Online Privacy and Information Disclosure by Consumers: Online Appendix

Shota Ichihashi*

August 27, 2019

1 Proof of Proposition 3

Proposition 3 follows from a series of lemmas. Lemma 1 proves vertical efficiency. Lemma 2 proves that any equilibrium is horizontally inefficient whenever x^{MAX} has a unique monopoly price. Lemma 3 proves that this condition is true for generic x_0. The existence of an equilibrium is separately proved later in this Appendix.

Lemma 1. *Under the no-commitment regime, any equilibrium is vertically efficient.*

Proof. Take any disclosure rule (M^*, ϕ^*) that leads to a vertically inefficient allocation given the seller's best response and the consumer's optimal purchase decision with the tie-breaking. (Hereafter, I omit the caveat "given the tie-breaking rule.") Then, ϕ^* draws a posterior $x \in \Delta(V^K)$ at which trade fails to occur with positive probability.[1] Without loss of generality, suppose that given x, the seller recommends product 1 at price v_n. Consider the following disclosure rule ϕ^{**}: On top of the information that ϕ^* discloses, ϕ^{**} also discloses $u_1 \geq v_n$ or $u_1 < v_n$ whenever posterior x is realized.

*Bank of Canada (email: shotaichihashi@gmail.com).

[1]Because $|V^K| < +\infty$, without loss of generality, I can assume $|M^*| < +\infty$. Then, each message is realized with a positive probability from the ex-ante perspective. This implies that there is an ex-ante positive probability event such that some posterior $x \in \Delta(V^K)$ is realized and trade fails to occur.

I show that ϕ^{**} yields a weakly greater consumer surplus and a strictly greater total surplus than ϕ^* does. Let x^+ and $x^- \in \Delta(V^K)$ denote the posterior beliefs of the seller when the consumer discloses $u_1 \geq v_n$ and $u_1 < v_n$ (following x), respectively. Note that for some $\alpha \in (0, 1)$, $x = \alpha x^+ + (1 - \alpha)x^-$. First, consider the consumer's payoff and total surplus conditional on x^-. The consumer obtains a greater payoff under ϕ^{**} than under ϕ^* because consumer surplus is zero under ϕ^*. Total surplus is strictly greater under ϕ^{**} because trade occurs with a positive probability under ϕ^{**} but occurs with zero probability under ϕ^*. Second, I show that the seller continues to recommend product 1 at price v_n given x^+. Suppose to the contrary that the seller strictly prefers to recommend product m at price v_ℓ where $(m, \ell) \neq (1, n)$. Let $x_1^+ \in \Delta(V)$ and $x_m^+ \in \Delta(V)$ denote the marginal distributions of u_1 and u_m given x^+, respectively. Because the seller strictly prefers recommending product m at price v_ℓ to recommending product 1 at price v_n, we get

$$v_\ell \sum_{j=\ell}^{N} x_m^+(v_j) > v_n \sum_{j=n}^{N} x_1^+(v_j),$$

which implies

$$\begin{aligned} &v_\ell \sum_{j=\ell}^{N} \left[\alpha x_m^+(v_j) + (1-\alpha) x_m^-(v_j)\right] \geq v_\ell \sum_{j=\ell}^{N} \alpha x_m^+(v_j) \\ >&v_n \sum_{j=n}^{N} \alpha x_1^+(v_j) = v_n \sum_{j=n}^{N} \left[\alpha x_1^+(v_j) + (1-\alpha) x_1^-(v_j)\right]. \end{aligned}$$

The last equality follows from $x_1^-(v) = 0$ for any $v \geq v_n$. The resulting inequality

$$v_\ell \sum_{j=\ell}^{N} \left[\alpha x_m^+(v_j) + (1-\alpha) x_m^-(v_j)\right] > v_n \sum_{j=n}^{N} \left[\alpha x_1^+(v_j) + (1-\alpha) x_1^-(v_j)\right]$$

contradicts the fact that the seller prefers to recommend product 1 at price v_n at x. Thus, the seller continues to recommend the same product at the same price between x and x^+. Overall, disclosing $u_1 \geq v_n$ or $u_1 < v_n$ at x leads to a weakly greater consumer surplus and a strictly greater total surplus.

To show that any equilibrium is vertically efficient, take any equilibrium disclosure rule ϕ^*. (Later, I prove the existence of an equilibrium.) Suppose to the contrary that ϕ^* is vertically inefficient. Then, I can apply the modification described above to create ϕ^{**}. ϕ^{**} gives the consumer a weakly greater payoff than ϕ^*. Because ϕ^* is optimal for the consumer, he is indifferent between ϕ^* and ϕ^{**}. However, ϕ^{**} yields a strictly greater total surplus, which implies that the seller strictly prefers ϕ^{**}. This contradicts the tie breaking rule, which requires that ϕ^* maximizes the seller's payoffs among all consumer-optimal disclosure rules. Therefore, ϕ^* is vertically efficient. □

Lemma 2. *Suppose that the prior distribution x^0 satisfies Assumption 1 and there is a unique monopoly price given value distribution F^{MAX}, which is the CDF of $\max(u_1, \ldots, u_K)$ where each u_k is an IID draw from x^0. Then, any equilibrium is horizontally inefficient.*

Proof. I construct a disclosure rule that maximizes the consumer's ex ante expected payoff among $\mathcal{E} \subset \mathcal{D}$, where $\mathcal{E}$ is the set of all disclosure rules that lead to horizontally efficient outcomes given the optimal behavior of each player. Take any disclosure rule $\phi \in \mathcal{E}$. Since the recommended product belongs to $\arg\max_{\ell \in \mathcal{K}} u_\ell$ with (ex ante) probability 1, the consumer's value of the recommended product (unconditional on which product is recommended) is drawn according to F^{MAX}. This implies that under ϕ, the seller can obtain a revenue of at least $\underline{R} := \max_{p \in V} p[1 - F^{MAX}(p)]$ by setting a price of $\arg\max_{p \in V} p[1 - F^{MAX}(p)]$ for all realized posteriors. This implies that if ϕ^* achieves a vertically as well as horizontally efficient allocation and ϕ^* gives the seller a payoff of $\underline{R}$, then ϕ^* maximizes the consumer's payoff among $\mathcal{E}$.

Consider disclosure rule $\phi^E \in \mathcal{E}$ such that for any realized $\mathbf{u} \in V^K$, ϕ^E draws message $k \in \arg\max_\ell u_\ell$ with probability $\frac{1}{|\arg\max_\ell u_\ell|}$. Two remarks are in order. First, u_k is distributed according to F^{MAX} conditional on message k. Second, the seller prefers to recommend product k after observing message k no matter what additional information she learns, because she can maximize the probability of trade by recommending product k.

Next, I create $\phi^* \in \mathcal{E}$ by modifying ϕ^E as follows: For each $k \in \mathcal{K}$, conditional on that message k is realized under ϕ^E, ϕ^* discloses additional information about u_k according to a *consumer surplus maximizing segmentation (CSMS)* characterized by Bergemann, Brooks and Morris (2015).[2] In our context, the information disclosed according to (any) CSMS ensures that the trade occurs with probability 1 whereas the seller's resulting revenue is $\underline{R}$. Thus, under ϕ^*, the seller

[2]In single product monopoly pricing, a consumer surplus maximizing segmentation is equivalent to a disclosure rule that has the following property. First, at each realized posterior, the seller is willing to set the price equal to the minimum of its support, which implies that the trade occurs with probability 1. Second, at each posterior, the seller is indifferent between charging the minimum of each posterior and charging the monopoly price for the prior.

recommends the highest value product and the trade occurs with probability 1, whereas the seller's revenue is $\underline{R}$. Thus, ϕ^* maximizes the consumer's payoff among $\mathcal{E}$.

Hereafter, I focus on a particular ϕ^* where the additional information about the highest value product is disclosed according to a CSMS constructed by the *greedy algorithm* in Bergemann, Brooks and Morris (2015). This has the following implication. Let $\left\{x^k_{S_1}, \ldots, x^k_{S_L}\right\}$ denote the set of posteriors induced by ϕ^* conditional on ϕ^E drawing message k. Without loss of generality, regard $x^k_{S_1}, \ldots, x^k_{S_L}$ as messages drawn by ϕ^*. Let us also regard each $x^k_{S_\ell}$ as a marginal distribution of u_k instead of a joint distribution of $(u_1, \ldots, u_K)$. The greedy algorithm guarantees that each $x^k_{S_\ell}$ has support $S_\ell \subset V$, $S_1 \subset S_2 \subset \cdots \subset S_L = V$, and the set of all optimal prices against $x^k_{S_\ell}$ is S_ℓ. Moreover, it holds that $S_1 = \{v^*\}$ with $v^* > v_1$. To see this, note that $|S_1| \geq 2$ implies that two prices in S_1 are optimal against all posteriors in $\left\{x^k_{S_1}, \ldots, x^k_{S_L}\right\}$, which in turn implies that these prices are optimal under F^{MAX} because the expected revenue is linear in the value distribution. This contradicts the assumption that there is a unique optimal price under F^{MAX}. Thus, $|S_1| = 1$, which implies that $S_1 = \{v^*\}$. Following the proof of Proposition 2, we can show that Assumption 1 implies $v^* > v_1$.

I modify ϕ^* to create a horizontally inefficient ϕ^I that yields a strictly greater consumer surplus than ϕ^*. From now on, I treat each $x^k_{S_\ell}$ as a joint distribution of $(u_1, \ldots, u_K)$. To simplify exposition, I use the following terminologies. First, I regard a distribution $x \in \Delta(V^K)$ as consisting of a unit mass of consumers, where mass $x(\mathbf{u})$ of consumers have value vector u. Second, I call any set of (a continuum of) consumers a "segment."

To construct ϕ^I, I make three observations. First, a positive mass of consumers in $x^1_{S_1}$ have value v^* for product 1 and the lowest possible value $v_1 < v^*$ for product 2. Call this mass of consumers "segment (v^*, v_1)." Second, a positive mass of consumers in $x^1_{S_1}$ have value v^* for both products 1 and 2. Call these consumers "segment (v^*, v^*)."

First, I take a small but positive (say ε_1) mass of segment (v^*, v_1) from $x^1_{S_1}$ and pool this segment with $x^2_{S_L}$.[3] Let $\hat{x}^2_{S_L}$ denote the posterior created by this pooling. For a sufficiently small $\varepsilon_1 > 0$, at $\hat{x}^2_{S_L}$, the seller recommends product 2, and she *strictly* prefers to set price v_1 for product 2. The reason is as follows. Under the original posterior $x^2_{S_L}$, it is optimal for the seller to recom-

[3] In terms of a disclosure rule, this means that I modify ϕ^* so that it draws message $x^2_{S_L}$ with probability $\varepsilon_1 > 0$ not only following message 2 but also when ϕ^* draws segment (v^*, v_1) in $x^1_{S_1}$.

mend product 2 at any price in V because $S_L = V$. After the modification, $\hat{x}^2_{S_L}$ contains a strictly greater mass of consumers who have value v_1 for product 2 (i.e., segment (v^*, v_1)). Thus, the seller strictly prefers to set price v_1 for product 2. Moreover, for a small $\varepsilon_1 > 0$, the seller does not strictly prefer to recommend other products. Indeed, if the seller recommended product $k \neq 2$ at $x^2_{S_L}$, then she would strictly prefer to set price v_1. Thus, for a small $\varepsilon_1 > 0$, the seller's pricing incentive does not change under $\hat{x}^2_{S_L}$. This implies that the optimal revenue from recommending other products (at the new posterior $\hat{x}^2_{S_L}$) is v_1, which is no greater than the revenue from recommending product 2. Importantly, this modification does not change the consumer's payoff, because consumers in segment (v^*, v_1) obtain zero payoffs under $x^1_{S_1}$. Let ϕ^H denote the resulting disclosure rule.

Finally, I modify ϕ^H by pooling a small but positive (say ε_2) mass of segment (v^*, v^*) in $x^1_{S_1}$ with $\hat{x}^2_{S_L}$. Let $\tilde{x}^2_{S_L}$ denote the posterior following this pooling. If ε_2 is small, the seller continues to recommend product 2 at price v_1 under $\tilde{x}^2_{S_L}$, because she strictly prefers to set price v_1 for product 2 at $\hat{x}^2_{S_L}$. This modification strictly increases the consumer's payoff relative to ϕ^*, because consumers in segment (v^*, v^*) obtain a positive payoff $v^* - v_1$ while they obtain zero payoff under ϕ^*. Let ϕ^I denote the resulting disclosure rule.

ϕ^I gives the consumer a strictly greater expected payoff than ϕ^* but leads to an inefficient recommendation at $\tilde{x}^2_{S_L}$. This implies that any equilibrium is horizontally inefficient, because for any disclosure rule leading to horizontal efficiency, the consumer can find strictly more profitable disclosure rules that lead to horizontal inefficiency. □

Lemma 3. *Fix a finite support $V \subset \mathbb{R}_+$ with $|V| \geq 2$. There is a Lebesgue measure-zero set $X_0 \subset \Delta(V)$ such that, for any $x_0 \in \Delta(V) \setminus X_0$, the induced F^{MAX} has a unique monopoly price (i.e., F^{MAX} satisfies the condition of Lemma 2).*

Proof. First, let D_2 denote the set of all distributions $x \in \Delta(V)$ such that there are two or more optimal prices. I show that D_2 has measure zero. Let $D_{v,v'}$ denote the set of all distributions in $\Delta(V)$ such that both prices v and v' are optimal. Then, it holds that $D_{v,v'} \subset \{x \in \Delta(V) : v \sum_{v_n \geq v, v_n \in V} x(v_n) = v' \sum_{v_n \geq v', v_n \in V} x(v_n)\}$. The right hand side is a subset of an $N-1$-dimensional hyperplane, which has measure zero in $\mathbb{R}^N$, which, in turn, implies that $D_{v,v'}$ has measure zero. Thus, $D_2 = \cup_{(v,v') \in V^2} D_{v,v'}$ has measure zero.

Consider a function φ that maps any distribution $x = (x_1, \ldots, x_N) \in \Delta(V)$ to the distribution of $\max(u_1, \dots, u_K)$, where each u_k is an IID draw from x. φ is written as follows.

$$\varphi(x) = K \cdot \begin{pmatrix} \frac{1}{K} x_1^K \\ x_2 \sum_{\ell=0}^{K-1} x_1^{K-1-\ell} x_1^{\ell} \cdot \frac{1}{\ell+1} \binom{K-1}{\ell} \\ x_3 \sum_{\ell=0}^{K-1} (x_1 + x_2)^{K-1-\ell} x_3^{\ell} \cdot \frac{1}{\ell+1} \binom{K-1}{\ell} \\ \vdots \\ x_N \sum_{\ell=0}^{K-1} (x_1 + \cdots + x_{N-1})^{K-1-\ell} x_N^{\ell} \cdot \frac{1}{\ell+1} \binom{K-1}{\ell} \end{pmatrix}.$$

φ is infinitely differentiable and its Jacobian matrix J_φ is a triangular matrix with the diagonal elements being positive as long as $x_n > 0$ for each $n = 1, \dots, N$. Thus, $J_\varphi(x)$ has full rank if x is *not* in the measure-zero set

$$\{(x_1, \dots, x_N) \in \Delta(V) : \exists n, x_n = 0\}.$$

By Theorem 1 of Ponomarev (1987), $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ has the "0-property": The inverse image of any measure-zero set by φ has measure zero. In particular, $X_0 := \varphi^{-1}(D_2)$ has measure zero. Clearly, X_0 has the desired property. □

2 Proof of Proposition 4

Proposition 4 relies on the following lemma.

Lemma 4. *Under the commitment regime, as $K \to +\infty$, the seller's equilibrium payoff converges to $\max V$ and the consumer's equilibrium payoff converges to 0. Under the no-commitment regime, if Assumption 2 holds, then there is $\underline{u} > 0$ such that the consumer's equilibrium payoff is at least $\underline{u}$ for any K.*

Proof. By the same argument as Proposition 2, the seller under the commitment regime recommends the most valuable product with probability 1. Let F denote the CDF of the value for each product (induced by x_0). Take any $\varepsilon > 0$. Suppose that the seller sets $p = \max V - \varepsilon/2$ for each product up front. As $K \to +\infty$, the probability $1 - F(p)^K$ that the consumer buys the recommended product goes to 1. Thus, there is $\underline{K}$ such that the seller's revenue is at least $\max V - \varepsilon$ if $K \geq \underline{K}$. This implies that the consumer's payoff is at most ε for any such K. This completes the proof of the first part.

To see that the consumer can always guarantee some positive payoff $\underline{u}$ under the no-commitment regime, observe that the consumer can choose to disclose no information and obtain a payoff of $\int_{p(x_0)}^{\max V} v - p(x_0) dF(v) > 0$, which is positive and independent of K. □

Proof of Proposition 4. The result under the commitment regime follows from the previous result, as total surplus is weakly greater than the seller's revenue.

I show that total surplus under the no-commitment regime is uniformly bounded away from $\max V$. Suppose to the contrary that for any $n \in \mathbb{N}$, there exists K_n such that when the seller sells K_n products, some equilibrium under the no-commitment regime achieves total surplus of at least $\max V - \frac{1}{n}$. Then, I can take a subsequence $(K_{n_\ell})_\ell$ such that $K_{n_\ell} < K_{n_{\ell+1}}$ for any $\ell \in \mathbb{N}$. Next, I show that for any $p < \max V$ and $\varepsilon < 1$, there exists $\ell^* \in \mathbb{N}$ such that for any $\ell \geq \ell^*$,

$$\mathbf{P}_\ell(\text{the consumer's value for the recommended product} \geq p) \geq \varepsilon. \tag{1}$$

where $\mathbf{P}_\ell(\cdot)$ is the probability measure on the consumer's value for the recommended product in equilibrium of K_{n_ℓ}-product model. To show inequality (1), suppose to the contrary that there is some (p, ε) and a subsequence $(K'_m)_m$ of $(K_{n_\ell})_\ell$ such that the inequality is violated. Then, given any K'_m in this subsequence, the total surplus is at most $(1 - \varepsilon)p + \varepsilon \max V < \max V$. This contradicts the assumption that the equilibrium total surplus converges to $\max V$ as $K'_m \to +\infty$.

Now, I use inequality (1) to show that the seller's equilibrium revenue converges to $\max V$ along $(K_{n_\ell})_\ell$. Take any $r < \max V$. If the seller sets price $\frac{r + \max V}{2}$, then for a sufficiently large ℓ, the consumer accepts the price with probability greater than $\frac{2r}{r + \max V} < 1$. That is, for a large ℓ, the seller's expected revenue exceeds r. Since this holds for any $r < \max V$, the seller's revenue converges to $\max V$ as $\ell \to +\infty$. This contradicts the observation that the consumer's payoff is bounded from below by a positive number independent of K, as in Lemma 4. □

3 Existence of Equilibrium under the No-commitment Regime

I prove the existence of an equilibrium under the no-commitment regime. Recall that for the commitment regime, I have proved the existence by explicitly constructing an equilibrium.

Restricted Model

Claim 1. In the restricted model, there exists an equilibrium under the no-commitment regime.

The result follows from two lemmas.

Lemma 5. *Given a disclosure level δ, the lowest optimal price $p(\delta)$ in equation (1) exists and is lower semicontinuous in δ.*[4]

Proof. Define $G(p) := \delta F^{MAX}(p) + (1-\delta)F^{MIN}(p)$. Recall that I define a CDF as a left-continuous function. Take any p^* and c with $G(p^*) > c$. For some $\delta > 0$ and for all $p \in (p^* - \delta, p^*]$, $G(p) > c$. Since $G(p)$ is increasing in p, for all $p \in (p^* - \delta, p^* + \delta)$, $G(p) > c$. Thus, G is lower semicontinuous. Then, $1 - G$ is upper semicontinuous, and thus $p[1 - G(p)]$, which is a product of two nonnegative upper semicontinuous functions, is upper semicontinuous in p.[5] This implies that $P(\delta) := \arg\max_{p \in V} p[1 - G(p)]$ is nonempty and compact (Theorem 2.43 of Aliprantis and Border (2006)). Thus, $p(\delta) := \min P(\delta)$ exists.

Next, suppose to the contrary that $p(\delta)$ is not lower semicontinuous at some δ^*. Then, there is $\varepsilon > 0$ such that we can construct a sequence $\delta_n \to \delta^*$ so that $p(\delta_n) < p(\delta^*) - \varepsilon$ for all n.[6] Then, we can find a convergent subsequence of $(p(\delta_n))_n$ because $p(\delta_n) \in V$ and V is compact. Without loss of generality, assume that $(p(\delta_n))_n$ itself converges, so that there exists $p^* := \lim_n p(\delta_n) < p(\delta^*)$. Define $Y(p, \delta)$ as

$$Y(p, \delta) := p\left[1 - \delta F^{MAX}(p) - (1-\delta)F^{MIN}(p)\right] - p(\delta^*)\left[1 - \delta F^{MAX}(p(\delta^*)) - (1-\delta)F^{MIN}(p(\delta^*))\right].$$

[4] Depending on the context, I use one of the following two equivalent conditions as a definition of lower semicontinuity. Given a (first countable) topological space X and $f : X \to \mathbb{R}$, f is lower semicontinuous if for each $c \in \mathbb{R}$ the set $\{x \in X : f(x) \leq c\}$ is closed (or equivalently, the set $\{x \in X : f(x) > c\}$ is open). Equivalently, f is lower semicontinuous if $x_n \to x$ implies $\liminf_n f(x_n) \geq f(x)$. For the equivalence of the two conditions, see Lemma 2.42 of Aliprantis and Border (2006). f is upper semicontinuous if $-f$ is lower semicontinuous.

[5] To see this, if $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are nonnegative and upper semicontinuous, for any $x_n \to x$, we obtain $\limsup_n f(x_n)g(x_n) \leq \limsup_n f(x_n) \limsup_n g(x_n) \leq f(x)g(x)$. Thus, fg is upper semicontinuous.

[6] Formally, if $p(\cdot)$ is not lower semicontinuous, then for some c the set $S := \{\delta : p(\delta) > c\}$ is not open. This implies that we can take some $\delta^* \in S$ such that there is $\delta_n \to \delta^*$ with $p(\delta_n) \leq c < p(\delta^*)$. Define $\varepsilon := \frac{p(\delta^*) - c}{2}$, then $p(\delta_n) < p(\delta^*) - \varepsilon$.

Because $p(\delta_n)$ is optimal given δ_n, it holds $Y(p(\delta_n), \delta_n) \geq 0$. Also, $Y(p, \delta)$ is upper semicontinuous in (p, δ).[7] This implies that $Y^* := \{(p, \delta) : Y(p, \delta) \geq 0\}$ is closed. Thus, $(p^*, \delta^*) = \lim_n (p(\delta_n), \delta_n) \in Y^*$, or equivalently,

$$\begin{aligned}
&p^* \left[1 - \delta^* F^{MAX}(p^*) - (1 - \delta^*) F^{MIN}(p^*)\right] \\
\geq & p(\delta^*) \left[1 - \delta^* F^{MAX}(p(\delta^*)) - (1 - \delta^*) F^{MIN}(p(\delta^*))\right],
\end{aligned}$$

which implies $p^* \in P(\delta^*)$. This contradicts $p(\delta^*) = \min P(\delta^*)$ because $p^* < p(\delta^*)$. Therefore, $p(\delta)$ is lower semicontinuous. □

Lemma 6. *$\delta u^{MAX}(p(\delta)) + (1 - \delta) u^{MIN}(p(\delta))$ is upper semicontinuous in δ.*

Proof. $u^{MAX}(p) = \int_p^{+\infty} (x - p) dF^{MAX}(x) = \int_p^{+\infty} 1 - F^{MAX}(x) dx$ is continuous and decreasing in p. Since $p(\delta)$ is lower semicontinuous, $u^{MAX}(p(\delta))$ is upper semicontinuous in δ. (To see this, if g is continuous and decreasing, and f is lower semicontinuous, then for $x_n \to x$, we have $\lim_{k \to +\infty} \sup_{n > k} g(f(x_n)) = \lim_{k \to +\infty} g(\inf_{n > k} f(x_n)) = g(\lim_{k \to +\infty} \inf_{n > k} f(x_n)) \leq g(f(x))$. The last inequality is from the lower semi-continuity of f.) Similarly, we can show that $u^{MIN}(p(\delta))$ is upper semicontinuous. Therefore, $\delta u^{MAX}(p(\delta)) + (1 - \delta) u^{MIN}(p(\delta))$ is upper semicontinuous in δ. □

Proof of Claim 1. $\delta u^{MAX}(p(\delta)) + (1 - \delta) u^{MIN}(p(\delta))$ is upper semicontinuous, and the set of disclosure levels, $[1/2, 1]$, is compact. Thus, $D^* := \arg\max_{\delta \in [1/2, 1]} \delta u^{MAX}(p(\delta)) + (1 - \delta) u^{MIN}(p(\delta))$ is nonempty and compact (Theorem 2.43 of Aliprantis and Border (2006)). Thus, $\delta^* := \max D^*$ combined with the optimal on and off path actions consists of an equilibrium. □

Unrestricted Model

Claim 2. In the unrestricted model, there exists an equilibrium under the no-commitment regime.

Proof. I prepare some notations. Let $A := \mathcal{K} \times V$ denote the seller's (finite) action space, i.e., the set of all pairs of recommended products and prices. When the seller recommends product $k \in \mathcal{K}$

[7] This follows from the fact that the product of two nonnegative upper (lower) semicontinuous functions is upper (lower) semicontinuous, and the same thing holds for the sum. Since $F^{MAX}(p)$ and $F^{MIN}(p)$ are lower semicontinuous in p, $\delta F^{MAX}(p) + (1 - \delta) F^{MIN}(p)$ is lower semicontinuous. Thus, $p\left[1 - \delta F^{MAX}(p) - (1 - \delta) F^{MIN}(p)\right]$ is upper semicontinuous. Because the second term of $Y(p, \delta)$, $p(\delta^*)\left[1 - \delta F^{MAX}(p(\delta^*)) - (1 - \delta) F^{MIN}(p(\delta^*))\right]$, is continuous in (p, δ), overall, $Y(p, \delta)$ is upper semicontinuous.

at price $p \in V$, I say that the seller chooses $a = (k, p) \in A$. Given $(a, b) \in A \times \Delta(V^K)$, let $U(a, b)$ and $R(a, b)$ denote the expected payoffs of the consumer and the seller, respectively, when the seller chooses a, the consumer's values are drawn from b, and the consumer takes an optimal purchase decision (breaking ties in favor of the seller). Given the seller's belief $b \in \Delta(V^K)$, let $a(b) \in A$ denote the seller's optimal recommendation and price that break ties in favor of the consumer. $a(b)$ might not be unique, but the payoffs of the seller and the consumer are unique due to the tie-breaking rule. Also, $a(b)$ exists because A is finite.

The proof consists of two steps. First, I show that the consumer's payoff is upper semicontinuous in disclosure rules. Second, I show that the set of all disclosure rules is compact. I use weak* topology in $\Delta(\Delta(V^K))$.

Consider an information set where the seller sets a price and recommends a product. Let $b \in \Delta(V^K)$ denote the seller's belief about the value vector. If the seller and the consumer take optimal actions following the information set, the consumer's expected payoff is given by $U(a(b), b)$. I show that $U(a(b), b)$ is upper semicontinuous in $b \in \Delta(V^K)$. Suppose to the contrary that there exists $\varepsilon > 0$ and $(b_n)_{n=1}^{+\infty} \subset \Delta(V^K)$ such that $\lim_n b_n = b$ but $U(a(b_n), b_n) \geq U(a(b), b) + \varepsilon$ for all n. Because A is finite, we can choose a subsequence $(b_{n(m)})_{m=1}^{+\infty}$ so that for some $a' \in A$, $a(b_{n(m)}) = a'$ for all m. Without loss of generality, assume that $a(b_n) = a'$ for all n. Note that $R(a', b_n) \geq R(a(b), b_n)$ because $a' = a(b_n)$ is optimal for the seller given its belief b_n. Note also that $R(a, b)$ is continuous in b with a fixed a. Indeed, suppose that a is such that the seller recommends product k at price p, where the consumer's value for product k is distributed according to $b^k = (b_1^k, \dots, b_N^k) \in \Delta(V)$ under $b \in \Delta(V^K)$. Then, $R(a, b) = p \cdot \sum_{\ell=1}^{N} \mathbf{1}_{\{v_\ell \geq p\}} b_\ell^k$, which is continuous in b. ($\mathbf{1}_{\{v_\ell \geq p\}}$ is the indicator function that takes value 1 or 0 if $v_\ell \geq p$ or $v_\ell < p$, respectively.) Given the continuity of $R(a, b)$ in b, $R(a', b_n) \geq R(a(b), b_n)$ for all n implies $R(a', b) \geq R(a(b), b)$. Thus, a' is optimal for the seller given b. This implies that $U(a(b), b) \geq U(a', b)$ by the seller's tie-breaking rule. Also, $U(a(b_n), b_n) \geq U(a(b), b) + \varepsilon$ for all n implies $U(a', b) \geq U(a(b), b) + \varepsilon$, because $a(b_n) = a'$, and $U(a, b) = \sum_{\ell=1}^{N} \mathbf{1}_{\{v_\ell \geq p\}} (v_\ell - p) b_\ell^k$ is continuous in b. However, these two inequalities lead to $U(a(b), b) \geq U(a', b) \geq U(a(b), b) + \varepsilon$, which is a contradiction. Thus, $U(a(b), b)$ is upper semicontinuous in $b \in \Delta(V^K)$. By Theorem 15.5 of Aliprantis and Border (2006), $\int_{\Delta(V^K)} U(a(b), b) d\tau(b)$ is upper semicontinuous in $\tau \in \Delta(\Delta(V^K))$ when $\Delta(\Delta(V^K))$ is endowed with weak* topology. This completes the first part.

Next, I show that the set $\mathcal{D}$ of all disclosure rules is weak* compact. Let $b_0 := x_0 \times \cdots \times x_0$ denote the prior distribution of value vector. Also, for any disclosure rule $\phi \in \mathcal{D}$, define $\hat{\phi} \in \Delta(\Delta(V^K))$ as the distribution over posterior beliefs about value vector u induced by ϕ and b_0. Moreover, let $\hat{\mathcal{D}} := (\hat{\phi})_{\phi \in \mathcal{D}} \subset \Delta(\Delta(V^K))$. Proposition 1 ((iii) $\Rightarrow$ (ii)) of Kamenica and Gentzkow (2011) implies that if the consumer's payoff $\int_{\Delta(V^K)} U(a(b), b) d\tau(b)$ is maximized at some $\tau^* \in \hat{\mathcal{D}} = \left\{\tau \in \Delta(\Delta(V^K)) : \int_{\Delta(V^K)} b d\tau(b) = b_0\right\}$, then there is a disclosure rule (with a finite message space) that maximizes his payoff among all available disclosure rules. Now, $\Delta(\Delta(V^K))$ is weak* compact because $\Delta(V^K)$ is compact (e.g., Theorem 15.11 of Aliprantis and Border (2006)). Also, $\hat{\mathcal{D}}$ is closed. This is because if $\tau_\ell \in \hat{\mathcal{D}}$ for each $\ell \in \mathbb{N}$ and $\tau_\ell \to \tau$ in weak* topology, then $b_0 = \int_{\Delta(V^K)} b d\tau_\ell(b) \to \int_{\Delta(V^K)} b d\tau(b)$, which implies $\tau \in \hat{\mathcal{D}}$. Thus, $\hat{\mathcal{D}}$, which is a closed subset of a compact set $\Delta(\Delta(V^K))$, is weak* compact.

Finally, in equilibrium, the consumer solves $\max_{\tau \in \hat{\mathcal{D}}} \int_{\Delta(V^K)} U(a(b), b) d\tau(b)$. Since the objective function is upper semicontinuous in τ and $\hat{\mathcal{D}} \subset \Delta(\Delta(V^K))$ is compact with respect to weak* topology, the set $\mathcal{D}^* := \arg\max_{\tau \in \hat{\mathcal{D}}} \int_{\Delta(V^K)} U(a(b), b) d\tau(b)$ is nonempty and weak* compact. Moreover, as I prove below, the seller's expected payoff $\int_{\Delta(V^K)} R(a(b), b) d\tau(b)$ is upper semicontinuous in τ. Therefore, $\max_{\tau \in \mathcal{D}^*} \int_{\Delta(\Delta(V^K))} R(a(b), b) d\tau(b)$ has a maximizer. Any maximizer τ^*, combined with the optimal on and off path behavior of the seller and the consumer, consists of an equilibrium.

To see that $\int_{\Delta(\Delta(V^K))} R(a(b), b) d\tau(b)$ is upper semicontinuous, I show that $R(a(b), b)$ is upper semicontinuous in b. Suppose to the contrary that there exists $\varepsilon > 0$ and $(b_n)_{n=1}^{+\infty} \subset \Delta(V^K)$ such that $\lim_n b_n = b$ but $R(a(b_n), b_n) \geq R(a(b), b) + \varepsilon$ for all n. Because A is finite, we can choose a subsequence $(b_{n(m)})_{m=1}^{+\infty}$ so that for some $a' \in A$, $a(b_{n(m)}) = a'$ for all m. As $R(a, b)$ is continuous in b, we obtain $R(a', b) \geq R(a(b), b) + \varepsilon$. However, this contradicts $R(a(b), b) \geq R(a', b)$. Thus, $R(a(b), b)$ is upper semicontinuous in b, and thus $\int_{\Delta(\Delta(V^K))} R(a(b), b) d\tau(b)$ is upper semicontinuous by Theorem 15.5 of Aliprantis and Border (2006). □

References

Aliprantis, Charalambos D, and Kim Border. 2006. *Infinite dimensional analysis: a hitchhiker's guide.* Springer Science & Business Media.

Bergemann, Dirk, Benjamin Brooks, and Stephen Morris. 2015. "The limits of price discrimination." *The American Economic Review*, 105(3): 921–957.

Kamenica, Emir, and Matthew Gentzkow. 2011. "Bayesian persuasion." *American Economic Review*, 101(6): 2590–2615.

Ponomarev, Stanislav P. 1987. "Submersions and preimages of sets of measure zero." *Siberian Mathematical Journal*, 28(1): 153–163.