Online Appendix:

Strategyproof Choice of Social Acts

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We present here the proofs of our results. Let us begin with the formal definition of a SCF. Recall that each agent $i \in N$ is endowed with a SEU strict preference ordering $\succcurlyeq_i$ over X^Ω : there exist a valuation function $v_i : X \to \mathbb{R}$ (normalized to guarantee that $\min_X v_i = 0$ and $\max_X v_i = 1$) and a subjective probability measure p_i on the set of events such that for all $f, g \in X^\Omega$, $f \succcurlyeq_i g \Leftrightarrow E_{v_i}^{p_i}(f) \geq E_{v_i}^{p_i}(g)$. Although the valuation function v_i and the subjective probability measure p_i associated with $\succcurlyeq_i$ are not determined uniquely,[1] it is easy to see that if (v_i, p_i) and (w_i, q_i) both represent $\succcurlyeq_i$, then v_i, w_i generate the same ranking of the outcomes (i.e., $v_i(a) \geq v_i(b) \Leftrightarrow w_i(a) \geq w_i(b)$) and p_i, q_i generate the same ranking of the events (i.e., $p_i(E) \geq p_i(E') \Leftrightarrow q_i(E) \geq q_i(E')$). The assumption that $\succcurlyeq_i$ is a strict ordering implies that for any (v_i, p_i) representing $\succcurlyeq_i$, (i) v_i is injective and (ii) p_i is injective.[2] Because $p_i(\emptyset) = 0$, it follows from (ii) that $p_i(\omega) > 0$ for all $\omega \in \Omega$.

Let $\mathcal{V}$ be the set of normalized, injective valuation functions v_i. A *belief* is formally defined as a nonnegative, injective measure on 2^Ω, and $\mathcal{P}$ denotes the set of all beliefs. The domain of preferences $\mathcal{D}$ is the set of all pairs (v_i, p_i) that generate a strict ordering of the set of acts, that is to say,

$$\mathcal{D} = \left\{ (v_i, p_i) \in \mathcal{V} \times \mathcal{P} : E_{v_i}^{p_i}(f) \neq E_{v_i}^{p_i}(g) \text{ for all } f, g \in X^\Omega \text{ such that } f \neq g \right\}.$$

In our baseline model, a *social choice function* (or *SCF*) is a mapping $\varphi : \mathcal{D}^N \to X^\Omega$.

In our constrained model, a SCF is a mapping $\varphi : \mathcal{D}^N \to \times_{\omega \in \Omega} X_\omega$. We denote the ordered list $((v_1, p_1), ..., (v_n, p_n)) \in \mathcal{D}^N$ by (v, p). In principle, our formulation allows a SCF φ to choose different acts for profiles (v, p) and (v', p') that represent the same profile of preferences $(\succcurlyeq_1, ..., \succcurlyeq_n)$. The requirement of strategyproofness, however, rules this out. It is therefore convenient to refer to any $(v, p) \in \mathcal{D}^N$ as a *preference profile.* We call $v = (v_1, ..., v_n) \in \mathcal{V}^N$ a *valuation profile* and $p = (p_1, ..., p_n) \in \mathcal{P}^N$ a *belief profile.*

Appendices 2.A to 2.D contain the proofs of the results for the baseline model and Appendix 2.E contains the proofs of the results for the constrained model.

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[1] See for instance Haller (1985) for a discussion of this point.

[2] To see this, suppose (v_i, p_i) represents $\succcurlyeq_i$ but $p_i(E) = p_i(E')$ for two distinct events E, E'. Choose two outcomes a, b and consider two acts f, g such that $f(\omega) = g(\omega') = a$, $f(\omega') = g(\omega) = b$, and $f(\omega'') = g(\omega'')$ for all $\omega \in E \setminus E'$, $\omega' \in E' \setminus E$, and $\omega'' \in (E \cap E') \cup (\Omega \setminus (E \cup E'))$. We have $f \sim_i g$, and this indifference between distinct acts contradicts the linear ordering assumption.

Appendix 2.A: Proof of the Top Selection Lemma

Let $\varphi : \mathcal{D}^N \to X^\Omega$ be a strategyproof and unanimous SCF. For a given belief $p_i \in \mathcal{P}$, the set of valuation functions compatible with p_i is $\mathcal{V}_{p_i} := \{v \in \mathcal{V} : (v_i, p_i) \in \mathcal{D}\}$. For a given belief profile $p \in \mathcal{P}^N$, denote the set of compatible valuation profiles by $\mathcal{V}_p^N := \mathcal{V}_{p_1} \times \ldots \times \mathcal{V}_{p_n}$. For any $x, y \in X$ and $f \in X^\Omega$, we write $f^x := \{\omega \in \Omega : f(\omega) = x\}$ and $f^{xy} := f^x \cup f^y$. In particular, $\varphi^x(v,p) = \{\omega \in \Omega : \varphi(v,p;\omega) = x\}$ and $\varphi^{xy}(v,p) = \varphi^x(v,p) \cup \varphi^y(v,p)$ for any preference profile (v,p).

Our first lemma states that if the chosen act changes when an agent's valuation of some outcome increases (all else equal), then her subjective probability that the social act picks that outcome also increases.

Lemma 1. ***Monotonicity***
Let $i \in N$, $a \in X$, and let $(v,p), (w,p) \in \mathcal{V}^N$ be such that $v_i(a) > w_i(a)$, $v_i(x) = w_i(x)$ for all $x \neq a$, and $v_{-i} = w_{-i}$. If $\varphi(v,p) \neq \varphi(w,p)$, then $p_i(\varphi^a(v,p)) > p_i(\varphi^a(w,p))$.

Proof. Suppose i, a, and $(v,p), (w,p)$ satisfy the stated assumptions. Let $\varphi(v,p) = f$, $\varphi(w,p) = g$, and suppose $f \neq g$. For $z \in [0,1)$, define the valuation function v_i^z by $v_i^z(a) = z$ and $v_i^z(x) = v_i(x)$ for $x \neq a$. Define the function Δ_{fg} on $[0,1)$ by

$$\Delta_{fg}(z) = \sum_{\omega \in \Omega} p_i(\omega) \left[v_i^z(f(\omega)) - v_i^z(g(\omega))\right].$$

Factoring out z and reshuffling, we get

$$\Delta_{fg}(z) = [p_i(f^a) - p_i(g^a)] \cdot z + \sum_{\omega \notin f^a} p_i(\omega) v_i(f(\omega)) - \sum_{\omega \notin g^a} p_i(\omega) v_i(g(\omega)),$$

which is an affine function of $z \in [0,1)$.

Observe that $v_i^z = w_i$ if $z = w_i(a)$ and $v_i^z = v_i$ if $z = v_i(a)$. Therefore strategyproofness implies $\Delta_{fg}(w_i(a)) < 0$ and $\Delta_{fg}(v_i(a)) > 0$. Since $w_i(a) < v_i(a)$, the slope $[p_i(f^a) - p_i(g^a)]$ of the affine function Δ_{fg} must be positive, that is to say, $p_i(\varphi^a(v,p)) > p_i(\varphi^a(w,p))$. □

For all $v_i, w_i \in \mathcal{V}$, let us write $v_i \simeq w_i$ if $(v_i(x) - v_i(y))(w_i(x) - w_i(y)) > 0$ for all $x, y \in X$, that is, v_i and w_i generate the same ordering *over outcomes.* For $v, w \in \mathcal{V}^N$, we abuse notation and write $v \simeq w$ if $v_i \simeq w_i$ for all $i \in N$.

Our next lemma asserts that, given a belief profile, the same social act must be chosen at all preference profiles generating the same profile of orderings over outcomes.

Lemma 2. ***Ordinality***
If $(v,p), (w,p) \in \mathcal{D}^N$ and $v \simeq w$, then $\varphi(v,p) = \varphi(w,p)$.

Proof. Fix $(v,p),(w,p) \in \mathcal{D}^N$ such that $v \simeq w$. Without loss of generality, assume that there exist $i \in N$ and $a \in X$ such that $w_i(a) > v_i(a)$, $w_i(x) = v_i(x)$ for all $x \neq a$, and $v_{-i} = w_{-i}$. Let $f = \varphi(v,p)$, $g = \varphi(w,p)$.

If (v_i, p_i) represents the same preference over acts as (w_i, p_i), strategyproofness directly implies $f = g$. Suppose now that (v_i, p_i) and (w_i, p_i) represent different preferences. For each $z \in [v_i(a), w_i(a)]$, define the valuation function v_i^z by $v_i^z(a) = z$ and $v_i^z(x) = v_i(x)$ for $x \neq a$. Since the set of acts is finite, we may assume without loss of generality that there is a unique $z^* \in (v_i(a), w_i(a))$ such that (i) (v_i^z, p_i) belongs to $\mathcal{D}$ and represents the same preference as (v_i, p_i) if $z \in [v_i(a), z^*)$ and (ii) (v_i^z, p_i) belongs to $\mathcal{D}$ and represents the same preference relation as (w_i, p_i) whenever $z \in (z^*, w_i(a)]$.

Suppose, by way of contradiction, that $f \neq g$. By strategyproofness, $E^{p_i}_{v_i^z}(g) - E^{p_i}_{v_i^z}(f) < 0$ if $z \in [v_i(a), z^*)$ and $E^{p_i}_{v_i^z}(g) - E^{p_i}_{v_i^z}(f) > 0$ if $z \in (z^*, w_i(a)]$. By continuity of $E^{p_i}_{v_i^z}(g) - E^{p_i}_{v_i^z}(f)$ in z, we get $E^{p_i}_{v_i^{z^*}}(g) - E^{p_i}_{v_i^{z^*}}(f) = 0$. Defining $\Omega_+ := \left\{\omega \in \Omega : v_i^{z^*}(g(\omega)) > v_i^{z^*}(f(\omega))\right\}$ and $\Omega_- := \left\{\omega \in \Omega : v_i^{z^*}(f(\omega)) > v_i^{z^*}(g(\omega))\right\}$, this reads

$$\sum_{\omega \in \Omega_+} p_i(\omega) \left[v_i^{z^*}(g(\omega)) - v_i^{z^*}(f(\omega))\right] = \sum_{\omega \in \Omega_-} p_i(\omega) \left[v_i^{z^*}(f(\omega)) - v_i^{z^*}(g(\omega))\right]. \quad (9)$$

Since $v \simeq w$, we have $z^* \neq v_i(x)$ for all $x \in X$. It follows that $\Omega_+ \neq \emptyset$ and $\Omega_- \neq \emptyset$. Indeed, Lemma 1 implies $p_i(g^a) > p_i(f^a)$, hence $\emptyset \neq g^a \setminus f^a \subseteq \Omega_+ \cup \Omega_-$ because $v_i^{z^*}(a) = z^* \neq v_i(x)$ for all $x \in X$. Assuming that $\Omega_+ \neq \emptyset$ (or $\Omega_- \neq \emptyset$), (9) and the strict positivity of p_i imply $\Omega_- \neq \emptyset$ (or $\Omega_+ \neq \emptyset$).

Pick $\omega_1 \in \Omega_+$ and $\omega_2 \in \Omega_-$. For any $\alpha > 0$, define p_i^α by

$$p_i^\alpha(\omega) = \begin{cases} p_i(\omega) + \alpha & \text{if } \omega = \omega_1, \\ p_i(\omega) - \alpha & \text{if } \omega = \omega_2, \\ p_i(\omega) & \text{otherwise.} \end{cases}$$

Choose $\alpha > 0$ small enough to guarantee that $p_i^\alpha \in \mathcal{P}$. It comes from (9) that $\sum_{\omega \in \Omega_+} p_i^\alpha(\omega) \left[v_i^{z^*}(g(\omega)) - v_i^{z^*}(f(\omega))\right] > \sum_{\omega \in \Omega_-} p_i^\alpha(\omega) \left[v_i^{z^*}(f(\omega)) - v_i^{z^*}(g(\omega))\right]$, that is, $E^{p_i^\alpha}_{v_i^{z^*}}(g) - E^{p_i^\alpha}_{v_i^{z^*}}(f) > 0$. By continuity of $E^{p_i^\alpha}_{v_i^z}(g) - E^{p_i^\alpha}_{v_i^z}(f)$ in z, there exists $z \in [v_i(a), z^*)$ such that

$$E^{p_i^\alpha}_{v_i^z}(g) - E^{p_i^\alpha}_{v_i^z}(f) > 0. \quad (10)$$

For $\alpha > 0$ small enough, (v_i^z, p_i^α), represents the same preference as (v_i, p_i). Strategyproofness therefore implies $\varphi((v_i^z, v_{-i}), (p_i^\alpha, p_{-i})) = \varphi(v,p) = f$. Since $g = \varphi(w,p) = \varphi((w_i, v_{-i}), (p_i, p_{-i}))$, (10) gives

$$E^{p_i^\alpha}_{v_i^z}(\varphi((w_i, v_{-i}), (p_i, p_{-i}))) - E^{p_i^\alpha}_{v_i^z}(\varphi((v_i^z, v_{-i}), (p_i^\alpha, p_{-i}))) > 0.$$

a violation of strategyproofness. □

The rest of the proof does not require variations in the belief profile. **We therefore fix an arbitrary $p \in \mathcal{P}^N$ until the end of Appendix 2.A.** The statements of Lemmas 3 and 4 are valid for any such belief profile. For any $v \in \mathcal{V}_p^N$ and $i \in N$, we alleviate notation by writing $\varphi(v)$ and E_{v_i} instead of $\varphi(v,p)$ and $E_{v_i}^{p_i}$.

Given $v_i \in \mathcal{V}$, we call $a, b \in X$ *adjacent in* v_i if no $x \neq a, b$ has utility $v_i(x)$ between $v_i(a)$ and $v_i(b)$. We say that w_i *obtains by permuting the utilities of* a, b *in* v_i if $w_i(a) = v_i(b)$, $w_i(b) = v_i(a)$, and $w_i(x) = v_i(x)$ for $x \neq a, b$.

Our next lemma states that permuting the utilities of adjacent outcomes does not change the events where the remaining outcomes are selected.

Lemma 3. ***Permutation Invariance***
Let $v \in \mathcal{V}_p^N$, $i \in N$, and let $a, b \in X$ be adjacent in v_i. If w_i obtains by permuting the utilities of a, b in v_i, then $\varphi^x(w_i, v_{-i}) = \varphi^x(v)$ for all $x \neq a, b$.

Proof. Let v, i, a, b, and w_i satisfy the stated assumptions and suppose without loss of generality that $v_i(a) > v_i(b)$. For each integer $m > 1/(v_i(a) - v_i(b))$, define the valuation functions v_i^m, w_i^m by

$$\begin{aligned} v_i^m(b) &= v_i(a) - \frac{1}{m} \text{ and } v_i^m(x) = v_i(x) \text{ for } x \neq b, \\ w_i^m(a) &= w_i(b) - \frac{1}{m} \text{ and } w_i^m(x) = w_i(x) \text{ for } x \neq a. \end{aligned}$$

Step 1. There exist two acts $f, \widetilde{f} \in X^\Omega$ and an integer $m^* > 1/(v_i(a) - v_i(b))$ such that

$$\varphi(v_i^m, v_{-i}) = f \text{ and } \varphi(w_i^m, v_{-i}) = \widetilde{f} \text{ for all } m \geq m^*.$$

For each $m > 1/(v_i(a) - v_i(b))$, write $\varphi(v_i^m, v_{-i}) = f_m$ and $\varphi(w_i^m, v_{-i}) = \widetilde{f}_m$. By Lemma 1, $p_i(f_m^b) \leq p_i(f_{m+1}^b)$ for each m. Since p_i is injective, $f_m \neq f_{m+1}$ whenever $p_i(f_m^b) < p_i(f_{m+1}^b)$. Since the set of acts X^Ω is finite, it follows that $p_i(f_m^b) = p_i(f_{m+1}^b)$ for all m large enough. By Lemma 1 again, this means that $f_m = f_{m+1}$ for all m large enough. The same argument shows that $\widetilde{f}_m = \widetilde{f}_{m+1}$ for all m large enough, and the claim follows.

Step 2. $f^x = \widetilde{f}^x$ for all $x \neq a, b$.

Suppose, on the contrary, that $f^x \neq \widetilde{f}^x$ for some $x \neq a, b$. Define the acts g and $\tilde{g}$ by

$$g(\omega) = \begin{cases} a & \text{if } \omega \in f^{ab}, \\ f(\omega) & \text{otherwise,} \end{cases} \qquad \tilde{g}(\omega) = \begin{cases} a & \text{if } \omega \in \tilde{f}^{ab}, \\ \tilde{f}(\omega) & \text{otherwise.} \end{cases}$$

By construction, $g \neq \tilde{g}$. Since (v_i, p_i) defines a strict ordering over the set of acts, we must have $E_{v_i}(g) \neq E_{v_i}(\tilde{g})$. Assuming without loss that $E_{v_i}(g) > E_{v_i}(\tilde{g})$, we get

$$\sum_{x \neq a,b} \left(p_i(f^x) - p_i(\tilde{f}^x)\right) v_i(x) + \left(p_i(f^{ab}) - p_i(\tilde{f}^{ab})\right) v_i(a) =: \delta > 0.$$

Recalling that $v_i(a) = w_i(b)$, it follows that for all $m > 1/(v_i(a) - v_i(b))$,

$$\begin{aligned} E_{w_i^m}(f) - E_{w_i^m}(\tilde{f}) &= \sum_{x \in X} \left(p_i(f^x) - p_i(\tilde{f}^x)\right) w_i^m(x) \\ &= (p_i(f^a) - p_i(\tilde{f}^a))w_i^m(a) + (p_i(f^b) - p_i(\tilde{f}^b))w_i^m(b) \\ &\quad + \sum_{x \neq a,b} (p_i(f^x) - p_i(\tilde{f}^x))v_i(x) \\ &= \delta - \frac{1}{m}(p_i(f^a) - p_i(\tilde{f}^a)). \end{aligned}$$

Since $\delta > 0$ and $\lim_{m\to\infty} \frac{1}{m}(p_i(f^a) - p_i(\tilde{f}^a)) = 0$, it follows that $E_{w_i^m}(f) - E_{w_i^m}(\tilde{f}) > 0$ for m sufficiently large. By Step 1, this means that $E_{w_i^m}(\varphi(v_i^m, v_{-i})) > E_{w_i^m}(\varphi(w_i^m, v_{-i}))$ for m large, contradicting strategyproofness.

Step 3. $\varphi^x(w_i, v_{-i}) = \varphi^x(v)$ for all $x \neq a, b$.

By construction, $v_i^{m^*} \simeq v_i$ and $w_i^{m^*} \simeq w_i$. By Lemma 2 and Step 1, $\varphi(v) = \varphi(v_i^{m^*}, v_{-i}) = f$ and $\varphi(w_i, v_{-i}) = \varphi(w_i^{m^*}, v_{-i}) = \widetilde{f}$. Combining these equalities with Step 2 gives $\varphi^x(v) = f^x = \widetilde{f}^x = \varphi^x(w_i, v_{-i})$ for all $x \neq a, b$. □

The following corollary to Lemma 3 (whose obvious proof consists in a repeated application of Lemma 3) will be used in the proof of Lemma 4 below.

Corollary to Lemma 3. *Let $v \in \mathcal{V}_p^N$, $I \subseteq N$, and suppose $a, b \in X$ are adjacent in v_i for each $i \in I$. If w_i obtains by permuting the utilities of a, b in v_i for each $i \in I$, then $\varphi^x(w_I, v_{-I}) = \varphi^x(v)$ for all $x \neq a, b$.*

Some more notation and terminology is needed at this point. For any $v \in \mathcal{V}^N$, define $T(v) = \{\tau(v_i) : i \in N\}$ and $t(v) = |T(v)|$. Thus, $t(v)$ is the number of distinct top outcomes in the valuation profile v. For $\mathbf{a} = (a_1, \ldots, a_n) \in X^N$, define $X(\mathbf{a}) = \{a_i : i \in N\}$ and let $k(\mathbf{a}) = |X(\mathbf{a})|$. Thus, $k(\mathbf{a})$ is the number of distinct coordinates of $\mathbf{a}$. Finally, let $\mathcal{V}_p^N(\mathbf{a}) = \left\{v \in \mathcal{V}_p^N : (\tau(v_1), ..., \tau(v_n)) = \mathbf{a}\right\}$, the set of ($p$-compatible) valuation profiles generating a profile of top outcomes equal to $\mathbf{a}$.

Our next lemma establishes that, given p, (i) the act selected at any valuation profile must select top outcomes in all states of nature (the *tops* property) and (ii) the acts selected at two valuation profiles generating identical profiles of top outcomes must coincide (the *tops-only* property).

Lemma 4. ***Tops and Tops Only***
For all $\mathbf{a} \in X^N$,

$$\text{there exists } f \in X(\mathbf{a})^\Omega \text{ such that } \varphi(v) = f \text{ for all } v \in \mathcal{V}_p^N(\mathbf{a}). \tag{11}$$

Proof. The proof is by induction on $k(\mathbf{a})$.

Step 1. Assertion (11) holds for all $\mathbf{a} \in X^N$ such that $k(\mathbf{a}) = 1$.

If $k(\mathbf{a}) = 1$, there exists $a \in X$ such that $\mathbf{a} = (a, ..., a)$ and Unanimity implies $\varphi(v; \omega) = a$ for all $\omega \in \Omega$ and all $v \in \mathcal{V}_p^N(\mathbf{a})$.

Step 2. Let $\kappa > 1$ and make the induction hypothesis $\mathcal{H}1$ that assertion (11) holds for all $\mathbf{a} \in X^N$ such that $k(\mathbf{a}) \leq \kappa - 1$. We prove that assertion (11) holds for all $\mathbf{a} \in X^N$ such that $k(\mathbf{a}) = \kappa$.

Fix $\mathbf{a} \in X^N$ such that $k(\mathbf{a}) = \kappa$. Since $\kappa > 1$, assume without loss of generality that $a_1 \neq a_2$. For each $v \in \mathcal{V}_p^N(\mathbf{a})$, define $r_i(v_i) = |\{x \in X : 1 > v_i(x) > v_i(a_1)\}|$ for all $i \in N$ and let

$$r(v) = \sum_{i \in N} r_i(v_i).$$

This number may be interpreted as the *aggregate rank* of outcome a_1 in v. By definition, $r(v) = 0$ if a_1 is ranked first or second by every agent i at profile v. Let $\overline{r} = \max \left\{ r(v) : v \in \mathcal{V}_p^N(\mathbf{a}) \right\}$. For $\rho = 0, 1, ..., \overline{r}$, let

$$\mathcal{V}_p^N(\mathbf{a}, \rho) = \left\{ v \in \mathcal{V}_p^N(\mathbf{a}) : r(v) \leq \rho \right\}.$$

Choose an arbitrary valuation profile $v^0 \in \mathcal{V}_p^N(\mathbf{a}, 0)$. By definition of $\mathcal{V}_p^N(\mathbf{a}, 0)$, it holds that $(\tau(v_1^0), ..., \tau(v_n^0)) = \mathbf{a}$ and $r(v^0) = 0$, that is, a_1 is ranked first or second by every agent at v^0. Let

$$\varphi(v^0) = f.$$

Since $\mathbf{a} = (a_1, \ldots, a_n)$ is fixed, define $N_k = \{j \in N : a_j = a_k\}$, for all $k = 1, \ldots, n$.

Step 2.1. $f \in X(\mathbf{a})^\Omega$.

For each $j \in N_2$, recall that j ranks a_1 second (since $v^0 \in \mathcal{V}_p^N(\mathbf{a}, 0)$) and construct w_j^0 by permuting the utilities of a_1, a_2 in v_j^0. Let $g = \varphi(w_{N_2}^0, v_{-N_2}^0)$. By the Corollary to Lemma 3,

$$g^x = f^x \text{ for all } x \neq a_1, a_2. \tag{12}$$

By construction, $T(w_{N_2}^0, v_{-N_2}^0) = X(\mathbf{a}) \setminus a_2$. Therefore $t(w_{N_2}^0, v_{-N_2}^0) = \kappa - 1$ and the induction hypothesis $\mathcal{H}1$ implies that $g \in T(w_{N_2}^0, v_{-N_2}^0)^\Omega = [X(\mathbf{a}) \setminus a_2]^\Omega$. Together with (12), this implies that $f \in X(\mathbf{a})^\Omega$.

Step 2.2. $\varphi(v) = f$ for all $v \in \mathcal{V}_p^N(\mathbf{a})$.

The proof is by induction on $r(v)$.

Step 2.2.1. $\varphi(v) = f$ for all $v \in \mathcal{V}_p^N(\mathbf{a}, 0)$.

Let $v \in \mathcal{V}_p^N(\mathbf{a}, 0)$. Because of Lemma 2, we may assume without loss of generality that $v(X) = v^0(X)$. Hence there exist $v^1, ..., v^{\bar{t}} = v$ and, for each $t \in \{0, ..., \bar{t} - 1\}$, an agent i and two outcomes a, b which are adjacent in v_i^t, distinct from a_1, a_i, and such that (i) v_i^{t+1} obtains by permuting the utilities of a, b in v_i^t and (ii) $v_{-i}^{t+1} = v_{-i}^t$. Since X is finite, there is no loss of generality in assuming that this sequence is of length $\bar{t} = 1$, i.e., that v_i directly obtains by permuting the utilities of a, b in v_i^0. Also without loss, suppose $v_i^0(a) > v_i^0(b)$.

Let $\varphi(v) = h$. By Lemma 3,

$$h^x = f^x \text{ for } x \neq a, b. \tag{13}$$

Suppose, by contradiction, that $h \neq f$. By Lemma 1,

$$p_i(h^a) < p_i(f^a). \tag{14}$$

Since $f \in X(\mathbf{a})^\Omega$ (by Step 2.1), inequality (14) implies $a \in X(\mathbf{a})$. Since $a \neq a_1, a_i$, there exists $k \neq 1, i$ such that $a = a_k$. Now, (14) implies $p_k(h^{a_k}) \neq p_k(f^{a_k})$ and (13) implies $p_k(h^{a_1}) = p_k(f^{a_1})$, hence

$$p_k(h^{a_1 a_k}) \neq p_k(f^{a_1 a_k}). \tag{15}$$

For each $j \in N_k = \{j \in N : a_j = a_k\}$, a_k and a_1 are respectively ranked first and second in v_j^0. Construct the subprofile $w_{N_k}^0$ by permuting the utilities of a_1, a_k in v_j^0 for every $j \in N_k$. By definition of N_k, we have $T(w_{N_k}^0, v_{-N_k}^0) = T(w_{N_k}^0, v_i, v_{-N_k \cup i}^0) = T(v^0) \setminus a_k$. Since $|T(v^0) \setminus a_k| = \kappa - 1$, the induction hypothesis $\mathcal{H}1$ implies

$$\varphi(w_{N_k}^0, v_{-N_k}^0) = \varphi(w_{N_k}^0, v_i, v_{-N_k \cup i}^0). \tag{16}$$

For all $x \neq a_1, a_k$ we obtain $f^x = \varphi^x(v^0) = \varphi^x(w_{N_k}^0, v_{-N_k}^0) = \varphi^x(w_{N_k}^0, v_i, v_{-N_k \cup i}^0) = \varphi^x(v_i, v_{-i}^0) = \varphi^x(v) = h^x$, where the second and fourth equalities hold by the Corollary to Lemma 3 and the third equality holds by (16). It follows that $f^{a_1 a_k} = h^{a_1 a_k}$, contradicting (15).

Step 2.2.2. Let $\rho > 0$ and make the induction hypothesis $\mathcal{H}2$ that $\varphi(v) = f$ for all $v \in \mathcal{V}_p^N(\mathbf{a}, \rho - 1)$. We show that $\varphi(v) = f$ for all $v \in \mathcal{V}_p^N(\mathbf{a}, \rho)$.

Let $v \in \mathcal{V}_p^N(\mathbf{a}, \rho)$ and, to avoid triviality, assume $r(v) = \rho$. Suppose, by contradiction, that $\varphi(v) = h \neq f$. By definition of $r(v)$, there exists an agent $k \in N$ such that $r_k(v_k) \geq 1$. Let thus $b \neq a_1, a_k$ be such that a_1, b are adjacent in v_k and $v_k(b) > v_k(a_1)$. Define w_k by permuting the utilities of a_1, b in v_k. Since $(w_k, v_{-k}) \in \mathcal{V}_p^N(\mathbf{a}, \rho - 1)$, the induction hypothesis $\mathcal{H}2$ implies $\varphi(w_k, v_{-k}) = f$.

By Lemma 3,

$$h^x = f^x \text{ for all } x \neq a_1, b. \tag{17}$$

By Lemma 1, $p_k(h^{a_1}) < p_k(f^{a_1})$. Hence, $h^{a_1} \neq f^{a_1}$ and since (17) implies $h^{a_k} = f^{a_k}$,

$$h^{a_1 a_k} \neq f^{a_1 a_k}. \tag{18}$$

For each $j \in N_1 = \{j \in N : a_j = a_1\}$, pick $w_j \in \mathcal{V}_{p_j}$ such that $w_j(a_1) = 1 > w_j(a_k) > w_j(x)$ for all $x \neq a_1, a_k$. Observe that $r(w_{N_1}, w_k, v_{-N_1 \cup k}) = r(v_{N_1}, w_k, v_{-N_1 \cup k}) = r(w_k, v_{-k}) = \rho - 1$. By the induction hypothesis $\mathcal{H}2$,

$$\varphi(w_{N_1}, w_k, v_{-N_1 \cup k}) = f. \tag{19}$$

Combining this equality with Lemma 3,

$$\varphi^x(w_{N_1}, v_{-N_1}) = \varphi^x(w_{N_1}, v_k, v_{-N_1 \cup k}) = f^x \text{ for all } x \neq a_1, b. \tag{20}$$

Comparing (20) and (17), we note that $\varphi(w_{N_1}, v_{-N_1})$ and $\varphi(v) = h$ can only differ on those states where the outcome is a_1 or b. Since (w_{N_1}, v_{-N_1}) and v induce the same relative ranking of a_1 and b for all agents, strategyproofness requires

$$\varphi(w_{N_1}, v_{-N_1}) = \varphi(v) = h. \tag{21}$$

For each $j \in N_1$, define now $u_j \in \mathcal{V}_{p_j}$ by permuting the utilities of the two adjacent outcomes a_1, a_k in w_j. By Lemma 3 and (19), $\varphi^x(u_{N_1}, w_k, v_{-N_1 \cup k}) = f^x$ for all $x \neq a_1, a_k$. By Lemma 3 and (21), $\varphi^x(u_{N_1}, v_{-N_1}) = h^x$ for all $x \neq a_1, a_k$. Hence,

$$\varphi^{a_1 a_k}(u_{N_1}, w_k, v_{-N_1 \cup k}) = f^{a_1 a_k} \text{ and } \varphi^{a_1 a_k}(u_{N_1}, v_{-N_1}) = h^{a_1 a_k}. \tag{22}$$

But $T(u_{N_1}, w_k, v_{-N_1 \cup k}) = T(u_{N_1}, v_{-N_1}) = X(\mathbf{a}) \setminus a_1$. Since $|X(\mathbf{a}) \setminus a_1| = \kappa - 1$, the induction hypothesis $\mathcal{H}1$ implies $\varphi(u_{N_1}, w_k, v_{-N_1 \cup k}) = \varphi(u_{N_1}, v_{-N_1})$. Combining this equality with (22) gives $f^{a_1 a_k} = h^{a_1 a_k}$, contradicting (18). □

Conclusion of the proof of the Top Selection lemma

Now that we have established the *tops* and *tops only* properties of Lemma 4, we abuse notation and write $\varphi(x_1, ..., x_n)$ to refer to the act $\varphi(v, p)$ chosen at any profile $v \in \mathcal{V}_p^N$ where $\tau(v_i) = x_i$, for all $i = 1, \ldots, n$. We are now ready to construct $s(p)$, the assignment of states to agents at the belief profile p. Since p is fixed, we write s instead of $s(p)$. For any distinct $a, b \in X$, define

$$s_1^{ab} := \varphi^a(a, b, \ldots, b),$$

that is, s_1^{ab} is the set of states of nature where the social act yields outcome a when agent 1's top is a and every other agent's top is b. Define s_i^{ab} in a similar way for every agent $i \in N$; and write $s^{ab} = (s_1^{ab}, \ldots, s_n^{ab})$.

Step 1. For all $a, b, c, d \in X$, we have (i) $s^{ab} = s^{cb}$ if $b \neq a, c$ and (ii) $s^{ab} = s^{ad}$ if $a \neq b, d$.

To prove statement (i), fix $a, b, c \in X$ such that $b \neq a, c$. The case $a = c$ being trivial, assume $a \neq c$. By Lemma 3 and the tops only property, $\varphi^b(a, b, \ldots, b) = \varphi^b(c, b, \ldots, b)$. By the tops property, $\varphi^{ab}(a, b, \ldots, b) = \Omega = \varphi^{cb}(c, b, \ldots, b)$. Hence, $\varphi^a(a, b, \ldots, b) = \varphi^c(c, b, \ldots, b)$, that is, $s_1^{ab} = s_1^{cb}$. A similar argument gives $s_i^{ab} = s_i^{cb}$ for all $i \in N$, proving (i).

To prove statement (ii), apply Lemma 3 repeatedly to get $s_1^{ab} = \varphi^a(a, b, b, \ldots, b) = \varphi^a(a, d, b, \ldots, b) = \varphi^a(a, d, d, \ldots, b) = ... = \varphi^a(a, d, d, \ldots, d) = s_1^{ad}$. Likewise, $s_i^{ab} = s_i^{ad}$ for every $i \in N$, proving (ii).

Step 1 means that s^{ab} is in fact independent of the choice of a and b. For any agent i, we may therefore define i's share of the state space s_i at p to be the event in which i's top is selected at *any* profile v where that top is different from the common top of all other agents:

$$s_i = s_i^{ab} \text{ for any } a, b \in X \text{ such that } a \neq b.$$

To complete the proof of the Top Selection lemma, it remains to show that (i) s is a well-defined assignment (i.e., $s \in \mathcal{S}$) and (ii) at every valuation profile, every agent's top is selected in any state that is assigned to her (i.e., for all $v \in \mathcal{V}_p^N$ and $i \in N$, $\omega \in s_i \Rightarrow \varphi(v; \omega) = \tau(v_i)$).

Step 2. $\varphi^c(x_1, \ldots, x_{j-1}, c, x_{j+1}, \ldots, x_n) = s_j$ for all $j \in N$, $c \in X$, and $x_1, \ldots, x_{j-1}$, $x_{j+1}, \ldots, x_n \in X \setminus \{c\}$.

Without loss of generality, suppose $j = 1$. Fix $c \in X$ and $x_2, \ldots, x_n \neq c$. By repeated application of Lemma 3, $\varphi^c(c, x_2, x_3, \ldots, x_n) = \varphi^c(c, x_2, x_2, \ldots, x_n) = ... = \varphi^c(c, x_2, x_2, \ldots, x_2) = s_1$.

Step 3. $s_i \cap s_j = \emptyset$ for all distinct $i, j \in N$.

Without loss of generality, we prove that $s_1 \cap s_2 = \emptyset$. Pick distinct $a, b, c \in X$ and consider the top profile $(a, b, c, \ldots, c)$. By Step 2, $\varphi^a(a, b, c, \ldots, c) = s_1$ and $\varphi^b(a, b, c, \ldots, c) = s_2$. The claim then follows because $a \neq b$ implies that $\varphi^a(a, b, c, \ldots, c) \cap \varphi^b(a, b, c, \ldots, c) = \emptyset$.

Step 4. $\varphi^a(x_1, \ldots, x_n) = \bigcup_{i \in N : x_i = a} s_i$ for all $a \in X$ and $(x_1, \ldots, x_n) \in X^N$.

If $x_1, \ldots, x_n \neq a$, the tops property implies $\varphi^a(x_1, \ldots, x_n) = \emptyset$ and the result holds trivially (with the convention that $\cup_{i \in \emptyset} s_i = \emptyset$).

If $x_i = a$ for some $i \in N$, let us assume without loss of generality that $\{i \in N : x_i = a\} = \{1, \ldots, j\}$ with $1 \leq j \leq n$. We must prove that

$$\varphi^a(a, \ldots, a, x_{j+1}, \ldots, x_n) = \bigcup_{i=1}^{j} s_i.$$

We start by proving this claim when $x_{j+1} = \ldots = x_n = b \neq a$. By Step 1, $s_1 = \varphi^a(a, b, \ldots, b)$, which is the desired result when $j = 1$. Suppose now that $j \in \{2, \ldots, n\}$ and assume by induction that

$$\varphi^a(a, \ldots, a, \underbrace{b}_{x_j}, b, \ldots, b) = \bigcup_{i=1}^{j-1} s_i. \tag{23}$$

Choose $c \neq a, b$. Changing agent j's top from b to c, (23) and Lemma 3 give

$$\varphi^a(a, \ldots, a, \underbrace{c}_{x_j}, b, \ldots, b) = \bigcup_{i=1}^{j-1} s_i. \tag{24}$$

Changing now agent j's top from c to a, Lemma 3 gives $\varphi^b(a, \ldots, a, a, b, \ldots, b) = \varphi^b(a, \ldots, a, c, b \ldots, b)$. Combining this result with the tops property and the identity $\cup_{x \in X} \varphi^x = \Omega$, we get

$$\begin{aligned} \varphi^a(a, \ldots a, \underbrace{a}_{x_j}, b, \ldots, b) &= \varphi^{ac}(a, \ldots, a, c, b \ldots, b) \\ &= \varphi^a(a, \ldots, a, c, b \ldots, b) \cup s_j \\ &= \left(\bigcup_{i=1}^{j-1} s_i \right) \cup s_j \\ &= \bigcup_{i=1}^{j} s_i, \end{aligned}$$

where the second equality stems from Step 2 and the third from (24).

It is now easy to generalize this result to an arbitrary collection $x_{j+1}, \ldots, x_n \in X \setminus \{a\}$. By repeated application of Lemma 3, $\varphi^a(a, \ldots, a, x_{j+1}, x_{j+2}, \ldots, x_n) = \varphi^a(a, \ldots, a, x_{j+1}, x_{j+1}, \ldots, x_n) = \varphi^a(a, \ldots, a, x_{j+1}, x_{j+1}, \ldots, x_{j+1}) = \cup_{i=1}^{j} s_i$, completing the proof of Step 4.

Step 4 implies that for all $v \in \mathcal{V}_p^N$, $i \in N$, and $\omega \in \Omega$,

$$\begin{aligned} \omega \in s_i \quad &\Longrightarrow \quad \omega \in \varphi^{\tau(v_i)}(\tau(v)) \\ &\Longrightarrow \quad \varphi(v; \omega) = \tau(v_i), \end{aligned}$$

as asserted in the Top Selection lemma. To complete the proof, it only remains to be shown that s is a bona fide assignment.

Step 5. $s \in \mathcal{S}$.

In view of Step 3, we only need to argue that $\cup_{i=1}^{n} s_i = \Omega$. Indeed, note from Step 4 that $\varphi^a(a, \ldots, a) = \cup_{i=1}^{n} s_i$. Hence $\cup_{i=1}^{n} s_i = \varphi^a(a, \ldots, a) = \Omega$, where the last equality holds by unanimity.

The steps above prove that s is an assignment rule generating φ. It is obvious that any other assignment rule generates a SCF different from φ. The proof of the Top Selection lemma is therefore complete. □

We now turn to the proof of Theorem 1. It is easy to check that every locally bilateral top selection φ is strategyproof and unanimous. Conversely, fix a strategyproof and unanimous SCF φ. By the Top Selection lemma, φ is generated by an assignment rule s. It remains to prove that s is locally bilateral, i.e., is the union of a collection of constant, bilaterally dictatorial, or bilaterally consensual sub-rules.

Appendix 2.B: The Local Bilaterality Lemma

In the current section, we show that s satisfies a strong incentive-compatibility property –dubbed *super-strategyproofness*– and we use this property to characterize the local behavior of s. It turns out that this behavior is bilateral: an elementary change in an agent's belief may only affect her own share and that of *one* other agent.

Call s *strategyproof* if $p_i(s_i(p)) \geq p_i(s_i(p'_i, p_{-i}))$ for all $i \in N$, $p \in \mathcal{P}^N$, and $p'_i \in \mathcal{P}$: no agent can increase the likelihood of the event assigned to her by misrepresenting her belief.

For any assignment $\mathbf{A} = (A_1, ..., A_n) \in \mathcal{S}$ and any $M \subseteq N$, write $A_M = \cup_{i \in M} A_i$. Denote strict inclusion by the symbol $\subset$. Call s *super-strategyproof* if $p_i(s_M(p)) \geq p_i(s_M(p'_i, p_{-i}))$ for all i, M such that $i \in M \subset N$, all $p \in \mathcal{P}^N$, and all $p'_i \in \mathcal{P}$: by misrepresenting her belief, an agent can never increase the likelihood of the event assigned to any subset of agents to which she belongs.

For any $\omega \in \Omega$ and $p \in \mathcal{P}^N$, it will be convenient to let $a_\omega(p)$ denote the agent to whom s assigns ω at the belief profile p, that is,

$$a_\omega(p) = i \Leftrightarrow \omega \in s_i(p). \tag{25}$$

We call the condition $\cup_{i \in N} s_i(p) = \Omega$ the *feasibility constraint*.

Super-strategyproofness Lemma. *The assignment rule s is super-strategyproof.*

Proof. Suppose, by way of contradiction, that there exist i, M such that $i \in M \subset N$, $p \in \mathcal{P}^N$, and $p'_i \in \mathcal{P}$ such that $p_i(s_M(p'_i, p_{-i})) > p_i(s_M(p))$. Choose $v \in \mathcal{V}^N$ such that $(v, p), (v, (p'_i, p_{-i})) \in \mathcal{D}^N$ and $v_i(\tau(v_j)) = 1$ for all $j \in M$ and $v_i(\tau(v_j)) = 0$ for all

$j \in N \setminus M$. Then,

$$\begin{aligned}
\sum_{\omega\in\Omega} p_i(\omega) v_i(\varphi(v,(p_i',p_{-i});\omega)) &= \sum_{\omega\in\Omega} p_i(\omega) v_i(\tau(v_{a_\omega(p_i',p_{-i})})) \\
&= \sum_{\omega\in\Omega: a_\omega(p_i',p_{-i})\in M} p_i(\omega) \\
&= p_i(s_M(p_i',p_{-i})) \\
&> p_i(s_M(p)) \\
&= \sum_{\omega\in\Omega: a_\omega(p)\in M} p_i(\omega) \\
&= \sum_{\omega\in\Omega} p_i(\omega) v_i(\tau(v_{a_\omega(p)})) \\
&= \sum_{\omega\in\Omega} p_i(\omega) v_i(\varphi(v,p;\omega)),
\end{aligned}$$

contradicting the assumption that φ is strategyproof. □

An immediate consequence of the Super-strategyproofness lemma which will prove crucial in the remainder of the proof is that the assignment rule s must satisfy the well-known property of *non-bossiness*: for all $i \in N$, $p \in \mathcal{P}^N$, and $p_i' \in \mathcal{P}$, we have $s_i(p) = s_i(p_i',p_{-i}) \Rightarrow s(p) = s(p_i',p_{-i})$. In other words, non-bossiness says that no agent can affect another agent's share without affecting her own.

Non-Bossiness Corollary. *The assignment rule s is non-bossy.*

Proof. Given the Super-strategyproofness lemma, it suffices to show that s is non-bossy. S uppose, by way of contradiction, that there exist $i, j \in N$, $p \in \mathcal{P}^N$ and $p_i' \in \mathcal{P}$ such that $s_i(p) = s_i(p_i',p_{-i})$ and $s_j(p) \neq s_j(p_i',p_{-i})$. By super-strategyproofness applied to $M = \{i,j\}$ and because p_i is injective, $p_i(s_{ij}(p)) > p_i(s_{ij}(p_i',p_{-i}))$, hence $p_i(s_j(p)) > p_i(s_j(p_i',p_{-i}))$. Since such a strict inequality holds for every j such that $s_j(p) \neq s_j(p_i',p_{-i})$, we have $1 = \sum_{j\in N} p_i(s_j(p)) > \sum_{j\in N} p_i(s_j(p_i',p_{-i})) = 1$, a contradiction. □

We are now ready to characterize the *local* behavior of the assignment rule s. Define $\mathcal{H} = \{\{A,B\} : \emptyset \neq A, B \subset \Omega \text{ and } A\cap B = \emptyset\}$, the set of pairs of disjoint nonempty events. Two beliefs $p_i, q_i \in \mathcal{P}$ will be called $\{A,B\}$-*adjacent* if

$$\begin{aligned}
(p_i(A)-p_i(B))(q_i(A)-q_i(B)) &< 0 \text{ and} \\
(p_i(C)-p_i(D))(q_i(C)-q_i(D)) &> 0 \text{ for any } \{C,D\} \in \mathcal{H}\setminus\{\{A,B\}\}.
\end{aligned}$$

If p_i, q_i are $\{A,B\}$-adjacent for some $\{A,B\} \in \mathcal{H}$, we say that they are *adjacent* and we write $p_i J q_i$. By definition, two beliefs are adjacent if the likelihood orderings they generate differ on a single pair of disjoint nonempty events.

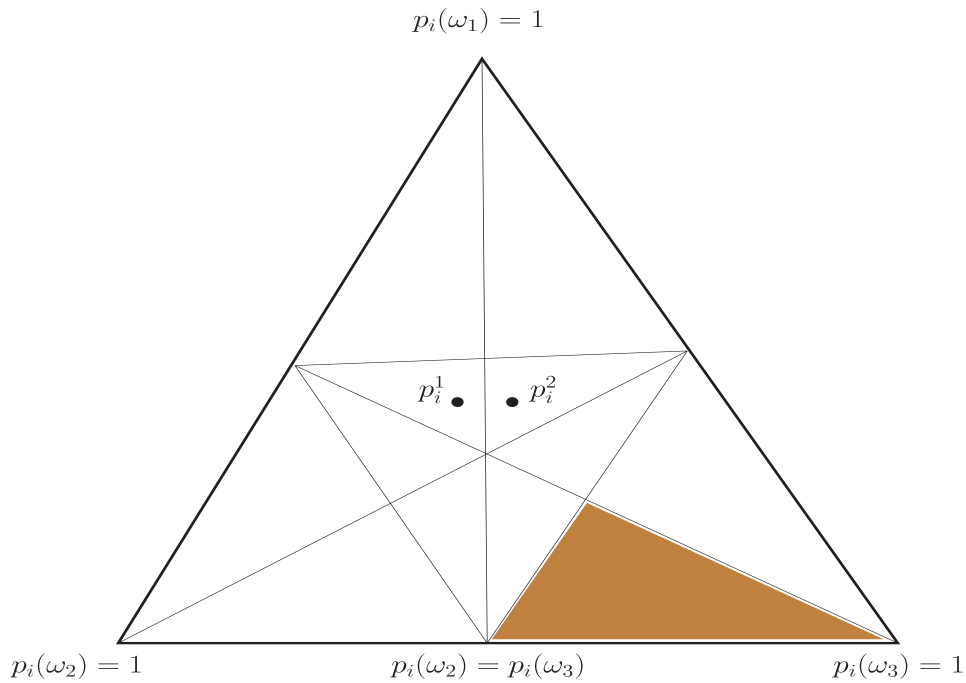


Figure 1: Beliefs, likelihood orderings, and adjacency

The adjacency relation J is obviously a symmetric binary relation. If $p_i, q_i \in \mathcal{P}' \subseteq \mathcal{P}$, *a* J*-path between* p_i *and* q_i *in* $\mathcal{P}'$ is a finite sequence $\mathbf{p}_i = (\mathbf{p}_i^t)_{t=1}^T$ such that $\mathbf{p}_i^1 = p_i$, $\mathbf{p}_i^T = q_i$, $\mathbf{p}_i^t J \mathbf{p}_i^{t+1}$ for $t = 1, ..., T-1$, and $\mathbf{p}_i^t \in \mathcal{P}'$ for $t = 1, ..., T$. We call $\mathcal{P}'$ *connected* if such a J-path exists between any two beliefs in $\mathcal{P}'$. The set $\mathcal{P}$ is connected.

Adjacency is an ordinal property. Every belief $p_i \in \mathcal{P}$ generates a likelihood ordering $R(p_i)$ over events defined by $A\, R(p_i)\, B \Leftrightarrow p_i(A) \geq p_i(B)$: event A is more likely than B according to p_i. If $R(p_i) = R(q_i)$, we call the two beliefs p_i, q_i *ordinally equivalent* and write $p_i \approx q_i$. We abuse this notation and, for any profiles $p, q \in \mathcal{P}^N$, we write $p \approx q$ if $p_i \approx q_i$ for all $i \in N$. If p_i, q_i are adjacent and p_i' is ordinally equivalent to p_i, then p_i', q_i are adjacent.

Example 1. *Let* $\Omega = \{\omega_1, \omega_2, \omega_3\}$ *and consider the simplex* Δ *depicted in Figure 1. Every point in* Δ *implicitly defines a measure* $p_i \in \overline{\mathcal{P}}$*, where* $\overline{\mathcal{P}}$ *denotes the closure of* $\mathcal{P}$ *in* $[0,1]^{2^\Omega}$*. Every line segment corresponds to (the intersection with* Δ *of) the hyperplane* $p_i(A) = p_i(B)$ *generated by some pair of disjoint events* $\{A, B\} \in \mathcal{H}$*. Each connected component of the complement of (the union of) these line segments defines a region of ordinally equivalent beliefs: the shaded area is an example. Two beliefs are adjacent if they lie on the same side of all but one hyperplane. For instance, the beliefs* p_i^1, p_i^2*, which lie on the same side of all hyperplanes except* $p_i(\{\omega_2\}) = p_i(\{\omega_3\})$*, are* $\{\{\omega_2\}, \{\omega_3\}\}$*-adjacent. These beliefs generate the likelihood relations*

$$
\begin{aligned}
R(p_i^1) &= \{\omega_1,\omega_2,\omega_3\},\{\omega_1,\omega_2\},\{\omega_1,\omega_3\},\{\omega_2,\omega_3\},\{\omega_1\},\{\omega_2\},\{\omega_3\}, \\
R(p_i^2) &= \{\omega_1,\omega_2,\omega_3\},\{\omega_1,\omega_3\},\{\omega_1,\omega_2\},\{\omega_2,\omega_3\},\{\omega_1\},\{\omega_3\},\{\omega_2\},
\end{aligned}
$$

where events are listed in decreasing order of likelihood. Note that $R(p_i^1)$ *and* $R(p_i^2)$ *disagree not only on* $\{\omega_2\},\{\omega_3\}$ *but, as a consequence, also on* $\{\omega_1,\omega_2\},\{\omega_1,\omega_3\}$*: this does not contradict the definition of adjacency because* $\{\omega_1,\omega_2\},\{\omega_1,\omega_3\}$ *intersect.*

Because it is strategyproof, the assignment rule s must be ordinal in the sense that $s(p) = s(q)$ whenever $p \approx q$: the assignment of states to agents cannot change as long as the likelihood relations associated with their beliefs remain the same. Our next result describes how the assignment of states changes when an agent's report switches between two adjacent beliefs.

Local Bilaterality Lemma. *Let* $\{A,B\} \in \mathcal{H}$ *and let* $i \in N$, $p \in \mathcal{P}^N$, $p_i' \in \mathcal{P}$ *be such that* p_i, p_i' *are* $\{A,B\}$*-adjacent and* $p_i(A) > p_i(B)$*. Then, either (i)* $s(p) = s(p_i', p_{-i})$ *or (ii) there exists* $j \in N \setminus i$ *such that*

$$
\begin{aligned}
s_i(p) \setminus s_i(p_i', p_{-i}) &= A = s_j(p_i', p_{-i}) \setminus s_j(p), \\
s_i(p_i', p_{-i}) \setminus s_i(p) &= B = s_j(p) \setminus s_j(p_i', p_{-i}), \\
s_k(p) &= s_k(p_i', p_{-i}) \text{ for all } k \in N \setminus \{i,j\}.
\end{aligned}
$$

This is a complete description of the local behavior of s. By reporting a belief adjacent to her own, an agent i can only change the event that is assigned to her and *one* other agent j. Moreover, if the assignment is indeed modified, i and j must precisely exchange the disjoint events that have been switched in i's likelihood ordering.

Proof. Let $\{A,B\} \in \mathcal{H}$ and let $i \in N$, $p \in \mathcal{P}^N$, $p_i' \in \mathcal{P}$ be such that p_i, p_i' are $\{A,B\}$-adjacent and $p_i(A) > p_i(B)$.

Step 1. We show that for all $M \subseteq N$ such that $i \in M$, either (i) $s_M(p) = s_M(p_i', p_{-i})$ or (ii) $s_M(p) \setminus s_M(p_i', p_{-i}) = A$ and $s_M(p_i', p_{-i}) \setminus s_M(p) = B$.

To see this, suppose (i) fails. Define $A_M = s_M(p) \setminus s_M(p_i', p_{-i})$ and $B_M = s_M(p_i', p_{-i}) \setminus s_M(p)$. These sets are disjoint and super-strategyproofness of s implies that both are nonempty; hence, they belong to $\mathcal{H}$. Suppose, by way of contradiction, that $A_M \neq A$ or $B_M \neq B$. Since p_i, p_i' are $\{A,B\}$-adjacent, their associated likelihood orderings must agree on the ranking of A_M, B_M : either (a) $p_i(A_M) > p_i(B_M)$ and $p_i'(A_M) > p_i'(B_M)$, or (b) $p_i(A_M) < p_i(B_M)$ and $p_i'(A_M) < p_i'(B_M)$. If (a) holds, then $p_i'(s_M(p)) > p_i'(s_M(p_i', p_{-i}))$ whereas if (b) holds, then $p_i(s_M(p_i', p_{-i})) > p_i(s_M(p))$. Each of these two inequalities contradicts super-strategyproofness.

Step 2. Applying Step 1 with $M = \{i\}$, either (i) $s_i(p) = s_i(p_i', p_{-i})$ or (ii) $s_i(p) \setminus s_i(p_i', p_{-i}) = A$ and $s_i(p_i', p_{-i}) \setminus s_i(p) = B$.

If (i) holds, non-bossiness of s implies $s(p) = s(p'_i, p_{-i})$, and we are done.

If (ii) holds, let $j \in N \setminus i$. Applying Step 1 with $M = \{i, j\} = ij$, we have either (a) $s_{ij}(p) = s_{ij}(p'_i, p_{-i})$ or (b) $s_{ij}(p) \setminus s_{ij}(p'_i, p_{-i}) = A$ and $s_{ij}(p'_i, p_{-i}) \setminus s_{ij}(p) = B$. If (a) holds, then (ii) implies

$$s_j(p'_i, p_{-i}) \setminus s_j(p) = A \text{ and } s_j(p) \setminus s_j(p'_i, p_{-i}) = B \tag{26}$$

whereas if (b) holds, (ii) implies

$$s_j(p) = s_j(p'_i, p_{-i}). \tag{27}$$

By feasibility, (26) can hold for at most one agent $j \in N \setminus i$. Because of (ii), it must hold for exactly one such agent. Since (27) holds for every other agent $j \in N \setminus i$, the proof is complete. □

Appendix 2.C: The Bilateral Consensus Lemma

This appendix and the next show how the local structure of the super-strategyproof rule s determines its global structure. Let $\Omega_0, \Omega_1, \Omega_2$ denote the sets of states whose assignment is either constant, varies with the belief of a single agent, or with the beliefs of at least two agents. Using the definition of a_ω in (25), we thus have:

(i) $\omega \in \Omega_0 \Leftrightarrow a_\omega$ is constant on $\mathcal{P}^N$;

(ii) $\omega \in \Omega_1 \Leftrightarrow$ [there exist $i \in N,\ p \in \mathcal{P}^N,$ and $p'_i \in \mathcal{P}$ such that $a_\omega(p) \neq a_\omega(p'_i, p_{-i})$] and [$a_\omega(., p_{-j})$ is constant on $\mathcal{P}$ for all $j \neq i$ and $p_{-j} \in \mathcal{P}^{N \setminus j}$];

(iii) $\omega \in \Omega_2 \Leftrightarrow$ there exist distinct agents $i, j \in N$, $p, q \in \mathcal{P}^N$, and $p'_i, q'_j \in \mathcal{P}$ such that $a_\omega(p) \neq a_\omega(p'_i, p_{-i})$ and $a_\omega(q) \neq a_\omega(q'_j, q_{-j})$.

By definition, $\{\Omega_0, \Omega_1, \Omega_2\}$ is a partition of Ω. This is because the definition in (iii) allows the assignment of states in Ω_2 to vary with the beliefs of more than two agents. Note also that the set of agents to whom a state in Ω_2 may potentially be assigned is a priori unrestricted.

The current appendix focuses exclusively on the states in Ω_2; the assignment of states in Ω_1 will be discussed in Appendix 2.D. We show here that each state in Ω_2 may only be assigned to two distinct agents, and its assignment must be based on the beliefs of these two agents only. More specifically, states in Ω_2 must be assigned through bilateral consensus:

Bilateral Consensus Lemma. *For every $\omega \in \Omega_2$ there exists a unique event $E^\omega \subseteq \Omega_2$ containing ω, and there exists a bilaterally consensual E^ω-assignment rule s^ω such that*

$$s_i(p) \cap E^\omega = s_i^\omega(p \mid E^\omega)$$

for all $p \in \mathcal{P}^N$ *and* $i \in N$.

Note that the Bilateral Consensus lemma fully determines the behavior of s on Ω_2. For any two states $\omega, \omega' \in \Omega_2$, since there exist a bilaterally consensual E^ω-rule s^ω and a bilaterally consensual $E^{\omega'}$-rule $s^{\omega'}$ such that $s_i(p) \cap E^\omega = s_i^\omega(p \mid E^\omega)$ and $s_i(p) \cap E^{\omega'} = s_i^{\omega'}(p \mid E^{\omega'})$ for all $i \in N$, we must have either (i) $E^\omega = E^{\omega'}$ and $s^\omega = s^{\omega'}$, or (ii) $E^\omega \cap E^{\omega'} = \emptyset$. This delivers at once the following corollary:

Bilateral Consensus Corollary. *There exists a partition* $\{\Omega^t\}_{t=1}^{T_2}$ *of* Ω_2 *and, for each* $t = 1, ..., T_2$*, a bilaterally consensual* Ω^t*-assignment rule* s^t *such that*

$$s_i(p) \cap \Omega_2 = \cup_{t=1}^{T_2} s_i^t(p \mid \Omega^t)$$

for all $p \in \mathcal{P}^N$ *and* $i \in N$.

Before diving into the long proof of the Bilateral Consensus lemma, let us sketch the main lines of the argument. Since we want to prove that the super-strategyproof rule s coincides on Ω_2 with a locally bilateral assignment rule of the form $s(p) = \cup_{t=1}^{T} s^t(p \mid \Omega^t)$, it is worth examining the behavior of such a rule in more detail. Fix a cell Ω^t on which the sub-rule s^t is bilaterally consensual –say, $\{1, 2\}$-consensual with default $B \subset \Omega^t$. Defining $A = \Omega^t \setminus B$, we have[3]

$$s(p) \cap \Omega^t = \begin{cases} (A, B, \emptyset, ..., \emptyset) & \text{if } p_1(A) > p_1(B) \text{ and } p_2(A) < p_2(B), \\ (B, A, \emptyset, ..., \emptyset) & \text{otherwise.} \end{cases}$$

The point we want to make is that $s(.) \cap \Omega^t$ varies differently with p_1, p_2 across different *regions* of $\mathcal{P}^N$. Let us say that $\{A, B\}$ *cuts* $\mathcal{Q}_i \subseteq \mathcal{P}$ if $\mathcal{Q}_i$ contains beliefs p_i, q_i that disagree on A, B in the sense that $p_i(A) > p_i(B)$ but $q_i(A) < q_i(B)$. Consider now a region $\times_{i \in N} \mathcal{Q}_i \subseteq \mathcal{P}^N$ of belief profiles.

(a) If $\{A, B\}$ cuts both $\mathcal{Q}_1$ and $\mathcal{Q}_2$, the assignment of A, B between 1 and 2 varies with both of their beliefs on $\times_{i \in N} \mathcal{Q}_i$ and we call the rule *actively* $\{1, 2\}$*-consensual with respect to* $\{A, B\}$ *on* $\times_{i \in N} \mathcal{Q}_i$.

(b) If $\{A, B\}$ cuts $\mathcal{Q}_1$ but not $\mathcal{Q}_2$, then either (i) $p_2(A) < p_2(B)$ for all $p_2 \in \mathcal{Q}_2$ or (ii) $p_2(A) > p_2(B)$ for all $p_2 \in \mathcal{Q}_2$. If (i) holds, then for all $p \in \times_{i \in N} \mathcal{Q}_i$ we have

$$s(p) \cap \Omega^t = \begin{cases} (A, B, \emptyset, ..., \emptyset) & \text{if } p_1(A) > p_1(B), \\ (B, A, \emptyset, ..., \emptyset) & \text{otherwise,} \end{cases}$$

and we say that s is *passively* $(1, 2)$*-consensual with respect to* $\{A, B\}$ *on* $\times_{i \in N} \mathcal{Q}_i$: although the assignment of A, B between 1 and 2 is in fact consensual, it varies only

[3] We slightly abuse notation and write $s(p) \cap \Omega^t$ for $(s_1(p) \cap \Omega^t, ..., s_n(p) \cap \Omega^t)$.

with p_1 on the considered region. If (ii) holds, then $s(p) \cap \Omega^t = (B, A, \emptyset, ..., \emptyset)$ for all $p \in \times_{i \in N} \mathcal{Q}_i$ and we say that the rule s is *constant on* $\times_{i \in N} Q_i$ *with respect to* $\{A, B\}$.
(c) If $\{A, B\}$ cuts $\mathcal{Q}_2$ but not $\mathcal{Q}_1$, then s may be either passively $(2,1)$-consensual with respect to $\{A, B\}$ or constant on $\times_{i \in N} \mathcal{Q}_i$.
(d) Finally, if $\{A, B\}$ cuts neither $\mathcal{Q}_1$ nor $\mathcal{Q}_2$, then s is again constant with respect to $\{A, B\}$ on $\times_{i \in N} Q_i$.

With the above comments in mind, let us now describe the structure of the proof of the Bilateral Consensus lemma. We are given the super-strategyproof assignment rule s. We fix $\widetilde{\omega} \in \Omega_2$, a state whose assignment varies with the beliefs of at least two agents. For simplicity, we write $\widetilde{\Omega}$ instead of $\Omega \setminus \widetilde{\omega}$ and $\widetilde{\mathcal{P}}$ instead of $\mathcal{P}(\widetilde{\Omega})$. We must show that there exists an event $E^{\widetilde{\omega}} \subseteq \Omega_2$ and a bilaterally consensual $E^{\widetilde{\omega}}$-assignment rule $s^{\widetilde{\omega}}$ such that $s_i(p) \cap E^{\widetilde{\omega}} = s_i^{\widetilde{\omega}}(p \mid E^{\widetilde{\omega}})$ for all $p \in \mathcal{P}^N$ and $i \in N$.

The strategy of the proof is to first partition $\mathcal{P}^N$ into a number of *regions* over which we will be able to pin down how the assignment of $\widetilde{\omega}$ varies with the belief profile p, and then patch the pieces together. For any profile $\pi \in \widetilde{\mathcal{P}}^N$, define $\mathcal{P}(\pi_i) = \left\{ p_i \in \mathcal{P} : p_i \mid \widetilde{\Omega} \ \approx \pi_i \right\}$ and let

$$\mathcal{P}^N(\pi) = \times_{i \in N} \mathcal{P}(\pi_i).$$

This is the region of belief profiles generating the same profile of likelihood orderings as π on the subsets of $\widetilde{\Omega}$.

Throughout Appendix 2.C.1, the profile π is fixed. The main result in that appendix is Lemma 7. It asserts that there exist two disjoint events A, B, whose union contains $\widetilde{\omega}$, such that $s(\cdot) \cap (A \cup B)$ coincides with an $\{i, j\}$-consensual $(A \cup B)$-assignment rule *on the region* $\mathcal{P}^N(\pi)$. We stress that this lemma determines how not only $\widetilde{\omega}$ but all the states in the entire event $A \cup B$ are assigned when the belief profile belongs to the region $\mathcal{P}^N(\pi)$. Of course, as explained earlier, the detailed behavior of s on $A \cup B$ depends on whether $\{A, B\}$ cuts one, both, or neither of $\mathcal{P}(\pi_i)$, $\mathcal{P}(\pi_j)$. In particular, the rule s need not be *actively* (i, j)-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$.

Lemma 7 is a "regional" result: it holds for a given profile π of beliefs over $\widetilde{\Omega}$. More importantly, it does not guarantee that the sets A, B or the agents i, j are independent of the profile π. The rest of Appendix 2.C shows that they are. The proof is "by contagion". The argument itself is presented in Appendix 2.C.4 but rests on a number of lemmas that we prove in Appendices 2.C.2 and 2.C.3.

Appendix 2.C.2 contains two types of *local contagion* results. We first prove an independence result asserting that if s is actively (i, j)-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$, this must also be true on any region $\mathcal{P}^N(\sigma_k, \pi_{-k})$ such that

σ_k is adjacent to π_k and k differs from i, j. This result is complemented by two contagion lemmas describing how the assignment of A, B on the regions $\mathcal{P}^N(\pi_i', \pi_{-i})$ and $\mathcal{P}^N(\pi_j', \pi_{-j})$ is linked to the assignment of A, B on $\mathcal{P}^N(\pi)$ when π_i' is adjacent to π_i and π_j' is adjacent to π_j. We show, for instance, that if s is actively (i, j)-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$, it is actively or passively (i, j)-consensual with respect to $\{A, B\}$ on the adjacent region $\mathcal{P}^N(\pi_i', \pi_{-i})$. But we cannot guarantee (and it is indeed not the case) that s is *actively* (i, j)-consensual on $\mathcal{P}^N(\pi_i', \pi_{-i})$. For that reason, we cannot directly use these local contagion results to prove the Bilateral Consensus lemma inductively: their contagion power fades away, so to speak, as the gap between the regions $\mathcal{P}^N(\pi)$ and $\mathcal{P}^N(\pi')$ increases.

In Appendix 2.C.3, we establish more powerful contagion lemmas describing how the assignment of A, B on the region $\mathcal{P}^N(\pi)$ is linked with their assignment on non-adjacent regions. We show that, if s is actively (i, j)-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$, its behavior on $\mathcal{P}^N(\pi_i', \pi_{-i})$ is determined by whether $\{A, B\}$ cuts $\mathcal{P}(\pi_i')$ or not. Likewise, its behavior on $\mathcal{P}^N(\pi_j', \pi_{-j})$ is determined by whether $\{A, B\}$ cuts $\mathcal{P}(\pi_j')$ or not.

Appendix 2.C.4 patches the pieces together. In an initialization step, we prove that there exists a profile $\pi \in \widetilde{\mathcal{P}}$ such that s is actively (i, j)-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$. Using the contagion results of Appendices 2.C.2 and 2.C.3 and the connectedness of the set of all beliefs on $\widetilde{\Omega}$, we show that s is an $\{i, j\}$-consensual $(A \cup B)$-assignment rule on the whole domain $\mathcal{P}^N$. The claim follows by setting $E^{\widetilde{\omega}} = A \cup B$.

Appendix 2.C.1: "Regional" Results

Throughout Appendices 2.C.1, 2.C.2 and 2.C.3, we fix a profile $\pi \in \widetilde{\mathcal{P}}^N$. For any $i \in N$, we define

$$\mathcal{P}(\pi_i) = \left\{ p_i \in \mathcal{P} : p_i \mid \widetilde{\Omega} \ \approx \pi_i \right\}.$$

This is the set of beliefs on Ω generating on $\widetilde{\Omega}$ a likelihood ordering that coincides with that generated by π_i. We write $\mathcal{P}^N(\pi) = \times_{k \in N} \mathcal{P}(\pi_k)$ and $\mathcal{P}^{N \setminus i}(\pi_{-i}) = \times_{k \neq i} \mathcal{P}(\pi_k)$.

The main result of Appendix 2.C.1, Lemma 7, describes the behavior of the assignment rule s on $\mathcal{P}^N(\pi)$. To prove Lemma 7, we begin by fixing an agent i and a profile $p_{-i} \in \mathcal{P}^{N \setminus i}(\pi_{-i})$: lemmas 5 and 6 describe the behavior of the function $s_i(., p_{-i})$ on $\mathcal{P}(\pi_i)$.

Define the relation $\widetilde{J}$ on $\mathcal{P}(\pi_i)$ by

$$p_i \widetilde{J} q_i \Leftrightarrow p_i, q_i \text{ are } \{A, B\}\text{-adjacent for some } \{A, B\} \in \mathcal{H},\ \widetilde{\omega} \in A, \text{ and } p_i(A) > p_i(B).$$

This is a sub-relation of the adjacency relation J. Contrary to J, the relation $\widetilde{J}$ is not symmetric. For an illustration, see Figure 2, where $\widetilde{\omega} = \omega_1$ and an arrow stands for $\widetilde{J}$. Observe that if two beliefs $p_i, q_i \in \mathcal{P}(\pi_i)$ are $\{A, B\}$-adjacent, then $\widetilde{\omega} \in A \cup B$: this is because the likelihood relations generated by p_i, q_i coincide on $\widetilde{\Omega}$. Just like J, the relation $\widetilde{J}$ is ordinal: if $p_i \widetilde{J} q_i$, $p_i' \approx p_i$ and $q_i' \approx q_i$, then $p_i' \widetilde{J} q_i'$. All its maximal elements in $\mathcal{P}(\pi_i)$ are ordinally equivalent; any such maximal element p_i^+ is characterized by the property that

$$p_i^+(\widetilde{\omega}) > p_i^+(\widetilde{\Omega}). \tag{28}$$

Likewise, all the minimal elements of $\widetilde{J}$ are ordinally equivalent and any such minimal element p_i^- is characterized by the property that

$$p_i^-(C \cup \widetilde{\omega}) < p_i^-(D) \text{ whenever } \pi_i(C) < \pi_i(D).$$

Example 2. *If* $\Omega = \{1, 2, 3\}$, $\widetilde{\omega} = 1$, *and* π_i *is a belief on* $\{2, 3\}$ *generating the ordering* $\{2, 3\}, \{2\}, \{3\}$, *then* $\mathcal{P}(\pi_i)$ *is the left half of the simplex on Figure 2. Any belief on* $\{1, 2, 3\}$ *generating the ordering*

$$\{\mathbf{1}, 2, 3\}, \{\mathbf{1}, 2\}, \{\mathbf{1}, 3\}, \{\mathbf{1}\}, \{2, 3\}, \{2\}, \{3\}$$

is a maximal element p_i^+ *of* $\widetilde{J}$ *on* $\mathcal{P}(\pi_i)$, *and any belief on* $\{1, 2, 3\}$ *generating the ordering*

$$\{\mathbf{1}, 2, 3\}, \{2, 3\}, \{\mathbf{1}, 2\}, \{2\}, \{\mathbf{1}, 3\}, \{3\}, \{\mathbf{1}\}.$$

is a minimal element p_i^- *of* $\widetilde{J}$ *on* $\mathcal{P}(\pi_i)$.

A *complete* $\widetilde{J}$*-path* in $\mathcal{P}(\pi_i)$, or simply a complete path, is a finite sequence $\mathbf{p}_i = (\mathbf{p}_i^t)_{t=1}^T$ such that $\mathbf{p}_i^1$ is a maximal element of $\widetilde{J}$ (in $\mathcal{P}(\pi_i)$), $\mathbf{p}_i^T$ is a minimal element, $\mathbf{p}_i^t \widetilde{J} \mathbf{p}_i^{t+1}$ for $t = 1, ..., T-1$, and $\mathbf{p}_i^t \in \mathcal{P}(\pi_i)$ for $t = 1, ..., T$.

The following three elementary observations will be useful.

Observation 1. *For each complete* $\widetilde{J}$*-path* $\mathbf{p}_i = (\mathbf{p}_i^t)_{t=1}^T$ *in* $\mathcal{P}(\pi_i)$, $T = \mid \{\{A, B\} \in \mathcal{H} : \widetilde{\omega} \in A \cup B\} \mid$.

Observation 2. *For each complete* $\widetilde{J}$*-path* $\mathbf{p}_i$ *in* $\mathcal{P}(\pi_i)$ *and each* $t \in \{1, ..., T-1\}$, *there is a unique* $\{A^t, B^t\} \in \mathcal{H}$ *such that* $\mathbf{p}_i^t, \mathbf{p}_i^{t+1}$ *are* $\{A^t, B^t\}$*-adjacent. Moreover,* $\{A^t, B^t\} \neq \{A^{t'}, B^{t'}\}$ *if* $t \neq t'$.

Observation 3. *Each belief* $p_i \in \mathcal{P}(\pi_i)$ *lies on some complete* $\widetilde{J}$*-path in* $\mathcal{P}(\pi_i)$: *there exist* $\mathbf{p}_i$ *and* $t \in \{1, ..., T\}$ *such that* $p_i = \mathbf{p}_i^t$.

Observation 1 follows from the fact that any maximal and minimal elements p_i^+, p_i^- lie (i) on opposite sides of every hyperplane $p_i(A) = p_i(B)$ such that $\widetilde{\omega} \in A \cup B$, and

(ii) on the same side of every hyperplane $p_i(A) = p_i(B)$ such that $\widetilde{\omega} \notin A \cup B$. The proofs of observations 2 and 3 are straightforward and left to the reader.

In the following lemma, we show that, for a given agent $i \in N$ and a given profile $p_{-i} \in \mathcal{P}^{N \setminus i}(\pi_{-i})$, the assignment map $s_i(\cdot, p_{-i})$ takes at most two values on $\mathcal{P}(\pi_i)$.

Lemma 5. *For all $i \in N$ and $p_{-i} \in \mathcal{P}^{N \setminus i}(\pi_{-i})$, either (a) $s_i(\cdot, p_{-i})$ is constant on $\mathcal{P}(\pi_i)$, or (b) there exist disjoint sets $A_i(p_{-i}), B_i(p_{-i}), C_i(p_{-i}) \subseteq \Omega$ such that $\widetilde{\omega} \in A_i(p_{-i})$, $\pi_i(A_i(p_{-i}) \setminus \widetilde{\omega}) < \pi_i(B_i(p_{-i}))$, and for all $p_i \in \mathcal{P}(\pi_i)$,*

$$s_i(p_i, p_{-i}) = \begin{cases} A_i(p_{-i}) \cup C_i(p_{-i}) & \text{if } p_i(A_i(p_{-i})) > p_i(B_i(p_{-i})), \\ B_i(p_{-i}) \cup C_i(p_{-i}) & \text{otherwise.} \end{cases}$$

The inequality $\pi_i(A_i(p_{-i}) \setminus \widetilde{\omega}) < \pi_i(B_i(p_{-i}))$ implies that the function $s_i(., p_{-i})$ in statement *(b)* is not constant: the assignment actually varies with agent i's beliefs.

Proof. Let $i \in N$ and $p_{-i} \in \mathcal{P}^{N \setminus i}(\pi_{-i})$. Since p_{-i} is fixed throughout the proof, we omit it from our notation. It is important to keep in mind, however, that the sets whose existence is asserted in Lemma 5 may depend on our choice of p_{-i}. Let $T = |\{\{A, B\} \in \mathcal{H} : \widetilde{\omega} \in A \cup B\}|$.

Step 1. We claim that for any complete $\widetilde{J}$-path $\mathbf{p}_i = (\mathbf{p}_i^t)_{t=1}^T$ in $\mathcal{P}(\pi_i)$, one of the following statements hold:

(α) $s_i(\mathbf{p}_i^1) = s_i(\mathbf{p}_i^2) = ... = s_i(\mathbf{p}_i^T)$,
(β) there exist disjoint sets $A_i(\mathbf{p}_i), B_i(\mathbf{p}_i), C_i(\mathbf{p}_i) \subseteq \Omega$ such that $\widetilde{\omega} \in A_i(\mathbf{p}_i)$, $\pi_i(A_i(\mathbf{p}_i) \setminus \widetilde{\omega}) < \pi_i(B_i(\mathbf{p}_i))$, and there exists $t^*(\mathbf{p}_i) \in \{1, ..., T-1\}$ such that

$$s_i(\mathbf{p}_i^t) = \begin{cases} A_i(\mathbf{p}_i) \cup C_i(\mathbf{p}_i) & \text{if } t \leq t^*(\mathbf{p}_i), \\ B_i(\mathbf{p}_i) \cup C_i(\mathbf{p}_i) & \text{if } t > t^*(\mathbf{p}_i). \end{cases} \tag{29}$$

To prove this claim, fix a complete $\widetilde{J}$-path $\mathbf{p}_i$ in $\mathcal{P}(\pi_i)$. For each $t = 1, ..., T-1$, let $\{A^t, B^t\}$ be the unique pair in $\mathcal{H}$ such that $\mathbf{p}_i^t, \mathbf{p}_i^{t+1}$ are $\{A^t, B^t\}$-adjacent. By definition of $\widetilde{J}$, $\widetilde{\omega} \in A^t$ and $\mathbf{p}_i^t(A^t) > \mathbf{p}_i^t(B^t)$. By the Local Bilaterality lemma, one of the following statements holds:

(i) $s_i(\mathbf{p}_i^t) = s_i(\mathbf{p}_i^{t+1})$,
(ii) $s_i(\mathbf{p}_i^t) \setminus s_i(\mathbf{p}_i^{t+1}) = A^t$ and $s_i(\mathbf{p}_i^{t+1}) \setminus s_i(\mathbf{p}_i^t) = B^t$.

If (i) holds for $t = 1, ..., T-1$, then statement (α) is true. Otherwise, let t^* be the smallest $t \in \{1, ..., T-1\}$ such that $s_i(\mathbf{p}_i^t) \neq s_i(\mathbf{p}_i^{t+1})$. By (ii), $s_i(\mathbf{p}_i^{t^*}) \setminus s_i(\mathbf{p}_i^{t^*+1}) = A^{t^*}$. Since $\widetilde{\omega} \in A^{t^*}$, we have $\widetilde{\omega} \notin s_i(\mathbf{p}_i^{t^*+1})$. This means that statement (ii) cannot hold for any $t = t^* + 1, ..., T$. Hence, $s_i(\mathbf{p}_i^t) = s_i(\mathbf{p}_i^{t^*+1})$ for $t = t^* + 1, ..., T$. Defining $A_i(\mathbf{p}_i) = A^{t^*}$, $B_i(\mathbf{p}_i) = B^{t^*}$, $C_i(\mathbf{p}_i) = s_i(p_i^1) \setminus A^{t^*}$, we obtain (29).

Step 2. Let p_i^+ and p_i^- be maximal and minimal elements of $\widetilde{J}$ in $\mathcal{P}(\pi_i)$.

If $s_i(p_i^+) = s_i(p_i^-)$, define $C_i = s_i(p_i^+) = s_i(p_i^-)$. For any $p_i \in \mathcal{P}(\pi_i)$ there exists some path $\mathbf{p}_i$ and some $t \in \{1, ..., T\}$ such that $p_i = \mathbf{p}_i^t$ (Observation 3). By Step 1, $s_i(p_i) = s_i(\mathbf{p}_i^t) = C_i$, that is, statement *(a)* in Lemma 5 holds.

If $s_i(p_i^+) \neq s_i(p_i^-)$, we know from Step 2 that statement (β) holds for every complete $\widetilde{J}$-path $\mathbf{p}_i = (\mathbf{p}_i^t)_{t=1}^T$ in $\mathcal{P}(\pi_i)$. We claim that the sets $A_i(\mathbf{p}_i), B_i(\mathbf{p}_i), C_i(\mathbf{p}_i)$ do not change with $\mathbf{p}_i$. To see why, let $\mathbf{p}_i, \mathbf{q}_i$ be two paths. If $A_i(\mathbf{p}_i) \neq A_i(\mathbf{q}_i)$ or $C_i(\mathbf{p}_i) \neq C_i(\mathbf{q}_i)$, then $s_i(p_i^+) = s_i(\mathbf{p}_i^1) = A_i(\mathbf{p}_i) \cup C_i(\mathbf{p}_i) \neq A_i(\mathbf{q}_i) \cup C_i(\mathbf{q}_i) = s_i(\mathbf{q}_i^1) = s_i(p_i^+)$, a contradiction. Thus $A_i(\mathbf{p}_i) = A_i(\mathbf{q}_i)$ and $C_i(\mathbf{p}_i) = C_i(\mathbf{q}_i)$. Next, if $B_i(\mathbf{p}_i) \neq B_i(\mathbf{q}_i)$, then $s_i(p_i^-) = s_i(\mathbf{p}_i^T) = B_i(\mathbf{p}_i) \cup C_i(\mathbf{p}_i) = B_i(\mathbf{p}_i) \cup C_i(\mathbf{q}_i) \neq B_i(\mathbf{q}_i) \cup C_i(\mathbf{q}_i) = s_i(\mathbf{q}_i^T) = s_i(p_i^-)$, again a contradiction.

Let A_i, B_i, C_i be the sets such that $A_i(\mathbf{p}_i) = A_i, B_i(\mathbf{p}_i) = B_i$, and $C_i(\mathbf{p}_i) = C_i$ for all complete $\widetilde{J}$-paths $\mathbf{p}_i$ in $\mathcal{P}(\pi_i)$. For any $p_i \in \mathcal{P}(\pi_i)$ there exist some path $\mathbf{p}_i$ and some $t \in \{1, ..., T\}$ such that $p_i = \mathbf{p}_i^t$, and, by Step 1, an integer $t^*(\mathbf{p}_i) \in \{1, ..., T-1\}$ such that

$$s_i(\mathbf{p}_i^t) = \begin{cases} A_i \cup C_i & \text{if } t \leq t^*(\mathbf{p}_i), \\ B_i \cup C_i & \text{if } t > t^*(\mathbf{p}_i). \end{cases} \tag{30}$$

This integer may –and typically does– change with the path $\mathbf{p}_i$, as Figure 2 illustrates.

If $p_i(A_i) = \mathbf{p}_i^t(A_i) > \mathbf{p}_i^t(B_i) = p_i(B_i)$, then $t \leq t^*(\mathbf{p}_i)$: otherwise (30) would imply $s_i(p_i) = B_i \cup C_i$, hence $p_i(s_i(\mathbf{p}_i^1)) = p_i(A_i \cup C_i) > p_i(B_i \cup C_i) = p_i(s_i(p_i))$, contradicting strategyproofness. Since $t \leq t^*(\mathbf{p}_i)$, (30) implies $s_i(p_i) = A_i \cup C_i$.

Likewise, if $p_i(A_i) < p_i(B_i)$, then $t > t^*(\mathbf{p}_i)$ and (15) implies $s_i(p_i) = B_i \cup C_i$. We conclude that statement *(b)* in Lemma 5 holds. □

We record below two immediate consequences of Lemma 5 that will be used later.

Corollary 1. *For all $i \in N$, all $p_i, p'_i \in \mathcal{P}(\pi_i)$, and all $p_{-i} \in \mathcal{P}^{N\setminus i}(\pi_{-i})$,*
(a) $\widetilde{\omega} \in s_i(p_i, p_{-i}) \cap s_i(p'_i, p_{-i}) \Rightarrow s_i(p_i, p_{-i}) = s_i(p'_i, p_{-i})$,
(b) $\widetilde{\omega} \notin s_i(p_i, p_{-i}) \cup s_i(p'_i, p_{-i}) \Rightarrow s_i(p_i, p_{-i}) = s_i(p'_i, p_{-i})$.

Given the other agents' beliefs, i's assignment is fully determined by whether it contains $\widetilde{\omega}$ or not.

Corollary 2. *For all $i \in N$, all $p_{-i} \in \mathcal{P}^{N\setminus i}(\pi_{-i})$, and all maximal and minimal elements p_i^+, p_i^- of $\widetilde{J}$ in $\mathcal{P}(\pi_i)$, if $s(\cdot, p_{-i})$ is not constant on $\mathcal{P}(\pi_i)$, then $\widetilde{\omega} \in s_i(p_i^+, p_{-i}) \setminus s_i(p_i^-, p_{-i})$.*

The next step consists in showing that the sets $A_i(p_{-i})$, $B_i(p_{-i})$, $C_i(p_{-i})$ in Lemma 5 do not vary with p_{-i}.

Lemma 6. *For all $i \in N$, there exist disjoint sets A_i, B_i, $C_i \subseteq \Omega$ such that $\widetilde{\omega} \in A_i$, $\pi_i(A_i \setminus \widetilde{\omega}) < \pi_i(B_i)$, and, for all $p_{-i} \in \mathcal{P}^{N\setminus i}(\pi_{-i})$, either (a) $s_i(\cdot, p_{-i})$ is constant on*

$\mathcal{P}(\pi_i)$, *or (b) for all* $p_i \in \mathcal{P}(\pi_i)$,

$$s_i(p_i, p_{-i}) = \begin{cases} A_i \cup C_i & \textit{if } p_i(A_i) > p_i(B_i), \\ B_i \cup C_i & \textit{otherwise.} \end{cases}$$

We emphasize that Lemma 6 does *not* assert that $s_i(p_i, \cdot)$ is constant over $\mathcal{P}^{N \setminus i}(\pi_{-i})$.

Proof. Let $i \in N$ and define the set

$$\mathcal{P}_*^{N\setminus i}(\pi_{-i}) = \left\{ p_{-i} \in \mathcal{P}^{N\setminus i}(\pi_{-i}) : s_i(\cdot, p_{-i}) \text{ is not constant on } \mathcal{P}(\pi_i) \right\}. \tag{31}$$

Let $p_{-i}, q_{-i} \in \mathcal{P}_*^{N\setminus i}(\pi_{-i})$. By Lemma 5, there exist disjoint sets $A_i(p_{-i})$, $B_i(p_{-i})$, $C_i(p_{-i}) \subseteq \Omega$ such that $\widetilde{\omega} \in A_i(p_{-i})$, $\pi_i(A_i(p_{-i}) \setminus \widetilde{\omega}) < \pi_i(B_i(p_{-i}))$, and

$$\text{for all } p_i \in \mathcal{P}(\pi_i),\ s_i(p_i, p_{-i}) = \begin{cases} A_i(p_{-i}) \cup C_i(p_{-i}) & \text{if } p_i(A_i(p_{-i})) > p_i(B_i(p_{-i})), \\ B_i(p_{-i}) \cup C_i(p_{-i}) & \text{otherwise,} \end{cases} \tag{32}$$

and there exist disjoint sets $A_i(q_{-i})$, $B_i(q_{-i})$, $C_i(q_{-i}) \subseteq \Omega$ such that $\widetilde{\omega} \in A_i(q_{-i})$, $\pi_i(A_i(q_{-i}) \setminus \widetilde{\omega}) < \pi_i(B_i(q_{-i}))$, and

$$\text{for all } p_i \in \mathcal{P}(\pi_i),\ s_i(p_i, q_{-i}) = \begin{cases} A_i(q_{-i}) \cup C_i(q_{-i}) & \text{if } p_i(A_i(q_{-i})) > p_i(B_i(q_{-i})), \\ B_i(q_{-i}) \cup C_i(q_{-i}) & \text{otherwise.} \end{cases} \tag{33}$$

We must prove that $A_i(p_{-i}) = A_i(q_{-i})$, $B_i(p_{-i}) = B_i(q_{-i})$, and $C_i(p_{-i}) = C_i(q_{-i})$.

There is obviously no loss of generality in assuming that there exists some $j \neq i$ such that $p_k = q_k$ for all $k \in N \setminus \{i, j\}$. We therefore drop the beliefs of the agents other than i, j from our notation. Moreover, since $\mathcal{P}(\pi_j)$ is connected, there is no loss in assuming that p_j, q_j are adjacent.

Let p_i^+, p_i^- be maximal and minimal elements of $\widetilde{J}$ in $\mathcal{P}(\pi_i)$. By Corollary 2,

$$\begin{aligned} \widetilde{\omega} &\in s_i(p_i^+, p_j) \setminus s_i(p_i^-, p_j), \\ \widetilde{\omega} &\in s_i(p_i^+, q_j) \setminus s_i(p_i^-, q_j). \end{aligned}$$

Since $\widetilde{\omega} \notin s_j(p_i^+, p_j) \cup s_j(p_i^+, q_j)$, Corollary 1 implies $s_j(p_i^+, p_j) = s_j(p_i^+, q_j)$. By non-bossiness, $s_i(p_i^+, p_j) = s_i(p_i^+, q_j)$. Since $\widetilde{\omega} \in s_i(p_i^+, p_j) \cap s_i(p_i^+, q_j)$, it follows from (32) and (33) that

$$A_i(p_j) \cup C_i(p_j) = A_i(q_j) \cup C_i(q_j).$$

Because p_j and q_j agree on $\widetilde{\Omega}$, the Local Bilaterality lemma implies that (i) $\widetilde{\omega} \in s_j(p_i^-, p_j) \setminus s_j(p_i^-, q_j)$ or (ii) $\widetilde{\omega} \in s_j(p_i^-, q_j) \setminus s_j(p_i^-, p_j)$ or (iii) $s_i(p_i^-, p_j) = s_i(p_i^-, q_j)$. Since $\widetilde{\omega} \notin s_i(p_i^-, p_j) \cup s_i(p_i^-, q_j)$, (iii) must hold. It follows from (32) and (33) that

$$B_i(p_j) \cup C_i(p_j) = B_i(q_j) \cup C_i(q_j).$$

Since $A_i(p_j)$, $B_i(p_j)$, $C_i(p_j)$ are disjoint and $A_i(q_j)$, $B_i(q_j)$, $C_i(q_j)$ are disjoint, these equalities imply $A_i(p_j) = A_i(q_j)$, $B_i(p_j) = B_i(q_j)$, and $C_i(p_j) = C_i(q_j)$. $\square$

We are finally ready to describe the behavior of s on $\mathcal{P}^N(\pi)$.

Terminology. We say that s *varies only with agent* i*'s beliefs (on* $\mathcal{P}^N(\pi)$*)* if there exists $p_{-i} \in \mathcal{P}^{N\setminus i}(\pi_{-i})$ such that $s(\cdot, p_{-i})$ is not constant on $\mathcal{P}(\pi_i)$ but $s(\cdot, p_{-j})$ is constant on $\mathcal{P}(\pi_j)$ for every $j \neq i$ and every $p_{-j} \in \mathcal{P}^{N\setminus j}(\pi_{-j})$. We say that s *varies with the beliefs of agents* i *and* j *(on* $\mathcal{P}^N(\pi)$*)* if there exist $p_{-i} \in \mathcal{P}^{N\setminus i}(\pi_{-i})$ such that $s(\cdot, p_{-i})$ is not constant on $\mathcal{P}(\pi_i)$ and there exists $p_{-j} \in \mathcal{P}^{N\setminus j}(\pi_{-j})$ such that $s(\cdot, p_{-j})$ is not constant on $\mathcal{P}(\pi_j)$. We emphasize that this second definition allows s to potentially vary with the beliefs of agents other than i, j as well.

We say that $\{A, B\} \in \mathcal{H}$ *cuts* $\mathcal{P}(\pi_i)$ if there exist $p_i, q_i \in \mathcal{P}(\pi_i)$ such that $(p_i(A) - p_i(B))(q_i(A) - q_i(B)) < 0$. Observe that if $\widetilde{\omega} \in A$, then $\{A, B\}$ cuts $\mathcal{P}(\pi_i)$ if and only if $\pi_i(A \setminus \widetilde{\omega}) < \pi_i(B)$.

Lemma 7. *There exists a partition* $\{A, B, C_1, ..., C_n\}$ *of* Ω *such that* $\widetilde{\omega} \in A \cup B$ *and*
(a) if s *varies only with agent* 1*'s beliefs on* $\mathcal{P}^N(\pi)$*, then* $\{A, B\}$ *cuts* $\mathcal{P}(\pi_1)$ *and there exists an agent* $i \in N \setminus 1$*, say agent* 2*, such that for all* $p \in \mathcal{P}^N(\pi)$*,*

$$s(p) = \begin{cases} (A \cup C_1,\ B \cup C_2,\ C_3, ...,\ C_n) & \textit{if } p_1(A) > p_1(B), \\ (B \cup C_1,\ A \cup C_2,\ C_3, ...,\ C_n) & \textit{otherwise,} \end{cases}$$

(b) if s *varies with the beliefs of agents* 1 *and* 2 *on* $\mathcal{P}^N(\pi)$*, then* $\{A, B\}$ *cuts* $\mathcal{P}(\pi_1), \mathcal{P}(\pi_2)$ *and for all* $p \in \mathcal{P}^N(\pi)$*,*

$$s(p) = \begin{cases} (A \cup C_1,\ B \cup C_2,\ C_3, ...,\ C_n) & \textit{if } p_1(A) > p_1(B) \textit{ and } p_2(A) < p_2(B), \\ (B \cup C_1,\ A \cup C_2,\ C_3, ...,\ C_n) & \textit{otherwise.} \end{cases}$$

Remark 1. *(a) We stated Lemma 7 with reference to agents* 1 *and* 2 *for notational convenience but of course the result holds, up to a relabeling, for any pair of agents.*

(b) Statement (b) does not assume that the assignment is independent of the beliefs of agents $3, ..., n$*. Rather, it is a corollary to Lemma 7 that, on* $\mathcal{P}^N(\pi)$*, (i) the assignment may vary with the beliefs of at most two agents and (ii) only the events assigned to two agents may change.*

Proof.
Step 1. Suppose first that s varies only with agent 1's beliefs on $\mathcal{P}^N(\pi)$.

Recall the definition of $\mathcal{P}_*^{N\backslash 1}(\pi_{-1})$ in (31). By Lemma 6, there exist disjoint sets A_1, B_1, C_1 such that for all $p_1 \in \mathcal{P}(\pi_1)$ and all $p_{-1} \in \mathcal{P}_*^{N\backslash 1}(\pi_{-1})$,

$$s_1(p_1, p_{-1}) = \begin{cases} A_1 \cup C_1 & \text{if } p_1(A_1) > p_1(B_1), \\ B_1 \cup C_1 & \text{otherwise.} \end{cases}$$

Moreover, $\widetilde{\omega} \in A_1$ and $\pi_1(A_1 \setminus \widetilde{\omega}) < \pi_1(B_1(\pi))$, implying that $\{A_1, B_1\}$ cuts $\mathcal{P}(\pi_1)$.

Since s does not vary with the beliefs of agents $2, ..., n$, the above expression must, in fact, hold for all $(p_1, p_{-1}) \in \mathcal{P}^N(\pi)$. Statement *(a)* now follows from the Local Bilaterality lemma and non-bossiness.

Step 2. Suppose next that s varies with the beliefs of agents 1 and 2 on $\mathcal{P}^N(\pi)$.

Since $\mathcal{P}(\pi_1)$, $\mathcal{P}(\pi_2)$ are connected, there are adjacent beliefs $p_1, p_1' \in \mathcal{P}(\pi_1)$, adjacent beliefs $p_2, p_2' \in \mathcal{P}(\pi_2)$, and sub-profiles $p_{-1} \in \mathcal{P}^{N\backslash 1}(\pi_{-1})$, $q_{-2} \in \mathcal{P}^{N\backslash 2}(\pi_{-2})$ such that

$$s(p_1, p_{-1}) = \alpha \neq \alpha' = s(p_1', p_{-1}), \tag{34}$$

$$s(q_2, q_{-2}) = \beta \neq \beta' = s(q_2', q_{-2}). \tag{35}$$

Sub-step 2.1. We show that the assignment varies locally with two agents' beliefs: there exist two agents $i, j \in N$, two adjacent beliefs $p_i, p_i' \in \mathcal{P}(\pi_i)$, two adjacent beliefs $p_j, p_j' \in \mathcal{P}(\pi_j)$, and a sub-profile $p_{-ij} \in \mathcal{P}^{N\backslash ij}(\pi_{-ij})$ such that $s(p_i', p_j, p_{-ij}) \neq s(p_i, p_j, p_{-ij}) \neq s(p_i, p_j', p_{-ij})$.

Suppose not. Then (34) implies

$$s(p_1, p_j', p_{-1j}) = \alpha \neq \alpha' = s(p_1', p_j', p_{-1j})$$

for all $j \neq 1$ and all p_j' adjacent to p_j. Since $\mathcal{P}(\pi_j)$ is connected, it follows that

$$s(p_1, p_{-1}') = \alpha \neq \alpha' = s(p_1', p_{-1}') \tag{36}$$

for all $p_{-1}' \in \mathcal{P}^{N\backslash 1}(\pi_{-1})$.

By the same token, (35) implies

$$s(q_2, q_{-2}') = \alpha \neq \alpha' = s(q_2', q_{-2}') \tag{37}$$

for all $q_{-2}' \in \mathcal{P}^{N\backslash 2}(\pi_{-2})$.

Statement (36) implies $s(p_1, q_2, p_{-12}) = s(p_1, q_2', p_{-12})$ and statement (37) implies $s(p_1, q_2, p_{-12}) \neq s(p_1, q_2', p_{-12})$, a contradiction.

Sub-step 2.2. We show that there exist disjoint sets $A, B, C_1, ..., C_n$ such that $A, B \neq \emptyset$, $\widetilde{\omega} \in A \cup B$, and, for all $k \neq i, j$,

$$\begin{aligned} (s_i, s_j, s_k)(p_i, p_j, p_{-ij}) = (A \cup C_i,\ B \cup C_j,\ C_k), \\ (s_i, s_j, s_k)(p'_i, p_j, p_{-ij}) = (s_i, s_j, s_k)(p'_i, p_j, p_{-ij}) = (B \cup C_i,\ A \cup C_j,\ C_k). \end{aligned} \tag{38}$$

Since p_{-ij} is fixed, let us drop it from the notation. By Sub-step 2.1 and Lemma 6, there exist disjoint sets A_i, B_i, C_i and disjoint sets A_j, B_j, C_j such that $\widetilde{\omega} \in A_i \cap A_j$, $B_i, B_j \neq \emptyset$, and

$$[s_i(p_i, p_j) = A_i \cup C_i,\ s_i(p'_i, p_j) = B_i \cup C_i] \text{ or } [s_i(p_i, p_j) = B_i \cup C_i,\ s_i(p'_i, p_j) = A_i \cup C_i]$$

and

$$\left[s_j(p_i, p_j) = A_j \cup C_j,\ s_j(p_i, p'_j) = B_j \cup C_j\right] \text{ or } \left[s_j(p_i, p_j) = B_j \cup C_j,\ s_j(p_i, p'_j) = A_j \cup C_j\right].$$

Since $\widetilde{\omega} \in A_i \cap A_j$ and $s_i(p_i, p_j) \cap s_j(p_i, p_j) = \emptyset$, we need only consider three cases.

Case 1. (i) $s_i(p_i, p_j) = A_i \cup C_i$, (ii) $s_i(p'_i, p_j) = B_i \cup C_i$, (iii) $s_j(p_i, p_j) = B_j \cup C_j$, (iv) $s_j(p_i, p'_j) = A_j \cup C_j$.

Define $A = A_i$, $B = B_j$, $C_k = s_k(p_i, p_j)$ for $k \neq i, j$. By the Local Bilaterality lemma, (i), (iii), and (iv) imply $A_j = A$, $B_i = B$, $s_i(p_i, p'_j) = B \cup C_i$, and $s_k(p_i, p'_j) = C_k$ for $k \neq i, j$.

Next, since $s_i(p_i, p_j) = A \cup C_i$, $s_i(p'_i, p_j) = B \cup C_i$, and $s_j(p_i, p_j) = B \cup C_j$, the Local Bilaterality lemma implies $s_j(p'_i, p_j) = A \cup C_j$ and $s_k(p'_i, p_j) = C_k$ for $k \neq i, j$, establishing (38).

Case 2. (i) $s_i(p_i, p_j) = B_i \cup C_i$, (ii) $s_i(p'_i, p_j) = A_i \cup C_i$, (iii) $s_j(p_i, p_j) = A_j \cup C_j$, (iv) $s_j(p_i, p'_j) = B_j \cup C_j$.

Define $A = B_i$, $B = A_j$, $C_k = s_k(p_i, p_j)$ for $k \neq i, j$. Statement (38) follows by the same argument as in Case 1, mutatis mutandis.

Case 3. (i) $s_i(p_i, p_j) = B_i \cup C_i$, (ii) $s_i(p'_i, p_j) = A_i \cup C_i$, (iii) $s_j(p_i, p_j) = B_j \cup C_j$, (iv) $s_j(p_i, p'_j) = A_j \cup C_j$.

This case is impossible. To see why, note first that (i), (ii), (iii), and the Local Bilaterality lemma imply $s_j(p'_i, p_j) = B_j \cup C_j$ whereas (i), (iii), (iv) and the Local Bilaterality lemma imply $s_i(p_i, p'_j) = B_i \cup C_i$.

Since $(s_i, s_j)(p'_i, p_j) = (A_i \cup C_i, B_j \cup C_j)$ and $(s_i, s_j)(p_i, p'_j) = (B_i \cup C_i, A_j \cup C_j)$, Lemma 3 implies that one of the following statements holds:

$$\begin{aligned} (s_i, s_j)(p'_i, p'_j) &= (A_i \cup C_i,\ B_j \cup C_j), \\ (s_i, s_j)(p'_i, p'_j) &= (B_i \cup C_i,\ A_j \cup C_j). \end{aligned}$$

In either case, the Local Bilaterality lemma requires $A_i = A_j$ and $B_i = B_j$. The latter equality implies that $s_i(p_i, p_j) \cap s_j(p_i, p_j) \neq \emptyset$, violating feasibility.

Sub-step 2.3. Assume from now on that $\widetilde{\omega}$ belongs to the set A in (38). The case where $\widetilde{\omega}$ belongs to B is identical up to a permutation of agents i and j. We show that for all $(q_i, q_j) \in \mathcal{P}(\pi_i) \times \mathcal{P}(\pi_j)$ and all $k \neq i, j$,

$$(s_i, s_j, s_k)(q_i, q_j, p_{-ij}) = \begin{cases} (A \cup C_i,\ B \cup C_j,\ C_k) & \text{if } q_i(A) > q_i(B) \text{ and } q_j(A) < q_j(B), \\ (B \cup C_i,\ A \cup C_j,\ C_k) & \text{otherwise.} \end{cases} \tag{39}$$

Since p_{-ij} is fixed, let us drop it again from the notation. By Sub-step 2.2 and Lemma 6, $p_i(A) > p_i(B)$ and $p_j(A) < p_j(B)$, and it follows that (39) holds for the case where $q_i = p_i$ or $q_j = p_j$.

Next, for any q_i such that $q_i(A) < q_i(B)$, the fact that $s_j(q_i, p_j) = A \cup C_j$ implies that $s_j(q_i, .)$ is constant, hence, by non-bossiness, $(s_i, s_j, s_k)(q_i, q_j) = (B \cup C_i,\ A \cup C_j,\ C_k)$.

Similarly, for any q_j such that $q_j(A) > q_j(B)$, the fact that $s_i(p_i, q_j) = B \cup C_i$ implies that $s_i(., q_j)$ is constant, hence, by non-bossiness, $(s_i, s_j, s_k)(q_i, q_j) = (B \cup C_i,\ A \cup C_j,\ C_k)$.

Finally, for any (q_i, q_j) such that $q_i(A) > q_i(B)$ and $q_j(A) < q_j(B)$, the fact that $s_i(., q_j)$ and $s_j(., q_i)$ are not constant, together with non-bossiness, implies $(s_i, s_j, s_k)(q_i, q_j) = (A \cup C_i,\ B \cup C_j,\ C_k)$, completing the proof of (39).

Sub-step 2.4. We show that for all $q \in \mathcal{P}^N(\pi)$ and all $k \neq i, j$,

$$(s_i, s_j, s_k)(q) = \begin{cases} (A \cup C_i, B \cup C_j, C_k) & \text{if } q_i(A) > q_i(B) \text{ and } q_j(A) < q_j(B), \\ (B \cup C_i,\ A \cup C_j,\ C_k) & \text{otherwise.} \end{cases} \tag{40}$$

Let $q \in \mathcal{P}^N(\pi)$. Given Sub-step 2.3 and because each $\mathcal{P}(\pi_k)$ is connected, we may assume without loss of generality that there exists some $k \neq i, j$ such that q_k is adjacent to p_k and $q_{k'} = p_{k'}$ for all $k' \neq i, j, k$. In what follows, we drop $q_{-ijk} = p_{-ijk}$ from our notation. Suppose, by way of contradiction, that $s(q_i, q_j, q_k) \neq s(q_i, q_j, p_k)$.

If $(s_i, s_j, s_k)(q_i, q_j, p_k) = (A \cup C_i,\ B \cup C_j,\ C_k)$, non-bossiness implies $s_k(q_i, q_j, q_k) \neq s_k(q_i, q_j, p_k)$. Since $p_k, q_k \in \mathcal{P}(\pi_k)$, the pair of events $\{E, E'\}$ for which p_k, q_k are $\{E, E'\}$-adjacent is such that $\widetilde{\omega} \in E \cup E'$. Since $\widetilde{\omega} \in A \cup C_i = s_i(q_i, q_j, p_k)$, we must therefore have $s_i(q_i, q_j, q_k) \neq s_i(q_i, q_j, p_k)$ and Lemma 6 implies $s_i(q_i, q_j, q_k) = B \cup C_i$. By the Local Bilaterality lemma, $s_j(q_i, q_j, q_k) = s_i(q_i, q_j, p_k) = B \cup C_j$. This means that $s_i(q_i, q_j, q_k) \cap s_j(q_i, q_j, q_k) \neq \emptyset$, contradicting feasibility.

If $(s_i, s_j, s_k)(q_i, q_j, p_k) = (B \cup C_i,\ A \cup C_j,\ C_k)$, exchanging the roles of i and j in the above argument yields the same contradiction.

Sub-step 2.5. Since s varies with the beliefs of agents 1 and 2 on $\mathcal{P}^N(\pi)$, (40) must hold with $\{i,j\} = \{1,2\}$, completing the proof of statement *(b).* $\square$

Terminology. If the rule s is of the type identified in part *(a)* of Lemma 7, we call it *passively* $(1,2)$*-consensual (with respect to* $\{A,B\}$*) on* $\mathcal{P}^N(\pi)$*.* In that case, there is no loss of generality in assuming that $\widetilde{\omega} \in A$: we maintain that convention throughout.

If s is of the type identified in part *(b)*, we call it *actively* $\{1,2\}$*-consensual (with respect to* $\{A,B\}$*) on* $\mathcal{P}^N(\pi)$*.* We call it *actively* $(1,2)$*-consensual* if $\widetilde{\omega} \in B$ and *actively* $(2,1)$*-consensual* if $\widetilde{\omega} \in A$: under an actively (i,j)-consensual rule, the "default option" assigns state $\widetilde{\omega}$ to agent i.

We call the sets $C_1, ..., C_n$ *residuals.*

Appendix 2.C.2: Local Contagion Results

Lemma 7 described the behavior of $s(\cdot) \cap (A \cup B)$ on the region $\mathcal{P}^N(\pi)$. In the current Appendix 2.C.2, we study how that behavior varies locally with π. The three main results are the Independence lemma, the First Contagion lemma, and the Second Contagion lemma. These local contagion results will be used in the main contagion argument in Appendix 2.C.4.

In order to proceed, we first need to extend the notion of adjacency to beliefs defined over an arbitrary *subset* of Ω. For any $\Omega' \subseteq \Omega$ (e.g., $\Omega' = \widetilde{\Omega}$), let $\mathcal{H}(\Omega') = \{\{A,B\} : \emptyset \neq A, B \subset \Omega' \text{ and } A \cap B = \emptyset\}$ and say that $\pi_i, \sigma_i \in \mathcal{P}(\Omega')$ are $\{A,B\}$-*adjacent* if $(\pi_i(A) - \pi_i(B))(\sigma_i(A) - \sigma_i(B)) < 0$ and $(\pi_i(C) - \pi_i(D))\,(\sigma_i(C) - \sigma_i(D)) > 0$ for all $\{C,D\} \in \mathcal{H}(\Omega') \backslash \{\{A,B\}\}$. With a slight abuse of notation, we use J to denote the adjacency relation between beliefs on any Ω'. Connectedness of a subset of $\mathcal{P}(\Omega')$ is defined in the obvious way.

The first main result of this Appendix 2.C.2 states an independence property saying that a local change in the beliefs of agents $3, ..., n$, who have no say in allocating A, B, does not matter.

Independence Lemma. *Let* $k \in N \setminus \{1,2\}$*, and let* $\sigma_k \in \widetilde{\mathcal{P}}$ *be adjacent to* π_k*. If* s *is actively* $(2,1)$*-consensual with respect to* $\{A,B\}$ *on* $\mathcal{P}^N(\pi)$*, then* s *is actively* $(2,1)$*-consensual with respect to* $\{A,B\}$ *on* $\mathcal{P}^N(\sigma_k, \pi_{-k})$*.*

Proof. Suppose s is actively $(2,1)$-consensual with respect to $\{A,B\}$ on $\mathcal{P}^N(\pi)$: there exists a partition $\{A, B, C_1, ..., C_n\}$ of Ω such that $\widetilde{\omega} \in A$, $\{A,B\}$ cuts $\mathcal{P}(\pi_1), \mathcal{P}(\pi_2)$, and, for all $p \in \mathcal{P}^N(\pi)$,

$$s(p) = \begin{cases} (A \cup C_1, B \cup C_2, C_3, ..., C_n) & \text{if } p_1(A) > p_1(B) \text{ and } p_2(A) < p_2(B), \\ (B \cup C_1, A \cup C_2, C_3, ..., C_n) & \text{otherwise.} \end{cases} \tag{41}$$

Fix $k \in N \setminus \{1,2\}$, say, $k = 3$, and let $\sigma_3 \in \widetilde{\mathcal{P}}$ be adjacent to π_3.

By calibrating the probability assigned to $\widetilde{\omega}$, we can find $\{A, B\}$-adjacent beliefs $p_1, p_1' \in \mathcal{P}(\pi_1)$ and $\{A, B\}$-adjacent beliefs $p_2, p_2' \in \mathcal{P}(\pi_2)$ with, say, $p_1(A) > p_1(B)$ and $p_2(A) < p_2(B)$. Let $p_{-123} \in \mathcal{P}^{N\setminus 123}(\pi_{-123})$. This sub-profile is fixed throughout the argument and therefore omitted from the notation. Let p_3^+, q_3^+ be maximal elements of $\widetilde{J}$ in $\mathcal{P}(\pi_3), \mathcal{P}(\sigma_3)$.

By (41),

$$\begin{aligned} s(p_1, p_2, p_3^+) &= (A \cup C_1, B \cup C_2, C_3, ..., C_n), \\ s(p_1', p_2, p_3^+) &= (B \cup C_1, A \cup C_2, C_3, ..., C_n), \\ s(p_1, p_2', p_3^+) &= (B \cup C_1, A \cup C_2, C_3, ..., C_n). \end{aligned} \tag{42}$$

Step 1. We show that there exists a partition $\{C_1', ..., C_n'\}$ of $\Omega \setminus (A \cup B)$ such that

$$s(p_1, p_2, q_3^+) = (A \cup C_1', B \cup C_2', C_3', ..., C_n'). \tag{43}$$

By definition, p_3^+, q_3^+ are adjacent. By the Local Bilaterality lemma and the first equality in (42), there are only three cases.

Case 1. There exists some $j \neq 1, 2, 3$ such that $s_j(p_1, p_2, q_3^+) \cap s_3(p_1, p_2, p_3^+) \neq \emptyset$, $s_3(p_1, p_2, q_3^+) \cap s_j(p_1, p_2, p_3^+) \neq \emptyset$, and $s_i(p_1, p_2, q_3^+) = s_i(p_1, p_2, p_3^+)$ for all $i \neq j, 3$.

In this case (43) holds with $C_i' = C_i$ for all $i \neq j, 3$.

Case 2. $s_1(p_1, p_2, q_3^+) \cap s_3(p_1, p_2, p_3^+) \neq \emptyset$, $s_3(p_1, p_2, q_3^+) \cap s_1(p_1, p_2, p_3^+) \neq \emptyset$, and $s_i(p_1, p_2, q_3^+) = s_i(p_1, p_2, p_3^+)$ for all $i \neq 1, 3$.

If $A \nsubseteq s_1(p_1, p_2, q_3^+)$, then since p_1, p_1' are $\{A, B\}$-adjacent with $p_1(A) > p_1(B)$, the Local Bilaterality lemma implies $s(p_1', p_2, q_3^+) = s(p_1, p_2, q_3^+)$. Comparing with (42),

$$\begin{aligned} s_1(p_1', p_2, q_3^+) \cap B &= \emptyset \text{ and } s_1(p_1', p_2, p_3^+) \cap B \neq \emptyset, \\ s_2(p_1', p_2, q_3^+) \cap B &\neq \emptyset \text{ and } s_2(p_1', p_2, p_3^+) \cap B = \emptyset, \\ s_3(p_1', p_2, q_3^+) \cap A &\neq \emptyset \text{ and } s_3(p_1', p_2, p_3^+) \cap A = \emptyset, \end{aligned}$$

implying $s_i(p_1', p_2, q_3^+) \neq s_i(p_1', p_2, p_3^+)$ for $i = 1, 2, 3$, contradicting the Local Bilaterality lemma.

This shows that $A \subseteq s_1(p_1, p_2, q_3^+)$. Then (43) holds with $C_i' = C_i$ for all $i \neq 1, 3$.

Case 3. $s_2(p_1, p_2, q_3^+) \cap s_3(p_1, p_2, p_3^+) \neq \emptyset$, $s_3(p_1, p_2, q_3^+) \cap s_2(p_1, p_2, p_3^+) \neq \emptyset$, and $s_i(p_1, p_2, q_3^+) = s_i(p_1, p_2, p_3^+)$ for all $i \neq 2, 3$.

If $B \nsubseteq s_2(p_1, p_2, q_3^+)$, then since p_2, p_2' are $\{A, B\}$-adjacent with $p_2(A) < p_2(B)$, the Local Bilaterality lemma implies $s(p_1, p_2', q_3^+) = s(p_1, p_2, q_3^+)$. Comparing with (42),

$$\begin{aligned} s_1(p_1, p_2', q_3^+) \cap A &\neq \emptyset \text{ and } s_1(p_1, p_2', p_3^+) \cap A = \emptyset, \\ s_2(p_1, p_2', q_3^+) \cap A &= \emptyset \text{ and } s_2(p_1, p_2', p_3^+) \cap A \neq \emptyset, \\ s_3(p_1, p_2', q_3^+) \cap B &\neq \emptyset \text{ and } s_3(p_1, p_2', p_3^+) \cap B = \emptyset, \end{aligned}$$

implying $s_i(p_1, p_2', q_3^+) \neq s_i(p_1, p_2', p_3^+)$ for $i = 1, 2, 3$, contradicting the Local Bilaterality lemma again.

This shows that $B \subseteq s_2(p_1, p_2, q_3^+)$, Then (43) holds with $C_i' = C_i$ for all $i \neq 2, 3$.

Step 2. We show that

$$s(p_1', p_2, q_3^+) = s(p_1, p_2', q_3^+) = (B \cup C_1', A \cup C_2', C_3', ..., C_n'). \tag{44}$$

Since p_1, p_1' are $\{A, B\}$-adjacent, Step 1 and the Local Bilaterality lemma imply that either (i) $s(p_1', p_2, q_3^+) = (A \cup C_1',\ B \cup C_2',\ C_3', ..., C_n')$ or (ii) $s(p_1', p_2, q_3^+) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$. Statement (i) and the second statement in (42) together contradict the Local Bilaterality lemma, hence (ii) must hold. Likewise, the third statement in (42) and the Local Bilaterality lemma imply that $s(p_1, p_2', q_3^+) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$.

Step 3. Combining statements (43), (44), and statement *(b)* in Lemma 7, we obtain that for all $(q_1, q_2, q_3) \in \mathcal{P}(\pi_1) \times \mathcal{P}(\pi_2) \times \mathcal{P}(\sigma_3)$,

$$s(q_1, q_2, q_3) = \begin{cases} (A \cup C_1', B \cup C_2', C_3', ...,\ C_n') & \text{if } q_1(A) > q_1(B) \text{ and } q_2(A) < q_2(B), \\ (B \cup C_1', A \cup C_2', C_3', ...,\ C_n') & \text{otherwise.} \end{cases}$$

Since p_{-123} was chosen arbitrarily in $\mathcal{P}^{N\backslash 123}(\pi_{-123})$, this proves that s is actively $(2, 1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_3, \pi_{-3})$. □

We now examine how a local change in the beliefs of agents 1 and 2 affects the assignment of A, B. First, an intermediate result.

Lemma 8. *Let $\sigma_1, \sigma_2 \in \widetilde{\mathcal{P}}$ be adjacent to π_1, π_2, respectively, and suppose s is actively $(2, 1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$.*
(a) If s is actively $(2, 1)$-consensual with respect to some $\{A', B'\}$ on $\mathcal{P}^N(\sigma_2, \pi_{-2})$, then $\{A', B'\}$ cuts $\mathcal{P}(\pi_2)$ and $\{A, B\}$ cuts $\mathcal{P}(\sigma_2)$.
(b) If s is actively $(2, 1)$-consensual with respect to some $\{A', B'\}$ on $\mathcal{P}^N(\sigma_1, \pi_{-1})$, then $\{A', B'\}$ cuts $\mathcal{P}(\pi_1)$ and $\{A, B\}$ cuts $\mathcal{P}(\sigma_1)$.

Remark 2. *We stated Lemma 8 for the ordered pair $(2, 1)$ for notational simplicity only: up to a relabeling, the result applies to any ordered pair (i, j) of agents. This comment applies also to the results below.*

Proof. We only prove statement *(a)*. Although statement *(b)* is *not* a mere permutation of statement *(a)* (because s is actively $(2, 1)$-consensual in both cases), its proof is almost identical and therefore omitted. Fix $\sigma_2 \in \widetilde{\mathcal{P}}$ adjacent to π_2. Suppose s is actively $(2, 1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$, and actively $(2, 1)$-consensual with respect to $\{A', B'\}$ on $\mathcal{P}^N(\sigma_2, \pi_{-2})$ with residuals $C'_1, ..., C'_n$. Fix an arbitrary sub-profile $p_{-12} \in \mathcal{P}^{N\setminus 12}(\pi_{-12})$ and drop it from the notation. Then, for all $p = (p_1, p_2) \in \mathcal{P}(\pi_1) \times \mathcal{P}(\pi_2)$,

$$(s_1, s_2)(p_1, p_2) = \begin{cases} (A \cup C_1, B \cup C_2) & \text{if } p_1(A) > p_1(B) \text{ and } p_2(A) < p_2(B), \\ (B \cup C_1, A \cup C_2) & \text{otherwise,} \end{cases} \tag{45}$$

and for all $(p_1, q_2) \in \mathcal{P}(\pi_1) \times \mathcal{P}(\sigma_2)$,

$$(s_1, s_2)(p_1, q_2) = \begin{cases} (A' \cup C'_1, B' \cup C'_2) & \text{if } p_1(A') > p_1(B') \text{ and } q_2(A') < q_2(B'), \\ (B' \cup C'_1, A' \cup C'_2) & \text{otherwise,} \end{cases} \tag{46}$$

where $\widetilde{\omega} \in A \cap A'$, $\{A, B\}$ cuts $\mathcal{P}(\pi_1), \mathcal{P}(\pi_2)$, and $\{A', B'\}$ cuts $\mathcal{P}(\pi_1), \mathcal{P}(\sigma_2)$. In particular, writing $\widetilde{A} := A \setminus \widetilde{\omega}$, $\widetilde{A}' := A' \setminus \widetilde{\omega}$, we have

$$\pi_2(\widetilde{A}) < \pi_2(B). \tag{47}$$

$$\sigma_2(\widetilde{A}') < \sigma_2(B'). \tag{48}$$

Let p_1^+, p_2^+, q_2^+ and p_1^-, p_2^-, q_2^- be, respectively, maximal and minimal elements of $\widetilde{J}$ in, respectively, $\mathcal{P}(\pi_1), \mathcal{P}(\pi_2)$, and $\mathcal{P}(\sigma_2)$. Let $\{E, E'\} \in \mathcal{H}(\widetilde{\Omega})$ be the unique pair of disjoint subsets of $\widetilde{\Omega}$ such that π_2 and σ_2 are $\{E, E'\}$-adjacent with, say, $\pi_2(E) > \pi_2(E')$. Recall that π_2, σ_2 are beliefs on $\widetilde{\Omega} = \Omega \setminus \widetilde{\omega}$; this implies that $\widetilde{\omega} \notin E \cup E'$. Observe now that p_2^+, q_2^+ are $\{E, E'\}$-adjacent beliefs on Ω: this follows directly from the characteristic inequality (28). In contrast, p_2^-, q_2^- need not be adjacent, as Figure 2 illustrates.

We will only prove that $\{A', B'\}$ cuts $\mathcal{P}(\pi_2)$; the proof that $\{A, B\}$ cuts $\mathcal{P}(\sigma_2)$ is the same, mutatis mutandis. Suppose, by way of contradiction, that

$$\pi_2(\widetilde{A}') > \pi_2(B'). \tag{49}$$

We first claim that for every $\widehat{\omega} \in E \cup E'$,

$$p_2^- \mid \widehat{\Omega} \approx q_2^- \mid \widehat{\Omega}, \tag{50}$$

where $\widehat{\Omega} := \Omega \setminus \widehat{\omega}$. To see why, fix disjoint events $C, D \subseteq \widehat{\Omega}$ and observe that

$$\begin{aligned} p_2^-(C) < p_2^-(D) &\Leftrightarrow \pi_2(C \setminus \widetilde{\omega}) < \pi_2(D \setminus \widetilde{\omega}) \\ &\Leftrightarrow \sigma_2(C \setminus \widetilde{\omega}) < \sigma_2(D \setminus \widetilde{\omega}) \\ &\Leftrightarrow q_2^-(C) < q_2^-(D). \end{aligned}$$

The first equivalence holds by definition of p_2^-. The second holds because $\widehat{\omega} \in E \cup E'$ and $\widehat{\omega} \notin C \cup D$ imply that $\{C \setminus \widetilde{\omega}, D \setminus \widetilde{\omega}\}$ differs from $\{E, E'\}$, the unique pair of disjoints subsets of $\widetilde{\Omega}$ on which the likelihood orderings generated by π_2, σ_2 disagree. The third equivalence holds by definition of q_2^-.

Next, let $\widehat{\pi}_2$ be a belief on $\widehat{\Omega}$ such that $p_2^- \mid \widehat{\Omega} \approx q_2^- \mid \widehat{\Omega} \approx \widehat{\pi}_2$. We emphasize that the belief $\widehat{\pi}_2$ is not defined on the same event as π_2, σ_2, which are beliefs on $\widetilde{\Omega}$. Define $\mathcal{P}(\widehat{\pi}_2) = \left\{p_2 \in \mathcal{P} : p_2 \mid \widehat{\Omega} \approx \widehat{\pi}_2\right\}$. For every $\alpha \in [0,1]$, define

$$^{\alpha}q_2 = \alpha p_2^- + (1-\alpha) q_2^-.$$

Observe that $^{\alpha}q_2 \in \overline{\mathcal{P}(\widehat{\pi}_2)} \cap (\overline{\mathcal{P}(\sigma_2)} \cup \overline{\mathcal{P}(\pi_2)})$ for every $\alpha \in [0,1]$, where the upperbar denotes the closure operator. Furthermore, because we assumed that $\{A', B'\}$ does not cut $\mathcal{P}(\pi_2)$ (i.e., (49) holds), there exists some $\alpha \in [0,1]$ such that

$$^{\alpha}q_2 \in \mathcal{P}(\sigma_2) \text{ and } {}^{\alpha}q_2(A') > {}^{\alpha}q_2(B'). \tag{51}$$

We omit the easy proof for brevity.

Pick $p_1 \in \mathcal{P}(\pi_1)$ such $p_1(A) > p_1(B)$ and $p_1(A') > p_1(B')$. By definition of q_2^- and thanks to (48), $q_2^-(A') < q_2^-(B')$, hence from (46),

$$s_2(p_1, {}^0 q_2) = s_2(p_1, q_2^-) = B' \cup C_2'. \tag{52}$$

Choosing α such that (51) holds, (46) again implies

$$s_2(p_1, {}^{\alpha} q_2) = A' \cup C_2'. \tag{53}$$

But since $^{\beta}q_2 \in \overline{\mathcal{P}(\widehat{\pi}_2)}$ for all $\beta \in [0,1]$, (52), (53), and Lemma 6, applied with $\widehat{\Omega}$ instead of $\widetilde{\Omega}$, imply

$$s_2(p_1, {}^1 q_2) = s_2(p_1, p_2^-) = A' \cup C_2'.$$

However, by definition of p_2^- and thanks to (47), $p_2^-(A) < p_2^-(B)$, hence from (45),

$$s_2(p_1, p_2^-) = B \cup C_2,$$

contradicting the previous equality since $\widetilde{\omega} \in (A' \cup C_2') \setminus (B \cup C_2)$. □

We are now ready to prove the second main result of this Appendix 2.C.2. This result describes how a local change in agent 2's beliefs affects the assignment of events A, B.

First Contagion Lemma. *Let $\sigma_2 \in \widetilde{\mathcal{P}}$ be adjacent to π_2, and suppose s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$.*
(a) If $\{A, B\}$ cuts $\mathcal{P}(\sigma_2)$, then s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_2, \pi_{-2})$.
(b) If $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, then $s(p) = (B \cup C_1, A \cup C_2, C_3, ..., C_n)$ for all $p \in \mathcal{P}^N(\sigma_2, \pi_{-2})$.

Remark 3. *Statement (a) does not assert that the residuals* $C'_1, ..., C'_n$ *associated with the actively* $(2,1)$*-consensual rule* s *on* $\mathcal{P}^N(\sigma_2, \pi_{-2})$ *coincide with the residuals* $C_1, ..., C_n$ *on* $\mathcal{P}^N(\pi)$: *in fact, they generally do not.*

Statement (b), on the other hand, asserts that s *is constant on* $\mathcal{P}^N(\sigma_2, \pi_{-2})$ *and the residuals are the same as on* $\mathcal{P}^N(\pi)$*: the assignment outside* $A \cup B$ *remains constant when* 2*'s beliefs switch from* $\mathcal{P}(\pi_2)$ *to* $\mathcal{P}(\sigma_2)$*. It may be worth explaining why a locally bilateral assignment rule indeed possesses this property. The reason is the following. Since we have assumed that* s *is actively* $(2,1)$*-consensual with respect to* $\{A, B\}$ *on* $\mathcal{P}^N(\pi)$*, we know that* $\{A, B\}$ *cuts* $\mathcal{P}(\pi_2)$*, that is,* $\pi_2(\widetilde{A}) < \pi_2(B)$*. On the other hand, since* $\{A, B\}$ *does not cut* $\mathcal{P}(\sigma_2)$*, we have* $\sigma_2(\widetilde{A}) > \sigma_2(B)$*. It follows that the adjacent beliefs* π_2, σ_2 *must, in fact, be* $\{\widetilde{A}, B\}$*-adjacent. This means that any two beliefs* $p_2 \in \mathcal{P}(\pi_2), q_2 \in \mathcal{P}(\sigma_2)$ *agree on the ranking of all events* $C, D \subseteq \Omega \setminus (A \cup B)$*. As a result, the assignment outside* $A \cup B$ *remains unchanged under a locally bilateral assignment rule.*

Proof. Fix $\sigma_2 \in \widetilde{\mathcal{P}}$ such that π_2, σ_2 are adjacent. Suppose s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$: (45) holds for all $p \in \mathcal{P}^N(\pi)$, $\widetilde{\omega} \in A$, and $\{A, B\}$ cuts $\mathcal{P}(\pi_2)$, i.e., (47) holds. For any $k \in N$, let p_k^+, p_k^- denote maximal and minimal elements of $\widetilde{J}$ in $\mathcal{P}(\pi_k)$, q_2^+, q_2^- be maximal and minimal elements of $\widetilde{J}$ in $\mathcal{P}(\sigma_2)$, and let E, E' be the disjoint subsets of $\widetilde{\Omega}$ such that π_2 and σ_2 are $\{E, E'\}$-adjacent with $\pi_2(E) > \pi_2(E')$. Recall that $\widetilde{\omega} \notin E \cup E'$.

Step 1. We show that for every agent $k \neq 2$ and every $k' \neq k$, s is neither passively (k, k')-consensual nor actively (k, k')-consensual on $\mathcal{P}^N(\sigma_2, \pi_{-2})$.

Fix $k \neq 2$, $k' \neq k$. Fix a sub-profile $p_{-2k} \in \mathcal{P}^{N \setminus 2k}(\pi_{-2k})$ and drop it from the notation. Since s is actively $(2,1)$-consensual on $\mathcal{P}^N(\pi)$, we have $\widetilde{\omega} \in s_2(p_2^+, p_k^+)$. If s is passively (k, k')-consensual or actively (k, k')-consensual on $\mathcal{P}^N(\sigma_2, \pi_{-2})$, then $\widetilde{\omega} \in s_k(q_2^+, p_k^+)$. These two statements contradict the Local Bilaterality lemma because p_2^+, q_2^+ are $\{E, E'\}$-adjacent and $\widetilde{\omega} \notin E \cup E'$.

Step 2. We prove statement *(a)*.

Suppose $\{A, B\}$ cuts $\mathcal{P}(\sigma_2)$, that is,

$$\sigma_2(\widetilde{A}) < \sigma_2(B). \tag{54}$$

Sub-step 2.1. We show that s varies with the beliefs of agents 1 and 2 on $\mathcal{P}^N(\sigma_2, \pi_{-2})$.

Fix a sub-profile $p_{-12} \in \mathcal{P}^{N \setminus 12}(\pi_{-12})$ and drop it from the notation. Because $\{A, B\}$ cuts $\mathcal{P}(\sigma_2)$, there exist adjacent beliefs $\overline{p}_2 \in \mathcal{P}(\pi_2)$ and $\overline{q}_2 \in \mathcal{P}(\sigma_2)$ such that $\overline{p}_2(A) < \overline{p}_2(B)$. These beliefs are, in fact, $\{E, E'\}$-adjacent.

Choose $p_1 \in \mathcal{P}(\pi_1)$ such that $p_1(A) > p_1(B)$. From (45), $s_2(p_1, p_2^+) = A \cup C_2$ and $s_2(p_1, \overline{p}_2) = B \cup C_2$. By the Local Bilaterality lemma,

$$\begin{aligned} s_2(p_1, q_2^+) &= A \cup C_2 \text{ or } (A \cup C_2 \cup E') \setminus E, \\ s_2(p_1, \overline{q}_2) &= B \cup C_2 \text{ or } (B \cup C_2 \cup E') \setminus E. \end{aligned}$$

It follows that $\widetilde{\omega} \in s_2(p_1, q_2^+) \setminus s_2(p_1, \overline{q}_2)$: s varies with agent 2's beliefs.

Next, choose $q_1 \in \mathcal{P}(\pi_1)$ such that $q_1(A) < q_1(B)$. From (45), $s_2(q_1, \overline{p}_2) = A \cup C_2$. By the Local Bilaterality lemma,

$$s_2(q_1, \overline{q}_2) = A \cup C_2 \text{ or } (A \cup C_2 \cup E') \setminus E.$$

Thus $\widetilde{\omega} \in s_2(q_1, \overline{q}_2) \setminus s_2(p_1, \overline{q}_2)$: s varies with agent 1's beliefs.

Sub-step 2.2. Since s varies with the beliefs of agents 1 and 2 on $\mathcal{P}^N(\sigma_2, \pi_{-2})$, Lemma 7 and Step 1 imply that s is actively $(2,1)$-consensual with respect to some $\{A', B'\}$ on $\mathcal{P}^N(\sigma_2, \pi_{-2})$ with, say, residuals $C_1', ..., C_2'$. Thus, (46) holds for all $(p_1, q_2) \in \mathcal{P}(\pi_1) \times \mathcal{P}(\sigma_2)$, $\widetilde{\omega} \in A'$, and $\{A', B'\}$ cuts $\mathcal{P}(\pi_1), \mathcal{P}(\sigma_2)$. In particular, (48) holds. To complete the proof of statement *(a)*, it remains to prove that $\{A, B\} = \{A', B'\}$.

Suppose, contrary to our claim, that $\{A, B\} \neq \{A', B'\}$. Define the positive numbers

$$\begin{aligned} \delta &= \pi_1(B) - \pi_1(\widetilde{A}), \\ \delta' &= \pi_1(B') - \pi_1(\widetilde{A}'). \end{aligned}$$

Assume $\delta \neq \delta'$. This is without loss of generality: if $\delta = \delta'$, simply replace π_1 with an ordinally equivalent belief for which the two corresponding numbers differ. Either $\delta < \delta'$ or $\delta' < \delta$. We will only treat the former case; the latter is identical, mutatis mutandis.

For each $\alpha \in [0,1]$, define $p_1^\alpha \in \overline{\mathcal{P}(\pi_1)}$ by

$$p_1^\alpha(\widetilde{\omega}) = \alpha \text{ and } p_1^\alpha(\omega) = (1-\alpha)\pi_1(\omega) \text{ for all } \omega \in \widetilde{\Omega}.$$

Elementary algebra shows that $p_1^\alpha(A) < p_1^\alpha(B) \Leftrightarrow \alpha < \frac{\delta}{1+\delta}$ and $p_1^\alpha(A') < p_1^\alpha(B') \Leftrightarrow \alpha < \frac{\delta'}{1+\delta'}$. Since $\delta < \delta'$, we have $\frac{\delta}{1+\delta} < \frac{\delta'}{1+\delta'}$. Choosing $\frac{\delta}{1+\delta} < \alpha < \frac{\delta'}{1+\delta'}$, we have

$$p_1^\alpha(A) > p_1^\alpha(B) \text{ and } p_1^\alpha(A') < p_1^\alpha(B'). \tag{55}$$

Because of (47) and (54), there exist adjacent beliefs $p_2 \in \mathcal{P}(\pi_2)$ and $q_2 \in \mathcal{P}(\sigma_2)$ such that $p_2(A) < p_2(B)$. This is illustrated in Figure 3 with $A = \{1\}, B = \{2\}$; we omit the easy proof for brevity. From this inequality, (45), and the first inequality in (55), we obtain

$$s_2(p_1^\alpha, p_2) = B \cup C_2.$$

From (46) and the second inequality in (55),

$$s_2\left(p_1^{\alpha}, q_2\right) = A' \cup C_2'.$$

It follows that $\widetilde{\omega} \in s_2\left(p_1^{\alpha}, q_2\right) \setminus s_2\left(p_1^{\alpha}, p_2\right)$, contradicting the Local Bilaterality lemma because p_2, q_2 are $\{E, E'\}$-adjacent and $\widetilde{\omega} \notin E \cup E'$.

Step 3. We prove statement *(b).*

Suppose $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, that is,

$$\sigma_2(\widetilde{A}) > \sigma_2(B). \tag{56}$$

Sub-step 3.1. We prove that s is neither passively $(2,k)$-consensual nor actively $(2,k)$-consensual on $\mathcal{P}^N(\sigma_2, \pi_{-2})$ for any $k \neq 2$.

Suppose it is.

Case 1. $\{A', B'\}$ cuts $\mathcal{P}(\pi_2)$, that is, $\pi_2(\widetilde{A}') < \pi_2(B')$.

Fix a sub-profile $p_{-2k} \in \mathcal{P}^{N\setminus 2k}(\pi_{-2k})$ and drop it from the notation. Because of (56), there exist adjacent $p_2 \in \mathcal{P}(\pi_2)$ and $q_2 \in \mathcal{P}(\sigma_2)$ such that $p_2(A) > p_2(B)$ and $q_2(A') < q_2(B')$.

Choose $p_k \in \mathcal{P}(\pi_k)$ such that $p_k(A') > p_k(B')$. From (45), $\widetilde{\omega} \in s_2(p_2, p_k)$. But since s is passively $(2,k)$-consensual or actively $(2,k)$-consensual on $\mathcal{P}^N(\sigma_2, \pi_{-2})$, $\widetilde{\omega} \in s_k(q_2, p_k)$, contradicting the Local Bilaterality lemma.

Case 2. $\{A', B'\}$ does not cut $\mathcal{P}(\pi_2)$, that is, $\pi_2(\widetilde{A}') > \pi_2(B')$.

Fix a sub-profile $p_{-2} \in \mathcal{P}^{N\setminus 2}(\pi_{-2})$ such that $p_1(A) > p_1(B)$ and $p_k(A') > p_k(B')$ (where 1 and k may coincide). Drop this sub-profile from the notation.

We derive a contradiction using a variant of the argument in Lemma 8. Fix $\widehat{\omega} \in E \cup E'$. As we proved in Lemma 8, there exists a belief $\widehat{\pi}_2$ on $\Omega \setminus \widehat{\omega}$ such that $p_2^- \mid \widehat{\Omega} \approx q_2^- \mid \widehat{\Omega} \approx \widehat{\pi}_2$ and there exists $\alpha \in [0,1]$ such that ${}^{\alpha}q_2 := \alpha p_2^- + (1-\alpha) q_2^- \in \mathcal{P}(\sigma_2)$ and ${}^{\alpha}q_2(A') > {}^{\alpha}q_2(B')$.

Since $q_2^-(A') < q_2^-(B')$ and s is passively $(2,k)$-consensual or actively $(2,k)$-consensual on $\mathcal{P}^N(\sigma_2, \pi_{-2})$,

$$\begin{aligned} s_2({}^{0}q_2) &= s_2(q_2^-) = B' \cup C_2', \\ s_2({}^{\alpha}q_2) &= A' \cup C_2'. \end{aligned}$$

Since ${}^{\beta}q_2 \in \overline{\mathcal{P}(\widehat{\pi}_2)}$ for all $\beta \in [0,1]$, these equalities and Lemma 6 imply

$$s_2({}^{1}q_2) = s_2(p_2^-) = A' \cup C_2'.$$

But (45) implies $s_2(p_2^-) = B \cup C_2$, a contradiction.

Sub-step 3.2. Step 1, Sub-step 3.1, and Lemma 7 together imply that s is constant on $\mathcal{P}^N(\sigma_2, \pi_{-2})$. To complete the proof of statement *(b)*, we need to show that the constant assignment prescribed by s is $(B \cup C_1, A \cup C_2, C_3, ..., C_n)$.

Fix again $\widehat{\omega} \in E \cup E'$ and $\widehat{\pi}_2 \approx p_2^- \mid \widehat{\Omega} \approx q_2^- \mid \widehat{\Omega}$. Because $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, there exists $\alpha \in [0,1]$ such that ${}^{\alpha}q_2 := \alpha p_2^- + (1-\alpha) q_2^- \in \mathcal{P}(\pi_2)$ and ${}^{\alpha}q_2(A) > {}^{\alpha}q_2(B)$. Pick $\overline{p}_1 \in \mathcal{P}(\pi_1)$ such $\overline{p}_1(A) > \overline{p}_1(B)$. Fix p_{-12} and drop it from the notation. From (45),

$$
\begin{aligned}
s_2(\overline{p}_1, {}^1 q_2) &= s_2(\overline{p}_1, p_2^-) = B \cup C_2, \\
s_2(\overline{p}_1, {}^{\alpha} q_2) &= A \cup C_2.
\end{aligned}
$$

Since ${}^{\beta}q_2 \in \overline{\mathcal{P}(\widehat{\pi}_2)}$ for all $\beta \in [0,1]$, Lemma 6 implies

$$
s_2(\overline{p}_1, {}^0 q_2) = s_2(\overline{p}_1, q_2^-) = A \cup C_2,
$$

hence, since s is constant on $\mathcal{P}^N(\sigma_2, \pi_{-2})$, $s_2(p_1, q_2) = A \cup C_2$ for all $(p_1, q_2) \in \mathcal{P}(\pi_1) \times \mathcal{P}(\sigma_2)$. The claim now follows from non-bossiness. $\square$

The third main result of Appendix 2.C.2 describes how a local change in agent 1's beliefs affects the assignment of events A, B.

Second Contagion Lemma. *Let $\sigma_1 \in \widetilde{\mathcal{P}}$ be adjacent to π_1.*
(a) If s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ and $\{A, B\}$ cuts $\mathcal{P}(\sigma_1)$, then s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.
(b) If s is actively $(2,1)$-consensual or passively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ and $\{A, B\}$ does not cut $\mathcal{P}(\sigma_1)$, then s is passively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

Remark 4. *Statement (a) is not the permutation of statement (a) in the First Contagion lemma because the rule is assumed to be actively $(2,1)$-consensual in both cases.*

Proof. Fix $\sigma_1 \in \widetilde{\mathcal{P}}$ adjacent to π_1. For any $k \in N$, let p_k^+, p_k^- denote maximal and minimal elements of $\widetilde{J}$ in $\mathcal{P}(\pi_k)$, let q_1^+, q_1^- be maximal and minimal elements of $\widetilde{J}$ in $\mathcal{P}(\sigma_1)$, and let now E, E' denote the disjoint subsets of $\widetilde{\Omega}$ such that π_1 and σ_1 are $\{E, E'\}$-adjacent with $\pi_1(E) > \pi_1(E')$. Again, $\widetilde{\omega} \notin E \cup E'$.

Step 1. We show that if s is actively $(2,1)$-consensual or passively $(2,1)$-consensual on $\mathcal{P}^N(\pi)$, then for every $k \neq 2$ and $k' \neq k$, s is neither passively (k,k')-consensual nor actively (k,k')-consensual on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

Fix $k \neq 2$, $k' \neq k$. Fix a profile $p \in \mathcal{P}^N(\pi)$ such that $p_1 = p_1^+$, $p_2 = p_2^+$, and $p_k = p_k^+$ (where k may coincide with 1). Since s is actively $(2,1)$-consensual or passively $(2,1)$-consensual on $\mathcal{P}^N(\pi)$, we have $\widetilde{\omega} \in s_2(p)$. If s is passively (k,k')-consensual or actively (k,k')-consensual on $\mathcal{P}^N(\sigma_1, \pi_{-1})$, then $\widetilde{\omega} \in s_k(q_1^+, p_{-1})$. These two statements contradict the Local Bilaterality lemma because p_1^+, q_1^+ are $\{E, E'\}$-adjacent and $\widetilde{\omega} \notin E \cup E'$.

Step 2. We show that if s is actively $(2,1)$-consensual or passively $(2,1)$-consensual on $\mathcal{P}^N(\pi)$, then s it is not constant on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

Fix a sub-profile $p_{-12} \in \mathcal{P}^{N\setminus 12}(\pi_{-12})$ and drop it from the notation. If s is actively $(2,1)$-consensual or passively $(2,1)$-consensual on $\mathcal{P}^N(\pi)$, there exist disjoint sets A, B, C_2 such that $\widetilde{\omega} \in A$ and

$$\begin{aligned} s_2(p_1^+, p_2^+) &= A \cup C_2, \\ s_2(p_1^+, p_2^-) &= B \cup C_2 \end{aligned}$$

and the Local Bilaterality lemma implies

$$\begin{aligned} s_2(q_1^+, p_2^+) &= A \cup C_2 \text{ or } (A \cup C_2 \cup E) \setminus E', \\ s_2(q_1^+, p_2^-) &= B \cup C_2 \text{ or } (B \cup C_2 \cup E) \setminus E'. \end{aligned}$$

Hence, $\widetilde{\omega} \in s_2(q_1^+, p_2^+) \setminus s_2(q_1^+, p_2^-)$, proving that s is not constant on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

Step 3. We prove statement *(a)*.

Suppose s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with, say, residuals $C_1, ..., C_n$, and $\{A, B\}$ cuts $\mathcal{P}(\sigma_1)$. Fix $p_{-12} \in \mathcal{P}^{N\setminus 12}(\pi_{-12})$ and drop it from the notation. By assumption, (45) holds for all $(p_1, p_2) \in \mathcal{P}(\pi_1) \times \mathcal{P}(\pi_2)$ and $\sigma_1(\widetilde{A}) < \sigma_1(B)$.

Sub-step 3.1. We show that s varies with agent 1's beliefs on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

Because $\{A, B\}$ cuts $\mathcal{P}(\sigma_1)$, there exist adjacent beliefs $\overline{p}_1 \in \mathcal{P}(\pi_1)$ and $\overline{q}_1 \in \mathcal{P}(\sigma_1)$ such that $\overline{p}_1(A) < \overline{p}_1(B)$. These beliefs are, in fact, $\{E, E'\}$-adjacent.

Choose $p_2 \in \mathcal{P}(\pi_2)$ such that $p_2(A) < p_2(B)$. From (45), $s_2(p_1^+, p_2) = B \cup C_2$ and $s_2(\overline{p}_1, p_2) = A \cup C_2$. By the Local Bilaterality lemma,

$$\begin{aligned} s_2(q_1^+, p_2) &= B \cup C_2 \text{ or } (B \cup C_2 \cup E') \setminus E, \\ s_2(\overline{q}_1, p_2) &= A \cup C_2 \text{ or } (A \cup C_2 \cup E') \setminus E. \end{aligned}$$

It follows that $\widetilde{\omega} \in s_2(\overline{q}_1, p_2) \setminus s_2(q_1^+, p_2)$: s varies with agent 1's beliefs.

Sub-step 3.2. By Step 1, Sub-step 3.1, and Lemma 7, s is actively $(2,1)$-consensual on $\mathcal{P}^N(\sigma_1, \pi_{-1})$ with respect to some $\{A', B'\}$ and residuals $C'_1, ..., C'_n$. For all $(q_1, p_{-1}) \in \mathcal{P}^N(\sigma_1, \pi_{-1})$,

$$s(q_1, p_{-1}) = \begin{cases} (A' \cup C'_1, B' \cup C'_2, C'_3, ..., C'_n) & \text{if } q_1(A') > q_1(B') \text{ and } p_2(A') < p_2(B'), \\ (B' \cup C'_1, A' \cup C'_2, C'_3, ..., C'_n) & \text{otherwise,} \end{cases} \tag{57}$$

where $\widetilde{\omega} \in A'$ and $\{A', B'\}$ cuts $\mathcal{P}(\sigma_1), \mathcal{P}(\pi_2)$. It remains to prove that $\{A', B'\} = \{A, B\}$.

Fix $p_{-12} \in \mathcal{P}^{N\backslash 12}(\pi_{-12})$ and drop it from the notation. If $\{A', B'\} \neq \{A, B\}$, define the positive numbers

$$\begin{aligned} \delta &= \pi_2(B) - \pi_2(\widetilde{A}), \\ \delta' &= \pi_2(B') - \pi_2(\widetilde{A}') \end{aligned}$$

and assume without loss of generality $\delta \neq \delta'$.

If $\delta < \delta'$, there exists $p_2 \in \mathcal{P}(\pi_2)$ such that $p_2(A) > p_2(B)$ and $p_2(A') < p_2(B')$. From (45), $s_2(p_1^+, p_2) = A \cup C_2$ and from (57), $s_2(q_1^+, p_2) = B' \cup C_2'$, contradicting the Local Bilaterality lemma.

If $\delta' < \delta$, there exists $p_2 \in \mathcal{P}(\pi_2)$ such that $p_2(A) < p_2(B)$ and $p_2(A') > p_2(B')$. From (45), $s_2(p_1^+, p_2) = B \cup C_2$ and from (57), $s_2(q_1^+, p_2) = A' \cup C_2'$, contradicting the Local Bilaterality lemma again.

Step 4. We prove statement *(b)*.

Sub-step 4.1. Suppose first that s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ and $\{A, B\}$ does not cut $\mathcal{P}(\sigma_1)$.

By Steps 1, 2, and Lemmas 7 and 8, s is passively $(2,1)$-consensual on $\mathcal{P}^N(\sigma_1, \pi_{-1})$ with respect to some $\{A', B'\}$ and residuals $C_1', ..., C_n'$. For all $(q_1, p_{-1}) \in \mathcal{P}^N(\sigma_1, \pi_{-1})$,

$$s(q_1, p_{-1}) = \begin{cases} (A' \cup C_1', B' \cup C_2', C_3', ..., C_n') & \text{if } p_2(B') > p_2(A'), \\ (B' \cup C_1', A' \cup C_2', C_3', ..., C_n') & \text{otherwise,} \end{cases} \tag{58}$$

where $\widetilde{\omega} \in A'$ and $\{A', B'\}$ cuts $\mathcal{P}(\sigma_1)$. It remains to prove that $\{A', B'\} = \{A, B\}$.

If $\{A', B'\} \neq \{A, B\}$, consider again the numbers δ, δ' defined in Sub-step 3.2 and assume without loss of generality $\delta \neq \delta'$. Note that δ' may now be negative as $\{A', B'\}$ need no longer cut $\mathcal{P}(\pi_2)$. This, however, does not affect the rest of the argument: combining (45) with (58) rather than (57) delivers the same contradiction to the Local Bilaterality lemma.

Sub-step 4.2. Suppose next that s is passively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ and $\{A, B\}$ does not cut $\mathcal{P}(\sigma_1)$.

By Steps 1, 2, and Lemma 7, s is either actively $(2,1)$-consensual or passively $(2,1)$-consensual on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

If s is actively $(2,1)$-consensual on $\mathcal{P}^N(\sigma_1, \pi_{-1})$, it must be with respect to some $\{A', B'\} \neq \{A, B\}$ since $\{A, B\}$ does not cut $\mathcal{P}(\sigma_1)$.

Suppose first that $\{A', B'\}$ does not cut $\mathcal{P}(\pi_1)$: exchanging the roles of $\{A, B\}$, $\{A', B'\}$ and π_1, σ_1 in the argument in Sub-step 4.1 leads to the conclusion that s is passively $(2,1)$-consensual with respect to $\{A', B'\}$ on $\mathcal{P}^N(\pi)$, contradicting the assumption of the current sub-step.

Suppose next that $\{A', B'\}$ cuts $\mathcal{P}(\pi_1)$: exchanging the roles of $\{A, B\}, \{A', B'\}$ and π_1, σ_1 in statement *(a)* leads to the conclusion that s is actively $(2,1)$-consensual with respect to $\{A', B'\}$ on $\mathcal{P}^N(\pi)$, again a contradiction.

We conclude that s is passively $(2,1)$-consensual on $\mathcal{P}^N(\sigma_1, \pi_{-1})$. The proof that it must in fact be passively $(2,1)$-consensual with respect to $\{A, B\}$ proceeds in the same way as in Sub-step 4.1. □

Appendix 2.C.3: Global Contagion Results

As corollaries to the local contagion results of Appendix 2.C.2, we will now prove two results linking the behavior of s across regions that need not be adjacent. Our first result describes the effect of a change in agent 2' beliefs.

First Contagion Corollary. *Let $\sigma_2 \in \widetilde{\mathcal{P}}$, and suppose s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$.*
(a) If $\{A, B\}$ cuts $\mathcal{P}(\sigma_2)$, then s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_2, \pi_{-2})$.
(b) If $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, then there exists a partition $\{C_1', ..., C_n'\}$ of $\Omega \setminus (A \cup B)$ such that $s(p) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$ for all $p \in \mathcal{P}^N(\sigma_2, \pi_{-2})$.

Proof. Let $\sigma_2 \in \widetilde{\mathcal{P}}$, and suppose s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$. Define

$$\begin{aligned} \widetilde{\mathcal{P}}_+ &= \{\sigma_2 \in \widetilde{\mathcal{P}} : \sigma_2(\widetilde{A}) < \sigma_2(B)\}, \\ \widetilde{\mathcal{P}}_- &= \{\sigma_2 \in \widetilde{\mathcal{P}} : \sigma_2(\widetilde{A}) > \sigma_2(B)\}. \end{aligned}$$

These sets partition $\widetilde{\mathcal{P}}$: $\sigma_2 \in \widetilde{\mathcal{P}}_+$ if and only if $\{A, B\}$ cuts $\mathcal{P}(\sigma_2)$. Clearly, $\widetilde{\mathcal{P}}_+$ and $\widetilde{\mathcal{P}}_-$ are connected: any two beliefs in one set are linked by a J-path of adjacent beliefs in that set. Since s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$, we have $\pi_2 \in \widetilde{\mathcal{P}}_+$.

Step 1. We prove statement *(a)*.

Let $\sigma_2 \in \widetilde{\mathcal{P}}_+$. Let $(\sigma_2^t)_{t=1}^T$ be a J-path in $\widetilde{\mathcal{P}}_+$ with $\sigma_2^1 = \pi_2$ and $\sigma_2^T = \sigma_2$. Since s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_2^1, \pi_{-2})$, repeated application of statement *(a)* in the First Contagion lemma implies that s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_2^T, \pi_{-2}) = \mathcal{P}^N(\sigma_2, \pi_{-2})$.

Step 2. We prove statement *(b)*.

Call two distinct events $C, D \subseteq \widetilde{\Omega}$ *adjacent in* $\sigma_2 \in \widetilde{\mathcal{P}}$ if $(\sigma_2(C) - \sigma_2(E))(\sigma_2(D) - \sigma_2(E)) > 0$ for all $E \subseteq \widetilde{\Omega}$ different from C, D. Define

$$\begin{aligned} \widetilde{\mathcal{P}}^* &= \{\sigma_2 \in \widetilde{\mathcal{P}} : \widetilde{A}, B \text{ are adjacent in } \sigma_2\}, \\ \widetilde{\mathcal{P}}^*_+ &= \widetilde{\mathcal{P}}_+ \cap \widetilde{\mathcal{P}}^*, \\ \widetilde{\mathcal{P}}^*_- &= \widetilde{\mathcal{P}}_- \cap \widetilde{\mathcal{P}}^*. \end{aligned}$$

We will first prove that statement *(b)* holds if $\sigma_2 \in \widetilde{\mathcal{P}}^*_-$, then show that it holds for all $\sigma_2 \in \widetilde{\mathcal{P}}_-$. The argument is illustrated in Figure 4.

Sub-step 2.1. If $\sigma_2 \in \widetilde{\mathcal{P}}^*_-$, then σ_2 is $\{\widetilde{A}, B\}$-adjacent to some belief $\sigma_2' \in \widetilde{\mathcal{P}}^*_+$. By statement *(a)*, s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_2', \pi_{-2})$. Statement *(b)* now follows from statement *(b)* in the First Contagion lemma.

Sub-step 2.2. If $\sigma_2 \in \widetilde{\mathcal{P}}_- \setminus \widetilde{\mathcal{P}}^*_-$, recall first that, since $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, we have $\sigma_2(\widetilde{A}) > \sigma_2(B)$. Fix $p = (p_2, p_{-2}) \in \mathcal{P}^N(\sigma_2, \pi_{-2})$. Consider, for each $\alpha \in (0,1)$, the probability measure σ_2^α defined over the subsets of $\widetilde{\Omega}$ by

$$\sigma_2^\alpha(E) = \alpha \frac{\sigma_2\left(E \cap \widetilde{A}\right)}{\sigma_2\left(\widetilde{A}\right)} + (1-\alpha)\frac{\sigma_2\left(E \cap \overline{\widetilde{A}}\right)}{\sigma_2\left(\overline{\widetilde{A}}\right)} \text{ for all } E \subseteq \widetilde{\Omega}, \tag{59}$$

where $\overline{\widetilde{A}} := \widetilde{\Omega} \setminus \widetilde{A}$. Each σ_2^α is a variant of the belief σ_2 where the probability of the states in $\widetilde{A}$ relative to those outside $\widetilde{A}$ is modified, but the conditional beliefs on the subsets of $\widetilde{A}$, as well as on the subsets of $\overline{\widetilde{A}}$, are kept unchanged. If $\alpha = \sigma_2(\widetilde{A})$, then σ_2^α coincides with σ_2. If $\alpha = \frac{\sigma_2(B)}{1+\sigma_2(B)-\sigma_2(\widetilde{A})}$, then $\sigma_2^\alpha(\widetilde{A}) = \sigma_2^\alpha(B)$. This means that if α is sufficiently close to $\frac{\sigma_2(B)}{1+\sigma_2(B)-\sigma_2(\widetilde{A})}$, the belief σ_2^α belongs to $\widetilde{\mathcal{P}}^*_-$. Elementary algebra shows that $\sigma_2(\widetilde{A}) > \frac{\sigma_2(B)}{1+\sigma_2(B)-\sigma_2(\widetilde{A})}$.

Write $p_2(\widetilde{\omega}) = \gamma$ and define, for each $\alpha \in (0,1)$, the measure p_2^α over the subsets of Ω by

$$p_2^\alpha(E) = \gamma \ \mathit{1}(E \cap \{\widetilde{\omega}\}) + (1-\gamma)\ \sigma_2^\alpha\left(E \cap \widetilde{\Omega}\right) \text{ for all } E \subseteq \Omega, \tag{60}$$

where $\mathit{1}(E \cap \{\widetilde{\omega}\}) = 1$ if $\widetilde{\omega} \in E$ and 0 otherwise.

Choose an increasing sequence of numbers $\alpha(1), ..., \alpha(T)$ in $(0,1)$ such that (i) $\sigma_2^{\alpha(t)}$ is adjacent to $\sigma_2^{\alpha(t+1)}$ for all $t = 1, ..., T-1$, (ii) $\sigma_2^{\alpha(1)} \in \widetilde{\mathcal{P}}^*_-$, and (iii) $\sigma_2^{\alpha(T)} = \sigma_2$. Define the J-path $(\sigma_2^t)_{t=1}^T$ in $\widetilde{\mathcal{P}}_-$ by $\sigma_2^t = \sigma_2^{\alpha(t)}$ for $t = 1, ..., T$. Define the associated finite sequence $(\mathbf{p}_2^t)_{t=1}^T$ in $\mathcal{P}$ by $\mathbf{p}_2^t = p_2^{\alpha(t)}$ for $t = 1, ..., T$. Observe that $\mathbf{p}_2^T = p_2$ and $\mathbf{p}_2^t \in \mathcal{P}(\sigma_2^t)$ for each t, but $\mathbf{p}_2^t, \mathbf{p}_2^{t+1}$ need not be adjacent. Finally, for each $t = 1, ..., T$, let $\mathbf{y}_2^t$ be a maximal element of $\widetilde{J}$ in $\mathcal{P}(\sigma_2^t)$. Observe that $\mathbf{y}_2^t, \mathbf{y}_2^{t+1}$ *are* adjacent and write $\mathbf{y}_2^T = y_2$.

Since $y_2^1 \in \mathcal{P}(\sigma_2^1)$ and $\sigma_2^1 \in \widetilde{\mathcal{P}}^*_-$, Sub-step 2.1 implies that there exists a partition $\{C_1', ..., C_n'\}$ of $\Omega \setminus (A \cup B)$ such that $s(\mathbf{y}_2^1, p_{-2}) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$. We will show that $s(p) = s(p_2, p_{-2}) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$. By non-bossiness, it suffices to prove $s_2(p) = A \cup C_2'$.

We have

$$s_2(\mathbf{y}_2^1, p_{-2}) = A \cup C_2'.$$

Proceeding now by induction, fix $t \in \{1, ..., T-1\}$ and suppose that

$$s_2(\mathbf{y}_2^t, p_{-2}) = A \cup C_2'.$$

Let $\{E^t, E^{t+1}\} \in \mathcal{H}(\widetilde{\Omega})$ be the pair of disjoint events such that $\sigma_2^t, \sigma_2^{t+1}$ are $\{E^t, E^{t+1}\}$-adjacent with $\sigma_2^t(E^t) > \sigma_2^t(E^{t+1})$. Because $\sigma_2^t, \sigma_2^{t+1}$ coincide on $\widetilde{A}$ as well as on $\overline{\widetilde{A}}$,

$$E^t \cap \overline{\widetilde{A}} \neq \emptyset \text{ and } E^{t+1} \cap \widetilde{A} \neq \emptyset.$$

If $s_2(\mathbf{y}_2^{t+1}, p_{-2}) \neq s_2(\mathbf{y}_2^t, p_{-2})$, the Local Bilaterality lemma implies $s_2(\mathbf{y}_2^{t+1}, p_{-2}) \setminus s_2(\mathbf{y}_2^t, p_{-2}) = E^{t+1}$. Since $A \subseteq s_2(\mathbf{y}_2^t, p_{-2})$, we conclude $E^{t+1} \cap \widetilde{A} = \emptyset$, a contradiction. Therefore $s_2(\mathbf{y}_2^{t+1}, p_{-2}) = A \cup C_2'$, and finally

$$s_2(y_2, p_{-2}) = A \cup C_2'. \tag{61}$$

Next, we claim that

$$s_2(p) = s_2(p_2, p_{-2}) = A \cup C_2'.$$

First, observe that since $\mathbf{p}_2^1 \in \mathcal{P}(\sigma_2^1)$ and $\sigma_2^1 \in \widetilde{\mathcal{P}}_-^*$, we have

$$s_2(\mathbf{p}_2^1, p_{-2}) = A \cup C_2'$$

Next, suppose, by way of contradiction, that $s_2(p_2, p_{-2}) = D \neq A \cup C_2'$. By Lemma 7, $\widetilde{\omega} \notin D$.

Case 1. $\frac{\sigma_2(C_2' \setminus D)}{\sigma_2\left(\overline{\widetilde{A}}\right)} < \frac{\sigma_2\left(\widetilde{A} \setminus D\right)}{\sigma_2\left(\widetilde{A}\right)}$.

By strategyproofness, $p_2(s_2(p_2, p_{-2})) > p_2(s_2(y_2, p_{-2}))$, hence by (61), $\mathbf{p}_2^T(D) > \mathbf{p}_2^T(A \cup C_2')$. Given (60), this means

$$\frac{\sigma_2^T\left(\widetilde{A} \cup C_2'\right) - \sigma_2^T(D)}{1 + \sigma_2^T(D) - \sigma_2^T\left(\widetilde{A} \cup C_2'\right)} < -\gamma. \tag{62}$$

From (59),

$$\sigma_2^T(\widetilde{A} \cup C_2') - \sigma_2^T(D)) = \alpha(T)\left(\frac{\sigma_2\left(\widetilde{A} \setminus D\right)}{\sigma_2\left(\widetilde{A}\right)}\right) + (1 - \alpha(T))\left(\frac{\sigma_2(C_2') - \sigma_2\left(D \cap \overline{\widetilde{A}}\right)}{\sigma_2\left(\overline{\widetilde{A}}\right)}\right).$$

By assumption of Case 1, the second term of this convex combination is smaller than the first. Since $\alpha(1) < \alpha(T)$, it follows that $\sigma_2^1(\widetilde{A} \cup C_2') - \sigma_2^1(D) < \sigma_2^T(\widetilde{A} \cup C_2') - \sigma_2^T(D)$, hence from (62),

$$\frac{\sigma_2^1\left(\widetilde{A} \cup C_2'\right) - \sigma_2^1(D)}{1 + \sigma_2^1(D) - \sigma_2^1\left(\widetilde{A} \cup C_2'\right)} < -\gamma,$$

which, given (60), implies $\mathbf{p}_2^1(D) > \mathbf{p}_2^1(A \cup C_2')$, that is, $\mathbf{p}_2^1(s_2(q_2, p_{-2})) > \mathbf{p}_2^1(s_2(\mathbf{p}_2^1, p_{-2}))$, contradicting strategyproofness.

Case 2. $\frac{\sigma_2(C_2' \setminus D)}{\sigma_2\left(\overline{\widetilde{A}}\right)} \geq \frac{\sigma_2\left(\widetilde{A} \setminus D\right)}{\sigma_2\left(\widetilde{A}\right)}$.

Define $\overline{C_2'} := \widetilde{\Omega} \setminus C_2'$. Because $\sigma_2(C_2') < \sigma_2\left(\overline{\widetilde{A}}\right)$ and $\sigma_2\left(\widetilde{A}\right) < \sigma_2\left(\overline{C_2'}\right)$,

$$\frac{\sigma_2\left(\widetilde{A} \setminus D\right)}{\sigma_2\left(\overline{C_2'}\right)} < \frac{\sigma_2(C_2' \setminus D)}{\sigma_2(C_2')}.$$

Notice that this is the very same inequality as the one defining Case 1 –except that the roles of C_2' and $\widetilde{A}$ have been exchanged.

For each $\alpha \in (0,1)$, define the probability measure τ_2^α over the subsets of $\widetilde{\Omega}$ by

$$\tau_2^\alpha(E) = \alpha \frac{\sigma_2(E \cap C_2')}{\sigma_2(C_2')} + (1-\alpha) \frac{\sigma_2\left(E \cap \overline{C_2'}\right)}{\sigma_2\left(\overline{C_2'}\right)} \text{ for all } E \subseteq \widetilde{\Omega}$$

and the measure r_2^α over the subsets of Ω by

$$r_2^\alpha(E) = \gamma \; \mathit{1}(E \cap \{\widetilde{\omega}\}) + (1-\gamma) \; \tau_2^\alpha\left(E \cap \widetilde{\Omega}\right) \text{ for all } E \subseteq \Omega.$$

These constructions are the same as in (59) and (60), except that C_2' plays the role of $\widetilde{A}$.

Choose an increasing sequence $\alpha(1), ..., \alpha(T)$ in $(0,1)$ such that (i) $\tau_2^{\alpha(t)}$ is adjacent to $\tau_2^{\alpha(t+1)}$ for all t, (ii) $\tau_2^{\alpha(1)} \in \widetilde{\mathcal{P}}_-^*$, and (iii) $\tau_2^{\alpha(T)} = \sigma_2$. Define the path $(\tau_2^t)_{t=1}^T$ in $\widetilde{\mathcal{P}}_-$ by $\tau_2^t = \tau_2^{\alpha(t)}$ for all t, and define the sequence $(\mathbf{r}_2^t)_{t=1}^T$ in $\mathcal{P}$ by $\mathbf{r}_2^t = r_2^{\alpha(t)}$ for all t. Finally, for each t, let $\mathbf{z}_2^t$ be a maximal element of $\widetilde{J}$ in $\mathcal{P}(\tau_2^t)$ and let $\mathbf{z}_2^T = z_2$.

Since $\tau_2^1 \in \widetilde{\mathcal{P}}_-^*$, Sub-step 2.1 implies that there exists a partition $\{C_1'', ..., C_n''\}$ of $\Omega \setminus (A \cup B)$ such that $s(\mathbf{z}_2^1, p_{-2}) = (B \cup C_1'', A \cup C_2'', C_3'', ..., C_n'')$. In particular,

$$s_2(\mathbf{z}_2^1, p_{-2}) = A \cup C_2''.$$

By the same inductive argument as in Case 1, we obtain

$$s_2(z_2, p_{-2}) = A \cup C_2''.$$

But since both z_2 and y_2 are maximal elements of $\widetilde{J}$ in $\mathcal{P}(\sigma_2)$, we have $s_2(z_2, p_{-2}) = s_2(y_2, p_{-2})$, hence (61) implies

$$s_2(z_2, p_{-2}) = A \cup C_2'.$$

The proof that $s_2(p_2, p_{-2}) = A \cup C_2'$ now follows by the same argument as in Case 1, provided that we exchange the roles of $\widetilde{A}$ and C_2'. $\square$

The second result of this Appendix 2.C.3 describes the effect of a change in agent 1' beliefs.

Second Contagion Corollary. *Let $\sigma_1 \in \widetilde{\mathcal{P}}$, and suppose s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$.*
(a) If $\{A, B\}$ cuts $\mathcal{P}(\sigma_1)$, then s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.
(b) If $\{A, B\}$ does not cut $\mathcal{P}(\sigma_1)$, then s is passively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

Proof. Let $\sigma_1 \in \widetilde{\mathcal{P}}$, and let s be actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$. Define $\widetilde{\mathcal{P}}_+, \widetilde{\mathcal{P}}_-, \widetilde{\mathcal{P}}_+^*, \widetilde{\mathcal{P}}_-^*$ as in the proof of the previous corollary. By assumption, $\pi_1 \in \widetilde{\mathcal{P}}_+$. The argument below is illustrated in Figure 5.

Step 1. To prove statement *(a)*, let $\sigma_1 \in \widetilde{\mathcal{P}}_+$ and let $(\sigma_1^t)_{t=1}^T$ be a J-path in $\widetilde{\mathcal{P}}_+$ with $\sigma_1^1 = \pi_1$ and $\sigma_1^T = \sigma_1$. Since s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1^1, \pi_{-1})$, repeated application of statement *(a)* in the Second Contagion lemma implies that s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1^T, \pi_{-1}) = \mathcal{P}^N(\sigma_1, \pi_{-1})$.

Step 2. To prove statement *(b)*, we proceed again in two stages.

If $\sigma_1 \in \widetilde{\mathcal{P}}_-^*$, there exists a belief $\sigma_1' \in \widetilde{\mathcal{P}}_+^*$ to which σ_1 is $\{\widetilde{A}, B\}$-adjacent. By Step 1, s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1', \pi_{-1})$. By statement *(b)* in the Second Contagion lemma, it follows that s is passively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

If $\sigma_1 \in \widetilde{\mathcal{P}}_- \setminus \widetilde{\mathcal{P}}_-^*$, let $(\sigma_1^t)_{t=1}^T$ be a J-path in $\widetilde{\mathcal{P}}_-$ with $\sigma_1^1 \in \widetilde{\mathcal{P}}_-^*$ and $\sigma_1^T = \sigma_1$. Since s is passively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1^1, \pi_{-1})$, repeated application of statement *(b)* in the Second Contagion lemma implies that s is passively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1^T, \pi_{-1}) = \mathcal{P}^N(\sigma^1, \pi_{-1})$. $\square$

Appendix 2.C.4: Conclusion of the Proof of the Bilateral Consensus Lemma

We are finally ready to conclude the proof of the Bilateral Consensus lemma. We must show that there exist an event $E^{\widetilde{\omega}} \subseteq \Omega_2$ such that $\widetilde{\omega} \in E^{\widetilde{\omega}}$, and a bilaterally consensual $E^{\widetilde{\omega}}$-assignment rule $s^{\widetilde{\omega}}$ such that

$$s_i(p) \cap E^{\widetilde{\omega}} = s_i^{\widetilde{\omega}}(p \mid E^{\widetilde{\omega}}) \text{ for all } i \in N \tag{63}$$

and all $p \in \mathcal{P}^N$.

Recall the definition of $a_{\widetilde{\omega}}$ in (25). Throughout Appendix 2.C.4, we will use the shorthand notation $\widetilde{a} = a_{\widetilde{\omega}}$. Thus, $\widetilde{a}(p)$ is the agent to whom state $\widetilde{\omega}$ is assigned when the belief profile is p.

Step 1. There exist $\pi^0 \in \widetilde{\mathcal{P}}^N$, two distinct agents $i, j \in N$, $p, q \in \mathcal{P}^N(\pi^0)$, and $p'_i \in \mathcal{P}(\pi^0_i), q'_j \in \mathcal{P}(\pi^0_j)$ such that $\widetilde{a}(p) \neq \widetilde{a}(p'_i, p_{-i})$ and $\widetilde{a}(q) \neq \widetilde{a}(q'_j, q_{-j})$.

By definition of Ω_2, there exist two agents, say $1, 2$, profiles $p, q \in \mathcal{P}^N$, and beliefs $p'_1, q'_2 \in \mathcal{P}$ such that

$$\widetilde{a}(p) \neq \widetilde{a}(p'_1, p_{-1}) \text{ and } \widetilde{a}(q) \neq \widetilde{a}(q'_2, q_{-2}). \tag{64}$$

Because $\mathcal{P}$ is connected, we assume without loss of generality that p_1, p'_1 are adjacent and q_2, q'_2 are adjacent. Let $\{E, E'\}$ be the pair of events such that p_1, p'_1 are $\{E, E'\}$-adjacent. By the Local Bilaterality lemma and the first inequality in (64), $\widetilde{\omega} \in E \cup E'$, hence, $(p_1(C) - p_1(D))(p'_1(C) - p'_1(D)) > 0$ for all distinct $C, D \subseteq \widetilde{\Omega}$. This means that there exists $\pi^0_1 \in \widetilde{\mathcal{P}}$ such that $p_1 \mid \widetilde{\Omega} \approx p'_1 \mid \widetilde{\Omega} \approx \pi^0_1$, that is, $p_1, p'_1 \in \mathcal{P}(\pi^0_1)$. By the same token, there exists $\pi^0_2 \in \widetilde{\mathcal{P}}$ such that $p_2, p'_2 \in \mathcal{P}(\pi^0_2)$.

To keep notation simple, suppose $n = 3$; the argument is easily extended to any number of agents. Suppose first that $p_3 = q_3$. Dropping that belief from the notation, (64) reads

$$\widetilde{a}(p_1, p_2) \neq \widetilde{a}(p'_1, p_2) \text{ and } \widetilde{a}(q_1, q_2) \neq \widetilde{a}(q_1, q'_2).$$

Case 1. $\widetilde{a}(p'_1, q_2) \neq \widetilde{a}(p_1, q_2) \neq \widetilde{a}(p_1, q'_2)$. In this case the claim is trivially true.

Case 2. (i) $\widetilde{a}(p_1, q_2) = \widetilde{a}(p'_1, q_2)$ or (ii) $\widetilde{a}(p_1, q_2) = \widetilde{a}(p_1, q'_2)$.

Assume (i); the argument is the same, up to a relabeling, if (ii) holds. Let $(\mathbf{p}^t_2)^T_{t=1}$ be a J-path between $\mathbf{p}^1_2 = p_2$ and $\mathbf{p}^T_2 = q_2$. From (64) and (i), there exists an integer t such that

$$\widetilde{a}(p_1, \mathbf{p}^t_2) \neq \widetilde{a}(p'_1, \mathbf{p}^t_2) \text{ and } \widetilde{a}(p_1, \mathbf{p}^{t+1}_2) = \widetilde{a}(p'_1, \mathbf{p}^{t+1}_2) \tag{65}$$

Using the Local Bilaterality lemma, the same argument as before shows that there exists π^t_2 such that $\mathbf{p}^t_2 \mid \widetilde{\Omega} \approx \mathbf{p}^{t+1}_2 \mid \widetilde{\Omega} \approx \pi^t_2$, that is, $\mathbf{p}^t_2, \mathbf{p}^{t+1}_2 \in \mathcal{P}(\pi^t_2)$. Moreover, statement (65) implies

$$\widetilde{a}(p'_1, \mathbf{p}^t_2) \neq \widetilde{a}(p_1, \mathbf{p}^t_2) \neq \widetilde{a}(p_1, \mathbf{p}^{t+1}_2)$$

or

$$\widetilde{a}(p_1, \mathbf{p}^{t+1}_2) \neq \widetilde{a}(p'_1, \mathbf{p}^{t+1}_2) \neq \widetilde{a}(p'_1, \mathbf{p}^t_2).$$

In either case the claim is true.

Finally, let us drop the assumption that $p_3 = q_3$. Suppose that there exist $p_3 \neq q_3$ such that

$$\widetilde{a}(p_1, p_2, p_3) \neq \widetilde{a}(p_1', p_2, p_3) \text{ and } \widetilde{a}(q_1, q_2, q_3) \neq \widetilde{a}(q_1, q_2', q_3).$$

and

$$\widetilde{a}(p_1, p_2, q_3) = \widetilde{a}(p_1', p_2, q_3) \text{ and } \widetilde{a}(q_1, q_2, p_3) = \widetilde{a}(q_1, q_2', p_3).$$

Let $(\mathbf{p}_3^t)_{t=1}^T$ be a J-path between $\mathbf{p}_3^1 = p_3$ and $\mathbf{p}_3^T = q_3$. There exists an integer t such that

$$\widetilde{a}(p_1, p_2, \mathbf{p}_3^t) \neq \widetilde{a}(p_1', p_2, \mathbf{p}_3^t) \text{ and } \widetilde{a}(p_1, p_2, \mathbf{p}_3^{t+1}) = \widetilde{a}(p_1', p_2, \mathbf{p}_3^{t+1}). \tag{66}$$

By the Local Bilaterality lemma again, there exists π_3^0 such that $\mathbf{p}_3^t \mid \widetilde{\Omega} \approx \mathbf{p}_3^{t+1} \mid \widetilde{\Omega}$ $\approx \pi_3^0$, that is, $\mathbf{p}_3^t, \mathbf{p}_3^{t+1} \in \mathcal{P}(\pi_3^0)$. Moreover, statement (66) implies

$$\widetilde{a}(p_1, p_2, \mathbf{p}_3^t) \neq \widetilde{a}(p_1', p_2, \mathbf{p}_3^t) \neq \widetilde{a}(p_1', p_2, \mathbf{p}_3^{t+1})$$

or

$$\widetilde{a}(p_1, p_2, \mathbf{p}_3^{t+1}) \neq \widetilde{a}(p_1, p_2, \mathbf{p}_3^t) \neq \widetilde{a}(p_1', p_2, \mathbf{p}_3^t).$$

In either case the claim is again true.

Step 2. Step 1 has established that there is some $\pi^0 \in \widetilde{\mathcal{P}}^N$ such that s varies with the beliefs of two distinct agents, say 1 and 2, on $\mathcal{P}^N(\pi^0)$. By statement *(b)* in Lemma 7 (and Remark 2), we may assume without loss of generality that s is actively $(2,1)$-consensual on $\mathcal{P}^N(\pi^0)$: there exists a partition $\{A, B, C_1, ..., C_n\}$ of Ω such that $\widetilde{\omega} \in A$, $\{A, B\}$ cuts $\mathcal{P}(\pi_1^0)$, $\mathcal{P}(\pi_2^0)$, and for all $p \in \mathcal{P}^N(\pi^0)$,

$$s(p) = \begin{cases} (A \cup C_1, B \cup C_2, C_3, ..., C_n) & \text{if } p_1(A) > p_1(B) \text{ and } p_2(A) < p_2(B), \\ (B \cup C_1, A \cup C_2, C_3, ..., C_n) & \text{otherwise.} \end{cases} \tag{67}$$

Define $E^{\widetilde{\omega}} := A \cup B$ and define the bilaterally consensual $E^{\widetilde{\omega}}$-assignment rule $s^{\widetilde{\omega}}$ as follows: for all $\widetilde{p} \in \mathcal{P}(E^{\widetilde{\omega}})^N$,

$$s^{\widetilde{\omega}}(\widetilde{p}) = \begin{cases} (A, B, \emptyset, ..., \emptyset) & \text{if } \widetilde{p}_1(A) > \widetilde{p}_1(B) \text{ and } \widetilde{p}_2(A) < \widetilde{p}_2(B), \\ (B, A, \emptyset, ..., \emptyset) & \text{otherwise.} \end{cases}$$

We claim that (63) holds for all $p \in \mathcal{P}^N$.

By definition, statement (63) is true for all $p \in \mathcal{P}^N(\pi^0)$. Next, fix an arbitrary sub-profile $\pi_{-12} \in \widetilde{\mathcal{P}}^{N\setminus 12}$.

Sub-step 2.1. By repeated application of the Independence lemma, s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi_1^0, \pi_2^0, \pi_{-12})$, hence, (63) is true for all $p \in \mathcal{P}^N(\pi_1^0, \pi_2^0, \pi_{-12})$.

Sub-step 2.2. For any profile $(\pi_1, \pi_2) \in \widetilde{\mathcal{P}}_+ \times \widetilde{\mathcal{P}}_+$, combining Sub-step 2.1 with part *(a)* of the First Contagion Corollary and part *(a)* of the Second Contagion Corollary shows that s is actively $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi_1, \pi_2, \pi_{-12})$, hence, (63) is true for all $p \in \mathcal{P}^N(\pi_1, \pi_2, \pi_{-12})$.

Sub-step 2.3. For any profile $(\pi_1, \sigma_2) \in \widetilde{\mathcal{P}}_+ \times \widetilde{\mathcal{P}}_-$, Sub-step 2.2 and part *(b)* of the First Contagion Corollary imply that there is a partition $\{C_1', ..., C_n'\}$ of $\Omega \setminus (A \cup B)$ such that $s(p) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$ for all $p \in \mathcal{P}^N(\pi_1, \sigma_2, \pi_{-12})$. Since $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, we have $p_2(A) > p_2(B)$ for all $p_2 \in \mathcal{P}(\sigma_2)$, hence (63) is true for all $p \in \mathcal{P}^N(\pi_1, \sigma_2, \pi_{-12})$.

Sub-step 2.4. For any profile $(\sigma_1, \pi_2) \in \widetilde{\mathcal{P}}_- \times \widetilde{\mathcal{P}}_+$, Sub-step 2.2 and part *(b)* of the Second Contagion Corollary imply that s is passively $(2,1)$-consensual on $\mathcal{P}^N(\sigma_1, \pi_2, \pi_{-12})$. Since $\{A, B\}$ does not cut $\mathcal{P}(\sigma_1)$, we have $p_1(A) > p_1(B)$ for all $p_1 \in \mathcal{P}(\sigma_1)$, hence (63) is true for all $p \in \mathcal{P}^N(\sigma_1, \pi_2, \pi_{-12})$.

Sub-step 2.5. Consider finally a profile $(\sigma_1, \sigma_2) \in \widetilde{\mathcal{P}}_- \times \widetilde{\mathcal{P}}_-$. By definition, $\sigma_2(\widetilde{A}) > \sigma_2(B)$. For each $\alpha \in (0,1)$, consider again the measure ${}_\alpha\sigma_2$ defined on $\widetilde{\Omega}$ by (59). Recall that ${}_\alpha\sigma_2$ coincides with σ_2 for $\alpha = \sigma_2(\widetilde{A})$ and observe that ${}_\alpha\sigma_2 \in \widetilde{\mathcal{P}}_+$ for any generic $\alpha < \frac{\sigma_2(B)}{1+\sigma_2(B)-\sigma_2(\widetilde{A})}$.

Choose an increasing sequence of numbers $\alpha(1), ..., \alpha(T)$ such that (i) ${}_{\alpha(t)}\sigma_2$ is adjacent to ${}_{\alpha(t+1)}\sigma_2$ for all $t = 1, ..., T-1$, (ii) ${}_{\alpha(1)}\sigma_2 \in \widetilde{\mathcal{P}}_+$, and (iii) ${}_{\alpha(T)}\sigma_2 = \sigma_2$. Consider the J-path $(\sigma_2^t)_{t=1}^T$ in $\widetilde{\mathcal{P}}_-$ defined by $\sigma_2^t =_{\alpha(t)} \sigma_2$ for $t = 1, ..., T$.

Since $\sigma_2^1 \in \widetilde{\mathcal{P}}_+$, Sub-step 2.3 implies that there exists a partition $\{C_1', ..., C_n'\}$ of $\Omega \setminus (A \cup B)$ such that $s(p) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$ for all $p \in \mathcal{P}^N(\sigma_1, \sigma_2^1, \pi_{-12})$. The same argument as in Sub-step 2.2 of the proof of the First Contagion Corollary then establishes that $s(p) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$ for all $p \in \mathcal{P}^N(\sigma_1, \sigma_2^T, \pi_{-12}) = \mathcal{P}^N(\sigma_1, \sigma_2, \pi_{-12})$.

Since $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, we have $p_2(A) > p_2(B)$ for all $p_2 \in \mathcal{P}(\sigma_2)$, hence (63) is true for all $p \in \mathcal{P}^N(\sigma_1, \sigma_2, \pi_{-12})$.

Given that $\mathcal{P} = \cup_{\pi_i \in \widetilde{\mathcal{P}}} \mathcal{P}(\pi_i)$, the proof of the Bilateral Consensus lemma is complete. □

Appendix 2.D: Proof of the Bilateral Dictatorship Lemma and Conclusion

In this appendix we turn to the assignment of the states in Ω_1 and we complete the proof of Theorem 1. Let Ω_{11} be the subset of those states in Ω_1 whose assignment varies with the beliefs of agent 1. We show that these states are assigned by bilateral dictatorship of agent 1.

Bilateral Dictatorship Lemma. *There exist a set* $N_1 \subseteq N \setminus 1$, *a partition* $\left\{\Omega_{11}^j\right\}_{j \in N_1}$ *of* Ω_{11}, *and for each* $j \in N_1$ *a* $(1, j)$*-dictatorial* Ω_{11}^j*-assignment rule* s^j *such that*

$$s_i(p) \cap \Omega_{11} = \cup_{j \in N_1} s_i^j(p \mid \Omega_{11}^j) \tag{68}$$

for all $p \in \mathcal{P}^N$ *and* $i \in N$.

Before diving into the proof of this lemma, an outline may be helpful. Consider the family of all subsets of Ω_{11} that are assigned to agent 1 at some belief profile. We begin by showing that $s_1(p) \cap \Omega_{11}$ maximizes p_1 over that family whenever p_1 is a so-called Ω_{11}-dominant belief –one in which only the probability differences between events in Ω_{11} are large. We then use the Local Bilaterality lemma to extend this observation to all belief profiles p. The next and crucial step consists in proving that every state in Ω_{11} can only be allocated to a single agent other than 1. The set Ω_{11} can therefore be partitioned into a collection of subsets $\left\{\Omega_{11}^j\right\}$ such that every state in Ω_{11}^j is allocated to either 1 or j, and super-strategyproofness can be used to show that $s_1(p) \cap \Omega_{11}^j$ maximizes p_1 over the family of all subsets of Ω_{11}^j that are assigned to agent 1 at some belief profile. The argument is completed by appealing to non-bossiness.

Turning now to the formal argument, let Ω_{11} be the set of states whose assignment varies only with the beliefs of agent 1, namely,

$$\begin{aligned} \omega \in \Omega_{11} \Leftrightarrow & \left[\text{there exist } p \in \mathcal{P}^N \text{ and } p_1' \in \mathcal{P} \text{ such that } a_\omega(p) \neq a_\omega(p_1', p_{-1})\right] \text{ and} \\ & \left[a_\omega(., p_{-j}) \text{ is constant on } \mathcal{P} \text{ for all } j \neq 1 \text{ and } p_{-j} \in \mathcal{P}^{N \setminus j}\right]. \end{aligned}$$

To avoid triviality, assume $\Omega_{11} \neq \emptyset$. Let $\widetilde{\omega} \in \Omega_{11}$. We must show that there exist a set $N_1 \subseteq N \setminus 1$, a partition $\left\{\Omega_{11}^j\right\}_{j \in N_1}$ of Ω_{11}, and for each $j \in N_1$ a $(1, j)$-dictatorial Ω_{11}^j-assignment rule s^j such that

$$s_i(p) \cap \Omega_{11} = \cup_{j \in N_1} s_i^j(p \mid \Omega_{11}^j) \tag{69}$$

for all $p \in \mathcal{P}^N$ and $i \in N$.

Define the family

$$\begin{aligned} \mathcal{A}_{11} &= \left\{A \subseteq \Omega_{11} : \exists p \in \mathcal{P}^N \text{ such that } s_1(p) \cap \Omega_{11} = A\right\} \\ &= \left\{A \subseteq \Omega_{11} : \exists p_1 \in \mathcal{P} \text{ such that } s_1(p_1, p_{-1}) \cap \Omega_{11} = A \text{ for all } p_{-1} \in \mathcal{P}^{N \setminus 1}\right\}, \end{aligned}$$

where the first equality constitutes the definition and the second follows from the definition of Ω_{11}.

Let $\overline{\Omega}_{11} = \Omega \setminus \Omega_{11}$. Call a belief $p_1 \in \mathcal{P}$ Ω_{11}*-dominant* if $|p_1(A) - p_1(B)| > |p_1(A') - p_1(B')|$ for all distinct $A, B \subset \Omega_{11}$ and all distinct $A', B' \subset \overline{\Omega}_{11}$ (or, equivalently, $|p_1(\omega) - p_1(\omega')| > p_1(\overline{\Omega}_{11})$ for all distinct $\omega, \omega' \in \Omega_{11}$). In such a belief, the probability *differences* within Ω_{11} overwhelm the differences outside Ω_{11}. To see that such beliefs exist, write $\Omega_{11} = \{1, ..., m\}$ and observe that any belief p_1 such that $p_1(1) > p_1(\Omega \setminus 1)$, $p_1(2) > p_1(\Omega \setminus 12)$, ..., and $p_1(m) > p_1(\Omega \setminus 1...m-1)$, is Ω_{11}-dominant. Let $\mathcal{P}_{11}$ denote the set of Ω_{11}-dominant beliefs.

Step 1. We show that

$$s_1(p) \cap \Omega_{11} = \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1 \tag{70}$$

for all $p = (p_1, p_{-1}) \in \mathcal{P}_{11} \times \mathcal{P}^{N \setminus 1}$.

The claim is obviously true if $\Omega_{11} = \Omega$; in what follows we assume $\Omega_{11} \neq \Omega$. For any two beliefs $p_1, q_1 \in \mathcal{P}$ and for any $p_{-1} \in \mathcal{P}^{N \setminus 1}$, we claim that

$$\left[p_1 \mid \overline{\Omega}_{11} = q_1 \mid \overline{\Omega}_{11}\right] \Rightarrow \left[s_1(p_1, p_{-1}) \cap \overline{\Omega}_{11} = s_1(q_1, p_{-1}) \cap \overline{\Omega}_{11}\right]. \tag{71}$$

To see why this is true, fix $p_1, q_1 \in \mathcal{P}$, $p_{-1} \in \mathcal{P}^{N \setminus 1}$, and note that the definitions of Ω_0 and Ω_{1j} for $j \neq 1$ trivially imply

$$s_1(p_1, p_{-1}) \cap [\Omega_0 \cup \cup_{j \neq 1} \Omega_{1j}] = s_1(q_1, p_{-1}) \cap [\Omega_0 \cup \cup_{j \neq 1} \Omega_{1j}].$$

Moreover, by the Bilateral Consensus corollary, agent 1's share of Ω_2 is determined by bilateral consensus, hence does not depend on her belief outside Ω_2. Therefore,

$$\left[p_1 \mid \overline{\Omega}_{11} = q_1 \mid \overline{\Omega}_{11}\right] \Rightarrow \left[s_1(p_1, p_{-1}) \cap \Omega_2 = s_1(q_1, p_{-1}) \cap \Omega_2\right],$$

and (71) follows.

Let now $p = (p_1, p_{-1}) \in \mathcal{P}_{11} \times \mathcal{P}^{N \setminus 1}$. Since p_{-1} is fixed in the argument below, we drop it from the list of arguments of s_1. Suppose, contrary to the claim, that $s_1(p_1) \cap \Omega_{11} \neq \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1$. Choosing $q_1 \in \mathcal{P}$ such that $s_1(q_1) \cap \Omega_1 = \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1$, we have

$$p_1(s_1(q_1) \cap \Omega_{11}) > p_1(s_1(p_1) \cap \Omega_{11}).$$

Because p_1 is Ω_{11}-dominant,

$$\begin{aligned} & p_1(s_1(q_1) \cap \Omega_{11}) - p_1(s_1(p_1) \cap \Omega_{11}) \\ > \;& p_1(s_1(p_1) \cap \overline{\Omega}_{11}) - p_1(s_1(q_1) \cap \overline{\Omega}_{11}). \end{aligned}$$

Combining these inequalities yields $p_1(s_1(q_1)) > p_1(s_1(p_1))$, contradicting strategyproofness.

Step 2. We prove that (70) holds for all $p \in \mathcal{P}^N$.

Let $p = (p_1, p_{-1}) \in \mathcal{P}^N$ and drop again p_{-1} from the list of arguments of s_1. For each $\alpha \in (0,1)$, define the probability measure ${}_\alpha p_1$ over the subsets of Ω by

$${}_\alpha p_1(A) = \alpha \frac{p_1(A \cap \Omega_{11})}{p_1(\Omega_{11})} + (1-\alpha) \frac{p_1(A \cap \overline{\Omega}_{11})}{p_1(\overline{\Omega}_{11})} \text{ for all } A \subseteq \Omega. \tag{72}$$

If $\alpha = p_1(\Omega_{11})$, then ${}_\alpha p_1$ coincides with p_1. If α is sufficiently close to 1, then ${}_\alpha p_1$ is Ω_{11}-dominant. For every α, ${}_\alpha p_1 \mid \Omega_{11} = p_1 \mid \Omega_{11}$ and ${}_\alpha p_1 \mid \overline{\Omega}_{11} = p_1 \mid \overline{\Omega}_{11}$.

Choose an increasing sequence of numbers $\alpha(1), ..., \alpha(T)$ such that (i) ${}_{\alpha(t)}p_1$ is adjacent to ${}_{\alpha(t+1)}p_1$ for all $t = 1, ..., T-1$, (ii) ${}_{\alpha(1)}p_1 = p_1$, and (iii) ${}_{\alpha(T)}p_1$ is Ω_{11}-dominant. Consider the J-path $(\mathbf{p}_1^t)_{t=1}^T$ in $\mathcal{P}$ defined by $\mathbf{p}_1^t =_{\alpha(t)} p_1$ for $t = 1, ..., T$.

Let $A^t = s_1(\mathbf{p}_1^t) \cap \Omega_{11}$ for $t = 1, ..., T$. Suppose, contrary to the claim, that $A^1 \neq \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1$. Since $\mathbf{p}_1^T$ is Ω_{11}-dominant and $\mathbf{p}_1^T \mid \Omega_{11} = p_1 \mid \Omega_{11}$, Step 1 implies $A^T = \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1$. Let t be the largest integer in $\{1, ..., T-1\}$ such that $A^t \neq \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1$. Let $\{E^t, E^{t+1}\}$ be the pair of disjoint events such that $\mathbf{p}_1^t, \mathbf{p}_1^{t+1}$ are $\{E^t, E^{t+1}\}$-adjacent and $\mathbf{p}_1^t(E^t) > \mathbf{p}_1^t(E^{t+1})$. Because $\mathbf{p}_1^t \mid \Omega_{11} = \mathbf{p}_1^{t+1} \mid \Omega_{11}$ and $\mathbf{p}_1^t \mid \overline{\Omega}_{11} = \mathbf{p}_1^{t+1} \mid \overline{\Omega}_{11}$,

$$E^t \cap \overline{\Omega}_{11} \neq \emptyset \text{ and } E^{t+1} \cap \Omega_{11} \neq \emptyset.$$

By the Local Bilaterality lemma,

$$s_1(\mathbf{p}_1^t) \setminus s_1(\mathbf{p}_1^{t+1}) = E^t \text{ and } s_1(\mathbf{p}_1^{t+1}) \setminus s_1(\mathbf{p}_1^t) = E^{t+1}.$$

It follows that $(s_1(\mathbf{p}_1^t) \setminus s_1(\mathbf{p}_1^{t+1})) \cap \overline{\Omega}_{11} \neq \emptyset$, that is, $s_1(\mathbf{p}_1^t) \cap \overline{\Omega}_{11} \neq s_1(\mathbf{p}_1^{t+1}) \cap \overline{\Omega}_{11}$, contradicting (71).

Step 3. We show that for all $p, q \in \mathcal{P}_{11} \times \mathcal{P}^{N\setminus 1}$,

$$[p_1 \mid \Omega_{11} = q_1 \mid \Omega_{11}] \Rightarrow [s_i(p) \cap \Omega_{11} = s_i(q) \cap \Omega_{11} \text{ for all } i \in N].$$

Let $p, q \in \mathcal{P}_{11} \times \mathcal{P}^{N\setminus 1}$. Since we are only concerned with the restriction of s to Ω_{11}, we may assume $p_{-1} = q_{-1}$ and omit that sub-profile from the notation. Suppose $p_1 \mid \Omega_{11} = q_1 \mid \Omega_{11}$. By Step 1,

$$s_1(p_1) \cap \Omega_{11} = s_1(q_1) \cap \Omega_{11} = \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1. \tag{73}$$

Because $p_1, q_1 \in \mathcal{P}_{11}$, (73) and super-strategyproofness imply

$$s_i(p_1) \cap \Omega_{11} = s_i(q_1) \cap \Omega_{11} \text{ for all } i \in N.$$

Indeed, if, say, $s_2(p_1) \cap \Omega_{11} \neq s_2(q_1) \cap \Omega_{11}$, then (73) and the assumption $p_1 \mid \Omega_{11} = q_1 \mid \Omega_{11}$ imply that either (i) $p_1(s_{12}(p_1) \cap \Omega_{11}) > p_1(s_{12}(q_1) \cap \Omega_{11})$ and $q_1(s_{12}(p_1) \cap \Omega_{11}) > q_1(s_{12}(p_1) \cap \Omega_{11})$, or (ii) both of these two strict inequalities are reversed. Because $p_{1,}q_1$ are Ω_{11}-dominant, each of (i) and (ii) violates super-strategyproofness.

Step 4. We claim that for every $\omega \in \Omega_{11}$ there is a unique $j \neq 1$ such that $a_\omega(\mathcal{P}_{11} \times \mathcal{P}^{N\setminus 1}) = \{1, j\}$.

From Step 3, the assignment of all states in Ω_{11} depends only on the conditional beliefs of agent 1 over Ω_{11}. We may thus drop p_{-1} from the notation and regard s as a function from $\mathcal{P}(\Omega_{11})$ to $\mathcal{S}(\Omega_{11})$. By assumption, s is super-strategyproof (hence also non-bossy) and it is not constant on $\mathcal{P}(\Omega_{11})$.

We want to show that

$$s_j(p_1) \cap s_k(q_1) = \emptyset \text{ for any distinct } j, k \in N \setminus 1 \tag{74}$$

and any $p_1, q_1 \in \mathcal{P}(\Omega_{11})$. For any $\tilde{\Omega}_{11} \subset \Omega_{11}$, an $\tilde{\Omega}_{11}$-assignment rule $\tilde{s} : \mathcal{P}(\tilde{\Omega}_{11}) \to \mathcal{S}(\tilde{\Omega}_{11})$ will be called *1-C-BD union* if it is a *union of constant or bilaterally 1-dictatorial rules* on $\tilde{\Omega}_{11}$, namely, if there is a partition $\left\{\Omega_{11}^l\right\}_{t=1}^L$ of Ω_{11} such that, for all $p_1 \in \mathcal{P}(\tilde{\Omega}_{11})$,

$$\tilde{s}_i(p_1) = \cup_{l=1}^L s_i^l(p_1 \mid \Omega_{11}^l) \text{ for all } i \in N, \tag{75}$$

where each s^l is a constant or $(1, j^l)$-dictatorial Ω_{11}^l-assignment rule. With a slight abuse of terminology, we will call (the restriction to $\bar{\mathcal{P}}$ of) $\tilde{s}$ a *1-C-BD union over* $\bar{\mathcal{P}}$ if (75) is satisfied for all $p_1 \subset \bar{\mathcal{P}} \subset \mathcal{P}(\tilde{\Omega}_{11})$. We prove Step 4 by induction on the size of Ω_{11}.

Sub-step 4.1. Suppose that $|\Omega_{11}| = 2$ and consider a super-strategyproof assignment rule $\tilde{s} : \mathcal{P}(\Omega_{11}) \to \mathcal{S}(\Omega_{11})$. Then there exists $j \in N \setminus 1$ such that $\tilde{s}_{1j}(p) = \Omega_{11}$ for all $p_1 \in \mathcal{P}(\Omega_{11})$. It follows that $\tilde{s}$ is a 1-C-BD union.

Indeed, suppose that $\Omega_{11} = \{\omega_1, \omega_2\}$ and let $\tilde{p_1} \in \mathcal{P}(\Omega_{11})$. If we have either $\tilde{s}_1(\tilde{p_1}) = \emptyset$ or $\tilde{s}_1(\tilde{p_1}) = \Omega_{11}$, then $\tilde{s}$ is constant over $\mathcal{P}(\Omega_{11})$ and the result of Sub-step 4.1 trivially holds. Without loss of generality, suppose now that $\tilde{s}_1(\tilde{p_1}) = \{\omega_1\}$. Then there exists some agent $j \neq 1$ such that $\omega_2 \in s_j(\tilde{p_1})$ and obviously $\tilde{s}_{1j}(\tilde{p_1}) = \Omega_{11}$. By super-strategyproofness of $\tilde{s}$, we have $p_1(\tilde{s}_{1j}(p_1)) \geq p_1(\tilde{s}_{1j}(\tilde{p_1})) = p_1(\Omega_{11}) = 1$, hence, $p_1(\tilde{s}_{1j}(\tilde{p_1})) = 1$, for all $p \in \mathcal{P}(\Omega_{11})$, meaning that $\tilde{s}$ is $(1, j)$-dictatorial. Thus, in all possible cases, $\tilde{s}$ is a 1-C-BD union.

Suppose now that $|\Omega_{11}| = K \geq 3$ and assume by induction that every assignment rule $\tilde{s} : \mathcal{P}(\tilde{\Omega}_{11}) \to \mathcal{S}(\tilde{\Omega}_{11})$ such that $|\tilde{\Omega}_{11}| \leq K - 1$ is a 1-C-BD union.

Recalling that the range of $s_1(\cdot)$ is $\mathcal{E} \equiv \{E \subset \Omega_{11} : s_1(p_1) = E \text{ for some } p_1 \in \mathcal{P}(\Omega_{11})\}$, strategyproofness of s obviously implies $s_1(p_1) = \underset{\mathcal{E}}{\operatorname{argmax}}\, p_1$ for all $p_1 \in \mathcal{P}(\Omega_{11})$.

Given any $\omega \in \Omega_{11}$, define the set of ω-*lexicographic* beliefs $\mathcal{L}(\omega) := \{p_1 \in \mathcal{P}(\Omega_{11}) : p_1(\omega) > p_1(\Omega_{11} \setminus \omega)\}$. For any $q_1 \in \mathcal{P}(\Omega_{11}) \cup \mathcal{P}(\Omega_{11} \setminus \omega)$, let $\mathcal{L}^{q_1}(\omega) := \{p_1 \in \mathcal{L}(\omega) : p_1 \mid (\Omega_{11} \setminus \omega) = q_1 \mid (\Omega_{11} \setminus \omega)\}$ and, for any $\alpha \in (\frac{1}{2}, 1)$, define $q_1^{\omega,\alpha} \in \mathcal{L}^{q_1}(\omega)$ as follows: for all $\omega' \in \Omega_{11}$,

$$q_1^{\omega,\alpha}(\omega') := \begin{cases} \alpha & \text{if } \omega' = \omega, \\ \frac{q_1(\omega')}{1-\alpha} & \text{if } \omega' \neq \omega. \end{cases}$$

Sub-step 4.2. Consider $q_1 \in \mathcal{P}(\Omega_{11})$ s.t. $\omega \in s_1(q_1)$; and suppose that $p_1 \in \mathcal{L}^{q_1}(\omega)$. Then we have $s(p_1) = s(q_1)$.

The proof of Sub-step 4.2 is rather straightforward, and left to the reader. It follows from non-bossiness of s and the fact that $p_1(\omega) > 1/2$ for all $p_1 \in \mathcal{L}^{q_1}(\omega)$.

Sub-step 4.3. Fix $\bar{\omega} \in \Omega_{11}$ and $\alpha \in (\frac{1}{2}, 1)$. Define the mapping ${}_\alpha\tilde{s}^{-\bar{\omega}} : \mathcal{P}(\Omega_{11} \setminus \bar{\omega}) \to \mathcal{S}(\Omega_{11} \setminus \bar{\omega})$ as follows: **(i)** ${}_\alpha\tilde{s}_1^{-\bar{\omega}}(q_1) = s_1(q_1^{\bar{\omega},\alpha}) \setminus \bar{\omega}$; **(ii)** ${}_\alpha\tilde{s}_i^{-\bar{\omega}}(q_1) = s_i(q_1^{\bar{\omega},\alpha}), \forall i \neq 1$. Then ${}_\alpha\tilde{s}^{-\bar{\omega}}$ is an $(\Omega_{11} \setminus \bar{\omega})$-assignment rule and a 1-C-BD union.

To prove Sub-step 4.3, note first that $\bar{\omega} \in s_1(p_1)$ for all $p_1 \in \mathcal{L}(\bar{\omega})$. Indeed, since the range $\mathcal{E}$ of $s_1(\cdot)$ is a proper covering of Ω_{11}, there exists $\bar{p}_1 \in \mathcal{P}(\tilde{\Omega}_{11})$ such that $\bar{\omega} \in s_1(\bar{p}_1)$. Therefore, if $\bar{\omega} \notin s_1(p_1)$ for some $p_1 \in \mathcal{L}(\bar{\omega})$, we would have $p_1(s_1(\bar{p}_1)) \geq p_1(\bar{\omega}) > \frac{1}{2} > p_1(s_1(p_1))$, contradicting strategyproofness.

Building on this result, observe from (i)-(ii) above that the mapping ${}_\alpha\tilde{s}^{-\bar{\omega}}$ satisfies the feasibility constraint. Indeed, for any $q_1 \in \mathcal{P}(\Omega_{11} \setminus \bar{\omega})^1$, since $q_1^{\bar{\omega},\alpha} \in \mathcal{L}(\bar{\omega})$, we get from the feasibility of s that

$$\cup_{i \in N}\ {}_\alpha\tilde{s}_i^{-\bar{\omega}}(q_1) = \overbrace{(\underbrace{s_1(q_1^{\bar{\omega},\alpha})}_{\bar{\omega}\in} \setminus \bar{\omega})}^{=\ {}_\alpha\tilde{s}_1^{-\bar{\omega}}(q_1)} \cup [\cup_{i \in N \setminus i} \underbrace{s_i(q_1^{\bar{\omega},\alpha})}_{\bar{\omega}\notin}] = \Omega_{11} \setminus \bar{\omega}.$$

Thus, the mapping ${}_\alpha\tilde{s}^{-\bar{\omega}}$ is a well-defined $(\Omega_{11} \setminus \bar{\omega})$-assignment rule. Moreover, it is super-strategyproof (because s is), and since $|\Omega_{11} \setminus \bar{\omega}| = K - 1 < K$, our induction hypothesis implies that ${}_\alpha\tilde{s}^{-\bar{\omega}}$ is a 1-C-BD union.

Sub-step 4.4. Fix $\bar{\omega} \in \Omega_{11}$. The mapping $\bar{s}^{\bar{\omega}} : \mathcal{L}(\bar{\omega}) \to \mathcal{S}(\Omega_{11} \setminus \bar{\omega})$, defined as the restriction of s to $\mathcal{L}(\bar{\omega})$, is a 1-C-BD union over $\mathcal{L}(\bar{\omega})$. As a consequence, (74) must hold for all $p_1, q_1 \in \mathcal{L}(\bar{\omega})$.

This follows from the combination of Sub-step 4.2 and Sub-step 4.3. Indeed, fix any $\alpha > 1/2$; and note from Sub-step 4.2 that, for all $q_1 \in \mathcal{L}(\bar{\omega})$, we have $\bar{s}^{\bar{\omega}}(q_1) = s(q_1) = s(q_1^{\bar{\omega},\alpha})$ because $q_1^{\bar{\omega},\alpha} \in \mathcal{L}^{q_1}(\bar{\omega})$. That is to say,

$$\bar{s}_1^{\bar{\omega}}(q_1) = \bar{\omega} \cup\ {}_\alpha\tilde{s}_1^{-\bar{\omega}}(q_1 \mid (\Omega_{11} \setminus \bar{\omega})) \text{ and } \bar{s}_i^{\bar{\omega}}(q_1) =\ {}_\alpha\tilde{s}_i^{-\bar{\omega}}(q_1 \mid (\Omega_{11} \setminus \bar{\omega})), \forall i \neq 1. \quad (76)$$

Recalling from Sub-step 4.3 that ${}_\alpha\tilde{s}$ is a 1-C-BD union, there exists a partition $\{\Omega_{11}^1, \ldots, \Omega_{11}^L\}$ of $\Omega_{11} \setminus \bar{\omega}$ and L Ω^l-assignment rules $s^1, \ldots, s^L$ such that ${}_\alpha\tilde{s}_i^{-\bar{\omega}}(q_1 \mid (\Omega_{11} \setminus \bar{\omega})) = \cup_{l=1}^L s_i^l(q_1 \mid \Omega_{11}^l)$ and each s^l is constant or $(1, j^l)$-dictatorial for some $j^l \neq 1$. Substituting this in (76) thus gives: for all $q_1 \in \mathcal{L}(\bar{\omega})$ and $i \in N$,

$$\bar{s}_i^{\bar{\omega}}(q_1) = \begin{cases} \cup_{l=1}^L s_i^l(q_1 \mid \Omega_{11}^l) & \text{if } i \neq 1, \\ \bar{\omega} \cup \left(\cup_{l=1}^L s_i^l(q_1 \mid \Omega_{11}^l)\right) & \text{if } i = 1. \end{cases} \quad (77)$$

Observe from (77) that $\bar{s}^{\bar{\omega}}$, the restriction of s to $\mathcal{L}(\bar{\omega})$ is expressed as the union of the $L+1$ sub-rules $s^0, s^1, \ldots, s^L$, where s^0 is the constant Ω^0-assignment rule which always assigns $\Omega_{11}^0 := \{\bar{\omega}\}$ to agent 1. This concludes the proof of Sub-step 4.4.

We are now ready to proceed with the proof of Step 4. Since $\mathcal{P}(\Omega_{11})$ is connected, there is a J-path $(\mathbf{p}_1^t)_{t=1}^T$ in $\mathcal{P}(\Omega_{11})$ between any two beliefs $p_1, q_1 \in \mathcal{P}(\Omega_{11})$. If the *length* $T-1$ of this path is equal to 1, then p_1, q_1 are adjacent and the Local Bilaterality lemma implies $s_j(p_1) \cap s_k(q_1) = \emptyset$ for any distinct $j, k \in N \setminus 1$. Next, proceeding by induction, we assume that (74) is true whenever p_1, q_1 are connected by some J-path of length $T'-1 < T-1$ (with $T \geq 3$) and we prove that (74) also holds for any p_1, q_1 that are connected by some J-path of length $T-1$.

By contradiction, suppose that there exist $\omega^* \in \Omega_{11}$ and $p_1'', p_1''' \in \mathcal{P}(\Omega_{11})$ such that, say, $\omega^* \in s_2(p_1'') \cap s_3(p_1''')$ and p_1'', p_1''' are connected by some J-path $\mathbf{q}_1 = (\mathbf{q}_1^t)_{t=1}^T$. Combining the Local Bilaterality lemma with our induction hypothesis that (74) holds for all p_1, q_1 that are connected by some J-path of length $T' \leq T-1$, we obtain

$$w^* \in s_1(\mathbf{q}_1^{T-1}) \setminus s_1(\mathbf{q}_1^T) = s_3(\mathbf{q}_1^T) \setminus s_3(\mathbf{q}_1^{T-1}) \neq \emptyset \tag{78}$$

$$s_i(\mathbf{q}_1^{T-1}) = s_i(\mathbf{q}_1^T), \ \forall i \neq 1, 3 \tag{79}$$

$$s_3(\mathbf{q}_1^{T-1}) \cap s_i(p_1'') = \emptyset, \ \forall i \neq 1, 3. \tag{80}$$

To see why (78) holds, note that having $w^* \in s_k(\mathbf{q}_1^t)$ for some $k \neq 1, 2$ and $t \leq T-1$ would imply a violation of our induction hypothesis on the J-path $\{\mathbf{q}_1^1, \ldots, \mathbf{q}_1^t\}$, which is of length $t-1 < T-1$. Statement (80) holds for the same reason. Finally, (79) follows from (78) and the Local Bilaterality lemma. In addition, observe that combining (79) and (80) gives

$$s_i(p_1''') \cap s_3(p_1'') = s_i(\mathbf{q}_1^T) \cap s_3(p_1'') = s_i(\mathbf{q}_1^{T-1}) \cap s_3(p_1'') = \emptyset, \ \forall i \neq 1, 3. \tag{81}$$

Sub-step 4.5. There exist $\omega_3 \in s_1(p_1''') \cap s_3(p_1'')$ and $\omega_2 \in s_1(p_1'') \cap s_3(p_1''')$.

To prove Sub-step 4.5, first note that, together, $\omega^* \in s_2(p_1'') \cap s_3(p_1''')$ and the super-strategyproofness of s imply that $p_1'''(s_{N\setminus 3}(p_1''')) > p_1'''(s_{N\setminus 3}(p_1''))$. Thus, there exists $\hat{\omega} \in \Omega_{11}$ such that

$$\hat{\omega} \in s_{N\setminus 3}(p_1''') \setminus s_{N\setminus 3}(p_1'') = s_{N\setminus 3}(p_1''') \cap s_3(p_1''). \tag{82}$$

It thus suffices now to remark that $s_{N\setminus 3}(p_1''') \cap s_3(p_1'') = s_1(p_1''') \cap s_3(p_1'')$. Indeed, given that we have $s_{N\setminus 3}(p_1''') := \cup_{i \neq 3} s_i(p_1''')$, we can write

$$s_{N\setminus 3}(p_1''') \cap s_3(p_1'') = [s_1(p_1''') \cap s_3(p_1'')] \cup [\cup_{i \neq 1,3} \underbrace{(s_i(p_1''') \cap s_3(p_1''))}_{=\emptyset \text{ by } (81)}] = s_1(p_1''') \cap s_3(p_1''').$$

Thus, $\hat{\omega} \in s_{N\backslash 3}(p_1''') \cap s_{N\backslash 3}(p_1'') = s_1(p_1''') \cap s_3(p_1'')$. A symmetric argument shows that there exists $\omega_2 \in s_1(p_1'') \cap s_3(p_1''')$; and this ends the proof of Sub-step 4.4.

Recall from what precedes that $\omega^* \in s_2(p_1'') \cap s_3(p_1''')$, $\omega_3 \in s_1(p_1''') \cap s_3(p_1'')$ and $\omega_2 \in s_1(p_1'') \cap s_3(p_1''')$. The states $\omega^*, \omega_2, \omega_3$ are thus necessarily (pairwise) distinct. We show a few additional sub-steps below.

Fix any $q_1'' \in \mathcal{L}^{p_1''}(\omega_2)$ (see Figure 6) and $q_1''' \in \mathcal{L}^{p_1'''}(\omega_3)$, and define ${}^t q_1''' \in \mathcal{L}(\omega_3)$ by ${}^t q_1'''(\omega_3) = q_1'''(\omega_2)$, ${}^t q_1'''(\omega_2) = q_1'''(\omega_3)$ and ${}^t q_1'''(\omega) = q_1'''(\omega), \forall \omega \neq \omega_2, \omega_3$. In addition, call $\pi_{\omega_3}^{\omega_2}$ the probability measure over Ω_{11} defined by:[4]

$$\pi_{\omega_3}^{\omega_2}(\omega_2) = \pi_{\omega_3}^{\omega_2}(\omega_2) = 1/2; \quad \text{and } \pi_{\omega_3}^{\omega_2}(\omega) = 0 \text{ for all } \omega \neq \omega_2, \omega_3.$$

Define the two sequences $\{q_1^m\}_{m \geq \bar{m}_q}$ and $\{\bar{q}_1^m\}_{m \geq \bar{m}_{\bar{q}}}$ as follows: for any $\omega \in \Omega_{11}$,

$$\begin{aligned} q_1^m(\omega) =& \frac{1}{m} q_1''' + (1 - \frac{1}{m}) \pi_{\omega_3}^{\omega_2}; \\ \bar{q}_1^m(\omega) =& \frac{1}{m} \, {}^t q_1''' + (1 - \frac{1}{m}) \pi_{\omega_3}^{\omega_2}. \end{aligned} \tag{83}$$

Figure 6 gives an illustration of the construction of the beliefs $q_1^m, \bar{q}^m$ starting from $p_1'' \in \mathcal{L}(\omega_2)$. It is important to remark that, by definition, we have $q_1^m \in \mathcal{L}(\omega_2)$ and $\bar{q}_1^m \in \mathcal{L}(\omega_3)$.[5]

Sub-step 4.6. There exist $\tilde{m} \in \mathbb{N}$ (with $\tilde{m} \geq \bar{m}_q, \bar{m}_{\bar{q}}$) and $\mathbf{A}, \bar{\mathbf{A}} \in \mathcal{S}(\Omega_{11})$ such that

$$[m \geq \tilde{m}] \Rightarrow [s(q_1^m) = \mathbf{A} \text{ and } s(\bar{q}_1^m) = \bar{\mathbf{A}}].$$

The proof of Sub-step 4.6 is similar to that of Lemma 3-(i), and therefore left to the reader.

Sub-step 4.7. For any $m \geq \tilde{m}$, we have $\omega^* \in s_2(q_1^m)$; and it follows that $\mathbf{A} \neq \bar{\mathbf{A}}$. We showed in Sub-step 4.4 that $\bar{s}^{\omega_2}$, the restriction of s to $\mathcal{L}(\omega_2)$, can be written as

$$\bar{s}_i^{\bar{\omega_2}}(q_1) = \begin{cases} \cup_{l=1}^{L} s_i^l(q_1 \mid \Omega_{11}^l) & \text{if } i \neq 1, \\ \omega_2 \cup \left(\cup_{l=1}^{L} s_i^l(q_1 \mid \Omega_{11}^l)\right) & \text{if } i = 1, \end{cases} \tag{84}$$

where each s^l is constant or $(1, j^l)$-dictatorial for some $j^l \neq 1$. Call $\Omega_{11}^{w^*}$ the unique event in the partition $\{\underbrace{\Omega_{11}^0}_{=\{\omega_2\}}, \Omega_{11}^1, \ldots, \Omega_{11}^L\}$ of Ω_{11} such that $\omega^* \in \Omega_{11}^{w^*}$. Since $q_1'' \in$

[4] Obviously, $\pi_{\omega_3}^{\omega_2}$ is not an injective probability measure (i.e., $\pi_{\omega_3}^{\omega_2} \notin \mathcal{P}(\Omega_{11})$); but this does not affect the validity of our upcoming argument —which is based on the study of sequences of injective probability measures that converge to $\pi_{\omega_3}^{\omega_2}$.

[5] There may exist only a finite number of integers m such that $q_1^m, \bar{q}_1^m$ are not injective; and this issue is taken care of by conveniently starting the sequence at a rank $\bar{m}_q$ (or $\bar{m}_{\bar{q}}$) that is higher than any such integer.

$\mathcal{L}^{p''_1}(\omega_2) \subset \mathcal{L}(\omega_2)$, it follows from Sub-step 4.2 that $\omega^* \in s_2(p''_1) = s_2(q''_1) = \bar{s}_2^{\omega_2}(q''_1)$; and we may then conclude from (84) that $j^{\omega^*} = 2$ and s^{ω^*} is (1,2)-dictatorial over $\Omega_{11}^{\omega^*}$. We get in the same way that $j^{\omega_3} = 3$ and s^{ω_3} is (1,3)-dictatorial over $\Omega_{11}^{\omega_3}$. It thus follows that $\omega_3, \omega_2 \notin \Omega_{11}^{\omega^*}$ –obviously, $\omega^2 \notin \Omega_{11}^{\omega^*}$ since $\Omega_{11}^0 = \{\omega_2\}$. Using (84) and the fact that s^{ω^*} is (1,2)-dictatorial, we may assert that $\omega^* \in s_2(q_1)$ for any $q_1 \in \mathcal{L}(\omega_2)$ such that $q_1 \mid \Omega_{11}^{\omega^*} = q''_1 \mid \Omega_{11}^{\omega^*}$. One can then see that $\omega^* \in s_2(q_1^m)$ by combining (83) and $\omega_2, \omega_3 \notin \Omega_{11}^{\omega^*}$ to deduce that we indeed have: $q_1^m \mid \Omega_{11}^{\omega^*} = q''_1 \mid \Omega_{11}^{\omega^*}$, for all $m \geq \bar{m}_q$.

We conclude the proof of Sub-step 4.7 by noting that we necessarily have $\mathbf{A} \neq \bar{\mathbf{A}}$. Indeed, since $\tilde{m} \geq \bar{m}_q$, we have $\omega^* \in A_2 = s_2(q_1^{\tilde{m}})$. Assuming that $\mathbf{A} = \bar{\mathbf{A}}$ would thus give $\omega^* \in \mathbf{A}_2 = \bar{\mathbf{A}}_2 = s_2(\bar{q}^{\tilde{m}})$. But this would contradict the fact that $\bar{s}^{\omega_3}$ is a 1-C-BD union over $\mathcal{L}(\omega_3)$ (established in Sub-step 4.4), which requires (74) to hold for $\bar{q}^{\tilde{m}}, q'''_1 \in \mathcal{L}(\omega_3)$ —recall that $\omega^* \in s_3(q'''_1)$.

Sub-step 4.8. There exist disjoint subsets $E, \bar{E} \subset \Omega \setminus \{\omega_2, \omega_3, \omega^*\}$ such that

$$\mathbf{A}_1 \setminus \bar{\mathbf{A}}_1 = \omega_2 \cup E = \bar{\mathbf{A}}_3 \setminus \mathbf{A}_3,$$
$$\bar{\mathbf{A}}_1 \setminus \mathbf{A}_1 = \omega_3 \cup \bar{E} = \mathbf{A}_3 \setminus \bar{\mathbf{A}}_3,$$
$$\mathbf{A}_i = \bar{\mathbf{A}}_i \text{ for all } i \neq 1, 3.$$

We start the proof of Sub-step 4.8 by noting that: $\exists \hat{m} \geq \tilde{m}$ such that, for any $\{F, \bar{F}\} \in \mathcal{H}$ and any $m \geq \hat{m}$, $\left[w_2 \notin F \text{ or } \omega_3 \notin \bar{F}\right] \Rightarrow \left[(q_1^m(F) - q_1^m(\bar{F}))(\bar{q}_1^m(F) - \bar{q}_1^m(\bar{F})) > 0\right]$. This implication holds by construction since $\lim_{m\to\infty} q_1^m = \lim_{m\to\infty} \bar{q}_1^m = \pi_{\omega_3}^{\omega_2}$ and $\pi_{\omega_3}^{\omega_2}(\omega_2) = \pi_{\omega_3}^{\omega_2}(\omega_3) = 1/2$. In words: when m is large enough, the segment $[q_1^m, \bar{q}_1^m]$ cuts only hyperplanes $\{F, \bar{F}\} \in \mathcal{H}$ such that $\omega_2 \in F$ and $\omega_3 \in \bar{F}$ (see Figure 7), and $q_1^m, \bar{q}_1^m$ are on the same side of all other hyperplanes.

Second, recall from (83) that $q_1^m \mid (\Omega_{11} \setminus \{\omega_2, \omega_3\}) = \bar{q}_1^m \mid (\Omega_{11} \setminus \{\omega_2, \omega_3\}) = q''_1 \mid (\Omega_{11} \setminus \{\omega_2, \omega_3\})$, for any $m \geq \hat{m}$. It hence follows that the set of hyperplanes of the form $\{\omega_2 \cup E, \omega_3 \cup \bar{E}\}$ is totally ordered along the segment $[q_1^{\hat{m}}, \bar{q}_1^{\hat{m}}]$. Calling T the number of such hyperplanes, we may thus write

$$\{\{F, \bar{F}\} \in \mathcal{H} \mid F = \omega_2 \cup E, \bar{F} = \omega_3 \cup \bar{E}\} = \{\{\omega_2 \cup E_1, \omega_3 \cup \bar{E}_1\}, \ldots, \{\omega_2 \cup E_T, \omega_3 \cup \bar{E}_T, \}\},$$

where E^t $[t = 1, \ldots, T]$ is the t^{th} hyperplane cut on the way from $q_1^{\hat{m}}$ to $\bar{q}_1^{\hat{m}}$. Using this notation, we may then consider a J-path $\{\mathbf{p}_1^t\}_{t=1}^{T+1}$ satisfying the properties: (i) $\mathbf{p}_1^1 = q_1^{\hat{m}}$, $\mathbf{p}_1^{T+1} = \bar{q}_1^{\hat{m}}$; (ii) $\mathbf{p}_1^t$ and $\mathbf{p}_1^{t+1}$ are $\{\omega_2 \cup E_t, \omega_3 \cup \bar{E}_t\}$-adjacent for any $t = 1, \ldots, T$.

We conclude the proof of Sub-step 4.8 by showing that there exists a unique $t^* \in \{1, T\}$ such that: (a) $s(\mathbf{p}_1^t) = s(q_1^{\hat{m}}), \forall t \in \{1, \ldots, t^*\}$ and (b) $s(\mathbf{p}_1^t) = s(\bar{q}_1^{\hat{m}}), \forall t \in \{t^* + 1, \ldots, T + 1\}$. First, note that the assignment may change only once along the J-path $\mathbf{p}$. Indeed, if $s(\mathbf{p}_1^{t^*}) \neq s(\mathbf{p}_1^{t^*+1})$ then we get from the Local Bilaterality lemma that $s_1(\mathbf{p}_1^{t^*}) \setminus s_1(\mathbf{p}_1^{t^*+1}) = \omega_2 \cup E_{t^*}$; and (given that $\omega_2 \notin s_1(\mathbf{p}_1^{t^*+1})$), the Local Bilaterality lemma requires that $s(\mathbf{p}_1^t) = s(\bar{q}_1^{\hat{m}}), \forall t \in \{t^*+1, \ldots, T+1\}$.

Second, recall from Sub-step 4.7 (and $\hat{m} \geq \tilde{m}$) that $s(q_1^{\hat{m}}) = \mathbf{A} \neq \bar{\mathbf{A}} = s(\bar{q}_1^{\hat{m}})$. Hence, there must indeed exist a unique $t^* \in \{1, \ldots, T\}$ such that $s(\mathbf{p}_1^{t^*}) \neq s(\mathbf{p}_1^{t^*+1})$. The Local Bilaterality lemma, applied to the adjacent beliefs $\mathbf{p}_1^{t^*}, \mathbf{p}_1^{t^*+1}$, then gives the desired result: $\mathbf{A}_1 \setminus \bar{\mathbf{A}}_1 = \omega_2 \cup E_{t^*} = \bar{\mathbf{A}}_3 \setminus \mathbf{A}_3$; $\bar{\mathbf{A}}_1 \setminus \mathbf{A}_1 = \omega_3 \cup \bar{E}_{t^*} = \mathbf{A}_3 \setminus \bar{\mathbf{A}}_3$; $\mathbf{A}_i = \bar{\mathbf{A}}_i, \forall i \neq 1, 3$. Recalling from Sub-step 4.7 that $\omega^* \in s_2(q_1^{\hat{m}}) = \mathbf{A}_2$, we obtain that $E_{t^*}, \bar{E}_{t^*} \subset \Omega \setminus \{\omega_2, \omega_3, \omega^*\}$.

We are finally ready to clinch the proof of Step 4. We have shown in Sub-step 4.8 that $\omega^* \in s_2(q_1^{\hat{m}}) = \mathbf{A}_2 = \bar{\mathbf{A}}_2 = s_2(\overbrace{\bar{q}_1^{\hat{m}}}^{\in \mathcal{L}(\omega_3)})$. But this is a contradiction given that $\omega^* \in s_3(\overbrace{q_1'''}^{\in \mathcal{L}(\omega_3)})$. Indeed, this violation of (74) contradicts the fact that (the restriction to $\mathcal{L}(\omega_3)$ of) s is a 1-C-BD union over $\mathcal{L}(\omega_3)$ —which was established in Sub-step 4.4. Thus, it never holds that $\omega^* \in s_j(p_1'') \cap s_k(p_1''')$ for any ω^*, p_1'', p_1''' and distinct $j, k \neq 1$. Given that s is not constant on $\mathcal{P}(\Omega_{11})$, for any $\omega \in \Omega_{11}$, we thus have, $a_\omega(\mathcal{P}(\Omega_{11})) = \{1, j\}$ for some $j \neq 1$.

Step 5. We show that for every $\omega \in \Omega_{11}$ there is a unique $j \neq 1$ such that $a_\omega(\mathcal{P}^N) = \{1, j\}$.

Let $\omega \in \Omega_{11}$. By Step 4, there is a unique $j \neq 1$ such that $a_\omega(\mathcal{P}_{11} \times \mathcal{P}^{N \setminus 1}) = \{1, j\}$. We claim that $a_\omega(\mathcal{P}^N) = \{1, j\}$. Suppose, by contradiction, that there exists some $k \neq 1, j$ and some $p \in \mathcal{P}^N$ such that $\omega \in s_k(p)$. Drop p_{-1} from the notation. Consider an Ω_{11}-dominant belief $p_1^* \in \mathcal{P}_{11}$ such that $p_1^* \mid \Omega_{11} = p_1 \mid \Omega_{11}$ and $p_1^* \mid \overline{\Omega}_{11} = p_1 \mid \overline{\Omega}_{11}$. Such a belief can be constructed by taking α close to 1 in (72). Since $a_\omega(\mathcal{P}_{11} \times \mathcal{P}^{N \setminus 1}) = \{1, j\}$, we have $\omega \notin s_k(p_1^*)$. By Step 2, $s_1(p_1) \cap \Omega_{11} = s_1(p_1^*) \cap \Omega_{11}$. By (71), $s_1(p_1) = s_1(p_1^*)$. By non-bossiness, $s(p_1) = s(p_1^*)$, contradicting $\omega \in s_k(p_1) \setminus s_k(p_1^*)$ and completing Step 5.

For every $j \neq 1$, define $\Omega_{11}^j = \left\{\omega \in \Omega_{11} : a_\omega(\mathcal{P}^N) = \{1, j\}\right\}$. Let $N_1 = \{j \in N \setminus 1 : \Omega_{11}^j \neq \emptyset\}$. By definition, $\left\{\Omega_{11}^j : j \in N_1\right\}$ is a partition of Ω_{11}. For each $j \in N_1$, let

$$\mathcal{A}_{11}^j = \left\{A^j \subseteq \Omega_{11}^j : \exists p \in \mathcal{P}^N \text{ such that } s_1(p) \cap \Omega_{11}^j = A^j\right\}.$$

Step 6. We show that $\mathcal{A}_{11}$ is a product family. Namely, for any collection of events $\{A^j : j \in N_1\}$,

$$\left[A^j \in \mathcal{A}_{11}^j \text{ for all } j \in N_1\right] \Rightarrow \left[\cup_{j \in N_1} A^j \in \mathcal{A}_{11}\right].$$

Suppose $A^j \in \mathcal{A}_{11}^j$ for all $j \in N_1$ and write $N_1 = \{2, ..., n_1\}$. Call a belief p_1 *lexicographically* $(\Omega_{11}^2, ..., \Omega_{11}^{n_1})$*-dominant* if $|p_1(A) - p_1(B)| > |p_1(A') - p_1(B')|$ for all distinct $A, B \subset \Omega_{11}^j$, all $A', B' \subset \Omega \setminus (\cup_{k=1}^j \Omega_{11}^k)$, and all $j = 2, ..., n-1$. Consider a lexicographically $(\Omega_{11}^2, ..., \Omega_{11}^{n_1})$-dominant belief p_1 such that

$$\operatorname*{argmax}_{\mathcal{A}_{11}^j} p_1 = A^j$$

for all $j = 2, ..., n-1$. Fix $p_{-1} \in \mathcal{P}^{N\setminus 1}$ and drop it from the notation.

Strategyproofness implies

$$s_1(p_1) \cap \Omega_{11}^2 = A^2.$$

This is because there is some q_1 such that $s_1(q_1) \cap \Omega_{11}^2 = A^2$, $\operatorname*{argmax}_{\mathcal{A}_{11}^2} p_1 = A^2$, and p_1 is Ω_{11}^2-dominant.

Next, proceed inductively. Suppose we have shown that $s_1(p_1) \cap \Omega_{11}^j = A^j$ for $j = 2, ..., k-1$. We claim that

$$s_1(p_1) \cap \Omega_{11}^k = A^k. \tag{85}$$

Since $A^k \in \mathcal{A}_{11}^k$, there is some q_1 such that $s_1(q_1) \cap \Omega_{11}^k = A^k$. If $s_1(p_1) \cap \Omega_{11}^k = B^k \neq A^k$, then

$$\begin{aligned} p_1(s_{\{1,...,k-1\}}(p_1) \cap (\cup_{j=2}^k \Omega_{11}^j)) &= p_1(\cup_{j=2}^{k-1} \Omega_{11}^j \cup B^k) \\ &< p_1(\cup_{j=2}^{k-1} \Omega_{11}^j \cup A^k) \\ &= p_1(s_{\{1,...,k-1\}}(q_1) \cap (\cup_{j=2}^k \Omega_{11}^j)), \end{aligned}$$

contradicting super-strategyproofness and proving (85).

We conclude that $s_1(p_1) \cap \Omega_{11}^j = A^j$ for all $j \in N_1$, which implies that $s_1(p_1) \cap \Omega_{11} = \cup_{j \in N_1} A^j$, hence $\cup_{j \in N_1} A^j \in \mathcal{A}_{11}$.

Step 7. Step 6 ensures that $\operatorname*{argmax}_{\mathcal{A}_{11}} p_1 = \cup_{j \in N_1} \operatorname*{argmax}_{\mathcal{A}_{11}^j} p_1$ for all $p_1 \in \mathcal{P}$. Combining this with Step 2,

$$s_1(p) \cap \Omega_{11} = \cup_{j \in N_1} \operatorname*{argmax}_{\mathcal{A}_{11}^j} p_1$$

for all $p \in \mathcal{P}^N$. Defining for each $j \in N_1$ the $(1, j)$-dictatorial Ω_{11}^j-assignment rule s^j by

$$s_i^j(\widetilde{p}) = \begin{cases} \operatorname*{argmax}_{\mathcal{A}_{11}^j} \widetilde{p}_1 & \text{if } i = 1, \\ \Omega_{11}^j \setminus \operatorname*{argmax}_{\mathcal{A}_{11}^j} \widetilde{p}_1 & \text{if } i = j, \\ \emptyset & \text{if } i \neq 1, j \end{cases}$$

for all $\widetilde{p} \in \mathcal{P}(\Omega_{11}^j)^N$, statement (69) holds for $p \in \mathcal{P}^N$ and $i \in N$.

To complete the proof, it only remains to check that $\mathcal{A}_{11}^j$ is a proper covering of Ω_{11}^j for every $j \in N_1$.

Fix $j \in N_1$. To check that $\cup_{A^j \in \mathcal{A}_{11}^j} A^j = \Omega_{11}^j$, fix $\omega \in \Omega_{11}^j$. Since, by definition of Ω_{11}^j, $a_\omega(\mathcal{P}^N) = \{1, j\}$, there is some $p \in \mathcal{P}^N$ such that $\omega \in s_1(p)$, hence some $A^j \in \mathcal{A}_{11}^j$ such that $\omega \in A^j$.

To check that $A^j \setminus B^j \neq \emptyset$ for all distinct $A^j, B^j \in \mathcal{A}_{11}^j$, suppose on the contrary that $A^j \subset B^j$. By Step 6, this implies that there exist $A, B \in \mathcal{A}_{11}$ such that $A \subset B$. But by definition of $\mathcal{A}_{11}$ and Step 1, there is some p such that $A = \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1$, contradicting the fact that $p_1(A) < p_1(B)$.

To check that $\cap_{A^j \in \mathcal{A}_{11}^j} A^j = \emptyset$, suppose on the contrary that $\omega \in \cap_{A^j \in \mathcal{A}_{11}^j} A^j$. Then $\omega \in s_1(p)$ for all $p \in \mathcal{P}^N$, contradicting the fact that $a_\omega(\mathcal{P}^N) = \{1, j\}$. □

We have stated the Bilateral Dictatorship lemma for agent 1, but a corresponding lemma obviously holds for every agent. It now follows from these Bilateral Dictatorship lemmas, the Bilateral Consensus corollary, and the definition of Ω_0, that s is a locally bilateral top selection. Together with the Top Selection lemma, this completes the proof of Theorem 1.

Appendix 2.E: Proofs for the Constrained Model

Appendix 2.E.1: Proof of the Constrained-Top Selection lemma

The respective statements and proofs of Lemmas 1 to 3 carry over to the constrained model without the slightest change. As for Lemma 4, its statement and proof must be adjusted as explained below.

Lemma 4*. Tops and Tops Only
For all $p \in \mathcal{P}^N$ and $v, v' \in \mathcal{V}_p^N$, we have:

$$[\tau_\omega(v_i) = \tau_\omega(v_i'), \forall i \in N, \forall \omega \in \Omega] \Rightarrow [\varphi(v,p) = \varphi(v',p) \in \times_{\omega \in \Omega}\{\tau_\omega(v_1), \ldots, \tau_\omega(v_n)\}].$$

Proof. Given any $v \in \mathcal{V}_p^N$, write $\Omega_v := \{\omega \in \Omega : \tau_\omega(v_i) = \tau_\omega(v_j) \text{ for all } i, j \in N\}$, that is, Ω_v is the collection of states where the agents unanimously agree on the constrained tops. For all $\omega, \omega' \in \Omega$, write $\omega >_v \omega'$ if and only if $v_i(\tau_\omega(v_i)) \geq v_i(\tau_{\omega'}(v_i))$ for all $i \in N$ with a strict inequality for at least one agent i. In such a case, we say that ω dominates ω' (at v). Finally, let

$$\Omega_v^* := \{\omega \in \Omega_v : \omega >_v \omega' \text{ for all } \omega' \in \Omega \setminus \Omega_v\}, \tag{86}$$
$$\mu(v) := |\Omega_v^*|, \tag{87}$$
$$\beta(v) := |\{\tau_\omega(v_1) : \omega \in \Omega_v^*\}|. \tag{88}$$

In words, $\mu(v)$ is the number of states of nature (i) where all agents have the same constrained top and (ii) that dominate every state where the agents' constrained tops are not all identical. Note that $0 \leq \mu(v) \leq K = |\Omega|$ for any $v \in \mathcal{V}_p^N$; and $\mu(v) = K$ at any valuation profile where all agents have the same ranking of the outcomes in X. The number $\beta(v)$ stands for the number of *distinct* constrained tops associated with the respective states in Ω_v^*.

For any $i \in N$, $v_i \in \mathcal{V}$, and $x \in X$, denote by $O_{v_i}(x) \in \{1, \ldots, |X|\}$ the rank of outcome x when all outcomes in X are ranked in *decreasing* order of valuations (with i's top having rank 1). Observe that O_{v_i} is injective (because v_i is). For all $z \in \{1, \ldots, |X|\}$, we will therefore write $O_{v_i}^{-1}(z)$ to refer to the unique outcome $x \in X$ such that $O_{v_i}(x) = z$. We formally state a few direct consequences of the definitions given in (86)-(88).

Observation 1. For all $v \in \mathcal{V}_p^N$ and all $\omega \in \Omega$:

$$[\omega \in \Omega_v^*] \Leftrightarrow \quad [1 \leq O_{v_i}(\tau_\omega(v_i)) \leq \beta(v) \text{ for all } i \in N]; \tag{89}$$

$$[O_{v_i}(\tau_\omega(v_i)) > \beta(v) \text{ for some } i \in N] \Rightarrow \quad [X_\omega \cap \{\tau_\omega(v_1) : \omega \in \Omega_v^*\} = \emptyset]. \tag{90}$$

Observation 2. The set of (p-compatible) valuation profiles obtains as the disjoint union $\mathcal{V}_p^N = \bigcup_{k=0}^{K} \mathcal{V}_{p,k}^N$, where $\mathcal{V}_{p,k}^N := \{v \in \mathcal{V}_p^N : \mu(v) = k\}$ for all $k = 0, 1, \ldots, K$.

The proof of Lemma 4* proceeds by backward induction over $\mu(v)$. First remark that, for all $v \in \mathcal{V}_{p,K}^N$, the statement of Lemma 4* holds by unanimity: all agents agree on the best feasible act $f_v^* = (\tau_\omega(v_1))_{\omega \in \Omega}$, which must then be chosen regardless of the valuations of the outcomes that are not constrained tops.

Next, consider $v \in \mathcal{V}_{p,k}^N$ for some fixed $k \in \{0, \ldots, K-1\}$ and assume by induction that, for any $v, w \in \mathcal{V}_{p,k+1}^N$ such that $\tau_\omega(v_i) = \tau_\omega(w_i)$ for all $\omega \in \Omega$ and $i \in N$, we have $\varphi(v) = \varphi(w) \in \times_{\omega \in \Omega}\{\tau_\omega(v_1), \ldots, \tau_\omega(v_n)\}$.

For all $v \in \mathcal{V}_{p,k}^N$, let $a_1^v := O_{v_1}^{-1}(\beta(v)+1)$. That is, a_1^v is agent 1's next best outcome after all those that are unanimous constrained tops in the states belonging to Ω_v^*. Remark from (89)-(90) and $k < K$ that for all $v \in \mathcal{V}_{p,k}^N$,

$$\tau_\omega(v_1) = a_1^v \quad \text{for some } \omega \in \Omega,$$
$$O_{v_i}(a_1^v) \geq \beta(v) + 1 \quad \text{for all } i \in N.$$

For all $v \in \mathcal{V}_{p,k}^N$, define $r_i(v) = |\{x \in X : \beta(v)+1 > O_{v_i}(x) > O_{v_i}(a_1^v)\}|$ and $r(v) = \sum_{i \in N} r_i(v)$. By definition, we have $r(v) = 0$ if the outcome a_1^v is ranked $(\beta(v)+1)$th or $(\beta(v)+2)$th by every agent i at profile v. Letting $\bar{r} = \max\{r(v) : v \in \mathcal{V}_{p,k}^N\}$, define

$$\mathcal{V}_{p,k}^N(\rho) = \{v \in \mathcal{V}_{p,k}^N |\ r(v) \leq \rho\}$$

and note that we have the disjoint union $\mathcal{V}_{p,k}^N = \bigcup_{\rho=0}^{\bar{r}} \mathcal{V}_{p,k}^N(\rho)$.

The argument can now be completed by induction over $r(v)$. From this point on, one simply needs to repeat the procedures described in Step 2.1 and Step 2.2 of the proof of Lemma 4 (see Appendix 2.A) in order to conclude that, for any $v, w \in \mathcal{V}_{p,k}^N$ such that $\tau_\omega(v_i) = \tau_\omega(w_i)$ for all $\omega \in \Omega$ and $i \in N$, we have $\varphi(v) = \varphi(w) \in \times_{\omega\in\Omega}\{\tau_\omega(v_1), \ldots, \tau_\omega(v_n)\}$. □

Conclusion of the proof of the Constrained-Top Selection lemma

Given a collection of feasible acts $Y_1, ..., Y_n \in \times_{\omega\in\Omega} X_\omega$, Lemma 4* allows us to abuse notation and write $\varphi(Y_1, ..., Y_n)$ to refer to the act $\varphi(v, p)$ chosen at any profile $v \in \mathcal{V}_p^N$ such that $\tau_\omega(v_i) = Y_i(\omega)$ for all $\omega \in \Omega$ and $i \in N$. Call a feasible act $A \in \times_{\omega\in\Omega} X_\omega$ *minimal* if there exists no $B \in \times_{\omega\in\Omega} X_\omega$ such that $B(\Omega) \subset A(\Omega)$. Denote by $\mathcal{M}(\times_{\omega\in\Omega} X_\omega)$ the set of minimal acts.[6]

We are now ready to construct $s(p)$, the assignment of states to agents at the belief profile p. Given that p is fixed, we write s instead of $s(p)$. For all $A, B \in \mathcal{M}(\times_{\omega\in\Omega} X_\omega)$ such that $\{\omega \in \Omega : A(\omega) = B(\omega)\} = \emptyset$, let us define

$$s_1^{AB} := \{\omega \in \Omega : \varphi(A, B, ..., B; \omega) = A(\omega)\}.$$

In words, s_1^{AB} is the set of states of nature ω where the social act yields outcome $A(\omega)$ when agent 1's favorite (feasible) act is A and every other agent's is B, which disagrees with A in every state. Define s_i^{AB} in a similar way for every agent $i \in N$ and write $s^{AB} = (s_1^{AB}, \ldots, s_n^{AB})$. One can then generalize the five steps described in the conclusion of the proof of the Top Selection lemma as follows. We omit the proofs, which are easy adaptations of their counterparts.

Step 1. For all $A, B, C, D \in \mathcal{M}(\times_{\omega\in\Omega} X_\omega)$ such that $A(\omega), C(\omega) \neq B(\omega), D(\omega)$ for all $\omega \in \Omega$, we have (i) $s^{AB} = s^{CB}$ and (ii) $s^{AB} = s^{AD}$.

This means that s^{AB} is in fact independent of the choice of A and B. Define then

$$s = s^{AB} \text{ for all } A, B \in \mathcal{M}(\times_{\omega\in\Omega} X_\omega) \text{ s.t. } \{\omega \in \Omega : A(\omega) = B(\omega)\} = \emptyset.$$

Step 2. $\bigcup_{\omega\in\Omega} \varphi^{C(\omega)}(Y_1, \ldots, Y_{j-1}, C, Y_{j+1}, \ldots, Y_n) = s_j$ for all $j \in N$ and all $Y_1, \ldots, Y_{j-1}, C, Y_{j+1}, \ldots, Y_n \in \times_{\omega\in\Omega} X_\omega$ such that $Y_1(\omega), \ldots, Y_{j-1}(\omega), Y_{j+1}(\omega), \ldots, Y_n(\omega) \neq C(\omega)$ for all $\omega \in \Omega$.

Step 3. $s_i \cap s_j = \emptyset$ for all distinct $i, j \in N$.

Step 4. $\varphi^x(Y_1, \ldots, Y_n) = \bigcup_{i\in N} \{\omega \in s_i : Y_i(\omega) = x\}$ for all $x \in X$ and all $Y_1, \ldots, Y_n \in \times_{\omega\in\Omega} X_\omega$.

[6] Note in particular that a minimal act must be constant if $X_\omega = X$ for all $\omega \in \Omega$.

Step 5. $s \in \mathcal{S}$.

The steps above prove that s is an assignment rule generating φ. It is obvious that any other assignment rule generates a SCF different from φ, and the proof of the Constrained-Top Selection lemma is complete. □

Appendix 2.E.2: Proof of Theorem 2

The proof of the "if" statement in Theorem 2 is again just a matter of checking. To prove the "only if" statement, we first extend the Super-strategyproofness lemma. The definition of a super-strategyproof assignment rule is unchanged.

Constrained Super-strategyproofness Lemma. *The assignment rule s associated with a strategyproof and unanimous SCF $\varphi : \mathcal{D}^N \to \times_{\omega\in\Omega} X_\omega$ is super-strategyproof.*

Proof. Let φ: $\mathcal{D}^N \to \times_{\omega\in\Omega} X_\omega$ be a strategyproof and unanimous SCF and let s be the assignment rule associated with it. Suppose by way of contradiction that there exist $i \in M \subset N$, $p \in \mathcal{P}^N$ and $\hat{p}_i \in \mathcal{P}$ such that

$$p_i(s_M(\hat{p}_i, p_{-i})) > p_i(s_M(p)). \tag{91}$$

In the remainder of this argument, since p_{-i} is fixed, we write $p_i(s_M(p_i))$ and $p_i(s_M(\hat{p}_i))$ instead of $p_i(s_M(p))$ and $p_i(s_M(\hat{p}_i, p_{-i}))$.

Case 1. There exists an act $f \in \times_{\omega\in\Omega} X_\omega$ such that $X_\omega \setminus f(\Omega) \neq \varnothing$ for all $\omega \in \Omega$.

Pick such an act f. Fix $0 < \varepsilon < 1/2$. Consider a valuation profile $v^\varepsilon \in \mathcal{V}^N$ where all agents in M share a common valuation function v^ε_M, all agents in share a common valuation function $v^\varepsilon_{N\setminus M}$, and these two valuation functions are such that

$$\begin{aligned} v^\varepsilon_M(x) &> 1-\varepsilon \text{ for all } x \in f(\Omega), & (92)\\ v^\varepsilon_M(x) &< \varepsilon \text{ for all } x \in X \setminus f(\Omega), & (93)\\ v^\varepsilon_{N\setminus M}(x) &= 1 - v_M(x) \text{ for all } x \in X. & (94) \end{aligned}$$

Let $g = \varphi(v^\varepsilon, p_i)$ and $\widehat{g} = \varphi(v^\varepsilon, \hat{p}_i)$.

Since $f(\omega) \in X_\omega$ for all $\omega \in \Omega$, (92) guarantees that $v^\varepsilon_i(\tau_\omega(v^\varepsilon_M)) \geq v^\varepsilon_M(f(\omega)) > 1-\varepsilon$ for all $\omega \in \Omega$. Hence,

$$E^{p_i}_{v^\varepsilon_i}(\widehat{g}) > p_i(s_M(\hat{p}_i))(1-\varepsilon).$$

On the other hand, since $X_\omega \setminus f(\Omega) \neq \varnothing$ for all $\omega \in \Omega$, (92), (93), and (94) imply that $v^\varepsilon_i(\tau_\omega(v^\varepsilon_{N\setminus M})) < \varepsilon$ for all $\omega \in s_{N\setminus M}(p_i)$. Hence,

$$E^{p_i}_{v^\varepsilon_i}(g) \leq p_i(s_M(p_i))1 + p_i(s_{N\setminus M}(p_i))\varepsilon.$$

Therefore

$$E^{p_i}_{v^\varepsilon_i}(\widehat{g}) - E^{p_i}_{v^\varepsilon_i}(g) > [p_i(s_M(\hat{p}_i)) - p_i(s_M(p_i))] - \varepsilon \left[p_i(s_M(\hat{p}_i)) + p_i(s_{N\setminus M}(p_i))\right]$$

and (91) implies that $E^{p_i}_{v^\varepsilon_i}(\widehat{g}) - E^{p_i}_{v^\varepsilon_i}(g) > 0$ when ε is small enough, contradicting the assumption that φ is strategyproof.

Case 2. For every act $f \in \times_{\omega\in\Omega}$, there exists some $\omega \in \Omega$ such that $X_\omega \setminus f(\Omega) = \emptyset$.

Let $f_0 \in \times_{\omega\in\Omega}$ be a *minimal* feasible act (in the sense that there is no $f \in \times_{\omega\in\Omega}X_\omega$ such that $f(\Omega) \subset f_0(\Omega)$). By the assumption defining Case 2, there is a nonempty set of states $\Omega^* \subseteq \Omega$ such that $X_\omega \subseteq f_0(\Omega)$ for all $\omega \in \Omega^*$ and $X_\omega \nsubseteq f_0(\Omega)$ for all $\omega \in \Omega \setminus \Omega^*$.

Write $\Omega^* = \{\omega_1, ..., \omega_{T^*}\}$. For each $t = 1, ..., T^*$, choose two distinct outcomes $a_t, b_t \in X_{\omega_t}$ and define

$$\Omega_t := \{\omega \in \Omega : X_\omega \cap \bigcup_{t'=1}^{t-1}\{a_{t'}, b_{t'}\} = \varnothing \text{ and } X_\omega \cap \{a_t, b_t\} \neq \varnothing\}.$$

Note in particular that (i) $\Omega_1 = \{\omega \in \Omega : \{a_1, b_1\} \cap X_\omega \neq \emptyset\} \neq \emptyset$; (ii) some Ω_t may be empty (for $t = 2, \ldots, T^*$). Let then $T \in \{1, \ldots, T^*\}$ be the *number of nonempty subsets* Ω_t (for $t = 1, \ldots, T^*$) and, without loss of generality, label as $\Omega_1, \ldots, \Omega_T$ these T nonempty subsets of Ω. Furthermore, define the (possibly empty) set $\tilde{\Omega} := \Omega \setminus \cup_{t=1}^{T}\Omega_t$. By construction, $\left\{\Omega_1, ..., \Omega_T, \tilde{\Omega}\right\}$ is a partition of Ω and $\{a_t, b_t\} \cap \{a_{t'}, b_{t'}\} = \emptyset$ for all distinct $t, t' \in \{1, ..., T\}$. Moreover, by definition of Ω^*, we have

$$\bigcup_{t'=1}^{T}\{a_{t'}, b_{t'}\} \subseteq f_0(\Omega). \tag{95}$$

For any event $E \subseteq \Omega$ and any subset of agents $K \subseteq N$, we will use the shorthand notation $p^E_{iK} := p_i(s_K(p_i) \cap E)$ and $\hat{p}^E_{iK} := p_i(s_K(\hat{p}_i) \cap E)$. Let us rewrite (91) as follows:

$$\underbrace{(\hat{p}^{\Omega_1}_{iM} - p^{\Omega_1}_{iM})}_{=:\delta_1} + \ldots + \underbrace{(\hat{p}^{\Omega_T}_{iM} - p^{\Omega_T}_{iM})}_{=:\delta_T} + \underbrace{(\hat{p}^{\tilde{\Omega}}_{iM} - p^{\tilde{\Omega}}_{iM})}_{=:\tilde{\delta}} > 0. \tag{96}$$

Therefore, we have $\delta_t > 0$ (for some $t = 1, \ldots, T$) or $\tilde{\delta} > 0$.

Case 2.1. $\delta_t > 0$ for some $t = 1, \ldots, T$.

We claim first that

$$X \setminus \bigcup_{t'=1}^{t} \{a_{t'}, b_{t'}\} \neq \emptyset. \tag{97}$$

To see why (97) holds, suppose on the contrary that $X \subseteq \cup_{t'=1}^{t}\{a_{t'}, b_{t'}\}$. Then, for all $\omega \in \Omega$, we have $X_\omega \subseteq X \subseteq \cup_{t'=1}^{t}\{a_{t'}, b_{t'}\} \subseteq f_0(\Omega)$, where the last inclusion follows from (95). For each $\omega \in \Omega$, pick some $x_\omega \in X_\omega \setminus a_1 \subseteq f_0(\Omega)$. Define the feasible act $f \in \times_{\omega\in\Omega} X_\omega$ by $f(\omega) = x_\omega$ if $f_0(\omega) = a_1$ and $f(\omega) = f_0(\omega)$ otherwise. Then $f(\Omega) = f_0(\Omega) \setminus a_1$, which contradicts the minimality of f_0.

Let now $\Omega_t^0 := \{\omega \in \Omega_t : \{a_t, b_t\} \nsubseteq X_\omega\}$ and $\Omega_t^1 := \{\omega \in \Omega_t : \{a_t, b_t\} \subseteq X_\omega\}$. Define $\delta_t^0 = (\hat{p}_{iM}^{\Omega_t^0} - p_{iM}^{\Omega_t^0})$ and $\delta_t^1 = (\hat{p}_{iM}^{\Omega_t^1} - p_{iM}^{\Omega_t^1})$. Since $\delta_t = \delta_t^0 + \delta_t^1 > 0$, we have $\delta_t^0 > 0$ or $\delta_t^1 > 0$.

Subcase 2.1.1. $\delta_t^0 > 0$.
Fix $0 < \varepsilon < 1/3$. Consider a valuation profile $v^\varepsilon \in \mathcal{V}^N$ where all $j \in M$ share a common valuation function v_M^ε, all $j \in N \setminus M$ share a common valuation function $v_{N\setminus M}^\varepsilon$, and

$$v_M^\varepsilon(x) = v_{N\setminus M}^\varepsilon(x) > 1 - \varepsilon \qquad \text{for all } x \in \bigcup_{t'=1}^{t-1}\{a_{t'}, b_{t'}\}, \tag{98}$$

$$v_M^\varepsilon(x) \in (1 - 2\varepsilon, 1 - \varepsilon) \qquad \text{if } x \in \{a_t, b_t\}, \tag{99}$$

$$v_{N\setminus M}^\varepsilon(x) = 1 - v_M^\varepsilon(x) \qquad \text{if } x \in \{a_t, b_t\}, \tag{100}$$

$$v_M^\varepsilon(x) = v_{N\setminus M}^\varepsilon(x) < \varepsilon \qquad \text{for all } x \in X \setminus \bigcup_{t'=1}^{t}\{a_{t'}, b_{t'}\}. \tag{101}$$

Such a profile exists because (97) guarantees the existence of an outcome to which v_M^ε and $v_{N\setminus M}^\varepsilon$ may assign valuation zero.

Let $g = \varphi(v^\varepsilon, p_i)$ and $\widehat{g} = \varphi(v^\varepsilon, \hat{p}_i)$. From (98) we have $v_i^\varepsilon(\widehat{g}(\omega)) > 1 - \varepsilon$ for all $\omega \in \cup_{t'=1}^{t-1}\Omega_{t'}$. This is because φ is the constrained-top selection generated by s and, at the profile v^ε and in any state $\omega \in \cup_{t'=1}^{t-1}\Omega_{t'}$, agent i attaches a value of at least $1 - \varepsilon$ to the constrained top of *every* agent j —since this top belongs to $\cup_{t'=1}^{t-1}\{a_{t'}, b_{t'}\}$. Next, for all $\omega \in \Omega \setminus \cup_{t'=1}^{t}\Omega_{t'}$, (101) guarantees that $v_i^\varepsilon(g(\omega)) < \varepsilon$ since $X_\omega \cap \cup_{t'=1}^{t}\{a_{t'}, b_{t'}\} = \emptyset$. Finally, (99) and (100) imply that (i) $v_i^\varepsilon(\widehat{g}(\omega)) > 1 - 2\varepsilon$ for all $\omega \in (\Omega_t^0 \cap s_M(\widehat{p}_i)) \cup \Omega_t^1$ and $v_i^\varepsilon(\widehat{g}(\omega)) > \varepsilon$ for all $\omega \in \Omega_t^0 \cap s_{N\setminus M}(\widehat{p}_i)$, and (ii) $v_i^\varepsilon(g(\omega)) < 1 - \varepsilon$ for all $\omega \in (\Omega_t^0 \cap s_M(p_i)) \cup \Omega_t^1$ and $v_i^\varepsilon(g(\omega)) < 2\varepsilon$ for all

$\omega \in \Omega_t^0 \cap s_{N\setminus M}(p_i)$. Combining the above observations thus gives

$$E_{v_i^\varepsilon}^{p_i}(\widehat{g}) > \sum_{t'=1}^{t-1} p_i(\Omega_{t'})(1-\varepsilon) + \hat{p}_{iM}^{\Omega_t^0}(1-2\varepsilon) + \hat{p}_{iN\setminus M}^{\Omega_t^0}\varepsilon + p_i(\Omega_t^1)(1-2\varepsilon) + (1-\sum_{t'=1}^{t} p_i(\Omega_{t'}))0,$$

$$E_{v_i^\varepsilon}^{p_i}(g) < \sum_{t'=1}^{t-1} p_i(\Omega_{t'})1 \quad + \quad p_{iM}^{\Omega_t^0}(1-\varepsilon) \quad + p_{iN\setminus M}^{\Omega_t^0}2\varepsilon + p_i(\Omega_t^1)(1-\varepsilon) + (1-\sum_{t'=1}^{t} p_i(\Omega_{t'}))\varepsilon.$$

Taking the difference, one thus gets

$$E_{v_i^\varepsilon}^{p_i}(\widehat{g}) - E_{v_i^\varepsilon}^{p_i}(g) > \overbrace{(\hat{p}_{iM}^{\Omega_t^0} - p_{iM}^{\Omega_t^0})}^{\delta_t^0>0} - \varepsilon\theta(p_i, \widehat{p}_i) > 0$$

for ε small enough, which is a contradiction to the strategyproofness of φ.

Subcase 2.1.2. $\delta_t^1 > 0$.
In this case, consider a valuation profile $v^\varepsilon \in \mathcal{V}^N$ where all $j \in M$ share a common valuation function v_M^ε, all $j \in N \setminus M$ share a common valuation function $v_{N\setminus M}^\varepsilon$, and

$$\begin{aligned} v_M^\varepsilon = v_{N\setminus M}^\varepsilon(x) > 1-\varepsilon \qquad & \text{for all } x \in \bigcup_{t'=1}^{t-1}\{a_{t'}, b_{t'}\}, \\ v_M^\varepsilon(a^t) = 1-\varepsilon \qquad & \text{and } v_M^\varepsilon(b^t) \in (\varepsilon, 2\varepsilon), \\ v_{N\setminus M}^\varepsilon(x) = 1 - v_M^\varepsilon(x) \qquad & \text{if } x \in \{a_t, b_t\}, \\ v_M^\varepsilon = v_{N\setminus M}^\varepsilon(x) < \varepsilon \qquad & \text{for all } x \in X \setminus \bigcup_{t'=1}^{t}\{a_{t'}, b_{t'}\}. \end{aligned}$$

Using observations similar to those of Subcase 2.1.1, it is not difficult to verify that

$$E_{v_i^\varepsilon}^{p_i}(\varphi(v^\varepsilon, \hat{p}_i)) - E_{v_i^\varepsilon}^{p_i}(\varphi(v^\varepsilon, p_i)) > \overbrace{(\hat{p}_{iM}^{\Omega_t^1} - p_{iM}^{\Omega_t^1})}^{\delta_t^1>0} - \varepsilon\theta(p_i, \widehat{p}_i) > 0$$

for ε small enough, which contradicts the strategyproofness of φ.

Case 2.2. $\tilde{\delta} > 0$.
Recall from the definition of the partition $\left\{\Omega_1, \ldots, \Omega_T, \tilde{\Omega}\right\}$ that $X_\omega \cap \bigcup_{t'=1}^{T}\{a_{t'}, b_{t'}\} = \varnothing$ and $X_\omega \nsubseteq f_0(\Omega)$ for all $\omega \in \tilde{\Omega}$. For every $\omega \in \tilde{\Omega}$, select some $f(\omega) \in X_\omega \setminus f_0(\Omega)$ and consider a valuation profile $v^\varepsilon \in \mathcal{V}^N$ where all $j \in M$ share a common valuation

function v^{ε}_M, all $j \in N \setminus M$ share a common valuation function $v^{\varepsilon}_{N\setminus M}$, and

$$\begin{aligned} v^{\varepsilon}_M(x) = v^{\varepsilon}_{N\setminus M}(x) > 1-\varepsilon \quad & \text{for all } x \in \bigcup_{t'=1}^{T}\{a_{t'}, b_{t'}\}, \\ v^{\varepsilon}_M(x) \in (1-2\varepsilon, 1-\varepsilon) \quad & \text{for all } x \in f_0(\widetilde{\Omega}), \\ v^{\varepsilon}_M(x) \in (\varepsilon, 2\varepsilon) \quad & \text{for all } x \in f(\widetilde{\Omega}), \\ v^{\varepsilon}_{N\setminus M}(x) = 1 - v^{\varepsilon}_M(x) \quad & \text{for all } x \in f_0(\widetilde{\Omega}) \cup f(\widetilde{\Omega}); \\ v^{\varepsilon}_M(x) = v^{\varepsilon}_{N\setminus M}(x) < \varepsilon \quad & \text{for all } x \notin \bigcup_{t'=1}^{T}\{a_{t'}, b_{t'}\} \cup f_0(\widetilde{\Omega}) \cup f(\widetilde{\Omega}). \end{aligned}$$

Once again, one checks that

$$E^{p_i}_{v^{\varepsilon}_i}(\varphi(v^{\varepsilon}, \hat{p}_i)) - E^{p_i}_{v^{\varepsilon}_i}(\varphi(v^{\varepsilon}, p_i)) > \overbrace{(\hat{p}^{\tilde{\Omega}}_{iM} - p^{\tilde{\Omega}}_{iM})}^{\tilde{\delta}>0} - \varepsilon\theta(p_i, \widehat{p}_i) > 0$$

for ε small enough, which violates the strategyproofness of φ. □

The last part of the proof of Theorem 1 consisted in establishing the fact that *every super-strategyproof assignment rule is locally bilateral.* Note that this fact is "model-free" as the definition of an assignment rule is unaffected by the presence of a Cartesian constraint over the set of acts the social planner can choose from. The combination of the Constrained-Top Selection lemma, the Constrained Super-strategyproofness lemma, and the above fact yields that every strategyproof and unanimous SCF $\varphi : \mathcal{D}^N \to \times_{\omega\in\Omega} X_\omega$ is a constrained-top selection whose associated assignment rule is locally bilateral.

To complete the proof of Theorem 2, it remains to be shown that this locally bilateral assignment rule must be iso-constrained. This is the purpose of our last lemma.

Constraint Lemma. *Let s be a locally bilateral assignment rule with canonical partition $\left\{\Omega^1, ..., \Omega^T\right\}$ and let $\varphi : \mathcal{D}^N \to \times_{\omega\in\Omega} X_\omega$ be the constrained-top selection generated by s. If φ is strategyproof, $t \in \{1, ..., T\}$, and s^t is not a constant Ω^t-assignment rule, then $X_\omega = X_{\omega'}$ for all $\omega, \omega' \in \Omega^t$.*

Proof. Let s be a locally bilateral assignment rule with canonical partition $\left\{\Omega^1, ..., \Omega^T\right\}$ and let φ: $\mathcal{D}^N \to \times_{\omega\in\Omega} X_\omega$ be the constrained-top selection generated by s. Suppose φ is strategyproof, fix t, say, $t = 1$, and suppose s^1 is not a constant Ω^1-assignment rule.

Case 1. s^1 is a bilaterally dictatorial Ω^1-assignment rule, say, a $(1, 2)$-dictatorial one.

Let $\mathcal{A}^1 = \{A^1_1, ..., A^1_M\}$ be the proper covering of Ω^1 associated with s^1. We suppose that $X_\omega \neq X_{\omega'}$ for some $\omega, \omega' \in \Omega^1$ and show that φ is manipulable.

For all $x \in X$, let $\Omega^1_+(x) = \{\omega \in \Omega^1 : x \in X_\omega\}$ and $\Omega^1_-(x) = \{\omega \in \Omega^1 : x \notin X_\omega\}$. Let $\bar{x} \in X$ be such that $\Omega^1_+(\bar{x}) \neq \varnothing$ and $\Omega^1_-(\bar{x}) \neq \varnothing$.

Step 1. We show that there exist $m, m' \in \{1, ..., M\}$ such that

$$(A^1_m \bigtriangleup A^1_{m'}) \cap \Omega^1_+(\bar{x}) \neq \varnothing \text{ and } (A^1_m \bigtriangleup A^1_{m'}) \cap \Omega^1_-(\bar{x}) \neq \varnothing,$$

where $\bigtriangleup$ is the symmetric difference operator.

This is obvious if $M \leq 2$, so assume $M \geq 3$. Contrary to the claim, suppose that for all $m, m' \in \{1, ..., M\}$, we have

$$A^1_m \bigtriangleup A^1_{m'} \subseteq \Omega^1_+(\bar{x}) \text{ or } A^1_m \bigtriangleup A^1_{m'} \subseteq \Omega^1_-(\bar{x}). \tag{102}$$

Without loss of generality, assume

$$A^1_1 \bigtriangleup A^1_2 \subseteq \Omega^1_+(\bar{x}). \tag{103}$$

We begin by showing that

$$A^1_1 \cap A^1_2 \subseteq \Omega^1_+(\bar{x}). \tag{104}$$

Suppose, on the contrary, that there exists $\omega \in A^1_1 \cap A^1_2 \cap \Omega^1_-(\bar{x})$. Since by definition of a proper covering $\cap^M_{m=1} A^1_m = \varnothing$, there exists $m^* \in \{3, ..., M\}$ such that $\omega \notin A^1_{m^*}$. Since $\omega \in A^1_m \bigtriangleup A^1_{m^*} \cap \Omega^1_-(\bar{x})$ for $m = 1, 2$, (102) implies

$$A^1_m \bigtriangleup A^1_{m^*} \subseteq \Omega^1_-(\bar{x}) \text{ for } m = 1, 2. \tag{105}$$

Inclusions (103) and (105) imply $A^1_1 \bigtriangleup A^1_2 = \varnothing$, contradicting the fact that $\mathcal{A}^1$ is a proper covering of Ω^1.

Next, we show that

$$\Omega^1 \setminus (A^1_1 \cup A^1_2) \subseteq \Omega^1_+(\bar{x}). \tag{106}$$

Suppose, contrary to the claim, that there exists $\omega \in \Omega^1_-(\bar{x}) \setminus (A^1_1 \cup A^1_2)$. Then there exists $m^* \in \{3, ..., M\}$ such that $\omega \in A_{m^*}$. From (102), $A^1_1 \bigtriangleup A^1_{m^*} \subseteq \Omega^1_-(\bar{x})$. But (103) and (104) imply $A^1_1 \subseteq \Omega^1_+(\bar{x})$. Therefore $A^1_1 \setminus A^1_{m^*} = \varnothing$, contradicting the fact that $\mathcal{A}^1$ is a proper covering.

From (103), (104), and (106) we conclude $\Omega^1 = \Omega^1_+(\bar{x})$, contradicting the fact that $\Omega^1_-(\bar{x}) \neq \varnothing$.

Step 2. Given Step 1, we may assume without loss of generality that

$$(A^1_1 \bigtriangleup A^1_2) \cap \Omega^1_+(\bar{x}) \neq \varnothing \text{ and } (A^1_1 \bigtriangleup A^1_2) \cap \Omega^1_-(\bar{x}) \neq \varnothing.$$

Because $A^1_1 \setminus A^1_2$ and $A^1_2 \setminus A^1_1$ are nonempty, there is also no loss in further assuming that

$$(A^1_1 \setminus A^1_2) \cap \Omega^1_+(\bar{x}) \neq \varnothing \text{ and } (A^1_2 \setminus A^1_1) \cap \Omega^1_-(\bar{x}) \neq \varnothing.$$

Let thus $\omega_1 \in (A^1_1 \setminus A^1_2) \cap \Omega^1_+(\bar{x})$ and $\omega_2 \in (A^1_2 \setminus A^1_1) \cap \Omega^1_-(\bar{x})$.

Choose two distinct outcomes $x_1, x_2 \in X_{\omega_2}$. Since $\omega_2 \in \Omega^1_-(\bar{x})$, we have $\bar{x} \notin X_{\omega_2}$, so that $\bar{x}, x_1, x_2$ are all distinct. Since $\omega_1 \in \Omega^1_+(\bar{x})$, we have $\bar{x} \in X_{\omega_1}$.

Fix $\varepsilon > 0$ and consider a profile $(v,p) \in \mathcal{D}^N$ such that

$$\begin{aligned} p_1(\Omega \setminus \{\omega_1, \omega_2\}) &= \varepsilon, \\ \arg\max_{\mathcal{A}^1} p_1 &= A^1_1, \\ v_1(\bar{x}) &= 1 > v_1(x_1) > 0 = v_1(x_2), \\ v_2(\bar{x}) &= 1 > v_2(x_2) > v_2(x) \text{ for all } x \in X \setminus \{\bar{x}, x_2\}. \end{aligned}$$

By reporting truthfully (v_1, p_1), agent 1 gets an expected utility of at most

$$\begin{aligned} & p_1(\omega_1)v_1(\tau_{\omega_1}(v_1)) + p_1(\omega_2)v_1(\tau_{\omega_2}(v_2)) + p_1(\Omega \setminus \{\omega_1, \omega_2\}) \\ = \; & p_1(\omega_1)v_1(\bar{x}) + p_1(\omega_2)v_1(x_2) + \varepsilon \\ = \; & p_1(\omega_1) + \varepsilon. \end{aligned}$$

Consider a belief q_1 such that $(v_1, q_1) \in \mathcal{D}$ and $\arg\max_{\mathcal{A}^1} q_1 = A^1_2$. By reporting (v_1, q_1), agent 1 gets an expected utility of at least

$$\begin{aligned} & p_1(\omega_1)v_1(\tau_{\omega_1}(v_2)) + p_1(\omega_2)v_1(\tau_{\omega_2}(v_1)) \\ = \; & p_1(\omega_1)v_1(\bar{x}) + p_1(\omega_2)v_1(x_1) \\ = \; & p_1(\omega_1) + p_1(\omega_2)v_1(x_1). \end{aligned}$$

Thus φ is manipulable at (v,p) when $\varepsilon < p_1(\omega_2)v_1(x_1)$.

Case 2. s^1 is a bilaterally consensual Ω^1-assignment rule, say, a $(1,2)$-consensual Ω^1-assignment rule with default $A^1 \subset \Omega^1$. Again, we suppose that $X_\omega \neq X_{\omega'}$ for some $\omega, \omega' \in \Omega^1$ and show that φ is manipulable. Let $\bar{x} \in X$ be such that $\Omega^1_+(\bar{x}) \neq \varnothing$ and $\Omega^1_-(\bar{x}) \neq \varnothing$.

Case 2.1. $A^1 \cap \Omega^1_+(\bar{x}) \neq \varnothing$ and $(\Omega^1 \setminus A^1) \cap \Omega^1_-(\bar{x}) \neq \varnothing$.

Let $\omega_1 \in A^1 \cap \Omega^1_+(\bar{x})$ and $\omega_2 \in (\Omega^1 \setminus A^1) \cap \Omega^1_-(\bar{x})$. Choose $x_1, x_2 \in X_{\omega_1}$, fix $\varepsilon > 0$, and let $(v,p) \in \mathcal{D}^N$ be a profile such that

$$\begin{aligned} p_1(\Omega \setminus \{\omega_1, \omega_2\}) &= \varepsilon, \\ p_1(A^1) &> p_1(\Omega^1 \setminus A^1), \\ p_2(A^1) &> p_2(\Omega^1 \setminus A^1), \\ v_1(\bar{x}) &= 1 > v_1(x_1) > 0 = v_1(x_2), \\ v_2(\bar{x}) &= 1 > v_2(x_2) > v_2(x) \text{ for all } x \in X \setminus \{\bar{x}, x_2\}. \end{aligned}$$

Let q_1 be a belief such that $(v_1, q_1) \in \mathcal{D}$ and $q_1(A^1) < q_1(\Omega^1 \setminus A^1)$ and check that agent 1 gains from reporting (v_1, q_1) instead of (v_1, p_1) when $\varepsilon < p_1(\omega_2)v_1(x_1)$.

Case 2.2. $A^1 \cap \Omega^1_-(\bar{x}) \neq \varnothing$ and $(\Omega^1 \setminus A^1) \cap \Omega^1_+(\bar{x}) \neq \varnothing$.

Let $\omega_1 \in A^1 \cap \Omega^1_-(\bar{x})$ and $\omega_2 \in (\Omega^1 \setminus A^1) \cap \Omega^1_+(\bar{x})$. Choose $x_1, x_2 \in X_{\omega_2}$, fix $\varepsilon > 0$, and let $(v, p) \in \mathcal{D}^N$ be a profile where

$$\begin{aligned} p_1(\Omega \setminus \{\omega_1, \omega_2\}) &= \varepsilon, \\ p_1(A^1) &< p_1(\Omega^1 \setminus A^1), \end{aligned}$$

and p_2, v_1, v_2 satisfy the same conditions as in Case 2.1. Check that if q_1 is a belief such that $(v_1, q_1) \in \mathcal{D}$ and $q_1(A^1) > q_1(\Omega^1 \setminus A^1)$, then agent 1 gains from reporting (v_1, q_1) instead of (v_1, p_1) when $\varepsilon < p_1(\omega_2)v_1(x_1)$. $\square$

Appendix 2.F: Figures

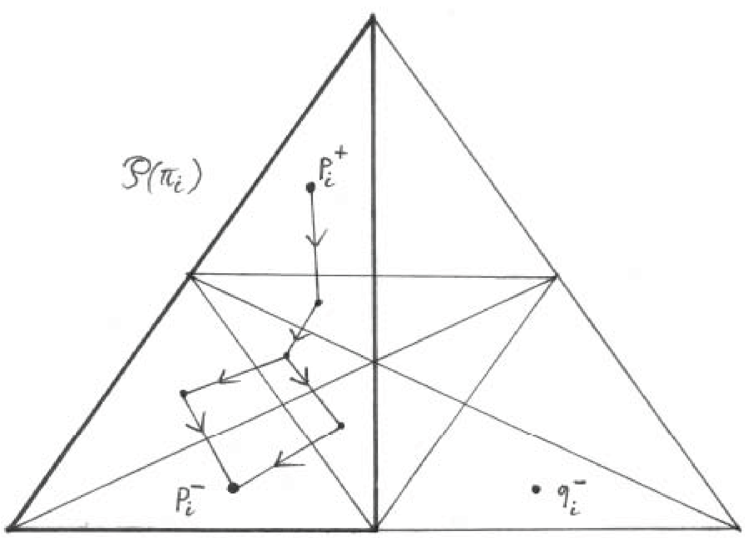


Figure 2: The binary relation $\widetilde{J}$

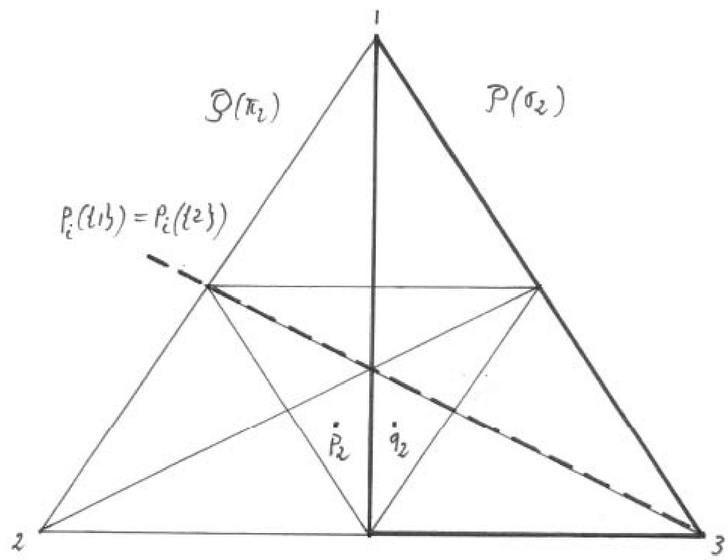


Figure 3: Illustration of the proof of the first contagion lemma

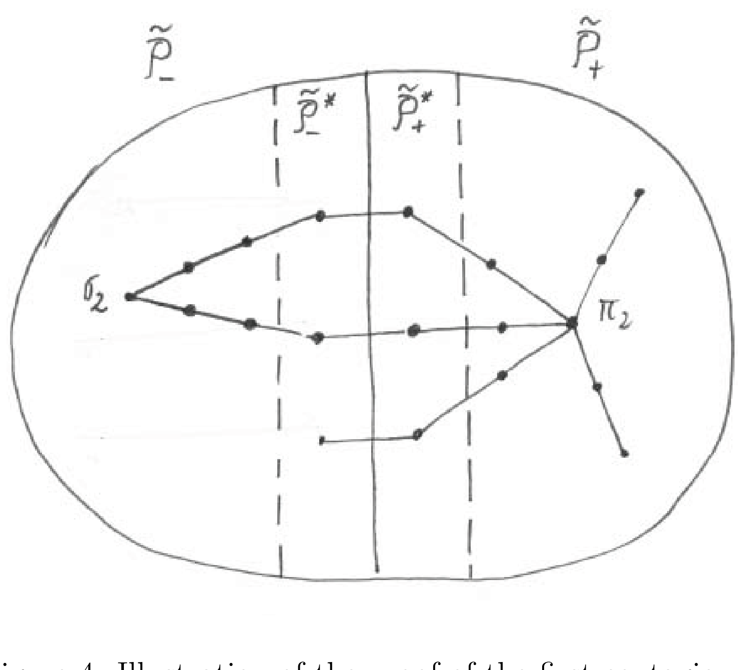


Figure 4: Illustration of the proof of the first contagion corollary

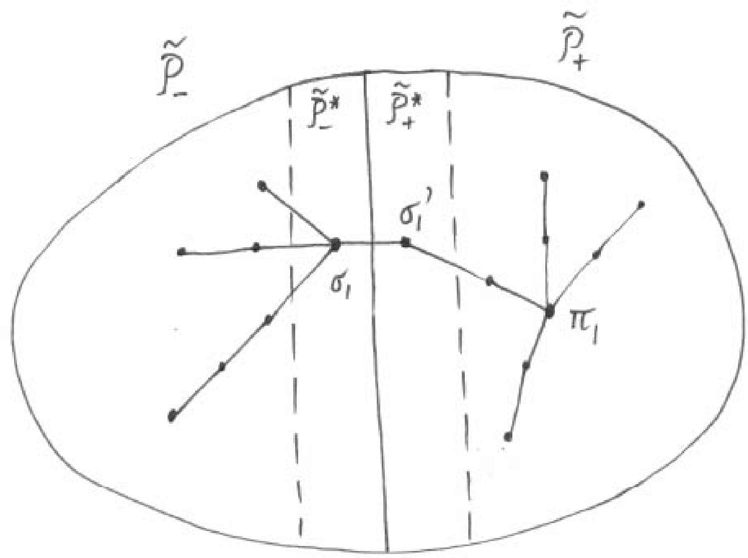


Figure 5: Illustration of the proof of the second contagion corollary

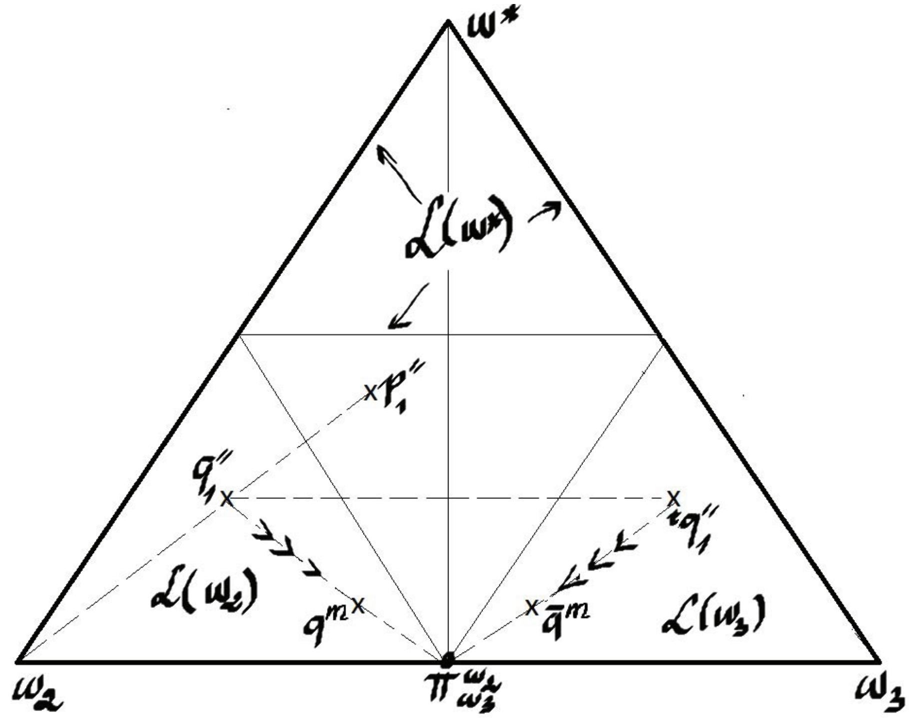
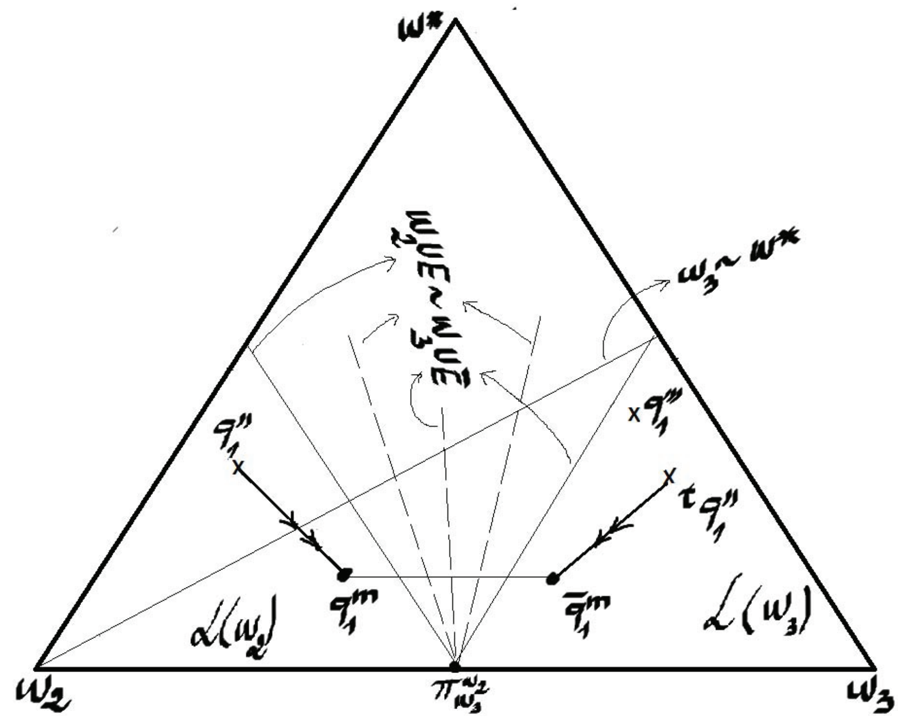


Figure 6: Construction of q_1^m and $\bar{q}_1^m$.



For m large, $[q_1^m, \bar{q}_1^m]$ cuts only hyperplanes of the form $\{\omega_2 \cup E, \omega_3 \cup \bar{E}\}$.
Note in this example that $[q_1'', q_1''']$ — but not $[q_1^m, \bar{q}_1^m]$– cuts $\{\omega_3, \omega^*\} \in \mathcal{H}$.

Figure 7: Hyperplanes cut by $[q_1^m, \bar{q}_1^m]$.