Online Appendix to "The Race to the Base"

Dan Bernhardt, Peter Buisseret and Sinem Hidir*

Appendix A: Proofs of Results

Let $x^*(z_L, z_R, \rho)$ denote the preferred policy of the swing voter, given party L's platform z_L, party R's platform z_R, and party R's net valence advantage, $\rho \in [\rho_0 - \psi, \rho_0 + \psi]$. Assumption 2 says that $\rho_0 - \psi < -1$ and $\theta > \rho_0 + \psi + 1$. Thus $\theta > 2$, implying that for any pair $(z_L, z_R) \in \mathbb{R}^2$, a voter with preferred policy $x_i > x^*(z_L, z_R, \rho)$ strictly prefers R, and a voter with preferred policy $x_i \leq x^*(z_L, z_R, \rho)$ weakly prefers L. Recall our convention that when a voter is indifferent between the parties, she votes for party L, that when the parties tie in a district, L wins the district, and that when each party wins one half of the districts, L wins the majority. Party L therefore wins a district with median m if and only if $x^*(\rho, z_L, z_R) \geq m$. Using the fact that district medians are uniformly distributed on $[-1, 1]$, party L's share of districts is given by $d_L = \frac{1+x^*(z_L,z_R,\rho)}{2}$, and party L therefore wins the election if and only if $d_L \geq \frac{1}{2}$, i.e., if and only if $x^*(z_L, z_R, \rho) \geq 0$. It is immediate that a platform $z_J < -1$ or $z_J > 1$ is strictly dominated, for either party $J \in \{L, R\}$. Thus, we restrict attention to $(z_L, z_R) \in [-1, 1]^2$ in the arguments that follow.

We first state three intermediate results that streamline our subsequent proofs of existence and uniqueness.

Lemma 1. *For any $z_L \in [-1, 1]$, z_R is a best response to z_L only if $z_R \leq \max\{0, z_L\}$.*

Proof. We establish that party R's payoff strictly decreases in $z_R \geq \max\{0, z_L\}$. Because arguments used in the proof of this result are repeated throughout the Appendix, we provide some commentary, to guide the reader. Recall from expression (7) that the swing voter type $x^*(z_L, z_R, \rho)$ solves $\Delta(x^*; z_L, z_R, \rho) = 0$, where

$$\Delta(x^*; z_L, z_R, \rho) = |z_R - x^*| - |z_L - x^*| - \theta x^* - \rho. \tag{A1}$$

Whenever $z_R \geq z_L$, the swing voter $x^*(z_L, z_R, \rho)$ may be drawn from one of at most three intervals.

* Bernhardt: Departments of Economics and Finance, University of Illinois, Champaign, IL 61801 and Department of Economics, University of Warwick, Coventry CV4 7AL (email: danber@illinois.edu) ; Buisseret: Harris School of Public Policy, 1307 E 60th Street, Chicago, IL 60637 (email: pbuisseret@uchicago.edu); Hidir: Department of Economics, University of Warwick, Coventry CV4 7AL (email: s.hidir@warwick.ac.uk).

1. The swing voter's type is $x^*(z_L, z_R, \rho) \geq z_R$ if $\Delta(z_R; z_L, z_R, \rho) \geq 0$, i.e., if $z_L - z_R - \theta z_R - \rho \geq 0$, i.e., if $\rho \leq z_L - z_R - \theta z_R$. Thus, $x^*(z_L, z_R, \rho)$ solves $(x^* - z_R) - (x^* - z_L) - \theta x^* - \rho = 0$, i.e.,

$$x^*(z_L, z_R, \rho) = \frac{z_L - z_R - \rho}{\theta} \equiv x_1^*. \tag{A2}$$

2. The swing voter's type is $x^*(z_L, z_R, \rho) \in (z_L, z_R)$ if both $\Delta(z_L; z_L, z_R, \rho) > 0$, i.e., $z_R - z_L - \theta z_L - \rho > 0$ and also $\Delta(z_R; z_L, z_R, \rho) < 0$, i.e., $z_L - z_R - \theta z_R - \rho < 0$. Thus, $x^*(z_L, z_R, \rho)$ solves $(z_R - x^*) - (x^* - z_L) - \theta x^* - \rho = 0$, i.e.,

$$x^*(z_L, z_R, \rho) = \frac{z_L + z_R - \rho}{2 + \theta} \equiv {x_2}^*. \tag{A3}$$

3. The swing voter's type is $x^*(z_L, z_R, \rho) \leq z_L$ if $\Delta(z_L; z_L, z_R, \rho) \leq 0$, i.e., $z_R - z_L - \theta z_L - \rho \leq 0$, i.e., $\rho \geq z_R - z_L - \theta z_L$, i.e., $(z_R - x^*) - (z_L - x^*) - \theta x^* - \rho = 0$, i.e.,

$$x^*(z_L, z_R, \rho) = \frac{z_R - z_L - \rho}{\theta} \equiv {x_3}^*. \tag{A4}$$

Assumption 2 that $\theta > \rho_0 + \psi + 1$ implies that $x_1^* < 1$ and $x_3^* > -1$ for all $\rho \in [\rho_0 - \psi, \rho_0 + \psi]$, whenever $z_L \leq z_R$.[1] In words: $x_1^* < 1$ states that even if the net valence shock in favor of party R is drawn most *unfavorably* to R, i.e., $\rho = \rho_0 - \psi$, the district median voter type $+1$ strictly prefers party R, for any pair (z_L, z_R) such that $z_L \leq z_R$. This implies that R always wins a positive share of districts whenever $z_L \leq z_R$. Likewise, $x_3^* > -1$ states that even if the net valence shock ρ in favor of party R is drawn most *favorably* to R, i.e., $\rho = \rho_0 + \psi$, the district median voter type -1 strictly prefers party L, for any pair (z_L, z_R) such that $z_L \leq z_R$. This implies that L always wins a positive share of districts whenever $z_L \leq z_R$.

We consider two possible cases for the location of party L's platform: weakly to the left of the median voter, i.e., $z_L \leq 0$, or strictly to the right of the median voter, i.e., $z_L > 0$.

Case 1: $z_L \leq 0$. If party R locates at $z_R \geq 0$, party R wins a majority if and only if ${x_2}^* < 0$, i.e., if and only if $\rho > z_L + z_R$. Party R's expected payoff from $z_R \geq 0$ is:

$$\pi_R(z_L, z_R) = \frac{\alpha}{2\psi} \int_{\rho_0 - \psi}^{\max\{z_L - z_R - \theta z_R, \rho_0 - \psi\}} \left(\frac{1}{2} - \frac{x_1^*}{2}\right) d\rho + \frac{\alpha}{2\psi} \int_{\max\{z_L - z_R - \theta z_R, \rho_0 - \psi\}}^{z_L + z_R} \left(\frac{1}{2} - \frac{x_2^*}{2}\right) d\rho$$

[1] In fact, $\theta > \rho_0 + \psi$ is sufficient, but we impose the stronger parameter restriction streamline our subsequent proofs.

$$+\frac{1}{2\psi}\int_{z_L+z_R}^{\min\{z_R-z_L-\theta z_L,\rho_0+\psi\}}\left(r-\beta\frac{x_2^*}{2}\right)d\rho+\frac{1}{2\psi}\int_{\min\{z_R-z_L-\theta z_L,\rho_0+\psi\}}^{\rho_0+\psi}\left(r-\beta\frac{x_3^*}{2}\right)d\rho. \tag{A5}$$

We explain how this expected payoff is constructed, taking each of the four integrals in turn.

<u>First term</u>. The first integral reflects R's share of districts for realizations of ρ such that the swing voter's type is $x^*(z_L, z_R, \rho) \in (z_R, 1]$. Since $x^*(z_L, z_R, \rho) > z_R \geq 0$, R wins a minority of districts. We have already shown that, with probability one, $x^*(z_L, z_R, \rho) < 1$, i.e., we have shown that with probability one the swing voter type $x^*(z_L, z_R, \rho)$ is realized strictly to the left of the district median with ideal policy 1. However, we have *not* shown that with positive probability the swing voter type $x^*(z_L, z_R, \rho)$ is realized strictly to the right of party R's platform, z_R. The net value that a voter with ideal policy x_i receives from party L is $\Delta(x_i; z_L, z_R, \rho)$, defined in (3). Thus, $x^*(z_L, z_R, \rho) > z_R$ with positive probability if and only

$$\begin{aligned}\Delta(z_R; z_L, z_R, \rho_0-\psi) > 0 &\iff |z_R - z_R| - |z_R - z_L| - \theta z_R - (\rho_0-\psi) > 0\\ &\iff \rho_0 - \psi < z_L - z_R - \theta z_R.\end{aligned} \tag{A6}$$

When (A6) fails, the first integral in expression (A5) is zero, since with probability one R wins every district with median $m \in [z_R, 1]$.

<u>Second term</u>. The second integral reflects party R's share of districts for realizations of ρ such that the swing voter's type is $x^*(z_L, z_R, \rho) \in [0, z_R]$, in which case party R wins a minority of districts. The median voter in the median district weakly prefers party L for some shock realization if and only if

$$\Delta(0; z_L, z_R, \rho_0-\psi) \geq 0 \iff |z_R - 0| - |z_L - 0| - \theta \times 0 - (\rho_0-\psi) \geq 0 \iff z_R + z_L \geq \rho_0 - \psi. \tag{A7}$$

Likewise, the median voter strictly prefers party R for some shock realization if and only if

$$\Delta(0; z_L, z_R, \rho_0+\psi) < 0 \iff |z_R - 0| - |z_L - 0| - \theta \times 0 - (\rho_0+\psi) < 0 \iff z_R + z_L < \rho_0 + \psi, \tag{A8}$$

Since $-1 < z_R + z_L < 1$ for any (z_L, z_R) such that $-1 \leq z_L \leq 0 \leq z_R \leq 1$, Assumption 2 that $\rho_0 - \psi < -1$ implies that $\rho_0 - \psi < z_R + z_L < \rho_0 + \psi$, implying that with positive probility each party wins a strict majority of districts. This yields the upper limit of integration in the second term of expression (A5).

Third term. The third integral reflects party R's share of districts for realizations of ρ such that $x^*(z_L, z_R, \rho) \in [z_L, 0)$. Because $x^*(z_L, z_R, \rho) < 0$, party R wins a majority of districts. While we have shown that with positive probability $x^*(z_L, z_R, \rho) < 0$, we have *not* shown that with positive probability $x^*(z_L, z_R, \rho) < z_L$. The net value that a voter with ideal policy x_i receives from party L is $\Delta(x_i; z_L, z_R, \rho)$, defined in (3). Thus, with positive probability $x^*(z_L, z_R, \rho) < z_L$ if and only if $x^*(z_L, z_R, \rho_0 + \psi) < z_L$, i.e., if and only if

$$\begin{aligned}\Delta(z_L; z_L, z_R, \rho_0 + \psi) < 0 &\iff |z_R - z_L| - |z_L - z_L| - \theta z_L - (\rho_0 + \psi) < 0 \\ &\iff \rho_0 + \psi > z_R - z_L - \theta z_L. \end{aligned} \tag{A9}$$

When this condition fails, the upper limit of integration in the third term of expression (A5) is $\rho_0 + \psi$, the highest value taken by the preference shock.

Fourth term. The fourth integral reflects party R's share of districts for realizations of ρ such that $x^*(z_L, z_R, \rho) < z_L$. Since $z_L \leq 0$, $x^*(z_L, z_R, \rho) < z_L$ implies that party R wins a majority of districts. As we highlighted in the previous paragraph, $x^*(z_L, z_R, \rho) < z_L$ occurs with positive probability if and only if $\rho_0 + \psi > z_R - z_L - \theta z_L$. Otherwise, the fourth integral in (A5) is zero.

We first argue that a platform z_R is not a best response if with probability one the swing voter's type $x^*(z_L, z_R, \rho)$ is realized weakly to the left of z_R. That is, we argue that z_R is not a best response if $z_L - z_R - \theta z_R \leq \rho_0 - \psi$, i.e., if the first integral in expression (A5) is zero. To prove this, we first observe that for any $z_L \leq 0$, party R can select a platform $z_R \geq 0$ such that $z_L - z_R - \theta z_R > \rho_0 - \psi$. This follows from the fact that party R can select $z_R = 0$: Assumption 2 that $\rho_0 - \psi < -1$ implies that $z_L - 0 - \theta \times 0 = z_L \geq -1 > \rho_0 - \psi$.

Suppose, however, that party R locates at $z_R > 0$ such that $z_L - z_R - \theta z_R \leq \rho_0 - \psi$, i.e., so that with probability one the swing voter's type $x^*(z_L, z_R, \rho)$ is realized weakly to the left of z_R. If, in addition, with positive probability $x^*(z_L, z_R, \rho)$ is realized strictly to the left of z_L, i.e., if $z_R - z_L - \theta z_L < \rho_0 + \psi$, then differentiation of (A5) yields:

$$2\psi \frac{\partial \pi_R(z_L, z_R)}{\partial(-z_R)} = \frac{\alpha}{2} \int_{\rho_0 - \psi}^{z_L + z_R} \frac{\partial x_2^*}{\partial z_R} d\rho + \frac{\beta}{2} \left[\int_{z_L + z_R}^{z_R - z_L - \theta z_L} \frac{\partial x_2^*}{\partial z_R} d\rho + \int_{z_R - z_L - \theta z_L}^{\rho_0 + \psi} \frac{\partial x_3^*}{\partial z_R} d\rho \right] + \left(r - \frac{\alpha}{2} \right). \tag{A10}$$

Because $\frac{\partial x_2^*}{\partial z_R} > 0$ and $\frac{\partial x_3^*}{\partial z_R} > 0$, (A10) is strictly positive, and thus R's platform is not a best response. The argument if $z_L - z_R - \theta z_R \leq \rho_0 - \psi$ and $z_R - z_L - \theta z_L \geq \rho_0 + \psi$ is the same. We conclude that

R's expected payoff (A5) strictly decreases in z_R whenever with probability one $x^*(z_L, z_R, \rho) \le z_R$.

We then verify via straightforward algebra (omitted) that for any $z_R \ge 0$ such that with positive probability $x^*(z_L, z_R, \rho) > z_R$ and with positive probability $x^*(z_L, z_R, \rho) < z_L$, party R's payoff strictly decreases in z_R under Assumption 1 that $r > \frac{\alpha}{2} + \frac{\psi}{2\theta}(\alpha - \beta)$. That is, for any z_R such that $z_L - z_R - \theta z_R > \rho_0 - \psi$ and $z_R - z_L - \theta z_L < \rho_0 + \psi$, (A5) strictly decreases in z_R.

Finally, we argue that for any $z_R > 0$ such that with positive probability $x^*(z_L, z_R, \rho) > z_R$, and with probability one $x^*(z_L, z_R, \rho) \ge z_L$, party R's payoff strictly decreases in z_R. That is, we argue that for any z_R such that both $z_L - z_R - \theta z_R > \rho_0 - \psi$ and $z_R - z_L - \theta z_L \ge \rho_0 + \psi$, R's payoff strictly decreases in z_R. When $z_L - z_R - \theta z_R > \rho_0 - \psi$ and $z_R - z_L - \theta z_L \ge \rho_0 + \psi$, differentiation of (A5) with respect to z_R yields:

$$2\psi \frac{\partial \pi_R(z_L, z_R)}{\partial z_R} = -\frac{\alpha}{2}\left[\int_{\rho_0-\psi}^{z_L - z_R - \theta z_R} \frac{\partial x_1^*}{\partial z_R} d\rho + \int_{z_L - z_R - \theta z_R}^{z_L + z_R} \frac{\partial x_2^*}{\partial z_R} d\rho\right] - \frac{\beta}{2}\int_{z_L+z_R}^{\rho_0+\psi} \frac{\partial x_2^*}{\partial z_R} d\rho - \left(r - \frac{\alpha}{2}\right). \tag{A11}$$

Straightforward algebra reveals that (A11) strictly *increases* in z_L. The restriction that $z_R - z_L - \theta z_L \ge \rho_0 + \psi$ is equivalent to $z_L \le \frac{z_R - (\rho_0+\psi)}{1+\theta} \equiv \hat{z}_L(z_R)$. Evaluated at $z_L = \hat{z}_L(z_R)$, straightforward algebra verifies that (A11) is strictly negative evaluated for any $z_R \ge 0$, under Assumption 1 that $r > \frac{\alpha}{2} + \frac{\psi}{2\theta}(\alpha - \beta)$. We conclude that (A5) strictly decreases in z_R such that with positive probability $x^*(z_L, z_R, \rho) > z_R$ and with probability one $x^*(z_L, z_R, \rho) \ge z_L$. We have established that for any $z_L \le 0$, party R's expected payoff strictly decreases in $z_R \ge 0$.

Case 2: $z_L > 0$. We consider $z_R \ge z_L$. Again, party R wins if and only if $x^*(z_L, z_R, \rho) < 0$, i.e., if and only if $\rho > z_R - z_L$. Party R's expected payoff from $z_R \ge z_L$ is:

$$\begin{aligned}\pi_R(z_L, z_R) =& \frac{\alpha}{2\psi}\int_{\rho_0-\psi}^{\max\{z_L - z_R - \theta z_R, \rho_0 - \psi\}} \left(\frac{1}{2} - \frac{x_1^*}{2}\right) d\rho + \frac{\alpha}{2\psi}\int_{\max\{z_L - z_R - \theta z_R, \rho_0 - \psi\}}^{\min\{z_R - z_L - \theta z_L, \rho_0 + \psi\}} \left(\frac{1}{2} - \frac{x_2^*}{2}\right) d\rho \\ &+ \frac{\alpha}{2\psi}\int_{\min\{z_R - z_L - \theta z_L, \rho_0 + \psi\}}^{z_R - z_L} \left(\frac{1}{2} - \frac{x_3^*}{2}\right) d\rho + \frac{1}{2\psi}\int_{z_R - z_L}^{\rho_0+\psi} \left(r - \beta\frac{x_3^*}{2}\right) d\rho.\end{aligned} \tag{A12}$$

By similar arguments to those for Case 1, it is easy to verify that $z_R > z_L$ is a best response only if $z_L - z_R - \theta z_R > \rho_0 - \psi$. Assumption 2 that $\rho_0 - \psi < -1$ and $0 < z_L \le z_R \le 1$ implies that $z_R - z_L < \rho_0 + \psi$, i.e., that with positive probability $x^*(z_L, z_R, \rho) < 0$. This implies that with positive probability $x^*(z_L, z_R, \rho) < z_L$, i.e., $z_R - z_L - \theta z_L < \rho_0 + \psi$. Without loss of generality, we may therefore re-write (A12) as follows:

$$\pi_R(z_L, z_R) = \frac{\alpha}{2\psi}\int_{\rho_0-\psi}^{z_L-z_R-\theta z_R}\left(\frac{1}{2}-\frac{x_1^*}{2}\right)d\rho + \frac{\alpha}{2\psi}\int_{z_L-z_R-\theta z_R}^{z_R-z_L-\theta z_L}\left(\frac{1}{2}-\frac{x_2^*}{2}\right)d\rho$$
$$+\frac{\alpha}{2\psi}\int_{z_R-z_L-\theta z_L}^{z_R-z_L}\left(\frac{1}{2}-\frac{x_3^*}{2}\right)d\rho + \frac{1}{2\psi}\int_{z_R-z_L}^{\rho_0+\psi}\left(r-\beta\frac{x_3^*}{2}\right)d\rho. \tag{A13}$$

We obtain:

$$\frac{\partial \pi_R(z_L, z_R)}{\partial z_R} = \frac{\alpha(\theta-\rho_0+\psi)-\beta(\rho_0+\psi)-2\theta r+(z_L-z_R)(\alpha-\beta)-2\theta\alpha z_R}{4\theta\psi}. \tag{A14}$$

For all $z_R \geq z_L > 0$, $(z_L - z_R)(\alpha-\beta) - 2\theta\alpha z_R \leq 0$ under Assumption 1 that $\alpha \geq \beta$; the remainder of the numerator is strictly negative for all $\rho_0 \geq 0$ under Assumption 1 that $r > \frac{\alpha}{2} + \frac{\psi}{2\theta}(\alpha-\beta)$. □

Lemma 2. *(i) For any $z_R \leq 0$, a platform $z_L > \max\{0, z_R - \theta z_R - (\rho_0+\psi)\}$ is not a best response to z_R. (ii) For any $z_R > 0$, z_L is a best response to z_R only if $z_L \leq z_R$.*

Remark: A stronger version of part (i) of the Lemma would be: for any $z_R \leq 0$, a platform $z_L > 0$ is not a best response to z_R. If $z_R \in [\frac{\rho_0+\psi}{1-\theta}, 0]$, this is what part (i) states. Since we later show that, in equilibrium, $z_R > \frac{\rho_0+\psi}{1-\theta}$, it is sufficient, and simpler, to prove the Lemma as stated.

Proof of Lemma 2. The net value that a voter with ideal policy x_i receives from party L is $\Delta(x_i; z_L, z_R, \rho)$, defined in (3). The swing voter $x^*(z_L, z_R, \rho)$ solves $\Delta(x^*; z_L, z_R, \rho) = 0$. When $z_L \geq z_R$, the swing voter $x^*(z_L, z_R, \rho)$ may therefore be drawn from one of at most three intervals.

1. The swing voter's type is $x^*(z_L, z_R, \rho) < z_R$ if $\Delta(z_R; z_L, z_R, \rho) < 0$, i.e., $z_R - z_L - \theta z_R - \rho < 0$, i.e., $\rho > z_R - z_L - \theta z_R$. Thus, $x^*(z_L, z_R, \rho)$ solves $(z_R - x^*) - (z_L - x^*) - \theta x^* - \rho = 0$, i.e.,

$$x^*(z_L, z_R, \rho) = \frac{z_R - z_L - \rho}{\theta} \equiv x_4{}^*. \tag{A15}$$

2. The swing voter's type is $x^*(z_L, z_R, \rho) \in [z_R, z_L]$ if $\Delta(z_R; z_L, z_R, \rho) \geq 0 \geq \Delta(z_L; z_L, z_R, \rho)$, i.e. $z_R - z_L - \theta z_R - \rho \geq 0$, and $z_L - z_R - \theta z_L - \rho \leq 0$. Thus, $x^*(z_L, z_R, \rho)$ solves $(x^* - z_R) - (z_L - x^*) - \theta x^* - \rho = 0$, i.e.,

$$x^*(z_L, z_R, \rho) = \frac{-(z_L + z_R) - \rho}{\theta - 2} \equiv x_5{}^*. \tag{A16}$$

3. The swing voter's type is $x^*(z_L, z_R, \rho) > z_L$ if $\Delta(z_L; z_L, z_R, \rho) > 0$, i.e., $z_L - z_R - \theta z_L - \rho > 0$.

Thus, $x^*(z_L, z_R, \rho)$ solves $(x^* - z_R) - (x^* - z_L) - \theta x^* - \rho = 0$, i.e.,

$$x^*(z_L, z_R, \rho) = \frac{z_L - z_R - \rho}{\theta} \equiv {x_6}^*. \tag{A17}$$

Notice that $x_6^* \leq 1$ if and only if $z_L - z_R - \rho \leq \theta$, i.e., $\rho \geq z_L - z_R - \theta$; similarly, $x_4^* \geq -1$ if and only if $z_R - z_L - \rho \geq -\theta$, i.e., $\rho \leq z_R - z_L + \theta$. We consider two possible cases for the location of party R's platform: weakly to the left of the median voter, i.e., $z_R \leq 0$, or strictly to the right of the median voter, i.e., $z_R > 0$.

We now prove each part of the lemma.

Part (i). Suppose $z_R \leq 0$, and consider $z_L > \max\{0, z_R - \theta z_R - (\rho_0 + \psi)\}$. Party L wins if and only if $x^*(z_L, z_R, \rho) \geq 0$, i.e., if and only if $\rho \leq -z_L - z_R$. Party L's expected payoff is therefore:

$$\begin{aligned}\pi_L(z_L, z_R) &= \frac{1}{2\psi}\int_{\rho_0-\psi}^{\max\{z_L-z_R-\theta,\rho_0-\psi\}} \left(r + \frac{\beta}{2}\right) d\rho + \frac{1}{2\psi}\int_{\max\{z_L-z_R-\theta,\rho_0-\psi\}}^{\max\{z_L-z_R-\theta z_L,\rho_0-\psi\}} \left(r + \beta\frac{x_6^*}{2}\right) d\rho \\ &+ \frac{1}{2\psi}\int_{\max\{z_L-z_R-\theta z_L,\rho_0-\psi\}}^{-z_L-z_R} \left(r + \beta\frac{x_5^*}{2}\right) d\rho + \frac{\alpha}{2\psi}\int_{-z_L-z_R}^{z_R-z_L-\theta z_R} \left(\frac{1}{2} + \frac{x_5^*}{2}\right) d\rho \\ &+ \frac{\alpha}{2\psi}\int_{z_R-z_L-\theta z_R}^{\min\{z_R-z_L+\theta,\rho_0+\psi\}} \left(\frac{1}{2} + \frac{x_4^*}{2}\right) d\rho. \end{aligned} \tag{A18}$$

To understand the first term, note that Assumption 2 is not sufficient to ensure that either party wins a positive share of districts when $z_R < z_L$. Recalling that $\Delta(x_i; z_L, z_R, \rho)$ is voter type x_i's net value from party L, defined in (3), the voter type 1 weakly prefers party L for some ρ if and only if $\Delta(1; z_L, z_R, \rho_0 - \psi) \geq 0$, i.e., if and only if

$$|1 - z_R| - |1 - z_L| - \theta - (\rho_0 - \psi) \geq 0 \iff \rho_0 - \psi \leq z_L - z_R - \theta. \tag{A19}$$

If (A19) holds, then party L wins every district whenever $\rho \leq z_L - z_R - \theta$, in which case its payoff is $r + \beta/2$. The upper limit of integration in the final integration follows a similar derivation: if $z_R - z_L + \theta < \rho_0 + \psi$, then party R wins every district whenever the net preference shock in favor of R is $\rho > z_R - z_L + \theta$. If, instead, $z_R - z_L + \theta \geq \rho_0 + \psi$, L wins a positive share of districts for every realization of the preference shock. The remaining terms in (A18) follow a similar derivation to that for Lemma 1. Thus, the objective function becomes:

$$\begin{aligned}\pi_L(z_L, z_R) &= \frac{1}{2\psi}\int_{\rho_0-\psi}^{\max\{z_L-z_R-\theta,\rho_0-\psi\}}\left(r+\frac{\beta}{2}\right)d\rho + \frac{1}{2\psi}\int_{\max\{z_L-z_R-\theta,\rho_0-\psi\}}^{z_L-z_R-\theta z_L}\left(r+\beta\frac{x_6^*}{2}\right)d\rho \\ &+ \frac{1}{2\psi}\int_{z_L-z_R-\theta z_L}^{-z_L-z_R}\left(r+\beta\frac{x_5^*}{2}\right)d\rho + \frac{\alpha}{2\psi}\int_{-z_L-z_R}^{z_R-z_L-\theta z_R}\left(\frac{1}{2}+\frac{x_5^*}{2}\right)d\rho \\ &+ \frac{\alpha}{2\psi}\int_{z_R-z_L-\theta z_R}^{\min\{z_R-z_L+\theta,\rho_0+\psi\}}\left(\frac{1}{2}+\frac{x_4^*}{2}\right)d\rho. \end{aligned} \tag{A20}$$

We are left to consider cases depending on the order of $z_L-z_R-\theta$ and $\rho_0-\psi$ (the first term) and the order of $z_R-z_L+\theta$ and $\rho_0+\psi$ (the last term). We show that there are three relevant intervals from which z_L can be drawn. Recall that, by supposition, $z_L > \max\{0, z_R-\theta z_R-(\rho_0+\psi)\}$.

[1.] Suppose, first, that $z_L \in (\max\{0, z_R-\theta z_R-(\rho_0+\psi)\}, z_R+\theta-(\rho_0+\psi))$. This implies $z_R-z_L+\theta > \rho_0+\psi$, which further implies $z_L-z_R-\theta < \rho_0-\psi$. On this domain, L's objective (A20) is strictly concave, with first-order condition:

$$z_L'(z_R, \rho_0) = \frac{\alpha(\theta-\rho_0-\psi)-\beta(\rho_0-\psi)-2\theta r+z_R(\alpha-\beta)}{\alpha+\beta(2\theta-1)}, \tag{A21}$$

which is strictly negative for all $\rho_0 \geq 0$ and $z_R \leq 0$. We conclude that (A20) strictly decreases on this domain.

[2.] Suppose, second, that $z_L \in [z_R+\theta-(\rho_0+\psi), z_R+\theta+(\rho_0-\psi)]$. This implies $z_R-z_L+\theta \leq \rho_0+\psi$ and $z_L-z_R-\theta \leq \rho_0-\psi$. On this domain, L's objective (A20) is strictly concave, with associated first-order condition:

$$z_L'(z_R, \rho_0) = \frac{2\theta r+\beta(\rho_0-\psi+z_R)}{\beta(1-2\theta)}, \tag{A22}$$

which is strictly negative by Assumption 1 that $2r > \alpha \geq \beta$, and Assumption 2 that $\theta > \rho_0+\psi+1$, which implies that $\theta > -\rho_0+\psi-z_R$. We conclude that (A20) strictly decreases on this domain.

[3.] Suppose, finally, that $z_L > z_R+\theta+(\rho_0-\psi)$. This implies $z_R-z_L+\theta < \rho_0+\psi$ and $z_L-z_R-\theta > \rho_0-\psi$. We find that $\frac{\partial \pi_R(z_L,z_R)}{\partial z_R} = \frac{-2r+\beta-2z_L\beta}{4\psi} < 0$, implying that (A20) strictly decreases on this domain.

We have shown that L's payoff (A20) strictly decreases in $z_L \geq \max\{0, z_R-\theta z_R-(\rho_0+\psi)\}$ whenever $z_R \leq 0$, verifying the claim.

Case 2: $z_R > 0$. Suppose $z_L > z_R$ is a best response. Party L wins if and only if $x^*(z_L, z_R, \rho) \geq 0$, i.e., if and only if $\rho \leq z_R - z_L$. Party L's expected payoff from a platform $z_L \geq z_R$ is therefore:

$$\begin{aligned}\pi_L(z_L, z_R) &= \frac{1}{2\psi}\int_{\rho_0-\psi}^{\max\{z_L-z_R-\theta,\rho_0-\psi\}}\left(r+\frac{\beta}{2}\right)d\rho + \frac{1}{2\psi}\int_{\max\{z_L-z_R-\theta,\rho_0-\psi\}}^{\max\{z_L-z_R-\theta z_L,\rho_0-\psi\}}\left(r+\beta\frac{x_6^*}{2}\right)d\rho \\ &+ \frac{1}{2\psi}\int_{\max\{z_L-z_R-\theta z_L,\rho_0-\psi\}}^{\max\{z_R-z_L-\theta z_R,\rho_0-\psi\}}\left(r+\beta\frac{x_5^*}{2}\right)d\rho + \frac{1}{2\psi}\int_{\max\{z_R-z_L-\theta z_R,\rho_0-\psi\}}^{z_R-z_L}\left(r+\beta\frac{x_4^*}{2}\right)d\rho \\ &+ \frac{\alpha}{2\psi}\int_{z_R-z_L}^{\min\{z_R-z_L+\theta,\rho_0+\psi\}}\left(\frac{1}{2}+\frac{x_4^*}{2}\right)d\rho. \end{aligned} \tag{A23}$$

We first observe that for any $(z_L, z_R) \in [0,1]^2$, Assumption 2 that $\theta > \rho_0 + \psi + 1$ implies that for any $(z_L, z_R) \in [0,1]^2$, $z_R - z_L + \theta > \rho_0 + \psi$ and $z_L - z_R - \theta < \rho_0 - \psi$. This implies that the upper limit of integration in the final integral of (A23) is $\rho_0 + \psi$, and that the first integral in (A23) is zero.

Next, we observe that by a similar argument to that of Lemma 1, a pair $0 < z_R < z_L$ is not an equilibrium if with probability one the swing voter type $x^*(z_L, z_R, \rho)$ is drawn weakly to the left of z_L. This implies that (z_L, z_R) is an equilibrium only if $\Delta(z_L; z_L, z_R, \rho_0 - \psi) > 0$, i.e., only if

$$(z_L - z_R) - (z_L - z_L) - \theta z_L - (\rho_0 - \psi) > 0 \iff z_L - z_R - \theta z_L > \rho_0 - \psi. \tag{A24}$$

Thus, the objective (A23) becomes:

$$\begin{aligned}\pi_L(z_L, z_R) &= \frac{1}{2\psi}\int_{\rho_0-\psi}^{z_L-z_R-\theta z_L}\left(r+\beta\frac{x_6^*}{2}\right)d\rho + \frac{1}{2\psi}\int_{z_L-z_R-\theta z_L}^{z_R-z_L-\theta z_R}\left(r+\beta\frac{x_5^*}{2}\right)d\rho \\ &+ \frac{1}{2\psi}\int_{z_R-z_L-\theta z_R}^{z_R-z_L}\left(r+\beta\frac{x_4^*}{2}\right)d\rho + \frac{\alpha}{2\psi}\int_{z_R-z_L}^{\rho_0+\psi}\left(\frac{1}{2}+\frac{x_4^*}{2}\right)d\rho. \end{aligned} \tag{A25}$$

It is easily verified that $\frac{\partial \pi_L(z_L,z_R)}{\partial z_L}|_{z_L=z_R} < 0$ for all $z_R > 0$. Thus, a platform $z_L > z_R$ is not a best response by party L. $\square$

Lemma 3. *There does not exist an equilibrium in which $z_L > 0$, or in which $z_R > 0$.*

Proof. Suppose first that $z_L > 0$ in an equilibrium. We consider two possible cases for the location of party R's platform: weakly to the right of the median voter's ideal policy, i.e., $z_R \geq 0$, or strictly to the left of the median voter's ideal policy, i.e., $z_R < 0$.

Case 1: $z_R \geq 0$. The previous lemmata imply that if $z_R \geq 0$ and $z_L > 0$, then $z_R = z_L \equiv \hat{z}$ in any equilibrium. Consider a deviation by party L to $z_L \in [0, \hat{z})$. This yields the following payoff to party L:

$$\begin{aligned}\pi_L(z_L,\hat z) &= \frac{1}{2\psi}\int_{\rho_0-\psi}^{\max\{z_L-\hat z-\theta\hat z,\rho_0-\psi\}}\left(r+\beta\frac{x_1^*}{2}\right)d\rho+\frac{1}{2\psi}\int_{\max\{z_L-\hat z-\theta\hat z,\rho_0-\psi\}}^{\max\{\hat z-z_L-\theta z_L,\rho_0-\psi\}}\left(r+\beta\frac{x_2^*}{2}\right)d\rho \\ &+\frac{1}{2\psi}\int_{\max\{\hat z-z_L-\theta z_L,\rho_0-\psi\}}^{\hat z-z_L}\left(r+\beta\frac{x_3^*}{2}\right)d\rho+\frac{\alpha}{2\psi}\int_{\hat z-z_L}^{\rho_0+\psi}\left(\frac{1}{2}+\frac{x_3^*}{2}\right)d\rho. \end{aligned} \tag{A26}$$

By now standard arguments, if with probability one the swing voter type $x^*(\hat z,\hat z,\rho)$ is realized weakly to the left of $\hat z$, then L strictly prefers a platform strictly to the left of $\hat z$, and thus $z_L=\hat z$ is not a best response. Recalling that $\Delta(x_i;z_L,z_R,\rho)$ defined in (3) is voter type x_i's net value from party L, we observe that with probability one $x^*(\hat z,\hat z,\rho)\le\hat z$ if and only if $\Delta(\hat z;\hat z,\hat z,\rho_0-\psi)\le 0$, i.e.,

$$(\hat z-\hat z)-(\hat z-\hat z)-\theta\hat z-(\rho_0-\psi)\le 0 \iff -\theta\hat z\le\rho_0-\psi. \tag{A27}$$

We conclude that if (A27) is satisfied, we cannot have an equilibrium. Suppose, instead, $-\theta\hat z>\rho_0-\psi$. Straightforward algebra verifies that $\frac{\partial\pi_L(\hat z,\hat z)}{\partial(-z_L)}>0$ if $r>\frac{\alpha}{2}-\frac{\psi}{2\theta}(\alpha-\beta)$, which is true because $r>\frac{\alpha}{2}$. Thus a deviation by L to a platform $z_L<\hat z$ is profitable.

Case 2: $z_R<0$. Part (i) of Lemma 2 implies that $z_L\le\max\{0,z_R-\theta z_R-(\rho_0+\psi)\}$. If $z_R\ge\frac{\rho_0+\psi}{1-\theta}$, we are done. However, Assumption 2 that $\theta>\rho_0+\psi+1$ implies that $-1<\frac{\rho_0+\psi}{1-\theta}$. Suppose, therefore, there is an equilibrium profile (z_L,z_R) such that $z_R<\frac{\rho_0+\psi}{1-\theta}$, and that $z_L\in(0,z_R-\theta z_R-(\rho_0+\psi))$. This implies that $z_R-z_L-\theta z_R>\rho_0+\psi$. But, by a similar argument to the proof of Lemma 1, z_R is not a best response, since $z_R-z_L-\theta z_R>\rho_0+\psi$ implies that the swing voter is realized strictly to the right of z_R with probability one.

We conclude that $z_L\le 0$, in an equilibrium. This, together with Lemma 1, implies that $z_R\le 0$ in an equilibrium. □

Lemma 3 implies that to rule out the existence of equilibria that are not characterized in Propositions 1, 2, 3, it is sufficient to show that there is no equilibrium in which $z_R<z_L\le 0$.

Lemma 4. *There does not exist an equilibrium in which $z_R<z_L\le 0$.*

Proof. Party R's expected payoff from $z_R\le z_L$ is:

$$\pi_R(z_L,z_R)=\frac{\alpha}{2\psi}\int_{\rho_0-\psi}^{z_L-z_R}\left(\frac{1}{2}-\frac{x_6^*}{2}\right)d\rho+\frac{1}{2\psi}\int_{z_L-z_R}^{\min\{z_L-z_R-\theta z_L,\rho_0+\psi\}}\left(r-\beta\frac{x_6^*}{2}\right)d\rho$$

$$+\frac{1}{2\psi}\int_{\min\{z_L-z_R-\theta z_L,\rho_0+\psi\}}^{\min\{z_R-z_L-\theta z_R,\rho_0+\psi\}}\left(r-\beta\frac{x_5^*}{2}\right)d\rho+\frac{1}{2\psi}\int_{\min\{z_R-z_L-\theta z_R,\rho_0+\psi\}}^{\rho_0+\psi}\left(r-\beta\frac{x_4^*}{2}\right)d\rho. \tag{A28}$$

Similarly, party L's expected payoff from $z_L \in [z_R, 0]$ is:

$$\pi_L(z_L,z_R)=\frac{1}{2\psi}\int_{\rho_0-\psi}^{z_L-z_R}\left(r+\beta\frac{x_6^*}{2}\right)d\rho+\frac{\alpha}{2\psi}\int_{z_L-z_R}^{\min\{z_L-z_R-\theta z_L,\rho_0+\psi\}}\left(\frac{1}{2}+\frac{x_6^*}{2}\right)d\rho$$
$$+\frac{\alpha}{2\psi}\int_{\min\{z_L-z_R-\theta z_L,\rho_0+\psi\}}^{\min\{z_R-z_L-\theta z_R,\rho_0+\psi\}}\left(\frac{1}{2}+\frac{x_5^*}{2}\right)d\rho+\frac{\alpha}{2\psi}\int_{\min\{z_R-z_L-\theta z_R,\rho_0+\psi\}}^{\rho_0+\psi}\left(\frac{1}{2}+\frac{x_4^*}{2}\right)d\rho \tag{A29}$$

By now standard arguments, party R's platform $z_R < z_L$ is a best response only if with positive probability the swing voter type $x^*(z_L, z_R, \rho)$ is realized strictly to the left of z_R. Recalling that $\Delta(x_i; z_L, z_R, \rho)$ defined in (3) is voter type x_i's net value from party L, with positive probability $x^*(z_L, z_R, \rho) < z_R$ if and only if $\Delta(z_R; z_L, z_R, \rho + \psi) < 0$ i.e., if and only if

$$(z_R - z_R) - (z_L - z_R) - \theta z_R - (\rho_0 + \psi) < 0 \iff z_R - z_L - \theta z_R < \rho_0 + \psi. \tag{A30}$$

We therefore restrict attention to pairs (z_L, z_R) such that $z_R < z_L$ and that further satisfy $z_R - z_L - \theta z_R < \rho_0 + \psi$. Notice that since $z_R < z_L$, condition (A30) further implies that with positive probability $x^*(z_L, z_R, \rho) < z_L$, i.e., $z_L - z_R - \theta z_L < \rho_0 + \psi$. Party R's first-order condition on this implied domain is therefore:

$$\hat{z}_R(z_L) = \frac{-\alpha(\theta+\rho_0-\psi)-\beta(\rho_0+\psi)+2\theta r+z_L(\alpha-\beta)}{\alpha+\beta(2\theta-1)}. \tag{A31}$$

Similarly, party L's first-order condition is:

$$\hat{z}_L(z_R) = \frac{-\alpha(\theta+\rho_0+\psi)+\beta(\psi-\rho_0)+2\theta r+z_R(\alpha-\beta)}{2\alpha\theta+\alpha-\beta}. \tag{A32}$$

We consider two possible cases for an equilibrium in which $z_R < z_L \leq 0$. In the first case, party L's platform is strictly to the left of zero, i.e., $z_L < 0$. In the second case, party L's platform is zero, i.e., $z_L = 0$.

Case 1: $z_L < 0$. When $z_R < z_L < 0$, the platforms solve (A31) and (A32). This yields a unique pair (z_L^*, z_R^*), such that $z_L^* - z_R^* > 0$ if and only if $\rho_0 - \psi > \frac{(2r-\alpha)\theta}{\alpha+\beta}$, which contradicts Assumption 2 that

$\rho_0 - \psi < -1$, and Assumption 1 that $2r > \alpha$.

Case 2: $z_L = 0$. We obtain party R's best response to party L's platform by substituting $z_L = 0$ into (A31). This yields

$$\hat{z}_R(0) < 0 \iff \rho_0 > \frac{\theta(2r-\alpha) + \psi(\alpha-\beta)}{\alpha+\beta} \equiv \hat{\rho}_0. \tag{A33}$$

Expression (A32) reveals that $\hat{z}_L(z_R)$ strictly increases in $z_R < 0$. Thus, for any $z_R < 0$,

$$\hat{z}_L(z_R) < 0 \iff \rho_0 > \frac{\theta(2r-\alpha) - \psi(\alpha-\beta)}{\alpha+\beta} \equiv \underline{\rho}_0. \tag{A34}$$

We have shown that $\hat{z}_R(0) < 0$ if and only if $\rho_0 > \hat{\rho}_0$, and that for any $z_R < 0$, $\hat{z}_L(z_R) < 0$ if $\rho_0 \geq \underline{\rho}_0$. Because $\underline{\rho}_0 \leq \hat{\rho}_0$ for all $\alpha \geq \beta$, we conclude that there does not exist an equilibrium in which $z_R < z_L = 0$. □

Existence of equilibrium. We now verify that there exists an equilibrium in which $z_L \leq z_R \leq 0$. The (at most) three swing voter types are given by $x_1^* = \frac{z_L - z_R - \rho}{\theta}$, $x_2^* = \frac{z_L + z_R - \rho}{2+\theta}$ and $x_3^* = \frac{z_R - z_L - \rho}{\theta}$, defined in expressions (A2), (A3) and (A4). Assumption 2 that $\theta > \rho_0 + \psi$ implies that $x_1^* \leq 1$ and $x_3^* \geq -1$ for all $\rho \in [\rho_0 - \psi, \rho_0 + \psi]$. Finally, party R wins if and only if $x^*(z_L, z_R, \rho) < 0$, i.e., if and only if $\rho > z_L - z_R$. Given $z_L \leq 0$, R's expected payoff from $z_R \in [z_L, 0]$ is therefore:

$$\begin{aligned} \pi_R(z_L, z_R) =& \frac{\alpha}{2\psi}\int_{\rho_0-\psi}^{z_L-z_R}\left(\frac{1}{2} - \frac{x_1^*}{2}\right)d\rho + \frac{1}{2\psi}\int_{z_L-z_R}^{\min\{\rho_0+\psi, z_L-z_R-\theta z_R\}}\left(r - \beta\frac{x_1^*}{2}\right)d\rho \\ &+ \frac{1}{2\psi}\int_{\min\{\rho_0+\psi, z_L-z_R-\theta z_R\}}^{\min\{\rho_0+\psi, z_R-z_L-\theta z_L\}}\left(r - \beta\frac{x_2^*}{2}\right)d\rho + \frac{1}{2\psi}\int_{\min\{\rho_0+\psi, z_R-z_L-\theta z_L\}}^{\rho_0+\psi}\left(r - \beta\frac{x_3^*}{2}\right)d\rho. \end{aligned} \tag{A35}$$

Given $z_R \leq 0$, L's expected payoff from $z_L \leq z_R$ is:

$$\begin{aligned} \pi_L(z_L, z_R) =& \frac{1}{2\psi}\int_{\rho_0-\psi}^{z_L-z_R}\left(r + \beta\frac{x_1^*}{2}\right)d\rho + \frac{\alpha}{2\psi}\int_{z_L-z_R}^{\min\{\rho_0+\psi, z_L-z_R-\theta z_R\}}\left(\frac{1}{2} + \frac{x_1^*}{2}\right)d\rho \\ &+ \frac{\alpha}{2\psi}\int_{\min\{\rho_0+\psi, z_L-z_R-\theta z_R\}}^{\min\{\rho_0+\psi, z_R-z_L-\theta z_L\}}\left(\frac{1}{2} + \frac{x_2^*}{2}\right)d\rho + \frac{\alpha}{2\psi}\int_{\min\{\rho_0+\psi, z_R-z_L-\theta z_L\}}^{\rho_0+\psi}\left(\frac{1}{2} + \frac{x_3^*}{2}\right)d\rho. \end{aligned} \tag{A36}$$

By now standard arguments, we observe that for any $z_L \leq 0$, $z_R \in [z_L, 0]$ is a best response only if with positive probability the swing voter type $x^*(z_L, z_R, \rho)$ is realized strictly to the left of party R's platform z_R, i.e., only if $z_L - z_R - \theta z_R \geq \rho_0 + \psi$. Similarly, for any $z_R \leq 0$, $z_L \leq z_R$ is a best response only if with positive probability the swing voter type $x^*(z_L, z_R, \rho)$ is realized strictly to the left of z_L, i.e., only if $z_R - z_L - \theta z_L < \rho_0 + \psi$.

We therefore focus on platforms satisfying $z_L - z_R - \theta z_R < \rho_0 + \psi$ and $z_R - z_L - \theta z_L < \rho_0 + \psi$, subsequently verifying that these conditions hold at the solutions we characterize, below. Assumption 2 then implies that R's objective (A35) is strictly concave in z_R. Solving the first-order condition yields:

$$\hat{z}_R^{\text{int}}(z_L) = \frac{-\alpha(\theta + \rho_0 - \psi) - \beta(\rho_0 + \psi) + 2\theta r + z_L(\alpha - \beta)}{\alpha + \beta(2\theta - 1)}. \tag{A37}$$

Similarly, L's objective (A36) is strictly concave in z_L. Solving the first-order condition yields:

$$\hat{z}_L^{\text{int}}(z_R) = \frac{-\alpha(\theta + \rho_0 + \psi) + \beta(\psi - \rho_0) + 2\theta r + z_R(\alpha - \beta)}{2\alpha\theta + \alpha - \beta}. \tag{A38}$$

Let (z_L^*, z_R^*) denote an equilibrium pair of platforms.

First, we identify conditions under which $z_L^* = z_R^* = 0$. We observe that $\hat{z}_L^{\text{int}}(0)$ strictly decreases in ρ_0, and also that $\hat{z}_R^{\text{int}}(0)$ strictly decreases in ρ_0. We find that:

$$\hat{z}_L^{\text{int}}(0) \geq 0 \iff \rho_0 \leq \frac{\theta(2r - \alpha) - (\alpha - \beta)\psi}{\alpha + \beta} \equiv \underline{\rho}_0, \tag{A39}$$

and

$$\hat{z}_R^{\text{int}}(0) \geq 0 \iff \rho_0 \leq \frac{\theta(2r - \alpha) + (\alpha - \beta)\psi}{\alpha + \beta} = \rho_0', \tag{A40}$$

where Assumption 2 that $\alpha \geq \beta$ implies that $\rho_0' \geq \underline{\rho}_0$. Thus, $z_L^* = z_R^* = 0$ if $\rho_0 \leq \underline{\rho}_0$.

Second, we identify conditions for $z_L^* < z_R^* = 0$. In that case, we have

$$z_L^* = \hat{z}_L^{\text{int}}(0) = \frac{-\alpha(\theta + \rho_0 + \psi) + \beta(\psi - \rho_0) + 2\theta r}{2\alpha\theta + \alpha - \beta}, \tag{A41}$$

and further require that $\hat{z}_R^{\text{int}}(\hat{z}_L^{\text{int}}(0)) \geq 0$. We have already shown that $\hat{z}_L^{\text{int}}(0) < 0$ if and only if $\rho_0 > \underline{\rho}_0$. We also have that

$$\hat{z}_R^{\text{int}}(\hat{z}_L^{\text{int}}(0)) \geq 0 \iff \rho_0 \leq \frac{\theta\left(\alpha\left(\frac{\psi(\alpha-\beta)}{\alpha\theta+\alpha-\beta} - 1\right) + 2r\right)}{\alpha + \beta} \equiv \overline{\rho}_0. \tag{A42}$$

Therefore, $z_L^* < z_R^* = 0$ if $\rho \in (\underline{\rho}_0, \overline{\rho}_0]$.

Third, we identify conditions for $z_L^* < z_R^* < 0$. In that case, we may solve the system of interior solutions, directly, to obtain:

$$\begin{aligned} z_L^* &= \frac{\beta\theta\psi(\beta-\alpha) - (\alpha+\beta(\theta-1))(\alpha(\theta+\rho_0)+\beta\rho_0-2\theta r)}{\theta\left(\alpha^2+2\alpha\beta\theta-\beta^2\right)}, \\ z_R^* &= z_L^* + (\alpha-\beta)\frac{(\alpha+\beta)(\psi-\rho_0)+\theta(2r-\alpha)}{\alpha^2+2\alpha\beta\theta-\beta^2}. \end{aligned} \tag{A43}$$

We now verify that for all $\rho_0 \geq 0$, the solution (z_L^*, z_R^*) is an equilibrium. To establish this, we proceed in two steps.

Step 1: verifying interior solutions. We verify that the pair (z_L^*, z_R^*) always satisfies the restrictions that $z_R^* - z_L^* - \theta z_L^* < \rho_0 + \psi$, which, in turn, implies $z_L^* - z_R^* - \theta z_R^* < \rho_0 + \psi$. We write $(z_L^*(\rho_0), z_R^*(\rho_0))$ to emphasize the dependence on R's advantage, ρ_0.

First, consider $\rho_0 \leq \underline{\rho}_0$, so that $z_L^*(\rho_0) = z_R^*(\rho_0) = 0$. In this case, the claim is immediate from $\rho_0 + \psi > 0$.

Second, consider $\rho_0 \in [\underline{\rho}_0, \overline{\rho}_0]$, so that $z_R^*(\rho_0) = 0$, and $z_L^*(\rho_0)$ is given by (A41). We therefore want to verify that $\rho_0 + \psi - (-z_L^*(\rho_0) - \theta z_L^*(\rho_0)) > 0$. The left-hand side of this inequality is linear in ρ_0. Since $\rho_0 \in [\underline{\rho}_0, \overline{\rho}_0]$, it is sufficient to verify that

$$\underline{\rho}_0 + \psi - (0 - z_L^*(\underline{\rho}_0) - \theta z_L^*(\underline{\rho}_0)) = \frac{\theta(2r-\alpha)+2\beta\psi}{\alpha+\beta} > 0, \tag{A44}$$

and that

$$\overline{\rho}_0 + \psi - (0 - z_L^*(\overline{\rho}_0) - \theta z_L^*(\overline{\rho}_0)) = \frac{\theta\left(\frac{\psi\left(\alpha^2+\beta^2\right)}{\alpha\theta+\alpha-\beta} + 2r - \alpha\right)}{\alpha+\beta} > 0. \tag{A45}$$

Third, we consider $\rho_0 > \overline{\rho}_0$, so that $z_L^*(\rho_0)$ and $z_R^*(\rho_0)$ are given by (A43). Because $(\rho_0 + \psi) - (z_R^*(\rho_0) - z_L^*(\rho_0) - \theta z_L^*(\rho_0))$ strictly increases in ρ_0 it is sufficient to recall that, by expression (A45), the difference is strictly positive evaluated at $\overline{\rho}_0$, and thus strictly positive for all $\rho_0 > \overline{\rho}_0$.

Step 2: verifying no "jump" deviations. First, we highlight that if $z_R^*(\rho_0) - \theta z_R^*(\rho_0) < \rho_0 + \psi$, i.e., implying that for any $\tilde{z}_L > 0$, $z_R^*(\rho_0) - \tilde{z}_L - \theta z_R^*(\rho_0) < \rho_0 + \psi$ then lemmata 1 and 2 imply that the only deviations that we need to consider are by party R to $z_R < z_L^*(\rho_0)$, and by party L to $z_L \in (z_R^*(\rho_0), 0]$. If $\rho_0 \leq \overline{\rho}_0$, then $z_R^*(\rho_0) = 0$, and $z_R^*(\rho_0) - \theta z_R^*(\rho_0) < \rho_0 + \psi$, trivially. To verify that, indeed, $z_R^*(\rho_0) - \theta z_R^*(\rho_0) \leq \rho_0 + \psi$, for $\rho_0 > \overline{\rho}_0$, we observe that:

$$\rho_0 + \psi - (z_R^*(\rho_0) - \theta z_R^*(\rho_0)) = \frac{\theta\psi(\alpha\theta(\alpha+\beta) + \beta(\alpha-\beta)) + \rho_0\left(\alpha\left(\theta^2(\beta-\alpha) + \theta(\alpha+\beta) + \alpha\right) - \beta^2\right)}{\theta\left(\alpha^2 + 2\alpha\beta\theta - \beta^2\right)},$$

and thus the difference is linear in ρ_0. We have

$$\overline{\rho}_0 + \psi - (z_R^*(\overline{\rho}_0) - \theta z_R^*(\overline{\rho}_0)) = \overline{\rho}_0 + \psi > 0 \tag{A46}$$

and since Assumption 2 that $\rho_0 - \psi < -1$ implies that $\rho_0 < \psi$, it is sufficient to verify that

$$\psi + \psi - (z_R^*(\psi) - \theta z_R^*(\psi)) = \left(\frac{1}{\theta} + 1\right)\psi > 0. \tag{A47}$$

We may therefore invoke lemmata 1 and 2 and restrict attention to only two deviations: by party L to $z_L \in (z_R, 0]$, and by party R to $z_R < z_L$.

No profitable deviation by party L to $z_L \in (z_R^*, 0]$. We have $z_R^* < 0$ if and only if $\rho_0 > \overline{\rho}_0$. The (at most) three swing voter types are given by $x_4^* = \frac{z_R^* - z_L - \rho}{\theta}$, $x_5^* = \frac{-(z_L + z_R^*) - \rho}{\theta - 2}$, and $x_6^* = \frac{z_L - z_R^* - \rho}{\theta}$. Party R wins if $z_L - z_R^* - \rho < 0$, i.e., if $\rho > z_L - z_R^*$. Party L's expected payoff from this deviation is:

$$\begin{aligned}\pi_L(z_L, z_R) = &\frac{1}{2\psi}\int_{\rho_0-\psi}^{\max\{z_L - z_R^* - \theta, \rho_0 - \psi\}}\left(r + \frac{\beta}{2}\right)d\rho + \frac{1}{2\psi}\int_{\max\{z_L - z_R^* - \theta, \rho_0-\psi\}}^{z_L - z_R^*}\left(r + \beta\frac{x_6^*}{2}\right)d\rho \\ &+ \frac{\alpha}{2\psi}\int_{z_L - z_R^*}^{\min\{z_L - z_R^* - \theta z_L, \rho_0+\psi\}}\left(\frac{1}{2} + \frac{x_6^*}{2}\right)d\rho + \frac{\alpha}{2\psi}\int_{\min\{z_L - z_R^* - \theta z_L, \rho_0+\psi\}}^{\min\{z_R^* - z_L - \theta z_R^*, \rho_0+\psi\}}\left(\frac{1}{2} + \frac{x_5^*}{2}\right)d\rho \\ &+ \frac{\alpha}{2\psi}\int_{\min\{z_R^* - z_L - \theta z_R^*, \rho_0+\psi\}}^{\min\{z_R^* - z_L + \theta, \rho_0+\psi\}}\left(\frac{1}{2} + \frac{x_4^*}{2}\right)d\rho.\end{aligned} \tag{A48}$$

We have already shown that $z_R^* - z_L^* - \theta z_L^* < \rho_0 + \psi$. Notice that $z_R^* - z_L^* - \theta z_R^* < z_R^* - z_L^* - \theta z_L^*$. And, for $z_L \in (z_R^*, 0]$, we have $z_R^* - z_L - \theta z_R^* < z_R^* - z_L^* - \theta z_R^*$, since $z_L > z_R^*$ implies $z_L > z_L^*$. We conclude that $z_R^* - z_L - \theta z_R^* < \rho_0 + \psi$. This, in turn, implies $z_L - z_R^* - \theta z_L < \rho_0 + \psi$, since $\theta > 2$. Assumption 2 that $\theta > \rho_0 + \psi + 1$ is equivalent to $\theta - 1 > \rho_0 + \psi$. This implies that for any $z_R^* < 0$ and $z_L \in (z_R^*, 0]$, $z_R^* - z_L + \theta > \rho_0 + \psi$. This implies that $z_L - z_R^* - \theta < -\rho_0 - \psi < \rho_0 - \psi$, for $\rho_0 > \underline{\rho}_0 > 0$. We then verify that (A48) is strictly concave, and yields the same first-order condition as given in (A38), which implies that $z_L > z_R^*$ cannot be optimal when $\rho_0 > \overline{\rho}_0$.

No profitable deviation by party R to $z_R < z_L^*$. Party R wins if and only if $\rho > z_L^* - z_R$. R's payoff from this deviation is

$$\begin{aligned}\pi_R(z_L^*, z_R) &= \frac{\alpha}{2\psi}\int_{\max\{\rho_0-\psi, z_L^*-z_R-\theta\}}^{z_L^*-z_R}\left(\frac{1}{2}-\frac{x_6^*}{2}\right)d\rho + \frac{1}{2\psi}\int_{z_L^*-z_R}^{\min\{\rho_0+\psi, z_L^*-z_R-\theta z_L^*\}}\left(r-\beta\frac{x_6^*}{2}\right)d\rho \\ &+ \frac{1}{2\psi}\int_{\min\{\rho_0+\psi, z_L^*-z_R-\theta z_L^*\}}^{\min\{\rho_0+\psi, z_R-z_L^*-\theta z_R\}}\left(r-\beta\frac{x_5^*}{2}\right)d\rho + \frac{1}{2\psi}\int_{\min\{\rho_0+\psi, z_R-z_L^*-\theta z_R\}}^{\min\{\rho_0+\psi, z_R-z_L^*+\theta\}}\left(r-\beta\frac{x_4^*}{2}\right)d\rho \\ &+ \frac{1}{2\psi}\int_{\min\{\rho_0+\psi, z_R-z_L^*+\theta\}}^{\rho_0+\psi}\left(r+\frac{\beta}{2}\right). \end{aligned} \tag{A49}$$

By a similar argument to the previous paragraph, Assumption 2 implies that $z_L^* - z_R - \theta < \rho_0 - \psi$, and that $z_R - z_L^* + \theta > \rho_0 + \psi$. Also, by now familiar arguments, $z_R < z_L^*$ is not a best response if with probability one the swing voter's type $x^*(z_L^*, z_R, \rho)$ is realized weakly to the right of z_R. We therefore restrict attention to $z_R < z_L^*$ satisfying the restriction that $z_R - z_L^* - \theta z_R < \rho_0 + \psi$. Under these restrictions, (A49) is strictly concave, with first-order condition that is equivalent to the first-order condition identified in expression (A37), and which therefore implies that a deviation to $z_R < z_L^*$ is not profitable.

Proof of Corollary 1. In this case, we have $z_R^*(\rho_0) = 0$, so that

$$z_L^*(\rho_0) = \frac{-\alpha(\theta+\rho_0+\psi)+\beta(\psi-\rho_0)+2\theta r}{2\alpha\theta+\alpha-\beta}. \tag{A50}$$

We obtain comparative statics for each of the primitives, in turn.

Higher ρ_0. We have $\frac{\partial z_L^*}{\rho_0} = -\frac{\alpha+\beta}{2\alpha\theta+\alpha-\beta} < 0$. Thus, z_L^* *decreases* in ρ_0.

Higher θ. We have

$$\frac{\partial z_L^*}{\partial\theta} = \frac{\alpha(\alpha(2\rho_0+2\psi-1)+2\beta(\rho_0-\psi)+\beta)+2r(\alpha-\beta)}{(2\alpha\theta+\alpha-\beta)^2}. \tag{A51}$$

The numerator of this expression strictly increases in ρ_0, and is therefore positive if and only if $\rho_0 \geq -\frac{(\alpha-\beta)(\alpha(2\psi-1)+2r)}{2\alpha(\alpha+\beta)}$. This threshold is strictly negative and thus vacuously satisfied. We conclude that z_L^* *increases* in θ.

Higher α. We have

$$\frac{\partial z_L^*}{\partial\alpha} = \frac{\beta(2\theta\rho_0-2\theta\psi+\theta+2\rho_0)-2\theta(2\theta+1)r}{(2\alpha\theta+\alpha-\beta)^2}. \tag{A52}$$

Calling $\nu(\rho_0)$ the numerator of this expression, we find that $\nu(\rho_0)$ strictly increases in ρ_0, and that $\nu(\overline{\rho}_0) < 0$. Thus, z_L^* strictly *decreases* in α.

Higher ψ. $\frac{\partial z_L^*}{\partial \psi} = \frac{\beta-\alpha}{2\alpha\theta+\alpha-\beta} < 0$.

Higher r. We have $\frac{\partial z_L^*}{\partial r} = \frac{2\theta}{2\alpha\theta+\alpha-\beta} > 0$. $\square$

Proof of Corollaries 2, 3 and 4 and 5.

$$\begin{aligned} z_L^* &= \frac{\beta\theta\psi(\beta-\alpha) - (\alpha+\beta(\theta-1))(\alpha(\theta+\rho_0)+\beta\rho_0-2\theta r)}{\theta\left(\alpha^2+2\alpha\beta\theta-\beta^2\right)}, \\ z_R^* &= z_L^* + (\alpha-\beta)\frac{((\alpha+\beta)(\psi-\rho_0)+\theta(2r-\alpha))}{\alpha^2+2\alpha\beta\theta-\beta^2}. \end{aligned} \tag{A53}$$

We obtain comparative statics for each of the primitives, in turn.

Higher ρ_0. We find that $\frac{\partial z_L^*}{\partial \rho_0} = -\frac{(\alpha+\beta)(\alpha+\beta(\theta-1))}{\theta(\alpha^2+2\alpha\beta\theta-\beta^2)} < 0$. Moreover, $\frac{\partial[z_R^*-z_L^*]}{\partial \rho_0} = \frac{\beta^2-\alpha^2}{\alpha^2+2\alpha\beta\theta-\beta^2} < 0$, which implies that z_R^* also decreases in ρ_0, and faster than z_L^*. $\square$

Higher α. We start with the platform z_L^*. We find that $\frac{\partial z_L^*}{\partial \alpha}$ can be written as a quotient with a strictly positive denominator, and a numerator that we call $\nu(r,\rho_0)$, which strictly decreases in r. Recalling that Assumption 1 states $r > \frac{1}{2}\left(\alpha+\frac{\psi}{\theta}(\alpha-\beta)\right)$, we find that $\nu(\frac{1}{2}\left(\alpha+\frac{\psi}{\theta}(\alpha-\beta)\right),\rho_0)$ is linear in ρ_0. Straightforward algebra (omitted) verifies that $\nu(\frac{1}{2}\left(\alpha+\frac{\psi}{\theta}(\alpha-\beta)\right),\overline{\rho}_0) < 0$ and $\nu(\frac{1}{2}\left(\alpha+\frac{\psi}{\theta}(\alpha-\beta)\right),\psi) < 0$. Thus, $\frac{\partial z_L^*}{\partial \alpha} < 0$. Next, we consider the platform z_R^*. We find that $\frac{\partial z_R^*}{\partial \alpha}$ can be written as a quotient with a strictly positive denominator, and a numerator that we call $\mu(\rho_0,\psi)$, which strictly decreases in ρ_0. Therefore, there exists $\hat{\rho}_0$ such that $\mu(\rho_0,\psi) \geq 0$ if and only if $\rho_0 \leq \hat{\rho}_0$. Thus, $\rho_0 > \hat{\rho}_0$ implies that z_R^* *decreases* in α, while $\rho_0 \leq \hat{\rho}_0$ implies that z_R^* *increases* in α.

Next, we establish that $\hat{\rho}_0 < \overline{\rho}_0$. Straightforward substitution (omitted) verifies that $\overline{\rho}_0 - \hat{\rho}_0$ strictly increases in r, and that

$$\overline{\rho}_0 - \hat{\rho}_0 > 0 \iff r > \frac{1}{2}\beta\left(\frac{\psi\left(2\alpha^2\theta+(\alpha-\beta)^2\right)}{(\alpha\theta+\alpha-\beta)^2} - 1\right) \equiv \hat{r}. \tag{A54}$$

Assumption 1 says that $r > \frac{1}{2}\left(\alpha+\frac{\psi}{\theta}(\alpha-\beta)\right) \equiv r^*$. We establish that $r^* > \hat{r}$. We observe that $r^* - \hat{r}$ is linear in ψ, and strictly positive positive evaluated at $\psi=0$ and $\psi=\theta$. Because Assumption 2 that $\theta > \rho_0+\psi+1$ implies that $\psi < \theta$, we conclude that $r^* > \hat{r}$. Thus, $\overline{\rho}_0 > \hat{\rho}_0$, and z_R^* *decreases* in α.

Higher r. $\frac{\partial z_L^*}{\partial r} = \frac{2(\alpha+\beta(\theta-1))}{\alpha^2+2\alpha\beta\theta-\beta^2} > 0$, and $\frac{\partial z_R^*}{\partial r} = \frac{2(\alpha\theta+\alpha-\beta)}{\alpha^2+2\alpha\beta\theta-\beta^2} > 0$, and $\frac{\partial[z_R^*-z_L^*]}{\partial r} = \frac{2\theta(\alpha-\beta)}{\alpha^2+2\alpha\beta\theta-\beta^2} > 0$. $\square$

Higher ψ. $\frac{\partial z_L^*}{\partial \psi} = \frac{\beta(\alpha-\beta)}{\beta^2-\alpha(\alpha+2\beta\theta)} < 0$, and $\frac{\partial z_R^*}{\partial \psi} = \frac{\alpha(\alpha-\beta)}{\alpha^2+2\alpha\beta\theta-\beta^2} > 0$. $\square$

Online Appendix to "The Race to the Base"

Dan Bernhardt, Peter Buisseret and Sinem Hidir

Appendix B: "Returning to Base" Extension

Assumption 1 says that parties put a large premium $r > \frac{1}{2}\left(\alpha + \frac{\psi}{\theta}(\alpha - \beta)\right)$ on winning a majority. This reflects the majoritarian operation of legislative organization: the winning party enjoys control over the legislative timetable and the appointment of key positions such as committee chairs; in parliamentary democracies, the majority-winning party is also awarded control of the executive branch.

Parties may nonetheless face an election in which relative party popularity is especially volatile (high ψ), or long-standing party loyalties are in flux (low θ), or parties place an especially high premium on maintaining their core districts ($\alpha - \beta$ large), so that our assumption fails. In that case, if the initial advantage in favor of either party is not too large, we obtain a unique equilibrium in which *both* parties revert to their core districts.

In this Appendix, we focus on a setting in which $r < \frac{1}{2}\left(\alpha + \frac{\psi}{\theta}(\alpha - \beta)\right)$, i.e., in which Assumption 1 fails. We establish the following result for the case in which the initial advantage favoring party R is not too large.

Proposition 4. *If $r < \frac{1}{2}\left(\alpha + \frac{\psi}{\theta}(\alpha - \beta)\right)$, and party R's advantage is not too large in the sense that*

$$\rho_0 < \frac{\alpha\theta(\alpha\theta + \psi(\alpha - \beta) - 2r\theta)}{(\alpha + \beta)(\alpha - \beta + \alpha\theta)}, \tag{A55}$$

then there exists a unique pure strategy equilibrium, in which party L retreats to its base:

$$z_L^*(\rho_0) = \frac{\alpha\theta(\theta(2r - \alpha) - \alpha(\psi + \rho_0) + \beta(\psi - \rho_0)) - (\alpha^2 - \beta^2)\rho_0}{2\alpha\theta(\alpha\theta + \alpha - \beta)} < 0, \tag{A56}$$

and party R also retreats to its base, albeit to a more limited extent,

$$|z_L^*(\rho_0)| > z_R^*(\rho_0) = \; z_L^*(\rho_0) + \frac{\alpha\theta + \psi(\alpha - \beta) - 2\theta r}{\alpha\theta + \alpha - \beta} > 0. \tag{A57}$$

As each party's relative concern for its core districts α increases, both parties further retreat to their respective bases. Perhaps surprisingly, the stronger party retreats *more* quickly:

Corollary 6. *As α increases,* both *parties increasingly retreat to their respective bases, but the stronger party moves faster than the weaker party, and to an extent that increases in its initial advantage, ρ_0.*

When α increases, each party cares relatively more about catering to its core districts. On the one hand, this partly encourages a party to abandon centrist districts in favor of those whose medians are relatively more extreme than the party's platform—i.e., medians with preferred policies to the left of z_L^* for party L, or to the right of z_R^* for party R. On the other hand, each party also has core districts whose medians are relatively more moderate than the party's platform—i.e., medians with preferred policies between z_L^* and 0 for party L, or between 0 and z_R^* for party R. Increases in α also encourage each party to moderate further in order to increase its prospect of winning these districts. Critically, party R is initially positioned closer to the median voter than party L:

$$\frac{z_L^* + z_R^*}{2} = -\rho_0 \frac{\alpha + \beta}{2\alpha\theta} < 0. \tag{A58}$$

Thus, as α increases, a relatively higher proportion of R's core district medians are more extreme than the party's platform, vis-à-vis party L's. This encourages a relatively greater retreat to the base by party R. Thus, the parties become more polarized, and the midpoint of the parties' platforms also moves in the direction of the stronger party's core districts.

Proof of Proposition 4. We characterize the unique equilibrium, which satisfies $z_L \leq 0 \leq z_R$. First, we rule out other possible equilibria.

Step 1: No equilibrium in which $z_R \leq z_L \leq 0$, with at least one strict inequality. The proof replicates verbatim the proof of Lemma 4.

Step 2: No equilibrium in which $z_L \leq z_R \leq 0$. Suppose, first, $z_L = z_R = 0$. Letting $\hat{z}_L^{\text{int}}(z_R)$ denote L's interior solution on $[-1, z_R]$ given $z_R \leq 0$, we showed in our benchmark proofs that:

$$\hat{z}_L^{\text{int}}(0) \geq 0 \iff \rho_0 \leq \frac{\theta(2r - \alpha) - (\alpha - \beta)\psi}{\alpha + \beta} \equiv \underline{\rho}_0. \tag{A59}$$

If $r < \frac{1}{2}\left(\alpha + \frac{\psi}{\theta}(\alpha - \beta)\right)$, then $\underline{\rho}_0 < 0$, so that L strictly prefers to deviate to the left of zero, for all $\rho_0 \geq 0$. Suppose, next, that $z_L < z_R = 0$. Then, we showed in our benchmark proofs that L's best response (that we also showed is interior) is

$$\hat{z}_L^{\text{int}}(0) = \frac{-\alpha(\theta + \rho_0 + \psi) + \beta(\psi - \rho_0) + 2\theta r}{2\alpha\theta + \alpha - \beta}. \tag{A60}$$

Consider, however, R's value from a platform $z_R > 0$. For $z_R > 0$ sufficiently close to zero, this payoff is:

$$\pi_R(z_L, z_R) = \frac{\alpha}{2\psi}\int_{\rho_0-\psi}^{\hat{z}_L^{\text{int}}(0)-z_R-\theta z_R}\left(\frac{1}{2}-\frac{x_1^*}{2}\right)d\rho + \frac{\alpha}{2\psi}\int_{\hat{z}_L^{\text{int}}(0)-z_R-\theta z_R}^{\hat{z}_L^{\text{int}}(0)+z_R}\left(\frac{1}{2}-\frac{x_2^*}{2}\right)d\rho$$
$$+\frac{1}{2\psi}\int_{\hat{z}_L^{\text{int}}(0)+z_R}^{\min\{z_R-\hat{z}_L^{\text{int}}(0)-\theta\hat{z}_L^{\text{int}}(0),\rho_0+\psi\}}\left(r-\beta\frac{x_2^*}{2}\right)d\rho + \frac{1}{2\psi}\int_{\min\{z_R-\hat{z}_L^{\text{int}}(0)-\theta\hat{z}_L^{\text{int}}(0),\rho_0+\psi\}}^{\rho_0+\psi}\left(r-\beta\frac{x_3^*}{2}\right)d\rho. \tag{A61}$$

We show, first, that $-\hat{z}_L^{\text{int}}(0)-\theta\hat{z}_L^{\text{int}}(0) < \rho_0+\psi$. Straightforward substitution establishes:

$$-\hat{z}_L^{\text{int}}(0)-\theta\hat{z}_L^{\text{int}}(0)-(\rho_0+\psi) = \frac{2\beta\rho_0+\theta(-\psi(\alpha+\beta)+\alpha\theta-\alpha\rho_0+\alpha+\beta\rho_0-2(\theta+1)r)}{2\alpha\theta+\alpha-\beta}, \tag{A62}$$

which is linear in ρ_0, and easily verified to be strictly negative evaluated at $\rho_0 = 0$ and $\rho_0 = \frac{\alpha\theta(\alpha\theta+\psi(\alpha-\beta)-2r\theta)}{(\alpha+\beta)(\alpha-\beta+\alpha\theta)}$. Using the appropriate limits of integration in (A61), we have that

$$\frac{\partial\pi_R(\hat{z}_L^{\text{int}}(0),0)}{\partial z_R} = \frac{\alpha^2(\theta(\theta-\rho_0+\psi)-\rho_0)-\alpha\beta\theta(\rho_0+\psi)+\beta^2\rho_0-2\alpha\theta^2 r}{2\theta\psi(2\alpha\theta+\alpha-\beta)}, \tag{A63}$$

which strictly decreases in ρ_0, and satisfies

$$\left.\frac{\partial\pi_R(\hat{z}_L^{\text{int}}(0),0)}{\partial z_R}\right|_{\rho_0=\frac{\alpha\theta(\alpha\theta+\psi(\alpha-\beta)-2r\theta)}{(\alpha+\beta)(\alpha-\beta+\alpha\theta)}} = 0. \tag{A64}$$

Thus, for any $\rho_0 < \frac{\alpha\theta(\alpha\theta+\psi(\alpha-\beta)-2r\theta)}{(\alpha+\beta)(\alpha-\beta+\alpha\theta)}$, R strictly prefers to deviate to a platform strictly to the right of zero.

Suppose, finally, $z_L < z_R < 0$. We showed earlier that this implies $\rho_0 > \overline{\rho}_0$, where $\overline{\rho}_0$ is defined in (A42). Straightforward algebra verifies that $\overline{\rho}_0 > \hat{\rho}_0$, implying that $\rho_0 > \overline{\rho}_0$ violating the parameter restriction that $\rho_0 < \hat{\rho}_0$.

Step 3: No equilibrium in which $0 \le z_L \le z_R$, with at least one strict inequality. Party L's payoff from $z_L \in [0, z_R]$ is

$$\pi_L(z_L, z_R) = \frac{1}{2\psi}\int_{\rho_0-\psi}^{\max\{z_L-z_R-\theta z_R,\rho_0-\psi\}}\left(r+\beta\frac{x_1^*}{2}\right)d\rho + \frac{1}{2\psi}\int_{\max\{z_L-z_R-\theta z_R,\rho_0-\psi\}}^{\max\{z_R-z_L-\theta z_L,\rho_0-\psi\}}\left(r+\beta\frac{x_2^*}{2}\right)d\rho$$
$$+\frac{1}{2\psi}\int_{\max\{z_R-z_L-\theta z_L,\rho_0-\psi\}}^{z_R-z_L}\left(r+\beta\frac{x_3^*}{2}\right)d\rho + \frac{\alpha}{2\psi}\int_{z_R-z_L}^{\rho_0+\psi}\left(\frac{1}{2}+\frac{x_3^*}{2}\right)d\rho. \tag{A65}$$

Likewise, R's payoff from $z_R \geq z_L$ is:

$$\pi_R(z_L, z_R) = \frac{\alpha}{2\psi}\int_{\rho_0-\psi}^{\max\{z_L - z_R - \theta z_R, \rho_0-\psi\}} \left(\frac{1}{2} - \frac{x_1^*}{2}\right) d\rho + \frac{\alpha}{2\psi}\int_{\max\{z_L - z_R - \theta z_R, \rho_0-\psi\}}^{\max\{z_R - z_L - \theta z_L, \rho_0-\psi\}} \left(\frac{1}{2} - \frac{x_2^*}{2}\right) d\rho$$
$$+ \frac{\alpha}{2\psi}\int_{\max\{z_R - z_L - \theta z_L, \rho_0-\psi\}}^{z_R - z_L} \left(\frac{1}{2} - \frac{x_3^*}{2}\right) d\rho + \frac{1}{2\psi}\int_{z_R - z_L}^{\rho_0+\psi} \left(r - \beta\frac{x_3^*}{2}\right) d\rho. \tag{A66}$$

Similar arguments to those used in benchmark proofs imply that we must have $z_L - z_R - \theta z_R > \rho_0 - \psi$, and thus $z_R - z_L - \theta z_L > \rho_0 - \psi$, in an equilibrium. Both objectives are concave on their implied domains. We first argue that we cannot have $0 < z_L = z_R$. To see this, observe that

$$\frac{\partial \pi_L(z_R, z_R)}{\partial z_L} = -\frac{\alpha(-\theta + \rho_0 + \psi) + 2\theta r + \beta(\rho_0 - \psi + 2\theta z_R)}{4\theta\psi}, \tag{A67}$$

which strictly decreases in z_R and ρ_0, and satisfies $\frac{\partial \pi_L(z_R, z_R)}{\partial z_L}|_{\rho_0 = z_R = 0} = \frac{\alpha(\theta-\psi)+\beta\psi-2\theta r}{4\theta\psi}$, which is strictly negative if and only if $r > \frac{\alpha}{2} - \frac{\psi}{2\theta}(\alpha - \beta)$, which holds under $\alpha > \beta$ and $r > \frac{\alpha}{2}$.

We next argue that we cannot have $0 < z_L < z_R$. To show this, we characterize unique interior best responses, using (A65) and (A66):

$$\hat{z}_L(z_R) = \frac{\alpha(\theta - \rho_0 - \psi) - \beta(\rho_0 - \psi) - 2\theta r + z_R(\alpha - \beta)}{\alpha + \beta(2\theta - 1)}, \tag{A68}$$

and

$$\hat{z}_R(z_L) = \frac{\alpha(\theta - \rho_0 + \psi) - \beta(\rho_0 + \psi) - 2\theta r + z_L(\alpha - \beta)}{2\alpha\theta + \alpha - \beta}. \tag{A69}$$

Solving the pair of best responses, we obtain:

$$z_L^* = \frac{\alpha\theta\psi(\beta - \alpha) - (\alpha\theta + \alpha - \beta)(\rho_0(\alpha + \beta) + \theta(2r - \alpha))}{\theta\left(\alpha^2 + 2\alpha\beta\theta - \beta^2\right)} < 0, \tag{A70}$$

a contradiction. Suppose, finally, that $0 = z_L < z_R$. R's interior best response to $z_L = 0$ is:

$$\hat{z}_R(0) = \frac{\alpha(\theta - \rho_0 + \psi) - \beta(\rho_0 + \psi) - 2\theta r}{2\alpha\theta + \alpha - \beta}. \tag{A71}$$

It is straightforward to verify that L's payoff strictly decreases in $z_L \in [0, z_R]$. We further show that a platform $z_L < 0$ is strictly preferred to $z_L = 0$. To see this, note that L's payoff from a platform

$z_L < 0$ when $z_R > 0$ is:

$$\begin{aligned}\pi_L(z_L, z_R) \;&= \frac{1}{2\psi}\int_{\rho_0-\psi}^{\max\{z_L-z_R-\theta z_R,\rho_0-\psi\}}\left(r+\beta\frac{x_1^*}{2}\right)d\rho+\frac{1}{2\psi}\int_{\max\{z_L-z_R-\theta z_R,\rho_0-\psi\}}^{z_L+z_R}\left(r+\beta\frac{x_2^*}{2}\right)d\rho\\ &+\frac{\alpha}{2\psi}\int_{z_L+z_R}^{\min\{z_R-z_L-\theta z_L,\rho_0+\psi\}}\left(\frac{1}{2}+\frac{x_2^*}{2}\right)d\rho+\frac{\alpha}{2\psi}\int_{\min\{z_R-z_L-\theta z_L,\rho_0+\psi\}}^{\rho_0+\psi}\left(\frac{1}{2}+\frac{x_3^*}{2}\right)d\rho.\end{aligned} \tag{A72}$$

For $z_L < 0$ sufficiently close to zero, we have $z_R - z_L - \theta z_L < \rho_0 + \psi$. If $-\theta z_R \geq \rho_0 - \psi$, then R has a profitable deviation to $z_R' < z_R$. Suppose, instead, $-\theta z_R < \rho_0 - \psi$. We find that

$$\frac{\partial \pi_L(0, \frac{\alpha(\theta-\rho_0+\psi)-\beta(\rho_0+\psi)-2\theta r}{2\alpha\theta+\alpha-\beta})}{\partial z_L} = \frac{\alpha^2(-(\theta(\theta+\rho_0+\psi)+\rho_0))+\alpha\beta\theta(\psi-\rho_0)+\beta^2\rho_0+2\alpha\theta^2 r}{2\theta\psi(2\alpha\theta+\alpha-\beta)}, \tag{A73}$$

which strictly decreases in $\rho_0 \geq 0$, and is strictly negative when evaluated at $\rho_0 = 0$, under the parameter restriction $r < \frac{\alpha}{2} + \frac{\psi}{2\theta}(\alpha - \beta)$.

Step 4: No equilibrium in which $0 \leq z_R \leq z_L$*, with at least one strict inequality.* To rule out $z_L > z_R \geq 0$, we may replicate the argument of Lemma 2, Case 2. Similarly, to rule out $z_L = z_R > 0$, we may replicate the argument of Lemma 3, Case 1.

Step 5: No equilibrium in which $z_R \leq 0 \leq z_L$*, with at least one strict inequality.* If $z_R \leq 0$ is a best response to $z_L \geq 0$, then we must have $z_R - z_L - \theta z_R \leq \rho_0 + \psi$. It is then straightforward to extend the argument of Lemma 2 in our benchmark model to verify that L's payoff strictly decreases in $z_L \geq 0$. To rule out $z_R < z_L = 0$, it suffices to replicate verbatim the proof of Lemma 4.

Characterizing the equilibrium. We now verify that there exists an equilibrium in which $z_L \leq 0 \leq z_R$. The (at most) three swing voter types are given by $x_1^* = \frac{z_L - z_R - \rho}{\theta}$, $x_2^* = \frac{z_L + z_R - \rho}{2+\theta}$ and $x_3^* = \frac{z_R - z_L - \rho}{\theta}$. Assumption 2 that $\theta > \rho_0 + \psi$ implies that $x_1^* \leq 1$ and $x_3^* \geq -1$ for all $\rho \in [\rho_0 - \psi, \rho_0 + \psi]$. Finally, party R wins if and only if $\rho > z_L + z_R$. Given $z_L \leq 0$, R's expected payoff from $z_R \in [z_L, 0]$ is therefore:

$$\begin{aligned}\pi_R(z_L, z_R) \;&= \frac{\alpha}{2\psi}\int_{\rho_0-\psi}^{\max\{z_L-z_R-\theta z_R,\rho_0-\psi\}}\left(\frac{1}{2}-\frac{x_1^*}{2}\right)d\rho+\frac{\alpha}{2\psi}\int_{\max\{z_L-z_R-\theta z_R,\rho_0-\psi\}}^{z_L+z_R}\left(\frac{1}{2}-\frac{x_2^*}{2}\right)d\rho\\ &+\frac{1}{2\psi}\int_{z_L+z_R}^{\min\{z_R-z_L-\theta z_L,\rho_0+\psi\}}\left(r-\beta\frac{x_2^*}{2}\right)d\rho+\frac{1}{2\psi}\int_{\min\{z_R-z_L-\theta z_L,\rho_0+\psi\}}^{\rho_0+\psi}\left(r-\beta\frac{x_3^*}{2}\right)d\rho.\end{aligned} \tag{A74}$$

L's corresponding payoff is

$$\pi_L(z_L, z_R) = \frac{1}{2\psi}\int_{\rho_0-\psi}^{\max\{z_L-z_R-\theta z_R,\rho_0-\psi\}}\left(r+\beta\frac{x_1^*}{2}\right)d\rho + \frac{1}{2\psi}\int_{\max\{z_L-z_R-\theta z_R,\rho_0-\psi\}}^{z_L+z_R}\left(r+\beta\frac{x_2^*}{2}\right)d\rho$$
$$+\frac{\alpha}{2\psi}\int_{z_L+z_R}^{\min\{z_R-z_L-\theta z_L,\rho_0+\psi\}}\left(\frac{1}{2}+\frac{x_2^*}{2}\right)d\rho + \frac{\alpha}{2\psi}\int_{\min\{z_R-z_L-\theta z_L,\rho_0+\psi\}}^{\rho_0+\psi}\left(\frac{1}{2}+\frac{x_3^*}{2}\right)d\rho. \tag{A75}$$

By now standard arguments, we must have $z_L - z_R - \theta z_R > \rho_0 - \psi$ and $z_R - z_L - \theta z_L < \rho_0 + \psi$ in any equilibrium. We therefore solve for equilibrium under the presumption that both strict inequalities hold, and then verify that they indeed hold at the solutions we derive, below. Both objectives are strictly concave, and the corresponding system of first-order conditions yields a unique solution (z_L^*, z_R^*) as given in the statement of the Proposition. We observe that $z_R^*(\rho_0)$ decreases in ρ_0, and is strictly positive so long as ρ_0 is strictly less than the cut-off in the Proposition. Similarly, we observe that $z_L^*(\rho_0)$ strictly decreases in ρ_0, and $z_L^*(0) < 0$ so long as $r < \frac{\alpha}{2} + \frac{\psi}{2\theta}(\alpha - \beta)$.

Verifying interior solutions. We first verify that $z_L^*(\rho_0) - z_R^*(\rho_0) - \theta z_R^*(\rho_0) - (\rho_0 - \psi) > 0$. Straightforward algebra yields that this difference strictly *decreases* in ρ_0, and because $\rho_0 < \frac{\alpha\theta(\alpha(\theta+\psi)-\beta\psi-2\theta r)}{(\alpha+\beta)(\alpha\theta+\alpha-\beta)} \equiv \rho_0^*$, it is sufficient to observe that

$$z_L^*(\rho_0^*) - z_R^*(\rho_0^*) - \theta z_R^*(\rho_0^*) - (\rho_0^* - \psi) = \frac{2\alpha\beta\theta\psi + \theta(2r-\alpha)(\alpha\theta+\alpha+\beta)}{(\alpha+\beta)(\alpha\theta+\alpha-\beta)} > 0. \tag{A76}$$

We next verify that $\rho_0 + \psi - (z_R^*(\rho_0) - z_L^*(\rho_0) - \theta z_L^*(\rho_0)) > 0$. Straightforward algebra yields that this difference strictly *increases* in ρ_0, and because $\rho_0 \geq 0$, it is sufficient to observe that

$$0 + \psi - (z_R^*(0) - z_L^*(0) - \theta z_L^*(0)) = \frac{\theta\psi(\alpha+\beta) + \theta(\theta+2)(2r-\alpha)}{2(\alpha\theta+\alpha-\beta)} > 0. \tag{A77}$$

Verifying no "jump" deviations. We consider four possible deviations: to $z_L \in (0, z_R^*]$, to $z_L \in (z_R^*, 1]$, to $z_R \in [-1, z_L^*)$, and to $z_R \in [z_L^*, 0)$.

Case 1: $z_R \in [z_L^*, 0)$. R's payoff from this platform is:

$$\pi_R(z_L^*, z_R) = \frac{\alpha}{2\psi}\int_{\rho_0-\psi}^{z_L^*-z_R}\left(\frac{1}{2}-\frac{x_1^*}{2}\right)d\rho + \frac{1}{2\psi}\int_{z_L^*-z_R}^{\min\{\rho_0+\psi, z_L^*-z_R-\theta z_R\}}\left(r-\beta\frac{x_1^*}{2}\right)d\rho$$

$$+\frac{1}{2\psi}\int_{\min\{\rho_0+\psi,z_L^*-z_R-\theta z_R\}}^{\min\{\rho_0+\psi,z_R-z_L^*-\theta z_L^*\}}\left(r-\beta\frac{x_2^*}{2}\right)d\rho+\frac{1}{2\psi}\int_{\min\{\rho_0+\psi,z_R-z_L^*-\theta z_L^*\}}^{\rho_0+\psi}\left(r-\beta\frac{x_3^*}{2}\right)d\rho. \tag{A78}$$

Recall that $z_R^*-z_L^*-\theta z_L^*<\rho_0+\psi$, and $z_R^*>0$, implying that for any $z_R<0$, $z_R-z_L^*-\theta z_L^*<\rho_0+\psi$. In turn, this implies $z_L^*-z_R-\theta z_R<\rho_0+\psi$. R's optimal interior platform on this interval is therefore:

$$z_R(\rho_0)=\frac{-\alpha(\theta+\rho_0-\psi)-\beta(\rho_0+\psi)+2\theta r+z_L^*(\rho_0)(\alpha-\beta)}{\alpha+\beta(2\theta-1)}. \tag{A79}$$

It is straightforward to verify that $z_R(\rho_0)$ strictly *decreases* in ρ_0, and recalling that $\rho_0<\frac{\alpha\theta(\alpha(\theta+\psi)-\beta\psi-2\theta r)}{(\alpha+\beta)(\alpha\theta+\alpha-\beta)}\equiv\rho_0^*$, we verify $z_R(\rho_0^*)=\frac{2\theta(2r-\alpha)}{\alpha+\beta(2\theta-1)}>0$, so that a deviation to $z_R\in[z_L,0)$ cannot be optimal.

Case 2: $z_R\in[-1,z_L^*]$. Party R wins if and only if $\rho>z_L^*-z_R$. R's payoff from this deviation is

$$\begin{aligned}\pi_R(z_L^*,z_R)\ &=\frac{\alpha}{2\psi}\int_{\max\{\rho_0-\psi,z_L^*-z_R-\theta\}}^{z_L^*-z_R}\left(\frac{1}{2}-\frac{x_6^*}{2}\right)d\rho+\frac{1}{2\psi}\int_{z_L^*-z_R}^{\min\{\rho_0+\psi,z_L^*-z_R-\theta z_L^*\}}\left(r-\beta\frac{x_6^*}{2}\right)d\rho\\&+\frac{1}{2\psi}\int_{\min\{\rho_0+\psi,z_L^*-z_R-\theta z_L^*\}}^{\min\{\rho_0+\psi,z_R-z_L^*-\theta z_R\}}\left(r-\beta\frac{x_5^*}{2}\right)d\rho+\frac{1}{2\psi}\int_{\min\{\rho_0+\psi,z_R-z_L^*-\theta z_R\}}^{\min\{\rho_0+\psi,z_R-z_L^*+\theta\}}\left(r-\beta\frac{x_4^*}{2}\right)d\rho\\&+\frac{1}{2\psi}\int_{\min\{\rho_0+\psi,z_R-z_L^*+\theta\}}^{\rho_0+\psi}\left(r+\frac{\beta}{2}\right).\end{aligned} \tag{A80}$$

We first claim $z_L^*-z_R-\theta<\rho_0-\psi$, i.e., $\theta>z_L^*-z_R-\rho_0+\psi$. To see this, note that $\rho_0-\psi<-1$ and $\theta>\rho_0+\psi+1$, we have that $\theta>\rho_0+\psi+1>-\rho_0+\psi+(z_L^*-z_R)$. Similarly, $\theta-1>\rho_0+\psi$ implies $z_R-z_L^*+\theta>\rho_0+\psi$. If $-\theta z_L^*\geq\rho_0+\psi$, the non-profitability of $z_R<z_L^*$ is immediate. Suppose, instead, $-\theta z_L^*>\rho_0+\psi$. Then we may restrict attention to z_R such that $z_R-z_L^*-\theta z_R<\rho_0+\psi$, i.e., $z_R\in[\min\{-1,\frac{-(z_L+\rho_0+\psi)}{\theta-1}\},z_L]$. This implies that (A80) is strictly concave, with a first-order condition that is equivalent to the first-order condition identified in expression (A79), and which therefore implies that a deviation to $z_R<z_L^*$ is not profitable.

Case 3: $z_L\in(0,z_R^*]$. Party L's payoff from this deviation is

$$\begin{aligned}\pi_L(z_L,z_R^*)\ &=\frac{1}{2\psi}\int_{\rho_0-\psi}^{\max\{z_L-z_R^*-\theta z_R^*,\rho_0-\psi\}}\left(r+\beta\frac{x_1^*}{2}\right)d\rho+\frac{1}{2\psi}\int_{\max\{z_L-z_R^*-\theta z_R^*,\rho_0-\psi\}}^{\max\{z_R^*-z_L-\theta z_L,\rho_0-\psi\}}\left(r+\beta\frac{x_2^*}{2}\right)d\rho\\&+\frac{1}{2\psi}\int_{\max\{z_R^*-z_L-\theta z_L,\rho_0-\psi\}}^{z_R^*-z_L}\left(r+\beta\frac{x_3^*}{2}\right)d\rho+\frac{\alpha}{2\psi}\int_{z_R^*-z_L}^{\rho_0+\psi}\left(\frac{1}{2}+\frac{x_3^*}{2}\right)d\rho.\end{aligned} \tag{A81}$$

Because $z_L^*-z_R^*-\theta z_R^*>\rho_0-\psi$, we have $z_L-z_R^*-\theta z_R^*>\rho_0-\psi$, since $z_L^*<0<z_L$. This, in turn,

yields $z_R^* - z_L - \theta z_L > \rho_0 - \psi$. We obtain L's optimal platform on this domain:

$$z_L(\rho_0) = \frac{\alpha(\theta - \rho_0 - \psi) - \beta(\rho_0 - \psi) - 2\theta r + z_R^*(\rho_0)(\alpha - \beta)}{\alpha + \beta(2\theta - 1)}, \tag{A82}$$

It is straightforward to verify that $z_L(\rho_0)$ strictly *decreases* in $\rho_0 \geq 0$, and that $z_L(0) < 0$, so that $z_L > 0$ cannot be optimal.

Case 4: $z_L \in [z_R^*, 1]$. The argument ruling out a deviation to $z_L > z_R^*$ replicates that for Lemma 2, Case 2. Similarly, to rule out $z_L = z_R^*$, we may replicate the argument of Lemma 3, Case 1.

Proof of Corollary 6, and additional comparative statics.

Higher ρ_0. We have $\frac{\partial z_L^*}{\partial \rho_0} = \frac{\partial z_R^*}{\partial \rho_0} = -\frac{\alpha+\beta}{2\alpha\theta} < 0$.

Higher α. We have

$$\frac{\partial z_R^*}{\partial \alpha} = \frac{\alpha^2\theta^2(\beta(\psi - 1) + 2(\theta + 1)r) + \beta\rho_0(\alpha\theta + \alpha - \beta)^2}{2\alpha^2\theta(\alpha\theta + \alpha - \beta)^2}, \tag{A83}$$

which is strictly positive. However:

$$\frac{\partial z_L^*}{\partial \alpha} = \frac{\alpha^2\theta^2(\beta(-\psi) + \beta - 2(\theta + 1)r) + \beta\rho_0(\alpha\theta + \alpha - \beta)^2}{2\alpha^2\theta(\alpha\theta + \alpha - \beta)^2}. \tag{A84}$$

It is straightforward to observe that (A84) *increases* in ρ_0. We evaluate (A84) at $\rho_0 = \frac{\alpha\theta(\alpha(\theta+\psi) - \beta\psi - 2\theta r)}{(\alpha+\beta)(\alpha\theta+\alpha-\beta)}$, its highest value under the Proposition, obtaining:

$$\left.\frac{\partial z_L^*}{\partial \alpha}\right|_{\rho_0 = \frac{\alpha\theta(\alpha(\theta+\psi) - \beta\psi - 2\theta r)}{(\alpha+\beta)(\alpha\theta+\alpha-\beta)}} = \frac{\frac{\beta(\alpha\theta+\alpha-\beta)(\alpha(\theta+\psi)-\beta\psi-2\theta r)}{(\alpha+\beta)} + \alpha\theta(\beta(-\psi) + \beta - 2(\theta + 1)r)}{2(\alpha\theta + \alpha - \beta)^2}. \tag{A85}$$

Noting that (A85) *decreases* in r, we substitute $r = \frac{\alpha}{2}$, and after re-arranging, find that (A85) is strictly negative if and only

$$\beta\psi(\beta - \alpha)(\beta - \alpha(\theta + 1)) - \alpha\theta(\alpha + \beta)(\alpha(\theta + 1) + \beta(\psi - 1)) < 0. \tag{A86}$$

Expanding the LHS of (A86) yields:

$$-\alpha^2\theta(\alpha(\theta + 1) + \beta(\psi - 1)) - \alpha\beta\psi(\beta - \alpha(\theta + 1))$$

$$+\underbrace{\alpha(-\beta)\theta(\alpha(\theta+1)+\beta(\psi-1))+\beta^2\psi(\beta-\alpha(\theta+1))}_{<0}, \tag{A87}$$

so that it is sufficient to verify that the first term of (A87) is strictly negative. Rearranging the first term of (A87)

$$-\alpha\left(\alpha^2\theta^2+\alpha^2\theta-\alpha\beta(\theta+\psi)+\beta^2\psi\right), \tag{A88}$$

which is indeed strictly negative under $\alpha \geq \beta$ and $\theta > \psi$. We conclude that z_L^* strictly *decreases* α. Finally, we have:

$$\frac{\partial(.5(z_L^*+z_R^*))}{\partial\alpha}=\frac{\rho_0\beta}{2\alpha^2\theta}>0, \quad \frac{\partial^2(.5(z_L^*+z_R^*))}{\partial\alpha\partial\rho_0}=\frac{\beta}{2\alpha^2\theta}>0. \tag{A89}$$

Online Appendix to "The Race to the Base"

Dan Bernhardt, Peter Buisseret and Sinem Hidir

Appendix C: Policy-Motivated Justifications of $\alpha \geq \beta$.

C1: Parties Represent Their Constituents. Our benchmark model treats a party as a single, decisive agent. In practice, parties may consist of factions that are differentiated by their political goals. To illustrate how the reduced form payoff function in our main presentation can be justified, we recognize that a party's electoral strategy partly determines which voters support the party, but the set of voters that are expected to support a party also determine the party's electoral strategy. It is a perspective—and model formulation—that was introduced in theories of party formation and electoral competition with endogenous parties by Baron (1993) and Roemer (2001).[2]

We assume that whichever party wins the election implements its platform.[3] Recalling that $d_L \in [0, 1]$ is the share of districts (i.e., legislative seats) won by party L, we denote the winning policy

$$z^*(d_L) = \begin{cases} z_R & \text{if } d_L < \frac{1}{2} \\ z_L & \text{if } d_L \geq \frac{1}{2}. \end{cases} \tag{A90}$$

For any district with median $m \in [-1, 1]$, the total welfare of voters in that district is:

$$v(m, z_L, z_R, d_L) = \frac{1}{2Z} \int_{m-Z}^{m+Z} -|x - z^*(d_L)| \, dx. \tag{A91}$$

For any $x^* \in [-1, 1]$, a district with median type $m \in [-1, x^*]$ is subsequently represented by a member of party L, while a district with median type $m \in (x^*, 1]$ is represented by a member of party R. Because $d_L = \frac{1+x^*}{2}$, we may define the welfare of constituents served by L's representatives as:

$$W_L(d_L, z_L, z_R) = \frac{1}{x^* - (-1)} \int_{-1}^{x^*} v(m, z_L, z_R, d_L) \, dm = \frac{1}{2d_L} \int_{-1}^{2d_L - 1} v(m, z_L, z_R, d_L) \, dm. \tag{A92}$$

[2] Recent work includes Gomberg, Marhuenda and Ortuño-Ortín (2016). See also Caplin and Nalebuff (1997).

[3] We maintain our convention that if a voter is indifferent between the parties, she votes for party L. Similarly, in the event that the parties tie in a district, we specify that party L wins the district. Finally, if the parties each win one half of the districts, we specify that party L obtains the majority, and therefore wins the election. Party L therefore wins a majority if and only if its share of districts is $d_L \geq 1/2$. Since the set of indifferent voters has measure zero for any shock realization, this convention has no bearing on our results.

Likewise, the welfare of R's constituents is:

$$W_R(d_L, z_L, z_R) = \frac{1}{1-x^*}\int_{x^*}^{1} v(m, z_L, z_R, d_L)\, dm = \frac{1}{2(1-d_L)}\int_{2d_L-1}^{1} v(m, z_L, z_R, d_L)\, dm. \tag{A93}$$

We assume that each party $P \in \{L, R\}$ balances a concern for its constituents with a desire to increase its share of a fixed office rent normalized to one, that depends on its share of seats, $d_P \in [0, 1]$, according to the following specification:

$$\pi_p(d_P) = \mathbf{1}[d_P > .5]\eta + (1-\eta)d_P, \tag{A94}$$

where $\eta \in [0, 1)$ reflects the extent to which legislative power is concentrated in the hands of the majority. Critically, we do not assume that the marginal contribution of a seat above the majority threshold exceeds the marginal value of a seat below the majority threshold. To see this, observe that this formulation is a special case of our benchmark setting in which $\alpha = \beta = 1 - \eta$, and $r = \eta + \frac{1}{2}(1-\eta)$.

We assume that party J trades off the desire of party leaders to capture a share of office rents with the pressure to reflect the preferences of the party's electoral constituency:

$$u_P(d_P, z_L, z_R) = \pi_P(d_P) + \gamma W_P(d_P, z_L, z_R), \tag{A95}$$

where $\gamma > 0$. Finally, to simplify the analysis, we assume that voter preferences within each district are also uniformly distributed around their medians.

Assumption 3. In a district with median $m \in [-1, 1]$, voter types are uniformly distributed on $[m - Z, m + Z]$, where $Z > 2$.

The restriction that $Z > 2$ implies that there is more heterogeneity within districts than there is across district medians.

Introducing a seat motivation together with a policy-motivated component ensures that the marginal value of an additional seat is always positive. The subtlety in this formulation—that is absent in our benchmark—is that the party's trade-off over seats is partly a function of its platform and its opponent's platform.

Analysis. Define $\alpha_P(d_P) = \frac{\partial u_P}{\partial d_P}$ for $d_P \in [0, \frac{1}{2})$, and $\beta(d_P) = \frac{\partial u_P}{\partial d_P}$ for $d_P \in (\frac{1}{2}, 1]$.

Proposition 5. *For any $\eta \in [0, 1]$ and $\gamma > 0$, whenever $z_L \le z_R$: $d < \frac{1}{2} < d'$ implies $\alpha(d) > \beta(d')$.*

Proof. We prove the result for party L, because the extension to party R is immediate. By Assumption 3 that in any district with median m, voter types are distributed uniformly on $[m-Z, m+Z]$ for $Z > 2$, we obtain:

$$v(m, z_L, z_R, d_L) = \frac{1}{2Z}\int_{m-Z}^{m+Z} -|x - z^*(d_L)|\,dx = -\frac{Z}{2} - \frac{(m-Z)^2}{2Z}. \tag{A96}$$

It is therefore sufficient to observe that for any $d < \frac{1}{2} < d'$ and $z_L \leq z_R$,

$$\alpha(d) - \beta(d') = \frac{4\gamma(d'-d)}{3Z} + \frac{\gamma(z_R - z_L)}{Z} > 0. \tag{A97}$$

Finally, a sufficient condition for the marginal value of an additional seat always to be positive is that $\beta(1) > 0$. Straightforward substitution yields that

$$\beta(1) = 1 - \eta + \gamma\frac{z_L - \frac{1}{3}}{Z} > 1 - \eta - \gamma\frac{4}{3Z}, \tag{A98}$$

which is strictly positive if districts are sufficiently heterogeneous (Z large enough) or if parties are predominantly concerned with winning more seats (i.e., $\gamma < 1-\eta$). $\square$

Figure 2 illustrates L's value from winning additional districts. The discontinuous increase in L's value from winning a majority arises for any $\eta > 0$ and $\gamma > 0$, so long as $z_L \leq z_R$. Our piece-wise linear formulation can be interpreted as an approximation of u_P, and captures its key property that the marginal of winning a core district exceeds the marginal value of winning an opponent's core district.

To understand the real-world interpretation of this property, suppose that the R party offers a centrist platform, and wins the election. As its legislative majority advances from small to large, it represent districts whose voters increasingly dislike the party's platform. Party leaders may internalize this consequence for both non-instrumental and instrumental reasons. For instance, rank-and-file legislators from these districts may be more difficult to corral, and may require a larger share of side payments and transfers in exchange for their cooperation on other aspects of the party's legislative agenda. This idea is reflected in former Democratic House Minority Whip and Majority Leader Steny Hoyer's claim that "...the larger your majority, the harder it is to maintain your unity" (quoted in Poole, 2004.)

Note that Proposition 5 also applies if the average welfare of the party's constituents is evalu-

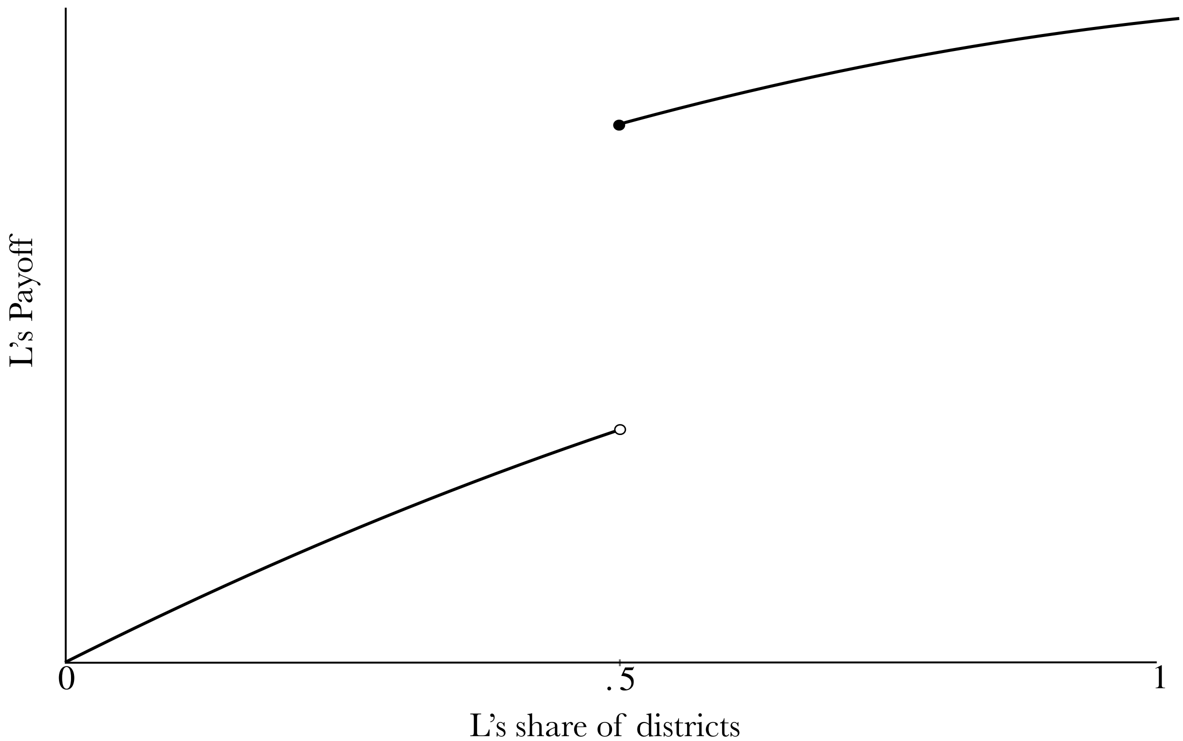

Figure 2 – L's induced preferences over seats when $z_L = -.25$ and $z_R = 0$. Primitives: $Z = 3$, $\gamma = 1.5$, $\eta = .5$

ated at the party's platform rather than the winning platform, i.e., if we replace $v(m, z^*(d_L))$ in expression (A91) with $v(m, z_J)$. This could reflect the preferences of a party faction that cares solely about the congruence of the party's platform with the preferences of the party's constituents—regardless of whether the party wins power and implements the platform. This is analogous to the preferences of the 'militant' party faction in Roemer (1999).

Equilibrium in Extended Model: Example. We close by highlighting an example of equilibrium under our extended model for a set of parameters. The main qualitative properties of the equilibrium platforms replicate those of our benchmark setting. All approximations are to three decimal places.

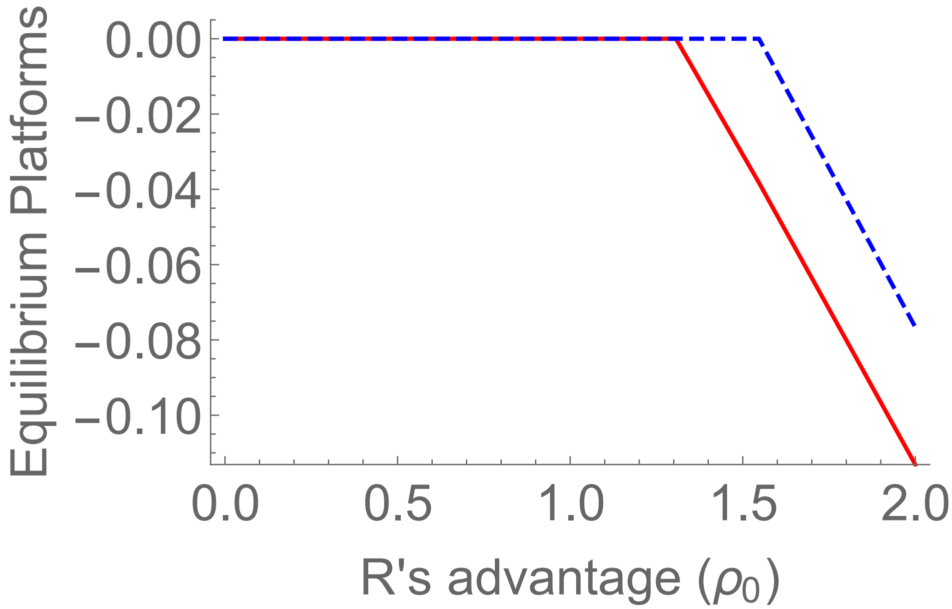

Figure 3 – Equilibrium platforms z_L^* (*red*) and z_R^* (*blue*) in the extended model when $\eta = \frac{1}{5}$, $\gamma = \frac{3}{10}$, $Z = 2$, $\psi = 3$, $\theta = 6$.

Example 1. *Set* $\eta = \frac{1}{5}$, $\gamma = \frac{3}{10}$, $Z = 2$, $\psi = 3$, $\theta = 6$. *Then, there exist thresholds* $\underline{\rho}_0 = 1.307$ *and* $\overline{\rho}_0 = 1.547$, *such that:*

[1.] if R*'s advantage is* ***small****, i.e.,* $\rho_0 \le \underline{\rho}_0$*, both parties locate at the ideal policy of the median voter in the median district:* $z_L^* = z_R^* = 0$,

[2.] if R*'s advantage is* ***intermediate****, i.e.,* $\rho_0 \in (\underline{\rho}_0, \overline{\rho}_0]$, $z_L^* < z_R^* = 0$*, then party* L *retreats to its base but* R *still locates at the ideal policy of the median voter in the median district:* $z_L^* < z_R^* = 0$*; and*

[3.] if R*'s advantage is* ***large****, i.e.,* $\rho_0 > \overline{\rho}_0$*, then party* L *retreats by more to its base, and party* R *advances towards* L*'s base:* $z_L^* < z_R^* < 0$.

The platforms are highlighted in Figure 3.

C2: A Policy Outcome Function, and Policy-Motivated Parties. We now provide an alternative justification based on post-election legislative policymaking employed in Grossman and Helpman (1996), Alesina and Rosenthal (1996), Ortuño-Ortín (1997) and Lizzeri and Persico (2001), in which the final policy outcome depends on both the parties' platforms, and the winner's margin of victory. The idea is that a party's vote share exerts marginal effects on the final policy outcome. Controlling a slight majority may give a party formal agenda-setting power, but winning more seats gives the party leadership a buffer to protect against defections and to weaken the negotiating leverage of the party's marginal legislators in shaping the final policy outcome.[4] Because this is a property of legislatures, not of election systems, the logic may hold in both majoritarian and proportional election contexts: parties with only small margins of victory will find it more challenging to implement their platforms than a party with an outsized victory. A larger majority may also be perceived as granting the majority party a greater electoral mandate to pursue its agenda, rather than mandating compromise with the minority party.

Party Platforms. Recall that, in addition to her policy payoffs from a party's platform choice z_L or z_R, a voter type x_i also derives a net value $-\theta x_i$ from party L. One interpretation is that the parties have fixed platforms y_L and y_R on a second policy. For example, let party R's fixed platform on this second policy be $y_R = 1$ and party L's fixed platform be $y_L = -1$. If a voter type x_i's relative value from party L on this policy dimension is $\theta|y_R - x_i| - \theta|y_L - x_i|$, then each district median voter's net value from L is $-2\theta x_i$.

In the analysis that follows, we adopt the interpretation that each party's platform is a vector with two components: party L's platform is $p_L = (y_L, z_L)$, and party R's platform is $p_R = (y_R, z_R)$. We admit any $(y_L, y_R) \in \mathbb{R}^2$, such that $y_L \neq y_R$.[5]

Policy Outcome Function. To capture the reality that a party's margin of victory affects its ability to implement its campaign promise, we assume that if the winning party's share of districts is d, the majority-winning party's platform is p^M and the minority party's platform is p^m, the final policy outcome is:

$$p^*(p^M, p^m, d) = \eta p^M + (1 - \eta)(dp^M + (1 - d)p^m). \tag{A99}$$

The parameter $\eta \in [0, 1)$ reflects the majoritarian organization of the legislature: higher values imply that the majority party increasingly dominates the policy outcome, regardless of its margin.

Party Goals. We assume that parties have both policy and office goals. Specifically, they divide a fixed office rent that's normalized to one—e.g., committee chairs, funding for districts—where the division is determined by the same rule specified in (A99).[6] Second, they aim to represent the

[4] For example, the Democratic leadership was forced to make many concessions to the Blue Dog Democrats, in shaping the final form of the Affordable Care Act. "Blue Dogs Delay, Water Down House Health Care Bill", *Huffington Post*, August 29 2009. https://www.huffpost.com/entry/blue-dogs-delay-water-dow_n_247177.

[5] This assumption ensures that even if the parties converge on the issue where they can adjust policies, i.e., by choosing $z_L = z_R$, the identity of the winner has a payoff consequence for voters.

[6] We adopt the same sharing rule on rents as for policy only for parsimony—we could allow for any division rule that increases with a party's share of seats.

entire polity, but may prioritize some districts over others. Specifically, party J's payoff is:

$$u_J(p_L, p_R, d_J) = M_J\eta + (1-\eta)d_J + \frac{1}{2}\int_{-1}^{1} w_J(m)v(m, p^*)\, dm, \tag{A100}$$

where, as before, M_J is an indicator taking the value 1 if J wins a majority, and for any policy outcome $p = (y, z)$:

$$v(m, p) = \int_{m-Z}^{m+Z} -|x-z| - \theta|x-y|\, f(x)dx \tag{A101}$$

is the welfare of citizens in a district with median m, and $w_J(m)$ is the weight that party J places on the welfare of voters in a district with median $m \in [-1, 1]$, satisfying $w_J(m) \geq 0$ for all m, and $\int_{-1}^{1} w_J(m)\, dm = 1$. We maintain Assumption 3 that voter types are uniformly distributed around their district medians.

Analysis. We analyze the model from party L's perspective, noting that the analysis for party R's is symmetric. Define $\alpha(d_L) = \frac{\partial u_L}{\partial d_L}$ for $d_L \in [0, \frac{1}{2})$, and $\beta(d_L) = \frac{\partial u_L}{\partial d_L}$ for $d_L \in (\frac{1}{2}, 1]$.

Proposition 6. *For any $(z_L, z_R) \in \mathbb{R}^2$, any $(y_L, z_R) \in \mathbb{R}^2$ such that $y_L \neq y_R$, and any $d < \frac{1}{2} < d'$, $\alpha(d) > \beta(d')$.*

Proof. Under Assumption 3 that in a district with median type $m \in [-1, 1]$, voter ideal points are uniformly distributed on $[m-Z, m+Z]$, we have that for any $d_L \in [0, 1]$:

$$u_L(p_L, p_R, d_J) = \eta\mathbf{1}[d_L \geq 1/2] + (1-\eta)d_L - \frac{1}{4Z}\int_{-1}^{-1} w_L(m)\Big[(Z^2 + (m-z^*)^2) + \theta(Z^2 + (m-y^*)^2)\Big]\, dm \tag{A102}$$

where

$$z^* = \begin{cases} \eta z_R + (1-\eta)[d_L z_L + (1-d_L)z_R] & \text{if } d_L < \frac{1}{2} \\ \eta z_L + (1-\eta)[d_L z_L + (1-d_L)z_R] & \text{if } d_L \geq \frac{1}{2} \end{cases} \tag{A103}$$

and

$$y^* = \begin{cases} \eta y_R + (1-\eta)[d_L y_L + (1-d_L)y_R] & \text{if } d_L < \frac{1}{2} \\ \eta y_L + (1-\eta)[d_L y_L + (1-d_L)y_R] & \text{if } d_L \geq \frac{1}{2}. \end{cases} \tag{A104}$$

Thus, for any $d_L \in [0, 1/2) \cup (1/2, 1]$:

$$\frac{\partial u_L}{\partial d_L} = 1-\eta + \frac{1}{2Z}\int_{-1}^{1} w_L(m)(m-z^*)(z_L-z_R)(1-\eta)\,dm + \frac{\theta}{2Z}\int_{-1}^{1} w_L(m)(m-y^*)(y_L-y_R)(1-\eta)\,dm, \tag{A105}$$

and

$$\frac{\partial^2 u_L}{\partial d_L^2} = -\frac{1}{2Z}\int_{-1}^{1} w_L(m)(z_L-z_R)^2(1-\eta)^2\,dm - \frac{\theta}{2Z}\int_{-1}^{1} w_L(m)(y_L-y_R)^2(1-\eta)^2\,dm < 0, \tag{A106}$$

First, inspection of (A106) reveals that for any pair $(z_L, z_R) \in [0,1]^2$, the party's payoff is strictly concave in $d_L \in [0, \frac{1}{2})$ and strictly concave in $d_L \in (\frac{1}{2}, 1]$, for any pair $(z_L, z_R) \in [-1,1]^2$.

Second, for any pair $(z_L, z_R) \in [0,1]^2$, we verify that for any $d_L^m < \frac{1}{2} < d_L^M$:

$$\left.\frac{\partial u_L}{\partial d_L}\right|_{d_L^m} > \left.\frac{\partial u_L}{\partial d_L}\right|_{d_L^M}. \tag{A107}$$

We prove this by observing that, by strict concavity of u_J in d_L^m and in d_L^M, it is sufficient to verify:

$$\lim_{d_L^m \uparrow \frac{1}{2}} \frac{\partial u_L}{\partial d_L} - \lim_{d_L^M \downarrow \frac{1}{2}} \frac{\partial u_L}{\partial d_L} = \frac{(z_L-z_R)^2\eta(1-\eta)}{2Z}\int_{-1}^{1} w_L(m)\,dm + \theta\frac{(y_L-y_R)^2\eta(1-\eta)}{2Z}\int_{-1}^{1} w_L(m)\,dm > 0. \tag{A108}$$

This yields the result. □

Note that we do not rely on any parameter or weighting function restrictions. Parameter restrictions do, however, ensure that the marginal value of an additional district is positive, i.e., that (A105) is positive. For example, this holds whenever districts are sufficiently heterogeneous (Z is large enough).

Figure 4 illustrates L's value from winning additional districts. Our piece-wise linear formulation can be interpreted as an approximation of u_P in this context, that again captures its key property that the marginal of winning a core district exceeds the marginal value of winning an opponent's core district.

Equilibrium in Extended Model: An Example. We close by highlighting an example of equilibrium under our extended model for a set of parameters. The main qualitative properties of the equilibrium platforms replicate those of our benchmark setting. All approximations are to three decimal places. The platforms are highlighted in figure 5.

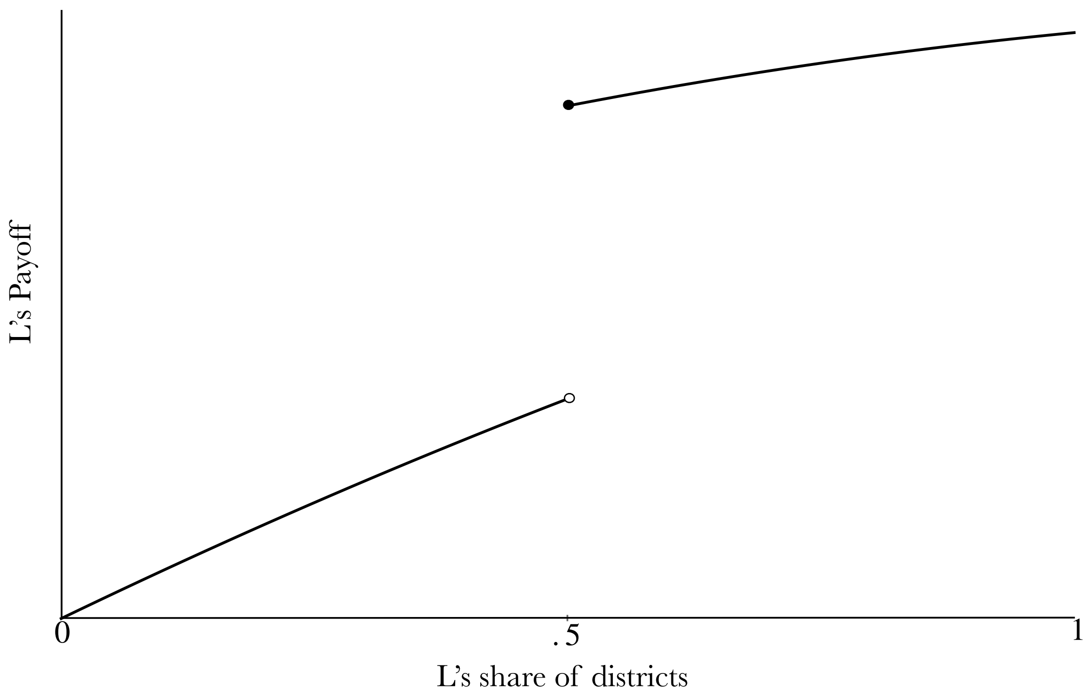

Figure 4 – L's induced preferences over seats when $z_L = 0$ and $z_R = 0$. Primitives: $Z = 3$, $\gamma = .5$, $\eta = .5$, $y_L = -1$, $y_R = 1$, $\theta = 6$, $w_L(m) = \frac{3}{8}(1-m)^2$.

Example 2. *Set* $\eta = \frac{1}{5}$, $\gamma = \frac{3}{10}$, $Z = 2$, $\psi = 3$, $\theta = 6$, $y_L = -1$, $y_R = 1$, *and* $w_L(m) = \frac{1-m}{2}$, *and* $w_R(m) = \frac{1+m}{2}$. *Then, there exist thresholds* $\underline{\rho}_0 = .650$ *and* $\overline{\rho}_0 = 1.211$, *such that:*

[1.] if R*'s advantage is* ***small****, i.e.,* $\rho_0 \leq \underline{\rho}_0$, *both parties locate at the ideal policy of the median voter in the median district:* $z_L^* = z_R^* = 0$,

[2.] if R*'s advantage is* ***intermediate****, i.e.,* $\rho_0 \in (\underline{\rho}_0, \overline{\rho}_0]$, $z_L^* < z_R^* = 0$, *then party* L *retreats to its base but* R *still locates at the ideal policy of the median voter in the median district:* $z_L^* < z_R^* = 0$*; and*

[3.] if R*'s advantage is* ***large****, i.e.,* $\rho_0 > \overline{\rho}_0$, *then party* L *retreats by more to its base, and party* R *advances towards* L*'s base:* $z_L^* < z_R^* < 0$.

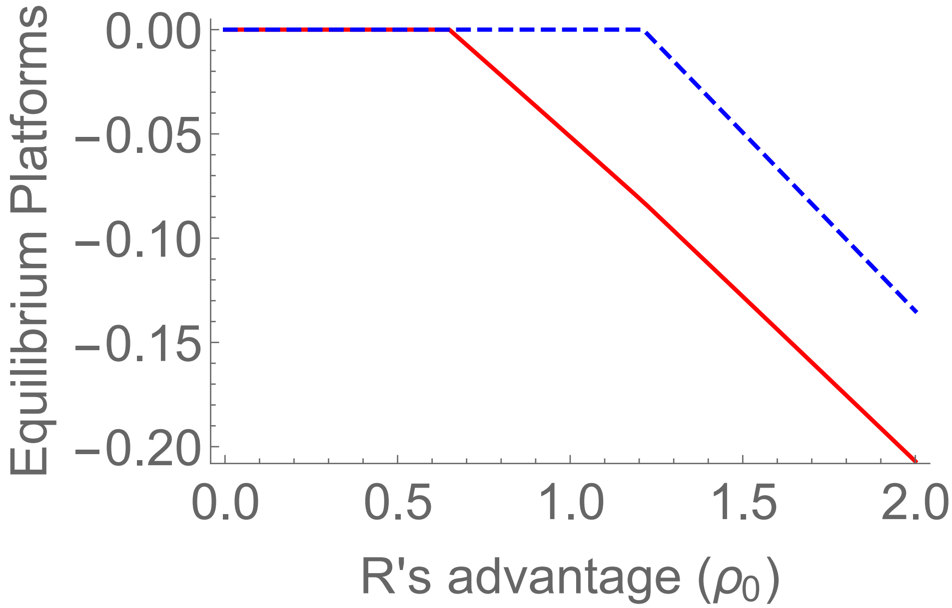

Figure 5 – Equilibrium platforms z_L^* (*red*) and z_R^* (*blue*) in the extended model when $\eta = \frac{1}{5}$, $\gamma = \frac{3}{10}$, $Z = 2$, $\psi = 3$, $\theta = 6$, $y_L = -1$, $y_R = 1$, $w_L(m) = \frac{1-m}{2}$, $w_R(m) = \frac{1+m}{2}$.

References

Alesina, A. and H. Rosenthal (1996): "A theory of divided government", *Econometrica*, 64, 1311-41.

Baron, D.P. (1993), "Government formation and endogenous parties," *American Political Science Review*, 87, 34-47

Gomberg, A. M., F. Marhuenda, and I. Ortunõ-Ortín. (2016): "Endogenous party platforms: 'stochastic' membership," *Economic Theory*, 62, 839-866

Grossman, G. M. and E. Helpman (1996): 'Electoral competition and special interest politics," *Review of Economic Studies*, 63, 265-286.

Lizzeri, A. and N. Persico (2001): "The provision of public goods under alternative electoral incentives," *American Economic Review*, 91, 25-239.

Ortunõ-Ortín, I. (1997): "A spatial model of political competition and proportional representation," *Social Choice and Welfare*, 14: 427-438

Roemer, J. E. (1999): "*Political Competition: Theory and Applications*", Harvard University Press.