \Delta_m, 2\Delta_m, \ldots$

$$\begin{aligned}\int_0^{v_t^{K_m,\Delta_m}} e^{-r\left(T^{K_m,\Delta_m}(x)-t\right)} J_t(x) dF^{(n)}(x) \geq & \frac{K_m+1}{2}\Pi^E(v_t^{K_m,\Delta_m}) \\ \geq & \Pi(\hat{p}^m, \hat{b}^m | v_t^{K_m,\Delta_m}).\end{aligned} \tag{14}$$

The left term is the continuation profit at time t on the candidate equilibrium path given by T^{K_m,Δ_m}. This is greater or equal than the second expression by Lemma 14. The term on the right is the continuation profit at time t if we switch to the punishment equilibrium. This continuation profit is smaller than the middle term by Proposition 4.(ii). Therefore, for each m, $(\hat{p}^m, \hat{b}^m)$ can be used to support T^{K_m,Δ_m} as an equilibrium outcome of the game indexed by Δ_m. Denote the equilibrium that supports T^{K_m,Δ_m} by $(p^m, b^m) \in \mathcal{E}(\Delta_m)$. It is defined as follows: On the equilibrium path, the seller posts reserve prices given by T^{K_m,Δ_m} and (12). A buyer with type v bids at time $T^{K_m,\Delta_m}(v)$ as long as the seller does not deviate. As argued in Section IV.B, this is a best response to the seller's on-path behavior because the prices given by (12) implement the reading time function T^{K_m,Δ_m}. After a deviation by the seller, she is punished by switching to the equilibrium $(\hat{p}^m, \hat{b}^m)$. Since the seller anticipates the switch to $(\hat{p}^m, \hat{b}^m)$ after a deviation, her deviation profit is bounded above by $\Pi(\hat{p}^m, \hat{b}^m | v_t^{K_m,\Delta_m})$. Therefore, (14) implies that the seller does not have a profitable deviation. To summarize, we have an approximation of the solution to the binding payoff floor constraint by discrete time equilibrium outcomes. This concludes the proof of Proposition 6.

[6]If the profit differs for different histories that lead to the same posterior, we could take the supremum, but this complication does not arise with weak-Markov equilibria.

C.2 Proof of Lemma 13

The key step of the approximation is to discretize the solution to the binding payoff floor constraint. In order to do that, we first need to find a feasible solution such that the payoff floor constraint is strictly slack. We have the following generalization of Lemma 9.

Lemma 15. *Suppose $T(x)$ satisfies (11) for all $t \in (a,b)$ and suppose T is continuously differentiable with $-\infty < T'(v) < 0$ for all $v \in (v_b, v_a)$ and v_t is continuously differentiable for all $t \in (a,b)$. Then v_t is twice continuously differentiable on (a,b) and is characterized by*

$$\frac{\ddot{v}_t}{\dot{v}_t} + g(v_t,K)\dot{v}_t + h(v_t,K)\left(\dot{v}_t\right)^2 + r = 0,$$

where

$$g(v_t,K) = \frac{f'(v_t)}{f(v_t)} - \frac{\left\{\left(2-\frac{1}{K}\right) v_t F^{n-1}(v_t) - 2\int_0^{v_t} F^{n-1}(v)\,dv\right\} f(v_t)}{(n-1)\int_0^{v_t}\left[F(v_t)-F(v)\right] F^{n-2}(v) f(v) v dv},$$

and

$$h(v_t,K) = \frac{K-1}{rK}\frac{F^{n-2}(v_t) f^2(v_t) v_t}{\int_0^{v_t}\left[F(v_t)-F(v)\right] F^{n-2}(v) f(v) v dv}.$$

Proof. The proof follows similar steps as the proof of Lemma 9. □

Repeatedly applying l'Hospital's rule yields

Lemma 16. *If Assumption 2 is satisfied, we have*

$$\kappa := \lim_{v\to 0} g(v)v = \phi - \frac{\left((n-1)\phi + n - 2\right)\left(n\phi + n + 1\right)}{(n-1)(1+\phi)}, \tag{15}$$

$$\lim_{v\to 0} g(v,K)v = \kappa - \frac{K-1}{K}\left(n\phi + n + 2 + \frac{\phi+2}{(n-1)(1+\phi)}\right), \tag{16}$$

and

$$\lim_{v\to 0} h(v,K)v^2 = \frac{1}{r}\frac{K-1}{K}(n+\phi n+1)(n+\phi n-\phi). \tag{17}$$

We use the change of variables $y = \dot{v}_t$ to rewrite the ODE obtained in Lemma 15 as

$$y'(v) = -r - g(v,K)y(v) - h(v,K)\left(y(v)\right)^2. \tag{18}$$

Any solution to the above ODE with $K > 1$ would lead to a strictly slack payoff floor constraint. Our goal is to show that the solution to the ODE exists for any K sufficiently close to zero and converges to the solution given by (7) as $K \searrow 1$. We will verify below that (7) satisfies the boundary condition $\lim_{v\to 0} y(v) = 0$. Given this observation, we want to show the existence of a solution $y_K(v) < 0$ of (18) that satisfies the same boundary condition. If the RHS is locally Lipschitz continuous in y for all $v \geq 0$ the Picard-Lindelöf Theorem would imply existence and uniqueness and moreover, Lipschitz continuity would imply that the $y_K(v)$ is continuous in K. Unfortunately, although the RHS is locally Lipschitz continuous for all $v > 0$, its Lipschitz continuity may fail at $v = 0$. Therefore, for v strictly away from 0, the standard argument applies given Lipschitz continuity, but for neighborhood around 0, we need a different argument. In what follows, we will center our analysis on the neighborhood of $v = 0$.

We start by rewriting (18) by changing variables again, $z(v) = y(v)v^m$:

$$z'(v) = -rv^m - (g(v,K)v - m)\frac{z(v)}{v} - h(v,K)\frac{z(v)^2}{v^m}. \tag{19}$$

First, we show that the operator

$$L_K(z)(v) = \int_0^v -rs^m - (g(s,K)s - m)\frac{z(s)}{s} - h(s,K)\frac{z(s)^2}{s^m}ds. \tag{20}$$

is a contraction mapping on a Banach space of solutions that includes (7). This extends the Picard-Lindelöf Theorem to our setting and thus implies existence and uniqueness. Next, we show that the fixed point of L_K converges uniformly to the fixed point of L_1 as $K \searrow 1$. Finally, we show that we can obtain a sequence of solutions T^K that converge (pointwise) to the solution of the binding payoff floor constraint (with $K = 1$) and show that the revenue of these solutions also converges to the value of the auxiliary problem.

Before we introduce the Banach space on which the contraction mapping is defined, we first derive bounds for the RHS of (19).

Lemma 17. *Suppose Assumption 2 is satisfied. For any $\kappa > -1$, there exist $\overline{K} > 1$; an integer $m \geq 0$ given by $m = \lfloor \kappa \rfloor$ if $\kappa \geq 1$, and $m = 0$ if $\kappa \in (-1, 1)$; and strictly positive real numbers α, η, ξ such that the following holds.*

(a) $m < |\kappa| + \eta$,

(b) $\frac{(|\kappa|+\eta-m)\alpha+\eta\alpha^2+r}{m+1} \in [0, \alpha]$,

(c) $\frac{|\kappa|+\eta(1+2\alpha)-m}{m+1} \in (0, 1)$,

(d) $\frac{\kappa+\eta(1+\alpha)-m}{m+1}, \frac{\kappa-\eta(1+\alpha)-m}{m+1} \begin{cases} \in (0,1) & \text{if } \kappa > m \\ \in (-\frac{1}{2}, \frac{1}{2}) & \text{if } \kappa = m \\ \in (-1, 0) & \text{if } \kappa < m \end{cases}.$

(e) $|h(v,K)v^2| < \eta$ *for any* $v < \xi$ *and* $K \in [1, \overline{K}]$, (*f*) $|g(v,K)v-\kappa|<\eta$ *for any* $v<\xi$ *and* $K\in[1,\overline{K}]$,

Proof. The choice of m implies $0\le m\le|\kappa|$ so that (a) is satisfied for any $\eta>0$. In addition, $0\le\frac{|\kappa|-m}{m+1}<1$ and $0\le|\kappa|<m+1$. Note that by the choice of m, $\kappa<m$ if and only if $\kappa<0$; $\kappa=m$ if and only if $\kappa=0,1,\ldots$; $\kappa>m$ if and only if $\kappa>0$ and κ is not an integer.

Next we choose α. Consider (b). By the choice of m, the expression in (b) is non-negative for any $\eta,\alpha>0$. Given this, Part (b) is equivalent to

$$\eta\alpha^2-(2m+1-|\kappa|-\eta)\alpha+r\le 0.$$

Hence, $\frac{(2m+1-|\kappa|-\eta)-\left[(2m+1-|\kappa|-\eta)^2-4r\eta\right]^{\frac{1}{2}}}{2\eta}\le\alpha\le\frac{(2m+1-|\kappa|-\eta)+\left[(2m+1-|\kappa|-\eta)^2-4r\eta\right]^{\frac{1}{2}}}{2\eta}$. Since $2m+1-|\kappa|>0$, as $\eta\to 0$, the upper bound of α goes to $+\infty$ while the lower bound converge to $\frac{r}{2m+1-|\kappa|}$ by L'Hospital's rule. We choose $\alpha=\frac{2r}{2m+1-|\kappa|}$. Then there exists $\eta_0>0$ such that Part (b) holds for any $\eta\in(0,\eta_0)$.

For m,α,and η_0 chosen above, since $0\le\frac{|\kappa|-m}{m+1}<1$, there exists $\eta_1\in(0,\eta_0)$ such that Part (c) holds for any $\eta\in(0,\eta_1)$.

For Part (d), consider the limit

$$\lim_{\eta\to 0}\frac{\kappa\pm\eta(1+\alpha)-m}{m+1}=\frac{\kappa-m}{m+1}\begin{cases}\in(0,1) & \text{if } \kappa>m\\ =0 & \text{if } \kappa=m\\ \in(-1,0) & \text{if } \kappa<m\end{cases}$$

By continuity in both cases there exists $\eta\in(0,\eta_1)$ such that Part (f) holds.

Finally, given η chosen for Part (f), it follows from Lemma 16 that we can choose ξ and $\overline{K}$ jointly such that (e) and (f) hold. The proof of Lemma 16 shows that ξ can be chosen independently of K if $K<\overline{K}$. □

Note that $(\overline{K},m,\alpha,\eta,\xi)$ in Lemma 17 only depend on the number of buyers n and the distribution function F. Since Lemma 13 is a statement for a fixed distribution and fixed n, we treat $(\overline{K},m,\alpha,\eta,\xi)$ as fixed constants for the rest of this section. In the following, we slightly abuse notation by using n as an index for sequences. The number of buyers does not show up in the notation in the remainder of this section except in the final proof of Lemma 13.

We define a space of real-valued functions

$$\mathcal{Z}_0=\left\{z:[0,\xi]\to\mathbb{R}\,\middle|\,\sup_v\left|\frac{z(v)}{v^{m+1}}\right|\in\mathbb{R}\right\},$$

and equip it with the norm

$$||z||_m = \sup_v \left| \frac{z(v)}{v^{m+1}} \right| .$$

Define a subset of $\mathcal{Z}_0$ by

$$\mathcal{Z} = \{ z : [0, \xi] \to \mathbb{R} \, | \, ||z||_m \leq \alpha \} .$$

Note that these definitions are independent of $K < \overline{K}$.

Lemma 18. *Suppose Assumption 2 is satisfied. $\mathcal{Z}_0$ is a Banach space with norm $||\cdot||_m$ and $\mathcal{Z}$ is a complete subset of $\mathcal{Z}_0$.*

Proof. For any $\gamma_1, \gamma_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathcal{Z}_0$ and $v \in [0, \xi]$, we have

$$\begin{aligned} \left| \frac{\gamma_1 z_1(v) + \gamma_2 z_2(v)}{v^{m+1}} \right| &\leq |\gamma_1| \left| \frac{z_1(v)}{v^{m+1}} \right| + |\gamma_2| \left| \frac{z_2(v)}{v^{m+1}} \right| \\ &\leq |\gamma_1| ||z_1||_m + |\gamma_2| ||z_2||_m \\ &< \infty. \end{aligned}$$

Therefore $\mathcal{Z}_0$ is a linear space. It's straight forward to see that $||\cdot||_m$ is a norm on $\mathcal{Z}_0$. We now show $\mathcal{Z}_0$ is complete. Consider a Cauchy sequence $\{z_n\} \subset \mathcal{Z}_0$: for any $\varepsilon > 0$, there exists N_ε such that $||z_{n'} - z_n||_m < \varepsilon$ for any $n', n \geq N_\varepsilon$.

First, notice that for any $n > 0$, $||z_n||_m \leq \beta := \max_{n' \leq N_\varepsilon} \{||z_{n'}||_m\} + \varepsilon < \infty$. Next we claim that z_n converges pointwise. To see this, note that $\sup_v |\frac{z_{n'}(v) - z_n(v)}{v^{m+1}}| < \varepsilon$ implies that $|\frac{z_{n'}(v) - z_n(v)}{v^{m+1}}| = |\frac{z_{n'}(v)}{v^{m+1}} - \frac{z_n(v)}{v^{m+1}}| < \varepsilon$ for any v. Since $|\frac{z_n(v)}{v^{m+1}}| \leq \beta$, completeness of real interval with the regular norm implies that there exists $x(\cdot)$ such that $\frac{z_n(v)}{v^{m+1}} \to x(v)$ pointwise and $|x(v)| \leq \beta$. Now define $z(v) = x(v) v^{m+1}$. It's straightforward that $z_n(v) \to z(v)$ pointwise.

Finally, we show that z_n converges under $||\cdot||_m$. To see this notice that $||z_n - z|| = \sup_v |\frac{z_n(v)}{v^{m+1}} - x(v)| \leq \varepsilon$ for any $n > N_\varepsilon$. In addition, since $|x(v)| \leq \beta$, $||z||_m \leq \beta$. This proves that $\mathcal{Z}$ is complete. The same argument shows that $\mathcal{Z}$ is complete, by replacing the bound β by α. □

To study the ODE (19) for each $K \in [1, \overline{K}]$,we define an operator L_K on $\mathcal{Z}$ as in (20).

Lemma 19. *Suppose Assumption 2 is satisfied. The operator L_K is a contraction mapping on $\mathcal{Z}$ with a common contraction parameter $\rho < 1$ for all $K \in [1, \overline{K}]$.*

Proof. First we show that $L_K \mathcal{Z} \in \mathcal{Z}$. For any $z \in \mathcal{Z}$ and $v \in [0, \xi]$,

$$|L_K(z)(v)| = \left| \int_0^v -r s^m - (g(s,K)s - m) \frac{z(s)}{s} - h(s,K) s^2 \frac{z(s)^2}{s^{m+2}} ds \right|$$

$$
\begin{aligned}
&\le \frac{rv^{m+1}}{m+1} + \left| \int_0^v (g(s,K)s-m)\frac{z(s)}{s} ds \right| + \eta\left(||z||_m\right)^2 \int_0^v s^{2m+2-m-2} ds \\
&\le \frac{rv^{m+1}}{m+1} + \sup_{s\in[0,\xi]} |g(s,K)s-m| \, ||z||_m \int_0^v \frac{s^{m+1}}{s} ds + \eta\alpha^2 \frac{v^{m+1}}{m+1} \\
&\le \frac{rv^{m+1}}{m+1} + (|\kappa| + \eta - m)\alpha \frac{v^{m+1}}{m+1} + \eta\alpha^2 \frac{v^{m+1}}{m+1} \\
&= \frac{(|\kappa| + \eta - m)\alpha + \eta\alpha^2 + r}{m+1} v^{m+1} \\
&\le \alpha v^{m+1}.
\end{aligned}
$$

The first inequality follows from the triangle inequality of real numbers, Part (e) of Lemma 17 and $|z(s)| \le ||z||_m s^{m+1}$. The second inequality follows from $|z(s)| \le ||z||_m s^{m+1}$ and $||z||_m \le \alpha$. The third inequality follows from Lemma 17: for any $s \in [0,\xi]$ and $K \in [1, \overline{K}]$:

$$
\begin{aligned}
|g(s,K)s - m| &\le |g(s,K)s - \kappa| + |\kappa - m| \\
&\le \begin{cases} \eta + \kappa - m \text{ if } \kappa \ge 1 \\ \eta + |\kappa| \text{ if } \kappa \in (-1,1) \end{cases} \\
&= |\kappa| + \eta - m.
\end{aligned}
$$

We now show $L_K : \mathcal{Z} \to \mathcal{Z}$ is a contraction mapping. For any $z_1, z_2 \in \mathcal{Z}$ and $v \in [0,\xi]$,

$$
\begin{aligned}
&|L_K(z_1)(v) - L_K(z_2)(v)| \\
=& \left| \int_0^v -(g(s,K)s-m)\frac{z_1(s)-z_2(s)}{s} - h(s,K)s^2 \frac{z_1(s)^2 - z_2(s)^2}{s^{m+2}} ds \right| \\
\le& \int_0^v \sup_{s\in[0,\xi]} |g(s,K)s-m| \frac{|z_1(s)-z_2(s)|}{s} \\
&+ \sup_{s\in[0,\xi]} |h(s,K)s^2| \frac{|z_1(s)+z_2(s)||z_1(s)-z_2(s)|}{s^{m+2}} ds \\
\le& (|\kappa| + \eta - m) \int_0^v ||z_1 - z_2||_m \frac{s^{m+1}}{s} ds \\
&+ \int_0^v \eta(||z_1||_m + ||z_2||_m)||z_1 - z_2||_m \frac{s^{2m+2}}{s^{m+2}} ds \\
\le& (|\kappa| + \eta - m)\frac{v^{m+1}}{m+1}||z_1 - z_2||_m + \eta 2\alpha \frac{v^{m+1}}{m+1}||z_1 - z_2||_m \\
=& v^{m+1} \frac{|\kappa| + \eta - m + \eta 2\alpha}{m+1} ||z_1 - z_2||_m
\end{aligned}
$$

The first inequality follows from the triangle inequality for real numbers. The second inequality follows from $\sup|g(s,K)s-m| < |\kappa|+\eta-m$ which was shown above, $|z_1(s)-z_2(s)| \le ||z_1-z_2||_m s^{m+1}$, $\sup|h(s,K)s^2| < \eta$, and $|z_1(s)+z_2(s)| \le |z_1(s)|+|z_2(s)| \le (||z_1||_m+||z_2||_m)s^{m+1}$. The third inequality follows from $||z||_m \le \alpha$.

It follows immediately that $||L_K(z_1)-L_K(z_2)||_m \le \frac{|\kappa|+\eta-m+\eta 2\alpha}{m+1}||z_1-z_2||_m$. By Part (c) of Lemma 17, $\rho := \frac{|\kappa|+\eta-m+\eta 2\alpha}{m+1} \in (0,1)$, which is independent of $K \in \overline{K}$. Hence L_K is contraction mapping on $\mathcal{Z}$, with a common contraction parameter for all $K \in [1,\overline{K}]$. □

Since $L_K : \mathcal{Z} \to \mathcal{Z}$ is a contraction mapping, the Banach fixed point theorem implies that there exists a unique fixed point of L_K in $\mathcal{Z}$. For any $K \in \left[1,\overline{K}\right]$, we denote the fixed point by z_K, i.e., $z_K = L_K(z_K) \in \mathcal{Z}$. By the Banach fixed point theorem we have $z_K = \lim_{n\to\infty} L_K^n(0)$.

Lemma 20. *Suppose Assumption 2 is satisfied. The fixed point of L_K on $\mathcal{Z}$, and hence the solution to the ODE* (19) *must be strictly negative for $v > 0$.*

Proof. Let $\rho_1 = \frac{\kappa+\eta-m+\eta\alpha}{m+1}$, $\rho_2 = \frac{\kappa-\eta-m-\eta\alpha}{m+1}$. We claim that there exists M_1, M_2 such that

$$M_1 \le \frac{L_K^n(0)\,(v)}{v^{m+1}} \le M_2 < 0 \tag{21}$$

for any $n \ge 1$.

For any $n > 1$,

$$\begin{aligned} & L_K^n(0)(v) \\ =& -\frac{r}{m+1}v^{m+1} - \int_0^v (g(s,K)s-m)\,\frac{L_K^{n-1}(0)\,(s)}{s} + h(s,K)s^2\frac{\left(L_K^{n-1}(0)\,(s)\right)^2}{s^{m+2}}ds \\ =& -\frac{r}{m+1}v^{m+1} \\ &+ \int_0^v \left((g(s,K)s-m)\frac{1}{s} + h(s,K)s^2\frac{L_K^{n-1}(0)\,(s)}{s^{m+2}}\right)\left(-L_K^{n-1}(0)\,(s)\right)ds \end{aligned} \tag{22}$$

We prove separate the three cases $\kappa > m$, $\kappa = m$, $\kappa < m$ (which is equivalent to $\kappa < 0$) separately.

Case 1: $\kappa > m$. In this case, $\rho_1, \rho_2 > 0$ by Lemma 17. Let $M_1 = -\frac{r}{m+1}$ and $M_2 = -\frac{r}{m+1}(1-\rho_1)$. By part (d) of Lemma 17: $M_1 \le \frac{-r}{m+1} \le M_2 < 0$. Therefore we have $L_K^1(0)\,(v) = -\frac{r}{m+1}v^{m+1}$ satisfying (21). We prove the desired result by induction. For $n > 1$, consider (22):

$$L_K^n(0)(v) \le -\frac{r}{m+1}v^{m+1} + \int_0^v \left(\frac{\kappa-m+\eta}{s} + \frac{\eta\alpha s^{m+1}}{s^{m+2}}\right)\left(-L_K^{n-1}(0)(s)\right)ds$$

$$
\begin{aligned}
&\leq -\frac{r}{m+1}v^{m+1} + (\kappa - m + \eta(1+\alpha)) \int_0^v \left(-M_1 \frac{s^{m+1}}{s}\right) ds \\
&= \left(-\frac{r}{m+1} - \rho_1 M_1\right) v^{m+1} \\
&= M_2 v^{m+1}
\end{aligned}
$$

The first inequality follows from $-L_K^{n-1}(0) > 0$, $L_K \in \mathcal{Z}$, and replacing the coefficient of $-L_K^{n-1}(0)$ by its upper bound. The second inequality follows from $\kappa - m + \eta(1+\alpha) > 0$ and replacing $-L_K^{n-1}(0)$ with its upper bound $-M_1 s^{m+1}$ (by the induction hypothesis). In addition,

$$
\begin{aligned}
L_K^n(0)(v) &\geq -\frac{r}{m+1}v^{m+1} + \int_0^v \left(\frac{\kappa - m - \eta}{s} - \frac{\eta\alpha s^{m+1}}{s^{m+2}}\right) \left(-L_K^{n-1}(0)\,(s)\right) ds \\
&\geq -\frac{r}{m+1}v^{m+1} + (\kappa - m - \eta(1+\alpha)) \int_0^v \left(-M_2 \frac{s^{m+1}}{s}\right) ds \\
&= \left(-\frac{r}{m+1} - \rho_2 M_2\right) v^{m+1} \\
&\geq M_1 v^{m+1}
\end{aligned}
$$

The first inequality follows from $-L_K^{n-1}(0)\,(s) > 0$ and replacing the coefficient of $\left(-L_K^{n-1}(0)\,(s)\right)$ by its lower bound. The second inequality follows from $\kappa - m - \eta(1+\alpha) > 0$ and replacing $-L_K^{n-1}(0)$ with its upper bound $-M_2 s^{m+1}$ (by the induction hypothesis). The last inequality follows from $-\rho_2 M_2 > 0$ and the choice of M_1.

Case 2: $\kappa < m$. In this case, $\rho_1, \rho_2 \in (-1, 0)$ by part (d) of Lemma 17. Let $M_1 = -\frac{r}{m+1}\frac{1}{1+\rho_2}$ and $M_2 = -\frac{r}{m+1}$. $\rho_2 < 0$ implies $M_1 \leq -\frac{r}{m+1} \leq M_2 < 0$. Therefore we have $L_K^1(0)\,(v) = -\frac{r}{m+1}v^{m+1}$ satisfying (21). For $n > 1$, consider (22) :

$$
\begin{aligned}
L_K^n(0)(v) &\leq -\frac{r}{m+1}v^{m+1} + (\kappa - m + \eta(1+\alpha)) \int_0^v \left(-M_2 \frac{s^{m+1}}{s}\right) ds \\
&= \left(-\frac{r}{m+1} - \rho_1 M_2\right) v^{m+1} \\
&\leq -\frac{r}{m+1}v^{m+1} \\
&= M_2 v^{m+1}
\end{aligned}
$$

The first inequality follows from a similar derivation as in the case $\kappa > m$. However here $\kappa - m + \eta(1+\alpha) < 0$, therefore $-L_K^{n-1}(0)$ is replaced by its lower bound $-M_2 s^{m+1}$. The second inequality follows because $\rho_1 M_2 > 0$. In addition,

$$\begin{aligned} L_K^n(0)(v) &\geq -\frac{r}{m+1}v^{m+1} + (\kappa - m - \eta(1+\alpha)) \int_0^v \left(-M_1 \frac{s^{m+1}}{s}\right) ds \\ &= \left(-\frac{r}{m+1} - \rho_2 M_1\right) v^{m+1} \\ &= M_1 v^{m+1}. \end{aligned}$$

Case 3: $\kappa = m$. Then $\rho_1 = -\rho_2 = \frac{\eta(1+\alpha)}{m+1} \in (-1/2, 1/2)$ by part (d) of Lemma 17. Let $M_1 = -\frac{r}{m+1}\frac{1}{1-\rho_1}$ and $M_2 = -\frac{r}{m+1}\frac{1-2\rho_1}{1-\rho_1}$. Since $m \geq 0$ we have $\rho_1 \in (0, 1/2)$. This implies $M_1 \leq -\frac{r}{m+1} \leq M_2 < 0$. Therefore we have $L_K^1(0)\,(v) = -\frac{r}{m+1}v^{m+1}$ satisfying (21). For $n > 1$, consider (22) :

$$\begin{aligned} L_K^n(0)(v) &\leq -\frac{r}{m+1}v^{m+1} + \eta(1+\alpha) \int_0^v \left(-M_1 \frac{s^{m+1}}{s}\right) ds \\ &= \left(-\frac{r}{m+1} - \frac{\eta(1+\alpha)}{m+1} M_1\right) v^{m+1} \\ &= M_2 v^{m+1} \end{aligned}$$

To obtain the first inequality, we replace $-L_K^{n-1}(0)$ by its upper bound $-M_1 s^{m+1}$ since $\eta(1+\alpha) > 0$. In addition,

$$\begin{aligned} L_K^n(0)(v) &\geq -\frac{r}{m+1}v^{m+1} - \eta(1+\alpha) \int_0^v \left(-M_1 \frac{s^{m+1}}{s}\right) ds \\ &= \left(-\frac{r}{m+1} + \frac{\eta(1+\alpha)}{m+1} M_1\right) v^{m+1} \\ &= M_1 v^{m+1} \end{aligned}$$

To obtain the first inequality, we replace $-L_K^{n-1}(0)\,(v)$ by its upper bound $-M_2 s^{m+1}$ since $-\eta(1+\alpha) < 0$. $\square$

Lemma 21. *Suppose Assumption 2 is satisfied. Then* $\sup_{v\in[0,\xi]} \left|\frac{z_K(v)}{v^m} - \frac{z_1(v)}{v^m}\right| \to 0$ *as* $K \to 1$.

Proof. First note that for any $\varepsilon > 0$, it follows from Lemma 16 that $g(v, K)v$ and $h(v, K)v^2$ are bounded over $v \in [0, \xi]$ and $K \in [1, \overline{K}]$. Hence there exists $\Gamma \in \left(1, \overline{K}\right)$ such that

$$\begin{aligned} \sup_{v\in[0,\xi], K\in[1,\Gamma]} &|g(v, K)v - g(v, 1)v| < \varepsilon, \\ \sup_{v\in[0,\xi], K\in[1,\Gamma]} &|h(v, K)v^2 - h(v, 1)v^2| < \varepsilon. \end{aligned}$$

Since $\sup_{v\in[0,\xi]}\left|\frac{z_K(v)}{v^m}-\frac{z_1(v)}{v^m}\right| \leq \sup_v ||z_K-z_1||_m \frac{v^{m+1}}{v^m} \leq \xi||z_K-z_1||_m$, it's sufficient to show that $\lim_{K\to 1}||z_K-z_1||_m = 0$. The proof follows from Lee and Liu (2013, Lemma 13(b)). Let $\rho = \frac{|\kappa|+\eta-m+\eta 2\alpha}{m+1} < 1$ be the contraction parameter, which is independent of K. For all $z \in \mathcal{Z}$ and $K \in [1,\Gamma]$,

$$
\begin{aligned}
&|L_K(z)(v)-L_1(z)(v)| \\
=&\left|\int_0^v (g(s,K)s-g(s,1)s)\frac{z(s)}{s} + (h(s,K)s^2-h(s,1)s^2)\frac{z(s)^2}{s^{m+2}}ds\right| \\
\leq&\varepsilon\int_0^v \frac{z(s)}{s}ds + \varepsilon\int_0^v \frac{z(s)^2}{s^{m+2}}ds \\
\leq&\varepsilon\left(||z||_m\frac{v^{m+1}}{m+1} + ||z||_m^2\frac{v^{m+1}}{m+1}\right) \\
\leq&\varepsilon\frac{\alpha+\alpha^2}{m+1}v^{m+1}
\end{aligned}
$$

Therefore, $||L_K(z)-L_1(z)||_m \leq \varepsilon\frac{\alpha+\alpha^2}{m+1}$.

For any $n > 1$,

$$
\begin{aligned}
&||L_K^n(z)-L_1^n(z)||_m \\
=&||L_K\left(L_K^{n-1}(z)\right)-L_1\left(L_K^{n-1}(z)\right)+L_1\left(L_K^{n-1}(z)\right)-L_1\left(L_1^{n-1}(z)\right)||_m \\
\leq&||L_K\left(L_K^{n-1}(z)\right)-L_1\left(L_K^{n-1}(z)\right)||+||L_1\left(L_K^{n-1}(z)\right)-L_1\left(L_1^{n-1}(z)\right)||_m \\
\leq&\varepsilon\frac{\alpha+\alpha^2}{m+1}+\rho||L_K^{n-1}(z)-L_1^{n-1}(z)||_m \\
\leq&\varepsilon\frac{\alpha+\alpha^2}{m+1}\sum_{k=0}^{n-1}\rho^k \\
\leq&\varepsilon\frac{\alpha+\alpha^2}{m+1}\frac{1}{1-\rho}
\end{aligned}
$$

Given $z_K = \lim_{n\to\infty} L_K^n(0)$, there exists N_ε s.t. $\forall n \geq N_\varepsilon$, $||z_K - L_K^n(0)|| \leq \varepsilon$:

$$
\begin{aligned}
||z_K-z_1||_m &\leq ||z_K-L_K^n(0)||_m + ||z_1-L_1^n(0)||_m + ||L_K^n(0)-L_1^n(0)||_m \\
&\leq 2\varepsilon+\varepsilon\frac{\alpha+\alpha^2}{m+1}\frac{1}{1-\rho} \\
&= \left(2+\frac{\alpha+\alpha^2}{m+1}\frac{1}{1-\rho}\right)\varepsilon
\end{aligned}
$$

Therefore $\lim_{K\to 1}||z_K-z_1||_m = 0$. □

Given definition $z(v) = y(v)v^m$, let $y_K(v) = \frac{z_K(v)}{v^m}$, where z_K is the fixed point of L_K. It follows from the previous two lemmas that $y_K(v)$ is negative and $\lim_{K\to 1} ||y_K - y_1|| = 0$ under standard sup norm. Now we have all the ingredients necessary to prove Lemma 13.

Proof of Lemma 13. The uniform convergence of y_K implies that the cutoff sequence v_t^K given by $v(t) = v(0) + \int_0^t y_K(v(s))\,ds$ converges pointwise to the cutoff sequence $v_t = v_t^1$ associated with the trading time function $T(v) = T^1(v)$. Since v_t is continuous and strictly decreasing (by Lemma 9), this implies that the trading time function

$$T^K(v) = \sup\left\{t : v_t^K \geq v\right\}$$

converges pointwise to $T(v)$. To see this, note that $\sup\{t : v_t \geq v\} = \sup\{t : v_t > v\}$, since v_t is continuous and strictly decreasing. Now, for all t such that $v_t > v$, there exists K^t such that $v_t^K > v$ for all $K < K^t$. Hence,

$$\lim_{K\searrow 1} \sup\left\{t : v_t^K \geq v\right\} \geq \sup\left\{t : v_t > v\right\}.$$

Similarly, for all t such that $v_t < v$, there exists K^t such that $v_t^K < v$ for $K < K^t$. Hence,

$$\lim_{K\searrow 1} \sup\left\{t : v_t^K \geq v\right\} \leq \sup\left\{t : v_t \geq v\right\}.$$

Therefore, for all v, we have

$$\lim_{K\searrow 1} \sup\left\{t : v_t^K \geq v\right\} = \sup\left\{t : v_t \geq v\right\},$$

or equivalently,

$$\lim_{K\searrow 1} T^K(v) = T(v).$$

It remains to show that the seller's ex ante revenue converges. Notice that the sequence $e^{-rT^K(v)}$ is uniformly bounded by 1. Therefore, the dominated convergence theorem implies that

$$\lim_{K\searrow 1} \int_0^1 e^{-rT^K(x)} J(x) dF^{(n)}(x) = \int_0^1 e^{-rT(x)} J(x) dF^{(n)}(x).$$

$\square$

C.3 Proof of Lemma 14

Proof. For $t \in \{0, \Delta, 2\Delta, \ldots\}$, define

$$\begin{aligned}
\mathcal{V}_t^{K,\Delta} &:= \left\{ v \in \left[0, v_t^{K,\Delta}\right] \middle| J(v|v \le v_t^{K,\Delta}) \ge 0 \right\}, \\
\overline{\mathcal{V}}_t^{K,\Delta} &:= \left[0, v_t^{K,\Delta}\right] \setminus \mathcal{V}_t^{K,\Delta}.
\end{aligned}$$

Consider the LHS of the payoff floor constraint at $t = k\Delta$, $k \in \mathbb{N}_0$. Notice that, for $k > 0$, the new posterior at this point in time is equal to the old posterior at $((k-1)\Delta)^+$. Therefore, we can approximate the LHS of the payoff floor at $t = k\Delta$ as:

$$\begin{aligned}
&\int_{\left[0, v_{k\Delta}^{K,\Delta}\right]} e^{-r\,(T^{K,\Delta}(v) - k\Delta)} J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv \\
= &\int_{\left[0, v_{k\Delta}^{K,\Delta}\right]} e^{-r\,(T^{K}(v) - (k-1)\Delta)} e^{-r\,(T^{K,\Delta}(v) - T^K(v) - \Delta)} J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv \\
= &\int_{\mathcal{V}_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v) - (k-1)\Delta)} e^{-r\,(T^{K,\Delta}(v) - T^K(v) - \Delta)} J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv \\
&+ \int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v) - (k-1)\Delta)} e^{-r\,(T^{K,\Delta}(v) - T^K(v) - \Delta)} J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv \\
\ge &\int_{\mathcal{V}_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v) - (k-1)\Delta)} J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv \\
&+ \int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v) - (k-1)\Delta)} e^{r\Delta} J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv \\
= &\int_{\mathcal{V}_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v) - (k-1)\Delta)} J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv \\
&+ \int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v) - (k-1)\Delta)} J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv \\
&- \int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v) - (k-1)\Delta)} \left(1 - e^{r\Delta}\right) J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv \\
\ge &\int_{\left[0, v_{k\Delta}^{K,\Delta}\right]} e^{-r\,(T^{K}(v) - (k-1)\Delta)} J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv \\
&- \int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v) - (k-1)\Delta)} \left(1 - e^{r\Delta}\right) J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv.
\end{aligned}$$

Where we have used that $T^{K,\Delta}(v) - T^K(v) - \Delta \le 0$ as well as the definitions of $\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}$ and $\mathcal{V}_{k\Delta}^{K,\Delta}$ to obtain the first inequality. The first term in the last expression is equal to the LHS of the payoff floor constraints at $((k-1)\Delta)^+$ for the original solution v^K. Hence it is equal to $K\Pi^E(v_{k\Delta}^{K,\Delta})$. Therefore, we have

$$\begin{aligned}
&\int_0^{v_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K,\Delta}(v)-k\Delta)} J(v|v\le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v\le v_{k\Delta}^{K,\Delta})dv \\
=&K\Pi^E(v_{k\Delta}^{K,\Delta}) + \left(e^{r\Delta}-1\right)\int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} e^{-r\,(T^K(v)-(k-1)\Delta)} J(v|v\le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v\le v_{k\Delta}^{K,\Delta})dv \\
\ge&K\Pi^E(v_{k\Delta}^{K,\Delta}) + \left(e^{r\Delta}-1\right)\int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} J(v|v\le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v\le v_{k\Delta}^{K,\Delta})dv \\
=&K\Pi^E(v_{k\Delta}^{K,\Delta}) - \left(e^{r\Delta}-1\right)\left[\widetilde{\Pi}^M(v_{k\Delta}^{K,\Delta}) - \Pi^E(v_{k\Delta}^{K,\Delta})\right] \\
=&K\Pi^E(v_{k\Delta}^{K,\Delta}) - \left(e^{r\Delta}-1\right)\left[\frac{\widetilde{\Pi}^M(v_{k\Delta}^{K,\Delta})}{\Pi^E(v_{k\Delta}^{K,\Delta})} - 1\right]\Pi^E(v_{k\Delta}^{K,\Delta}),
\end{aligned}$$

where

$$\widetilde{\Pi}^M(w) := \int_{[0,w]} \max\left\{0, J(v|v\le w)\right\} f^{(n)}(v|v\le w)dv \;\; < w.$$

Next we show that $\frac{\widetilde{\Pi}^M(v_{k\Delta}^{K,\Delta})}{\Pi^E(v_{k\Delta}^{K,\Delta})} - 1$ is uniformly bounded. Recall that by Assumption 3, there exist $0 < M \le 1 \le L < \infty$ and $\alpha > 0$ such that $Mv^\alpha \le F(v) \le Lv^\alpha$ for all $v \in [0,1]$. This implies that the rescaled truncated distribution

$$\tilde{F}_x(v) := \frac{F(vx)}{F(x)},$$

for all $v \in [0,1]$ is dominated by a function that is independent of x:

$$\tilde{F}_x(v) \le \frac{Lv^\alpha x^\alpha}{Mx^\alpha} = \frac{L}{M}v^\alpha.$$

Next, we observe that the revenue of the efficient auction can be written in terms of the rescaled expected value of the second-highest order statistic of the rescaled distribution:

$$\Pi^E(v) = \int_0^1 vsd\tilde{F}_v^{(n-1:n)}(s).$$

If we define $\hat{F}(v) := \min\left\{1, \frac{L}{M}v^\alpha\right\}$ and $B := \int_0^1 sd\hat{F}^{(n-1:n)}(s)$, then given $\tilde{F}_x(v) \le \frac{L}{M}v^\alpha$ we can apply Theorem 4.4.1 in David and Nagaraja (2003) to obtain $\Pi^E(v) \ge Bv > 0$. Since $\widetilde{\Pi}^M(v) \le v$, we have

$$\frac{\widetilde{\Pi}^M(v_{k\Delta}^{K,\Delta})}{\Pi^E(v_{k\Delta}^{K,\Delta})} - 1 \le \frac{1}{B} - 1.$$

Therefore, LHS of the payoff floor at $t = k\Delta$ is bounded below by

$$\left[K - \left(e^{r\Delta} - 1\right)\left(\frac{1}{B} - 1\right)\right] \Pi^E(v_{k\Delta}^{K,\Delta}).$$

Clearly, for Δ sufficiently small, the term in the square bracket is no less than $(K+1)/2$. $\square$

References

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