The Reversal Interest Rate: Online Appendix*

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Abstract

We present proofs and supplementary results for "The Reversal Interest Rate." Appendix A provides a definition and characterization of the model's equilibrium. Appendix B proves all of the main results. Appendix C provides additional discussion of issues in Section IV of the paper as well as sources for our benchmark parameters and robustness checks comparing the model's behavior to that of a standard New Keynesian model. Appendix D gives micro-foundations for the formulation of monopolistic competition among banks that we use in the setting of Section III. Appendix E micro-founds the process followed by bank net worth in the quantitative section.

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Contents

A General equilibrium

A.1 Formal treatment of standard model ingredients

We begin by setting up the problems faced by monopolistic retailers and final goods producers, which were omitted from the main text. There is a representative final goods producer that aggregates differentiated varieties supplied by monopolistic retailers $k \in [0,1]$ (at prices P_t^k) in order to produce output, which it sells at a competitive price P_t. Its production function is $Y_t = \left(\int_0^1 Y_t^{k\frac{\varepsilon-1}{\varepsilon}} dk\right)^{\frac{\varepsilon}{\varepsilon-1}}$, where $\varepsilon > 1$ is the elasticity of substitution. Thus, the final goods producer's problem is

$$\max_{Y_t^k} \ P_t\left(\int_0^1 Y_t^{k\frac{\varepsilon-1}{\varepsilon}} dk\right)^{\frac{\varepsilon}{\varepsilon-1}} - \int_0^1 P_t^k Y_t^k dk. \tag{A.1}$$

Accordingly, demand for a variety k with price P_t^k is $\left(\frac{P_t^k}{P_t}\right)^{-\varepsilon} Y_t$, where P_t is the usual CES price index.

Retailer k's production function is $Y_t^k = X_t^k$, where X_t^k denotes the quantity of intermediate inputs. Retailers set their prices P_t^k subject to Rotemberg (1982) adjustment costs (parameterized by $\theta > 0$), taking final goods producers' demand $Y_t^k = \left(\frac{P_t^k}{P_t}\right)^{-\varepsilon} Y_t$ as given. Their problem is

$$\max_{P_t^k} \sum_{t=0}^{\infty} \beta^t \Lambda_t \Big(\big(\frac{P_t^k}{P_t}\big)^{1-\varepsilon} Y_t - P_t^I \big(\frac{P_t^k}{P_t}\big)^{-\varepsilon} Y_t - \frac{\theta}{2}\big(\frac{P_t^k}{P_{t-1}^k} - 1\big)^2 Y_t \Big). \tag{A.2}$$

A.2 Optimality conditions

We now provide a list of optimality conditions that must hold in equilibrium.

Households: The first-order condition for the household's consumption decision is the Euler equation

$$1 = \beta \frac{1 + \tilde{i}_t}{1 + \pi_{t+1}} \frac{\Lambda_{t+1}}{\Lambda_t}, \tag{A.3}$$

where $1 + \pi_{t+1} = \frac{P_{t+1}}{P_t}$ denotes inflation from t to $t+1$, $\tilde{i}_t = \max\{i_t, 0\}$ denotes the rate at which households save, and

$$\Lambda_t = (C_t - hC_{t-1})^{-\sigma} - \beta h (C_{t+1} - hC_t)^{-\sigma} \tag{A.4}$$

is the household's marginal utility of consumption. The household's labor supply H_t^S satisfies

$$\chi H_t^{S\varphi} = \frac{W_t}{P_t} \Lambda_t. \tag{A.5}$$

The first-order conditions for the household's deposit and cash holdings are

$$\zeta \Phi'(\mathcal{L}_t) \frac{\partial \mathcal{L}_t}{\partial D_t} \leq \beta \Lambda_{t+1}(1 + i_t^D) - \Lambda_t, \tag{A.6}$$

$$\zeta \Phi'(\mathcal{L}_t) \frac{\partial \mathcal{L}_t}{\partial M_t} \leq \beta \Lambda_{t+1} - \Lambda_t, \tag{A.7}$$

which hold with equality when the household holds a positive quantity of the corresponding asset.

Intermediate goods firms and investment: Here we describe the solution to Problem (3) as well as the corresponding problem for non-bank-dependent firms. It will be convenient to define $\alpha_K \equiv \alpha\nu$, $\alpha_L \equiv (1-\alpha)\nu$ in describing intermediate goods firms' optimality conditions. The user cost of capital faced by a firm of type $z \in \{b, nb\}$ is defined as

$$1 + r_t^{k,z} = \frac{1 + i_t^z}{1 + \pi_{t+1}} Q_t^{K,z} - (1-\delta) Q_{t+1}^{K,z}, \tag{A.8}$$

where $i_t^z \in \{i_t^L, i_t\}$ depending on whether the firm is bank-dependent or non-bank-dependent. Capital demand for firms of type z is

$$K_t^z = \left(\frac{\alpha_K}{1-\alpha_L} \frac{A^{z\,\frac{1}{1-\alpha_L}} Z_t^K}{1 + r_{t-1}^{k,z}} \right)^{\frac{1-\alpha_L}{1-\alpha_K-\alpha_L}}, \tag{A.9}$$

where Z_t^K is defined as

$$Z_t^K \equiv (1-\alpha_L) \left(\frac{P_t^I}{P_t} \right)^{\frac{1}{1-\alpha_L}} \left(\frac{\alpha_L}{W_t/P_t} \right)^{\frac{\alpha_L}{1-\alpha_L}}. \tag{A.10}$$

Labor demand can then be derived as

$$H_t^z = \left(\frac{\alpha_L A^z P_t^I / P_t}{W_t/P_t} \right)^{\frac{1}{1-\alpha_L}} K_t^{z\,\frac{\alpha_K}{1-\alpha_L}}. \tag{A.11}$$

From these first-order conditions, it is possible to write firms' loan demand in terms of the nominal loan rate as

$$L_t(i_t^L) = Q_t^{K,b} \left(\frac{\alpha_K}{1-\alpha_L} \frac{A^{b\,\frac{1}{1-\alpha_L}} Z_{t+1}^K}{\frac{1+i_t^L}{1+\pi_{t+1}} Q_t^{K,b} - (1-\delta) Q_{t+1}^{K,b}} \right)^{\frac{1-\alpha_L}{1-\alpha_L-\alpha_K}}. \tag{A.12}$$

Banks: Banks' optimal loan and deposit rates are given by

$$i_t^L = i_t + \frac{1}{\varepsilon_t^L} + \frac{\partial \Psi^L(N_t, L_t)}{\partial L_t} - \frac{\partial \Psi^D(Q_t^B B_t^L, D_t)}{\partial (Q_t^B B_t^L)}, \tag{A.13}$$

$$i_t^D = i_t - \frac{1}{\varepsilon_t^D} + \frac{\partial \Psi^D(Q_t^B B_t^L, D_t)}{\partial (Q_t^B B_t^L)} + \frac{\partial \Psi^D(Q_t^B B_t^L, D_t)}{\partial D_t}, \tag{A.14}$$

where again ε_t^L, ε_t^D denote the semi-elasticities of loan and deposit rates with respect to i_t^L, i_t^D, respectively. Banks' long-term bond demand is pinned down by their balance sheet constraint (4).

Monopolistic retailers: Retailers set their prices P_t to solve

$$\Lambda_t\Bigg((1-\varepsilon)\big(\frac{P_t^k}{P_t}\big)^{-\varepsilon}\frac{Y_t}{P_t}-\varepsilon\frac{P_t^I}{P_t}\big(\frac{P_t^k}{P_t}\big)^{-(\varepsilon+1)}Y_t-\theta\big(\frac{P_t^k}{P_{t-1}^k}-1\big)\frac{Y_t}{P_{t-1}^k}\Bigg)=-\beta\Lambda_{t+1}\theta\big(\frac{P_{k,t+1}}{P_t^k}-1\big)\frac{Y_t}{{P_t^k}^2}. \quad (A.15)$$

Capital goods producers: The first-order condition for capital producers of type $z \in \{b, nb\}$ is

$$\Lambda_t Q_t^z\Bigg(1-\Xi(\frac{I_{t+1}^z}{I_t^z})-\frac{I_{t+1}^z}{I_t^z}\Xi'(\frac{I_{t+1}^z}{I_t^z})\Bigg)=\Lambda_t-\beta\Lambda_{t+1}Q_{t+1}^z\big(\frac{I_{t+1}^z}{I_t^z}\big)^2\Xi'(\frac{I_{t+2}^z}{I_{t+1}^z}). \quad (A.16)$$

Asset pricing: The Fisher equation implies that the real rate $1 + r_t$ satisfies

$$1+r_t=\frac{1+i_t}{1+\pi_{t+1}}. \quad (A.17)$$

The long-term bonds held by banks are priced according to

$$\frac{\frac{1}{\tau}+(1-\frac{1}{\tau})P_{t+1}Q_{t+1}^B}{P_tQ_t^B}=1+i_t. \quad (A.18)$$

which can be multiplied by $\frac{P_t}{P_{t+1}}=\frac{1}{1+\pi_{t+1}}$ to obtain the no-arbitrage relationship in real terms,

$$\frac{\frac{1}{\tau}\frac{1}{1+\pi_{t+1}}+(1-\frac{1}{\tau})Q_{t+1}^B}{Q_t^B}=1+r_t.$$

A.3 Government policy

The central bank sets the nominal rate i_t according to (9) and supplies cash elastically. The government sets transfers T_t^G to households to satisfy its budget constraint. Apart from taxes, government revenue comes from short-term and long-term bond issuance as well as seignorage, so

$$T_t^G=B_t-\frac{1+i_{t-1}}{1+\pi_t}B_{t-1}+Q_t^B(B_t^L-(1-\frac{1}{\tau})B_{t-1}^L)-\frac{1}{\tau}\frac{P_{t-1}}{P_t}B_{t-1}^L+\frac{M_t-M_{t-1}}{P_t}. \quad (A.19)$$

A.4 Market-clearing, aggregation, and consistency conditions

Below we list market-clearing, aggregation, and consistency conditions.

Goods market: The aggregate resource constraint is

$$Y_t=\Bigg(\int_0^1 {X_t^k}^{\frac{\epsilon-1}{\epsilon}}dk\Bigg)^{\frac{\epsilon}{\epsilon-1}}=C_t+I_t^b+I_t^{nb}. \quad (A.20)$$

The costs $\Psi^L(N_t, L_t)$ and $\Psi^D(Q_t^B B_t^L, D_t)$ paid by banks and firms' price adjustment costs are rebated back to households, so they do not enter in the aggregate resource constraint.

The aggregate price level P_t satisfies

$$P_t = \left(\int_0^1 P_t^{k\,1-\varepsilon} dk \right)^{\frac{1}{1-\varepsilon}}. \tag{A.21}$$

Labor market: The labor market clearing condition is standard:

$$H_t^S = \xi H_t^{D,b} + (1-\xi) H_t^{D,nb}. \tag{A.22}$$

Capital market: Firms' capital demand must equal the total quantity of capital of each type,

$$\xi K_t^{D,b} = K_t^b,\ (1-\xi) K_t^{D,nb} = K_t^{nb}. \tag{A.23}$$

The capital stock of each type $z \in \{b, nb\}$ evolves according to

$$K_{t+1}^z = (1-\delta) K_t^z + I_{t+1}^z \left(1 - \Xi(\frac{I_{t+1}^z}{I_t^z}) \right). \tag{A.24}$$

Intermediate goods market: The total supply of intermediate goods must equal the demand by monopolistic retailers:

$$\int_0^1 X_t^k dk = \xi A^b \left(\left(K_t^{D,b}\right)^\alpha \left(H_t^{D,b}\right)^{1-\alpha} \right)^\nu + (1-\xi) A^{nb} \left(\left(K_t^{D,nb}\right)^\alpha \left(H_t^{D,nb}\right)^{1-\alpha} \right)^\nu. \tag{A.25}$$

Loan and bond markets: The quantity of loans supplied by banks must be equal to bank-dependent firms' capital demand:

$$L_t = L_t(i_t^L) = Q_t^{K,b} K_{t+1}^{D,b}. \tag{A.26}$$

The central bank supplies short-term and long-term bonds elastically to clear the corresponding markets.

Deposit market: The quantity of deposits demanded by households is consistent with their demand function:

$$D_t^D = D_t(i_t^D, i_t). \tag{A.27}$$

A.5 Definition of equilibrium

In this section, we define a general equilibrium of the economy in Section I. The equilibrium for the extension of the model that we use for quantitative analysis, in Section III, is defined analogously.

Definition A.1. *A* ***perfect foresight equilibrium*** *consists of sequences* $\{C_t, H_t^S, M_t, D_t^D, B_t^H\}$, $\{H_t^{D,b}, H_t^{D,nb}, K_t^{D,b}, K_t^{D,nb}, I_t^b, I_t^{nb}\}$, $\{L_t, B_t^L, i_t^D, i_t^L, N_t\}$, $\{I_t^b, I_t^{nb}, K_t^b, K_t^{nb}, Y_t\}$, $\{\{P_t^k\}_{k\in[0,1]}, \{X_t^k\}_{k\in[0,1]}\}$,

$\{D_t^*(i^D, i), L_t^*(i^L)\}$, *and* $\{P_t, W_t, Q_t^K, P_t^I, \Lambda_t, \Pi_t^F, \Pi_t^B, T_t, Q_t^B, i_t, T_t^G\}$ *such that*

- *Decisions* $C_t, H_t^S, M_t, D_t^D, B_t^H$ *solve the household's problem, taking* W_t, T_t, i_t, i_t^D *as given, and deposit demand* $D_t(i_t^D, i_t)$ *and marginal utility* Λ_t *are consistent with the solution to the household's problem;*

- *Decisions* $H_t^{D,b}, K_t^b$ $(H_t^{D,nb}, K_t^{nb})$ *solve bank-dependent (non-bank-dependent) intermediate firms' problem, taking* W_t, i_t^L, i_t, P_t^I *as given, and loan demand* $L_t(i_t^L)$ *is consistent with the solution to intermediate firms' problem (Equation 3);*

- *Decisions* i_t^D, i_t^L, B_t^L *solve the bank's problem, taking* $i_t, D_t(i_t^D, i_t), L_t(i_t^L), \Lambda_t, Q_t^B$ *as given, and bank net worth* N_t *follows the law of motion (5), and* Π_t^B *is equal to aggregate bank dividends given these decisions;*

- *Investment* I_t^z *solves the problem of capital goods producers of type* $z \in \{b, nb\}$*, taking the price of capital* $Q_t^{K,z}$ *and the stochastic discount factor* Λ_t *as given, and the capital stock of each type evolves according to (A.24);*

- *Prices* $\{P_t^k\}_{k\in[0,1]}$ *and intermediate goods demand* $\{X_t^k\}_{k\in[0,1]}$ *solve monopolistically competitive retailers' problem, taking prices* P_t^I, P_t *and the stochastic discount factor* Λ_t *as given, and the price level* P_t *satisfies (A.21);*

- Π_t^F *is equal to aggregate profits of monopolistic retailers and capital producers given their decisions, and bank dividends are equal to a fraction* γ *of net worth,* $\Pi_t^B = \gamma N_t$*;*

- *Real rates satisfy the Fisher equation (A.17), and long-term bond prices* Q_t^B *satisfy the no-arbitrage condition (A.18);*

- *Monetary policy is set according to (9) and government transfers* T_t^G *are set to satisfy the government's intertemporal budget constraint (A.19);*

- *Total transfers* T_t *are equal to government transfers* T_t^G *plus the costs* $\Psi^L(L_t, N_t) + \Psi^D(Q_t^B B_t^L, D_t)$ *incurred by banks as well as the price adjustment costs incurred by retailers;*

- *Banks' initial net worth* N_0 *satisfies (7);*

- *All markets clear.*

A.6 Quantitative extension

In this section, we describe the modifications to the equilibrium equations in the quantitative model of Section III, which come from choosing specific functional forms and making the modified assumptions described in the text.

Households: First, we add the constraint $D_t + M_t \leq \overline{\mathcal{L}}$ to the household's problem. Households then discount at the risk-free rate, so the Euler equation reads

$$1 = \beta \frac{1 + i_t}{1 + \pi_{t+1}} \frac{\Lambda_{t+1}}{\Lambda_t}. \tag{A.28}$$

Deposits and cash are perfect substitutes, and banks never optimally choose to set a deposit rate less than zero in equilibrium. Without loss of generality, we assume that when households are indifferent between holding cash and deposits, they choose to invest exclusively in deposits. Their deposit holdings are therefore chosen according to (A.6), which becomes

$$D_t^D = \mathcal{L}^* - \frac{\Lambda_t}{\zeta} \max\left\{1 - \frac{1 + i_t^D}{1 + i_t}, 0\right\}, \tag{A.29}$$

where the Euler equation (A.28) is used to replace $\beta \frac{\Lambda_{t+1}}{\Lambda_t} = \frac{1+\pi_{t+1}}{1+i_t}$.

Intermediate goods firms: The optimality conditions for intermediate goods firms remain identical to those described in Section A.2. It is then possible to write the output produced by firms of type $z \in \{b, nb\}$ as

$$Y_t^z = A^z (K_t^{D,z^{\alpha}} H_t^{D,z^{1-\alpha}})^{\nu}. \tag{A.30}$$

Total output can then be written simply as

$$Y_t = (1 - \xi) Y_t^{nb} + \xi Y_t^b, \tag{A.31}$$

since, in equilibrium, all monopolistic retailers will set identical prices and produce the same quantity of goods.

Banks: Banks set their loan and deposit rates according to (**??**) and (**??**), respectively. Their net worth evolves according to

$$N_{t+1} = (1 - \gamma)\Big(CG_{t+1} + NII_t + N_t\Big) + \hat{N}, \tag{A.32}$$

where

$$CG_t \equiv \left(\frac{(1 - \frac{1}{\tau}) Q_t^B + \frac{1}{\tau}\frac{1}{1+\pi_t}}{Q_{t-1}^B} - (1 + r_{t-1})\right) Q_{t-1}^B B_{t-1}^L \tag{A.33}$$

denotes capital gains at time t,[1] and NII_t denotes net interest income at t. Net interest income is comprised of four components: the risk-free income on bank equity, profits from loan issuance Π_t^L, profits from deposit issuance Π_t^D, and marginal leverage costs Δ_t multiplied by lending L_t:

$$NII_t = \big(\frac{1 + i_t}{1 + \pi_t} - 1\big) N_t + \Pi_t^L + \Pi_t^D - \Delta_t L_t, \tag{A.34}$$

[1] Note that capital gains can be non-zero only at $t = 0$, when an unanticipated shock arrives.

where profits are defined by

$$\Pi_t^L = \frac{i_t^L - i_t}{1+\pi_{t+1}} L_t, \tag{A.35}$$

and

$$\Pi_t^D = \frac{i_t + \mu^D - i_t^D}{1+\pi_{t+1}} D_t. \tag{A.36}$$

The marginal costs of loan issuance are

$$\Delta_t = \kappa^L \max\left\{\frac{L_t}{N_t} - \frac{L^*}{N^*}, 0\right\}^2. \tag{A.37}$$

In the text, we also refer to the dividends paid by banks as well as their returns on accounting equity. The dividends Div_t paid by banks are

$$Div_t = CG_t + NII_{t-1} + N_{t-1} - N_t, \tag{A.38}$$

and accounting returns ROE_{t+1} between t and $t+1$ are therefore

$$ROE_{t+1} = \frac{Div_{t+1} + N_{t+1}}{N_t}. \tag{A.39}$$

Capital goods producers: It is easier to work with the stock of capital and investment normalized per firm in each sector $z \in \{b, nb\}$,

$$\tilde{K}_t^b \equiv \frac{K_t^b}{\xi}, \tilde{K}_t^{nb} \equiv \frac{K^{nb}}{1-\xi}$$

and

$$\tilde{I}_t^b \equiv \frac{I_t^b}{\xi}, \tilde{I}_t^{nb} \equiv \frac{I_t^{nb}}{1-\xi}.$$

With these normalizations, and the specification of adjustment costs $\Xi^z(\frac{I_{t+1}^z}{I_t^z}) = \frac{\kappa^I}{2}\big(\frac{I_{t+1}^z}{I_t^z} - 1\big)^2$, the first-order condition for capital goods producers of type $z \in \{b, nb\}$ becomes

$$\Lambda_t Q_t^{K,z}\left(1 - \frac{\kappa^I}{2}(\frac{\tilde{I}_t^z}{\tilde{I}_{t-1}^z} - 1)^2 - \kappa^I \frac{\tilde{I}_t^z}{\tilde{I}_{t-1}^z}(\frac{\tilde{I}_t^z}{\tilde{I}_{t-1}^z} - 1)\right) = \Lambda_t - \beta\Lambda_{t+1} Q_{t+1}^{K,z}\kappa^I \big(\frac{\tilde{I}_{t+1}^z}{\tilde{I}_t^z}\big)^2\big(\frac{\tilde{I}_{t+1}^z}{\tilde{I}_t^z} - 1\big), \tag{A.40}$$

whereas the capital accumulation equation remains

$$\tilde{K}_{t+1}^z = (1-\delta)\tilde{K}_t^z + \tilde{I}_{t+1}^z\left(1 - \frac{\kappa^I}{2}\big(\frac{\tilde{I}_{t+1}^z}{\tilde{I}_t^z} - 1\big)^2\right). \tag{A.41}$$

The first-order condition can be re-written as an asset pricing equation,

$$Q_t^{K,z} = \frac{\Lambda_t - \beta\kappa^I \Lambda_{t+1} Q_{t+1}^{K,z} \big(\frac{\tilde{I}_{t+2}^z}{\tilde{I}_{t+1}^z}\big)^2 \big(\frac{\tilde{I}_{t+2}^z}{\tilde{I}_{t+1}^z} - 1\big)}{\Lambda_t \bigg(1 - \frac{\kappa^I}{2}\big(\frac{\tilde{I}_{t+1}^z}{\tilde{I}_t^z} - 1\big)^2 - \kappa^I \frac{\tilde{I}_{t+1}^z}{\tilde{I}_t^z}\big(\frac{\tilde{I}_{t+1}^z}{\tilde{I}_t^z} - 1\big)\bigg)}. \tag{A.42}$$

Aggregate investment can be written as

$$I_t \equiv I_t^b + I_t^{nb} = \xi \tilde{I}_t^b + (1-\xi)\tilde{I}_t^{nb}. \tag{A.43}$$

Phillips curve: From monopolistic retailers' optimal price-setting rule (A.15), it is possible to derive the New Keynesian Phillips Curve, which (in logs) reads

$$\log(1+\pi_t) = \frac{\varepsilon - 1}{\theta} \log \frac{P_t^I}{P_t} + \beta \log(1+\pi_{t+1}). \tag{A.44}$$

B Proofs

In this section, we prove all of the results stated in the text. We begin with the results related to the solutions of the household and bank problems and then move on to the results characterizing the properties of the reversal rate.

B.1 Household and bank optimization

Lemma 1 is the only result about the household's problem that we must prove.

Proof of Lemma 1. Households' deposit holdings are chosen by solving Problem 12. For $i_t < 0$, holding cash strictly dominates holding safe bonds (since the function $\Phi(\mathcal{L}_t)$ is non-decreasing and $\mathcal{L}_t$ is increasing in real balances M_t). Thus, households always set $B_t = 0$, and the optimization problem reduces to

$$\max_{M_t, D_t} \ \zeta\Phi(\mathcal{L}(D_t, M_t)) + \beta\Lambda^*((1+i_t^D)D_t + M_t) \text{ s.t. } D_t + M_t \leq S^*, \ \ M_t, D_t \geq 0.$$

Note that the policy rate i_t appears nowhere in this problem, so deposit demand $D^*(i_t^D, i_t)$ must be independent of i_t for $i_t < 0$. □

We now solve the bank's problem. We begin by proving:

Claim B.1. *The solution to banks' problem (6) coincides with the problem of maximizing net interest income in each period t.*

Proof. Consider a plan (i_t^L, i_t^D, B_t) that maximizes net interest income each period and another arbitrary plan $(\tilde{i}_t^L, \tilde{i}_t^D, \tilde{B}_t)$. Denote by $\{N_t\}, \{\tilde{N}_t\}$ the sequence followed by the bank's net worth under these two plans, respectively. Initial net worth in the bank's problem is taken as given, so

$N_0 = \tilde{N}_0$. We show by induction that $N_t \geq \tilde{N}_t$ implies $N_{t+1} \geq \tilde{N}_{t+1}$. It follows that the plan (i_t^L, i_t^D, B_t) is optimal, since the bank's dividends are a fraction γ of its net worth in each period.

Suppose $N_t \geq \tilde{N}_t$. Using the balance sheet constraint (4), the problem of maximizing net interest income in each period is equivalent to maximizing

$$(i_t^L - i_t)L^*(i_t^L) + (i_t - i_t^D)D^*(i_t^D, i_t) - \Psi^L(N_t, L^*(i_t^L)) - \Psi^D(N_t + D^*(i_t^D, i_t) - L^*(i_t^L), D^*(i_t^D, i_t)).$$

with respect to i_t^L and i_t^D. Then we have

$$\begin{aligned}
N_{t+1} &= (1 + i_t)N_t + (i_t^L - i_t)L^*(i_t^L) + (i_t^D - i_t)D(i_t^D, i_t) \\
&- \Psi^L(N_t, L^*(i_t^L)) - \Psi^D(N_t + D^(i_t^D, i_t) - L^*(i_t^L), D^*(i_t^D, i_t)) \\
&\geq (1 + i_t)N_t + (\tilde{i}_t^L - i_t)L^*(\tilde{i}_t^L) + (\tilde{i}_t^D - i_t)D(\tilde{i}_t^D, i_t) \\
&- \Psi^L(N_t, L^*(\tilde{i}_t^L)) - \Psi^D(N_t + D^*(\tilde{i}_t^D, i_t) - L^*(\tilde{i}_t^L), D^*(\tilde{i}_t^D, i_t)) \\
&\geq (1 + i_t)\tilde{N}_t + (\tilde{i}_t^L - i_t)L^*(\tilde{i}_t^L) + (\tilde{i}_t^D - i_t)D(\tilde{i}_t^D, i_t) \\
&- \Psi^L(\tilde{N}_t, L^*(\tilde{i}_t^L)) - \Psi^D(\tilde{N}_t + D^*(\tilde{i}_t^D, i_t) - L^*(\tilde{i}_t^L), D^*(\tilde{i}_t^D, i_t)) = \tilde{N}_{t+1}
\end{aligned}$$

The first inequality follows from the fact that (i_t^L, i_t^D, B_t) maximizes net interest income. The second inequality follows from the facts that $N_t \geq \tilde{N}_t$ by assumption and that both Ψ^L and Ψ^D are decreasing in their first arguments. Hence, we have demonstrated that $N_t \geq \tilde{N}_t$ implies $N_{t+1} \geq \tilde{N}_{t+1}$, as desired. □

Next, we specialize to the setting of Section II.A. The Lagrangian of (13) can be written as

$$\begin{aligned}
\mathcal{L} = \max_{i^L, i^D, B^L} \; & iQ^B B^L + i^L L^*(i^L) - i^D D^*(i^D, i) - \zeta(L^*(i^L) + Q^B B^L - N - D^*(i^D, i)) \\
& - \lambda(L^*(i^L) - \psi^L N) - \mu(\psi^D D^*(i^D, i) - Q^B B^L)
\end{aligned}$$

The first-order conditions are

$$\begin{aligned}
(B^L): \; & i = \zeta - \mu, \\
(i^L): \; & i^L = \zeta + \frac{1}{\varepsilon^L} + \lambda, \\
(i^D): \; & i^D = \zeta - \frac{1}{\varepsilon^D} - \mu\psi^D.
\end{aligned}$$

where $\varepsilon^L, \varepsilon^D$ are the semi-elasticities of loan and deposit demand with respect to the loan and deposit rate, respectively (as defined in the text).

The Lagrange multipliers λ and μ on the net worth and liquidity constraints are equal to zero if and only if the respective constraints are slack. Otherwise, the multipliers are positive. Of course, the bank's balance sheet constraint always binds. The corresponding complementary slackness conditions are $\lambda(\psi^L N - L^*(i^L)) = 0$, $\mu(\psi^D D^*(i^D, i) - Q^B B^L) = 0$. There are four regimes, corresponding to each possible configuration of binding constraints. We treat each in turn.

Unconstrained regime: When neither the net worth nor liquidity constraint binds, the solution to the bank's problem is simply

$$i^L = i + \frac{1}{\varepsilon^L},\ i^D = i - \frac{1}{\varepsilon^D}.$$

Both i^L and i^D are increasing in i. Therefore, in this regime, total lending is decreasing in the policy rate (since loan demand is downward-sloping in i^L) and independent of net worth.

Net worth constraint binds: When only the net worth constraint binds, the loan rate is the solution to $L^*(i^L) = \psi^L N$, and the deposit rate is set as in the unconstrained regime. Thus, total lending is fully determined by bank net worth and is independent of the policy rate (holding the level of net worth fixed). The policy rate affects lending only through its effect on net worth: $\frac{dL}{di} = \psi^L \frac{dN}{di}$.

Liquidity constraint binds: It will be convenient to define the unconstrained optimal mark-up on loans (mark-down on deposits) as

$$\mu^L(i^L) \equiv \frac{1}{\varepsilon^L(i^L)} = \frac{L(i^L)}{\frac{\partial L(i^L)}{\partial i^L}},\quad \mu^D(i^D, i) \equiv \frac{1}{\varepsilon^D(i^D, i)} = \frac{D(i^D, i)}{\frac{\partial D(i^D, i)}{\partial i^D}},$$

where the partial derivatives are evaluated at i^L and (i^D, i), respectively. For the bank's problem to be guaranteed to have a unique solution, we require $\frac{\partial \mu^L}{\partial i^L} < 0$, $\frac{\partial \mu^D}{\partial i^D} > 0$. The optimization conditions in this regime are

$$i^D + \mu^D(i^D, i) = (1 - \psi^D)(i^L - \mu^L(i^L)) + \psi^D i,$$

$$L^*(i^L) = N + (1 - \psi^D) D^*(i^D, i).$$

These equations imply

$$\frac{di^D}{di} = \frac{\frac{\partial L}{\partial i^L}\frac{di^L}{di} - \frac{dN}{di} - (1 - \psi^D)\frac{\partial D}{\partial i}}{(1 - \psi^D)\frac{\partial D}{\partial i^D}},$$

$$\left((1 - \psi^D)(1 - \frac{\partial \mu^L}{\partial i^L}) - \frac{1 + \frac{\partial \mu^D}{\partial i^D}}{(1 - \psi^D)\frac{\partial D}{\partial i^D}}\right)\frac{di^L}{di} = -\frac{1 + \frac{\partial \mu^D}{\partial i^D}}{(1 - \psi^D)\frac{\partial D}{\partial i^D}}(\frac{dN}{di} + (1 - \psi^D)\frac{\partial D}{\partial i}) + \frac{\partial \mu^D}{\partial i} - \psi^D.$$

The term multiplying $\frac{di^L}{di}$ on the left-hand side of the above equation is positive, and the term multiplying $\frac{dN}{di}$ on the right-hand side is negative. Furthermore, when $i < 0$, deposit demand is independent of i (Lemma 1), so $\frac{\partial D}{\partial i} = \frac{\partial \mu^D}{\partial i} = 0$. Thus, in this regime, $\frac{dN}{di} > 0$ implies that $\frac{di^L}{di} < 0$.

Both constraints bind: In this regime, it is again the case that $L(i^L) = \psi^L N$. Therefore, just as in the regime in which only the net worth constraint is binding, $\frac{dL}{di} = \psi^L \frac{dN}{di}$.

B.2 General results

Recall that in our stylized theoretical model, the consequences of interest rate cuts are fully determined by the loan and deposit demand schedules faced by banks. In this section, we provide proofs of our main analytical results in a more general setting. Then, the next section proves that the assumptions of the general model are satisfied in the benchmark model of Section II.A.

The general model we consider is as follows. The loan and deposit demand schedules are exogenous functions $L(i^L, i)$ and $D(i^D, i)$ satisfying the following assumptions:

- Deposit demand $D^*(i^D, i)$ is continuous, differentiable, increasing in its first argument, and weakly decreasing in its second argument;
- Loan demand $L^*(i^L, i)$ is continuous, differentiable, decreasing in its first argument, and weakly increasing in its second argument. Moreover, $L^*(i^L, i) \to \infty$ as $i^L \to -\delta$ and $\frac{1}{\varepsilon^L}$ is bounded.

The central bank first announces an interest rate sequence (10), and banks' initial net worth is determined according to (7) and (11), where steady-state net worth N^* and bond holdings B^{L*} are parameters. In each period, banks solve the problem

$$NII(N_t, i_t) = \max_{B_t^L, i_t^L, i_t^D} \; i_t Q_t^B B_t^L + i_t^L L(i_t^L, i_t) - i_t^D D(i_t^D, i_t) \tag{B.1}$$
$$\text{s.t. (4)}, \; L(i_t^L, i_t) \le \psi^L N_t, \; Q_t^B B_t^L \ge \psi^D D(i_t^D, i_t).$$

taking the sequence of interest rates (10) as given as well as arbitrary loan and deposit demand schedules $L(i^L, i)$ and $D(i^D, i)$ satisfying the basic assumptions laid out at the beginning of Section II.F. The parameters on banks' net worth and liquidity constraints are $\psi^L \in [0, \infty)$ and $\psi^D \in [0, 1]$.[2] It will be convenient to define some notation related to the bank's problem. First, let

$$i^{L*}(i) = \operatorname*{argmax}_{i^L} \; (i^L - i)L(i^L, i), \quad i^{D*}(i) = \operatorname*{argmax}_{i^D} \; (i - i^D)D(i^D, i)$$

be the bank's optimal loan and deposit rates when it is unconstrained, respectively. Define $\hat{\delta}$ so that the bank's optimal loan rate converges to $-\delta$ as i approaches $-\hat{\delta}$ – that is, $\lim_{i \to -\hat{\delta}} i^{L*}(i) = -\delta$. Under our assumptions, then, the bank's optimal loan supply diverges as $i \to -\hat{\delta}$.[3]

Bank net worth accumulates in each period according to

$$N_{t+1} = (1 - \gamma)(N_t + NII(N_t, i_t)). \tag{B.2}$$

The other two exogenously given parameters in this problem are banks' payout rate $\gamma \in (i^*, 1)$ and the maturity of long-term bonds $\tau \in (0, \frac{1}{\hat{\delta}})$, which enter in the determination of banks' initial net worth and in their net worth accumulation equation. We define

$$\overline{N} \equiv \max_{i \in [-\hat{\delta}, i^*]} \frac{(i^{L*}(i) - i)L(i^{L*}(i), i) + (i - i^{D*}(i))D(i^{D*}(i), i)}{1 - (1 - \gamma)(1 + i)} + \left(Q_0^B - Q^{B*}\right)B^{L*},$$

[2] It does not make sense to allow $\psi^L < 0$ (since banks would have to *borrow* from firms) or $\psi^D > 1$ (since it would sometimes be impossible for banks to satisfy their liquidity constraints).

[3] Note that such a $\hat{\delta}$ always exists when the inverse elasticity $\frac{1}{\varepsilon^L}$ is bounded.

where bond prices are given by (11). It is easy to check that banks cannot reach a level of net worth greater than $\overline{N}$ in equilibrium.

We will also sometimes impose Properties 1 and 2 from Section II.F in order to prove our main results, Propositions 2, 3, and 5. Property 1 guarantees that for sufficiently low interest rates, profits are increasing in the policy rate, which is necessary for a net worth reversal to be possible. This is a restriction on all of the parameters of the banks' problem jointly, which are (N^*, B^*, $L(i^L, i)$, $D(i^D, i)$, ψ^L, ψ^D, γ, τ). Property 2 imposes that the parameter ψ^L on banks' net worth constraints is a positive real number $\psi^L \in (0, \infty)$, but this property imposes no restriction on the liquidity constraint parameter $\psi^D \in [0, 1]$.

In order to prove Proposition 4, we will need an additional property that guarantees that banks' liquidity constraint never binds, which in turn implies that *all* reversals will be triggered by the net worth constraint.

Property B.1. *Deposit demand $D^*(i^D, i)$ is large enough that banks' liquidity constraints never bind in equilibrium. Formally, for all $i \leq i^*$, $D^*(i^{D*}(i), i) > \frac{1}{1-\psi^D} \min\{\psi^L \overline{N}, L(i^{L*}(i), i)\}$.*

We proceed as follows. In this section, we impose the regularity conditions in Section II.F and prove results in the more general setting, indicating in which cases Properties 1, 2, and B.1 are imposed. Then, in the next section, we prove that the specific model of Section II.A satisfies the regularity conditions as well as Properties 1-2. We also show that when the liquidity demand parameter ζ is large enough, the model of Section II.A satisfies Property B.1 as well.

We begin by providing a characterization of the reversal rate in this setting.

Proposition B.1. *Suppose Property 2 holds, and that i is the highest interest rate satisfying the following two properties:*

1. *The net worth constraint binds (or both constraints bind) at t for all $i' \leq i$;*
2. *Time-t net worth is increasing in the interest rate, $\frac{dN_t(i)}{di} > 0$, for all $i' < i$.*

Then i is the time-t reversal rate i_t^{RR}.

Proof. If i is the greatest interest rate such that (1) either just the net worth constraint or both constraints bind, and (2) $\frac{dN_t}{di} > 0$ for all $i' < i$, then an analysis analogous to that in Appendix B.1 immediately proves that $\frac{dL_t}{di} > 0$ for all $i' < i$. (By inspection of each of the two cases in which the net worth constraint binds, the analysis in Appendix B.1 shows that if Property 2 holds, then $\frac{dL_t}{di} > 0$ if and only if $\frac{dN_t}{di} > 0$.) Therefore, i is the time-t reversal rate, since by hypothesis it is the greatest interest rate satisfying this property. □

It is useful to prove two lemmas before presenting our main results. The first is an elementary result: a bank's net worth in the next period is increasing in its current net worth.

Lemma B.1. *Banks' net worth plus interest income, $N + NII(N, i)$, is increasing in N for fixed i.*

Proof. Consider two levels of net worth $N' > N$. If (i^L, i^D) is a feasible choice for a bank with net worth N, then (13) implies it must also be feasible for a bank with net worth N'. Then if a bank with net worth N sets deposit rates i^{L*} and i^{D*} at an optimum, a bank with net worth N' must be able to achieve a value of at least

$$\begin{aligned}(1+i)N' + (i^{L*} - i)L(i^{L*}, i) + (i - i^{D*})D(i^{D*}, i) &> (1+i)N + (i^{L*} - i)L(i^{L*}, i) + (i - i^{D*})D(i^{D*}, i) \\ &= N + NII(N, i).\end{aligned}$$

□

The next lemma regards the convergence of N_t over time to a steady-state level.

Lemma B.2. *Suppose $i < i^*$. Then for $t \leq T$, bank net worth converges monotonically to a unique steady state value $\tilde{N}(i)$ characterized as the fixed point of*

$$\tilde{N}(i) = \frac{1-\gamma}{\gamma} NII(\tilde{N}(i), i). \tag{B.3}$$

Proof. It is immediately evident from (B.2) that a steady state is characterized by this fixed point equation. Given i, the sequence N_t is defined by

$$N_{t+1} = f(N_t) = (1-\gamma)(NII(N_t, i) + N_t).$$

Lemma B.1 implies $NII(N, i) + N$ is increasing in N, so f is a monotonic function of N. Thus, the sequence N_t is monotonic.

Now observe that banks' profits from deposit and loan taking, $\Pi_t^D = (i - i_t^D)D_t$ and $\Pi_t^L = (i_t^L - i)L_t$, are functions of N_t and i_t that are bounded in N_t (holding i_t fixed). This is because both $\Pi^D(N, i)$ and $\Pi^L(N, i)$ are independent of N when banks are unconstrained (i.e., when N is large enough). Thus, there exists $N_{\max}$ such that when $N \geq N_{\max}$, we have $\Pi^D(N, i) = \Pi_{\max}^D(i)$ and $\Pi^L(N, i) = \Pi_{\max}^L(i)$. However, these profits must remain positive (since the bank can always set deposit rates below i and invest in bonds that pay i). Therefore, we have

$$f(0) = (1-\gamma)(\Pi^D(0, i) + \Pi^L(0, i)) > 0,$$

$$f(N) = (1-\gamma)(1+i)N + \Pi_{\max}^D(i) + \Pi_{\max}^L(i) \text{ for } N \geq N_{\max}.$$

For large enough N, then, $f(N) < N$, since $(1-\gamma)(1+i) < (1-\gamma)(1-i^*) < 1$ by assumption.

Now we show that the function $\frac{NII(N,i)}{N}$ is (weakly) monotonically decreasing in N. By doing so, we finish the proof: if $\frac{NII(N,i)}{N}$ is monotonic in N, then there can be at most one fixed point. We consider four cases, corresponding to the possible configurations of binding constraints.

Case 1: Bank is unconstrained. In this case, the bank sets its deposit and loan rates to the unconstrained optimal values $i^{D*}(i)$ and $i^{L*}(i)$ defined above. Then

$$\begin{aligned}\frac{NII(N,i)}{N} &= \frac{iN + (i^{L*}(i) - i)L(i^{L*}(i), i) + (i - i^{D*}(i))D(i^{D*}(i), i)}{N} \\ &= i + \frac{(i^{L*}(i) - i)L(i^{L*}(i), i) + (i - i^{D*}(i))D(i^{D*}(i), i)}{N}.\end{aligned}$$

The numerator of the second term is independent of N, so $\frac{NII(N,i)}{N}$ is decreasing in N.

Case 2: Net worth constraint binds. The loan rate is a function $\hat{i}^L(N,i)$ in this regime, satisfying $L(\hat{i}^L(N,i), N) = \psi^L N$. Note that the loan rate must then be decreasing in N. On the other hand, the deposit rate is set to the unconstrained optimal value $i^{D*}(i)$. Then

$$\begin{aligned}\frac{NII(N,i)}{N} &= \frac{iN + (\hat{i}^L(N,i) - i)L(\hat{i}^L(N,i), i) + (i - i^{D*}(i))D(i^{D*}(i), i)}{N} \\ &= i + (\hat{i}^L(N,i) - i)\psi^L + \frac{(i - i^{D*}(i))D(i^{D*}(i), i)}{N}.\end{aligned}$$

The second term is decreasing in N, and the numerator of the third term is independent of N, so the right-hand side is decreasing in N.

Case 3: Liquidity constraint binds. In this regime, we must have $L(i^L, i) = N + (1 - \psi^D)D(i^D, i)$. The bank's return on net worth is

$$\begin{aligned}\frac{NII(N,i)}{N} &= \frac{iN + (i^L - i)L(i^L, i) - (i^D - i)D(i^D, i)}{N} \\ &= i^L + \big((1 - \psi^D)i^L + \psi^D i - i^D\big)\frac{D(i^D, i)}{N}.\end{aligned}$$

Thus, the bank's problem can be re-written as the problem of maximizing returns on net worth:

$$R(N,i) \equiv \max_{i^L, i^D}\; i^L + \big((1 - \psi^D)i^L + \psi^D i - i^D\big)\frac{D(i^D, i)}{N} \quad \text{s.t.} \quad L(i^L, i) = N + (1 - \psi^D)D(i^D, i).$$

By the envelope theorem,

$$\frac{\partial R(N,i)}{\partial R} = -\mu - \big((1 - \psi^D)i^L + \psi^D i - i^D\big)\frac{D(i^D, i)}{N^2},$$

where μ is the Lagrange multiplier on the constraint. The multiplier is positive, since a higher net worth implies a lower value of i^L for a given value of i^D.[4] Hence, $\frac{NII(N,i)}{N}$ is decreasing in N in this regime as well.

Case 4: Both constraints bind. When both constraints bind, we must have $L(i^L, i) = \psi^L N$ and $D(i^D, i) = \frac{\psi^L - 1}{1 - \psi^D}N$. It follows that $i^L = \hat{i}^L(N,i)$ is a decreasing function of N and $i^D = \hat{i}^D(N,i)$

[4] This statement can be verified from the first-order conditions.

is an increasing function of N. Then

$$\frac{NII(N,i)}{N} = i + (\hat{i}^L(N,i) - i)\psi^L + (i - \hat{i}^D(N,i))\frac{\psi^L - 1}{1 - \psi^D}.$$

The right-hand side is decreasing in N, as desired. $\square$

We now prove the key lemma. It will be useful to define a function $N_t(N_0, i)$ recursively as follows: $N_0(N_0, i) = N_0$, and

$$N_{t+1}(N_0, i) = (1-\gamma)\big(N_t(N_0, i) + NII(N_t(N_0, i), i)\big).$$

Lemma B.3. *Suppose Property 1 holds. Bank net worth $N_t(N_0, i)$ is strictly increasing as a function of i for all $i < \underline{i}$ (holding T and N_0 fixed).*

Proof. Assume by way of induction that we have shown $N_t(N_0, i)$ is weakly increasing in i for all $s \leq t$. For $i' < i$, we then have

$$\begin{aligned} N_{t+1}(N_0, i) &= (1-\gamma)\big(N_t(N_0, i) + NII(N_t(N_0, i), i)\big) \\ &\geq (1-\gamma)\big(N_t(N_0, i') + NII(N_t(N_0, i'), i)) \\ &> (1-\gamma)\big(N_t(N_0, i') + NII(N_t(N_0, i'), i')\big) = N_{t+1}(N_0, i') \end{aligned}$$

The second line uses the inductive assumption, $N_t(N_0, i') \leq N_t(N_0, i)$, and Lemma B.1. The third line uses the fact that $i' < i$ as well as the fact that $NII(N, i)$ is strictly increasing in i for $i < \underline{i}$ (Property 1). $\square$

Next, we can show that after accounting for capital gains, bank net worth is increasing in i if either (1) banks' initial bond holdings are sufficiently small, or (2) the horizon t considered is sufficiently long. In what follows, the following identity is useful:

$$\frac{dN_t(N_0, i)}{di} = \frac{\partial N_t}{\partial N_0}\frac{dN_0}{di} + \frac{\partial N_t}{\partial i}. \tag{B.4}$$

Lemma B.3 shows that $\frac{\partial N_t}{\partial i}$ is positive for $i < \underline{i}$. Therefore, it will be possible to prove the desired results by bounding the magnitude of the first term.

Lemma B.4. *Fix T, and suppose Property 1 holds. Then given $\epsilon > 0$, there exists a level of initial bond holdings $\underline{B}^L$ such that whenever $B^{L*} \leq \underline{B}^L$, $\frac{dN_t}{di} > 0$ for all $t > 0$ and $i \leq \underline{i} - \epsilon$.*

Proof. First, note that as $T \to \infty$, N_0 remains finite. This is because initial net worth is given by (7) and (11) and $Q_0 \to \frac{1}{\tau}\sum_{t=0}^{\infty}\left(\frac{1-\frac{1}{\tau}}{1+i}\right)^t$ as $T \to \infty$. We have assumed previously that $1 - \frac{1}{\tau}$ is greater than $1 - \hat{\delta}$, the minimum value that $1 + i$ that we consider, so Q_0 is bounded. Thus, for fixed i, there exists $\overline{N}(i)$ such that $N_0 \in [0, \overline{N}(i)]$ for all values of T. Additionally, observe that $\frac{dN_0}{di}$ is bounded, since

$$\left|\frac{dQ_0}{di}\right| \to \left| -(\frac{1}{1+i})(\frac{1}{\tau})\sum_{t=0}^{\infty}\left(\frac{1-\frac{1}{\tau}}{1+i}\right)^t\right| < \infty$$

as $T \to \infty$.

Recall that initial net worth is given by (7) and (11). Then $\frac{dN_0}{di} = (1-\gamma)(1-\frac{1}{\tau})B^{L*}\frac{dQ_0}{di}$. The derivative $\frac{dQ_0(i,T)}{di}$ is bounded for $i \in [-\hat{\delta}, i^*]$, so $\frac{dN_0}{di}$ can be bounded for i in that range. For a given N_0, as $t \to \infty$, N_t converges to some long-run value $\tilde{N}(i)$ that is independent of N_0 (Lemma B.2). Therefore, it must be that $\frac{\partial N_t}{\partial N_0}$ is bounded for $t \in [0, \infty)$ and $i \in [-\hat{\delta}, i^*]$.

The first term in (B.4) can then be taken arbitrarily close to zero for all $(i, T) \in (-\hat{\delta}, \underline{i}) \times [0, \infty)$ and $t > 0$ by choosing B^{L*} small enough. As long as $i < \underline{i}$, the second term is positive (Property 1). Moreover, it is bounded away from zero: we have

$$\begin{aligned}\frac{\partial N_{t+1}}{\partial i} &= (1-\gamma)\left(\left(1 + \frac{\partial NII(N_t(N_0,i),i)}{\partial N_t}\right)\frac{\partial N_t}{\partial i} + \frac{\partial NII(N_t(N_0,i),i)}{\partial i}\right)\\ &> (1-\gamma)\frac{\partial NII(N_t(N_0,i),i)}{\partial i}.\end{aligned}$$

The inequality follows from Lemma B.1, and for any $\epsilon > 0$, $\frac{\partial NII(N,i)}{\partial i}$ is positive on $(N, i) \in [0, \overline{N}] \times [-\hat{\delta}, \underline{i} - \epsilon]$ (Property 1) and therefore bounded away from zero. Hence, for fixed $\epsilon > 0$, B^{L*} can be taken small enough that $\frac{dN_t}{di} > 0$ for all t and $i \leq \underline{i}$. □

Lemma B.5. *Suppose Property 1 holds. Then given $\epsilon > 0$, there exists $\underline{T}$ such that for all $T \geq \underline{T}$, $\frac{dN_t(N_0(i,T),i)}{di} > 0$ for all $t \in [\underline{T}, T]$ and $i \leq \underline{i} - \epsilon$.*

Proof. Recall that, from the proof of Lemma B.4, $\frac{dN_0}{di}$ can be bounded for $i \in [-\hat{\delta}, i^*]$. Lemma B.2 implies that N_t converges to some long-run value $\tilde{N}(i)$ that is independent of N_0, so $\frac{\partial N_t}{\partial N_0} \to 0$ as $t \to \infty$. Moreover, as $t \to \infty$, $\frac{\partial N_t}{\partial i}$ converges to $\frac{1-\gamma}{\gamma}(\frac{\partial NII(\tilde{N}(i),i)}{\partial i} + \frac{\partial NII(\tilde{N}(i),i)}{\partial N}\frac{d\tilde{N}(i)}{di}) > 0$ (using Lemma B.1, Lemma B.2, and Property 1).

Define

$$\Delta_t = \max_{\hat{N}_0 \in [0, \overline{N}(i)]} \left|\frac{\partial N_t}{\partial N_0}\Big|_{\hat{N}_0, i}\right|,$$

$$\Gamma_t = \max_{\hat{N}_0 \in [0, \overline{N}(i)]} \left|\frac{\partial N_t(N_0,i)}{\partial i}\Big|_{\hat{N}_0,i} - \frac{1-\gamma}{\gamma}\left(\frac{\partial NII(N,i)}{\partial i} + \frac{\partial NII(N,i)}{\partial N}\frac{d\tilde{N}(i)}{di}\Big|_{\tilde{N}(i),i}\right)\right|.$$

We have shown that $\Delta_t, \Gamma_t \to 0$ as $t \to \infty$ for $i \in [-\hat{\delta}, \underline{i} - \epsilon]$. Then we can fix $\underline{T}$ such that $\frac{\partial N_t}{\partial N_0}\frac{dN_0}{di} + \frac{\partial N_t}{\partial i}$ is arbitrarily close to $\frac{1-\gamma}{\gamma}(\frac{\partial NII(\tilde{N}(i),i)}{\partial i} + \frac{\partial NII(\tilde{N}(i),i)}{\partial N}\frac{d\tilde{N}(i)}{di}) > 0$ for all $t \in [\underline{T}, T]$ and $i \in [-\hat{\delta}, \underline{i} - \epsilon]$ (making use of the fact that $\frac{dN_0}{di}$ is bounded). □

Observe that, in proving Lemmas B.5 and B.4, we have also proven Lemmas 3 and 4 stated in the text.

An analogue of Proposition 2 then follows almost immediately.

Proposition B.2. *Suppose Properties 1-2 hold. Then there exists $\underline{B}^L$ such that whenever banks' steady-state bond holdings are $B^{L*} \leq \underline{B}^L$, a time-t reversal rate exists for all $0 < t \leq T$.*

Proof. Proposition 1 holds regardless of the properties of loan and deposit demand – it simply comes from the characterization of the solution of the bank's problem under Property 2. We must then show that when B^{L*} is small enough, we can find $\tilde{i}_t$ for each $t \leq T$ such that

- The net worth constraint binds (or both constraints bind) at t for all $i \leq \tilde{i}_t$;
- $\frac{dN_t}{di} > 0$ for all $i < \tilde{i}_t$.

Then, after proving the existence of such an interest rate $\tilde{i}_t$ for each $t \leq T$, Proposition 1 implies that a time-t reversal rate i_t^{RR} exists (since it is the supremum of all interest rates satisfying both properties).

To see that the first point holds, note that as $i \to -\hat{\delta}$, the unconstrained optimal loan supply $L(i^{L*}(i), i)$ diverges. For all i sufficiently close to $-\hat{\delta}$, then, loan demand will be large enough that the net worth constraint must bind (by Property 2). Thus, for each t there exists $\hat{i}_t$ such that the bank is constrained at t whenever $i \leq \hat{i}_t$.

Fix $\epsilon > 0$. Lemma B.4 demonstrates that it is possible to choose $\underline{B}^L$ sufficiently small such that if $B^{L*} \leq \underline{B}^L$, then the second property is satisfied for all $i \leq \underline{i} - \epsilon$. Then for each t, $\tilde{i}_t = \min\{\hat{i}_t, \underline{i} - \epsilon\}$ satisfies both properties, as desired. $\square$

The proof of the analogue of Proposition 3 is almost exactly the same.

Proposition B.3. *Suppose Properties 1-2 hold. Then there exists $\underline{T}$ such that when the length of the policy shock (10) is $T \geq \underline{T}$, a time-t reversal rate exists for all $t \in [\underline{T}, T]$.*

Proof. By Proposition 1, it suffices to show that we can find $\underline{T}$ large enough such that whenever $T \geq \underline{T}$, there exists $\tilde{i}_t$ for each $t \in [\underline{T}, T]$ such that

- The net worth constraint binds (or both constraints bind) if $i \leq \tilde{i}_t$;
- $\frac{dN_t}{di} > 0$ for all $i < \tilde{i}_t$.

We can find $\hat{i}_t$ satisfying the first property in the same way as in the proof of Proposition B.2. Lemma B.5 proves that given $\epsilon > 0$, there exists $\underline{T}$ such that if $T \geq \underline{T}$, $\underline{i} - \epsilon$ satisfies the second property for all $t \in [\underline{T}, T]$,.

Then for each $t \in [\underline{T}, T]$, we can choose $\tilde{i}_t = \min\{\hat{i}_t, \tilde{i} - \epsilon\}$ such that banks are constrained at t whenever the interest rate is below $\tilde{i}_t$ and $\frac{dN_t}{di} > 0$ for all $i < \tilde{i}_t$, as desired. $\square$

So far, we have proven the existence results assuming only Properties 1-2. By assuming Property B.1, we can prove an analogue of Proposition 4 as well.

Proposition B.4. *Suppose Properties 1, 2, and B.1 hold. Then if the time-t reversal rate i_t^{RR} is below $\underline{i}$, i_{t+1}^{RR} exists and is (weakly) greater than i_t^{RR}.*

Proof. Property B.1 guarantees that banks' liquidity constraint will never bind. This comes from the characterization of the solution to the bank's problem under Property 2: it is possible for the liquidity constraint to bind only if, for some i, $N + D(i^{D*}(i), i) - \min\{\psi^L N, L(i^{L*}(i), i)\} < \psi^D D$, but Property B.1 implies this is never the case. Thus, if the time-t reversal rate exists, it is due to the fact that the net worth constraint binds for all $i < i_t^{RR}$ and $\frac{dN_t}{di} > 0$ for all $i < i_t^{RR}$ (Proposition B.1).

Fix $i < \min\{\underline{i}, i_t^{RR}\}$. We have

$$\begin{aligned}\frac{dN_{t+1}}{di} &= \frac{d}{di}(1-\gamma)\Big(N_t + NII(N_t, i)\Big) \\ &= (1-\gamma)\Big(\frac{\partial NII_t}{\partial i} + \big(1 + \frac{\partial NII_t}{\partial N}\big)\frac{dN_t}{di}\Big) \\ &> (1-\gamma)\frac{dN_t}{di} > 0.\end{aligned}$$

The inequality comes from an application of Property 1, which implies $\frac{\partial NII_t}{\partial i} \geq 0$, and Lemma B.1, which implies $1 + \frac{\partial NII_t}{\partial N_t} \geq 0$. Then $\frac{dN_{t+1}}{di} > 0$, as desired. Furthermore, as t increases, N_t converges monotonically to a long-run level $\tilde{N}(i)$ (Lemma B.2). Since Property 1 holds, $\tilde{N}$ (defined in (B.3)) must be decreasing in the interest rate, and $i < \underline{i} \leq i^*$, so the path of N_t is in fact monotonically decreasing. Then if the net worth constraint binds at t, it must bind at $t+1$ as well. In conjunction with Proposition B.1, then, $i < i_{t+1}^{RR}$. □

By Lemmas 4 and B.6, Properties 1 and B.1 hold in our benchmark model, so Proposition B.4 implies Proposition 4.

Now we can prove an analogue of the "low-for-long" result (under Property 1).

Proposition B.5. *Suppose Properties 1 and 2 hold, and fix i such that (18) holds. Then there exists $\underline{T}$ such that if $T > \underline{T}$, $L_t(i, T) < L^*$ for all $t \in [\underline{T}, T]$.*

Proof. First, we show that banks' capital gains are bounded as $T \to \infty$. Note that as $T \to \infty$, bond prices approach

$$\lim_{T\to\infty} Q_t = \frac{1}{\tau}\sum_{s=0}^{\infty}\left(\frac{1-\frac{1}{\tau}}{1+i}\right)^s = \frac{1}{\tau} \times \frac{1}{1 - \frac{1-\frac{1}{\tau}}{1+i}}$$

which is finite because $\tau < \frac{1}{\delta}$.

When banks' capital gains are bounded in T, we can consider the equilibrium with $T = \infty$. Lemma B.2 implies that there exists $\tilde{N}(i) = \frac{NII(\tilde{N}(i), i)}{\gamma}$ such that as $t \to \infty$, N_t converges to $\tilde{N}(i)$. By the hypothesis of the proposition, $\psi^L \tilde{N}(i)$ is less than steady-state lending L^*. Therefore, banks eventually become constrained at a level of lending below steady state when $T = \infty$. Let $\underline{T}$ be the first time at which $L_t(i, T) < L^*$.

Now observe that for any $T \in [\underline{T}, \infty]$, $L_t(i, T)$ must also be below L^* for $t \in [\underline{T}, T]$. This is because net worth $N_t(N_0, i)$ is an increasing function of N_0 (Lemma B.1), and $N_0(i, T)$ in the case of $T < \infty$ is less than $N_0(i,T)$ in the case $T = \infty$. Thus, for all $T \geq \underline{T}$, it will be the case that $L_t(i,T) < L^*$ for all $t \in [\underline{T}, T]$.

$\square$

B.3 Mapping the model to the general setting

We now must prove that our model's loan and deposit demand functions satisfy the assumptions laid out in Section II.F. We must show that loan demand in steady state is continuous, decreasing in i^L, weakly increasing in i, and divergent as $i^L \to -\infty$. Loan demand in the model is given by (**??**), which in steady state reduces to

$$L^*(i^L) = Q^{K,b*}\left(\nu(1-\alpha)\frac{\left(A^b\left(\frac{(1-\alpha)\nu}{W^*}\right)^{(1-\alpha)\nu}\right)^{\frac{1}{1-(1-\alpha)\nu}}}{(i^L-\delta)Q^{K,b*}}\right)^{\frac{1-(1-\alpha)\nu}{1-\nu}}, \tag{B.5}$$

where W^* is the steady-state wage and $Q^{K,b*}$ is the steady-state price of capital. (Here we assume that the steady-state nominal price level is equal to one.) Clearly, this function satisfies all the required properties (since $1-\nu(1-\alpha) > 0$ and loan demand does not depend on i).

Next, we prove that deposit demand is continuous, increasing in i^D, and weakly decreasing in i. The conditions determining the household's deposit and cash demand are:

$$\frac{\zeta}{\beta\Lambda^*}\Phi'(\mathcal{L}_t)\frac{\partial \mathcal{L}_t}{\partial D_t} = i_t - i_t^D + \mu_t, \tag{B.6}$$

$$\frac{\zeta}{\beta\Lambda^*}\Phi'(\mathcal{L}_t)\frac{\partial \mathcal{L}_t}{\partial M_t} = i_t + \mu_t, \tag{B.7}$$

Differentiating the deposit demand conditions (B.6) and (B.7) by i and rearranging, we have that $D(i^D, i)$ is continuous and differentiable (because Φ and $\mathcal{L}$ are assumed to be continuous and differentiable) and obtain

$$\frac{\partial D^*(i^D,i)}{\partial i} = \frac{1}{\det H^{\mathcal{L}}}\frac{1}{\zeta\Phi''(\mathcal{L})}\left(\frac{\partial^2\mathcal{L}}{\partial M^2} - \frac{\partial^2\mathcal{L}}{\partial D\partial M}\right),$$

$$\frac{\partial D^*(i^D,i)}{\partial i^D} = -\frac{1}{\det H^{\mathcal{L}}}\frac{1}{\zeta\Phi''(\mathcal{L})}\frac{\partial^2\mathcal{L}}{\partial M^2},$$

where

$$H^{\mathcal{L}} = \begin{pmatrix} \frac{\partial^2\mathcal{L}}{\partial D^2} & \frac{\partial^2\mathcal{L}}{\partial D\partial M} \\ \frac{\partial^2\mathcal{L}}{\partial D\partial M} & \frac{\partial^2\mathcal{L}}{\partial M^2} \end{pmatrix}$$

is the Hessian of the liquidity aggregator $\mathcal{L}$. Since $\mathcal{L}$ is a homothetic, concave aggregator, we have $\det H^{\mathcal{L}} < 0$, $\frac{\partial^2\mathcal{L}}{\partial M^2} < 0$, and $\frac{\partial^2\mathcal{L}}{\partial D\partial M} > 0$. Since Φ is concave, $\Phi''(\mathcal{L}) < 0$. We then have that $\frac{\partial D}{\partial i}$ is positive and $\frac{\partial D}{\partial i^D}$ is negative, as desired.

We then need to show that in our model, Properties 1 and 2 hold, since in the previous section we showed that those two properties suffice to prove analogues of Propositions 2, 3, and 5. Property

2 holds in our model simply by assumption: both the net worth constraint parameter ψ^L and the liquidity constraint parameter ψ^D are assumed to be positive real numbers. Lemma 4 is sufficient to prove that Property 1 holds, so we prove that lemma here. In the previous section, we used the general properties stated in Section II.F to prove all of the main results. It remains to prove that these properties are indeed satisfied in our model. Proving that Property 1 holds reduces to proving Lemma 2.

Proof of Lemma 4. Lemma 1 implies that $\frac{\partial D^*}{\partial i} = 0$ for $i \leq 0$. Equation (16) then becomes $\frac{\partial NII}{\partial i} = Q^B B^L$, which is positive in light of the liquidity constraint $Q^B B^L \geq \psi^D D$. We can therefore choose $\underline{i} = 0$. □

We would also like to show that Property B.1 holds when ζ is sufficiently large, so that the assumption that ζ is sufficiently large suffices to prove Proposition 4.

Lemma B.6. *There exists $\underline{\zeta}$ such that when $\zeta \geq \underline{\zeta}$, Property B.1 is satisfied.*

Proof. Note that the first-order conditions for deposit and money demand imply that deposit demand for *all* values of ζ can be written as a (continuous) function $g^D(\frac{i^D}{\zeta}, \frac{i}{\zeta})$. Rewriting in this form, the unconstrained optimal deposit rate $i^{D*}(i)$ solves

$$\max_{i^D} \frac{i - i^D}{\zeta} g^D(\frac{i^D}{\zeta}, \frac{i}{\zeta}).$$

As $\zeta \to \infty$, $\frac{i}{\zeta} \to 0$ for all i, so we have $i^{D*}(i) \to i^{D*}(0)$ for all i (due to the continuity of g^D). Hence, for ζ sufficiently large, $i^{D*}(i)$ can be made arbitrarily close to the optimal deposit rate when $i = 0$. Thus, for large ζ, households are arbitrarily close to being satiated in liquid assets (since they hit their satiation point once $i = 0$, and money begins to dominate bonds). We have assumed that ψ^D is small enough that households are not liquidity constrained when they are satiated in liquid assets, so $D(i^{D*}(0), 0) \geq \frac{1}{1-\psi^D} \min\{\psi^L \overline{N}, L^*(i^{L*}(0))\}$. But then for large enough ζ, banks are never liquidity constrained, since $D(i^{D*}(i), i) \to D(i^{D*}(0), 0)$ for all i. □

We can then describe how each of the main results in the text are proven. Above, we proved that Properties 1-2 hold in the model of Section II.A, whereas Property B.1 holds when the deposit demand parameter ζ is sufficiently large.

- Proposition 1 follows from Proposition B.1 and the analysis in Section II.C;
- Lemma 2 follows from Lemma B.6 and (16);
- Lemma 3 follows from Propositions B.4 and B.5;
- Lemma 4 follows from Lemma B.3;
- Propositions 2 and 3 follow from Propositions B.2 and B.3, respectively;
- Proposition 4 follows from the assumption that ζ is large and Proposition B.4;

- Proposition 5 follows from Proposition B.5.

C Additional sensitivity analysis and comparison to literature

This section analyzes the sensitivity of our benchmark estimates of the reversal rate to other parameters that were omitted in our original discussion, compares our model's behavior to that of a conventional medium-scale DSGE model, and provides additional results on the effectiveness of forward guidance relative to a standard model.

C.1 The model and the data

In this section, we provide additional results relating our model to the data that are not reported in the paper.

The deposit spread. Our modeling of the deposit spread is motivated by a key stylized fact: once interest rates entered negative territory, deposit rates remained stuck at zero (likely because banks feared that depositors would switch over to cash if they set negative rates). Heider et al. (2019) show that the median deposit rate in the Euro area stayed very close to (but above) zero following June 2014, even as the policy rate went negative. They also document that following 2014, deposit rates bunched at zero, with no bank setting a negative deposit rate for households (see also Eisenschmidt and Smets, 2019).[5] Since we are mainly concerned with the impact of interest rates in negative territory, banks' deposit rate-setting decision is deliberately simple in our benchmark quantitative model: when deposit rates are in positive territory, the pass-through from the policy rate is (nearly) one-for-one, whereas once the policy rate goes low enough, deposit rates get stuck at zero.

In the data, however, the pass-through from the policy rate to deposit rates is imperfect even when rates are in positive territory. To check whether our model's deviation from the data affects our main quantitative conclusions, we briefly consider an alternative model in which there is imperfect pass-through even when deposit rates are positive. Banks set deposit rates according to[6]

$$i_t^D = \max\{\omega(i_t + \mu^D), 0\}, \tag{C.1}$$

where $\omega \in [0, 1]$ is a pass-through parameter (and, as before, μ^D is the non-pecuniary benefit of issuing deposits). We calibrate the pass-through parameter $\omega = 0.39$ to minimize the (expected) distance between our modified model's predicted deposit spread (given the EONIA rate) and the data on average deposit spreads in the Euro area from 2003 to the present (from the ECB's Statistical Data Warehouse, ECB 2021). That is, if i_t^{Data}, $i_t^{D,Data}$ are the policy rate and the average

[5] Outside of the Euro area, Bech and Malkhozov (2016) and Basten and Mariathasan (2018) provide evidence that deposit rates were stuck at zero in Switzerland, whereas Hong and Kandrac (2022) document similar facts in Japan.

[6] This specification can be micro-founded by assuming that there is imperfect competition among a *finite* number of banks in the deposit market (see, e.g., Drechsler, Savov, and Schnabl 2017). However, given the difficulties involved in introducing such imperfect competition in an infinite-horizon macroeconomic model, we do not formally provide such foundations here.

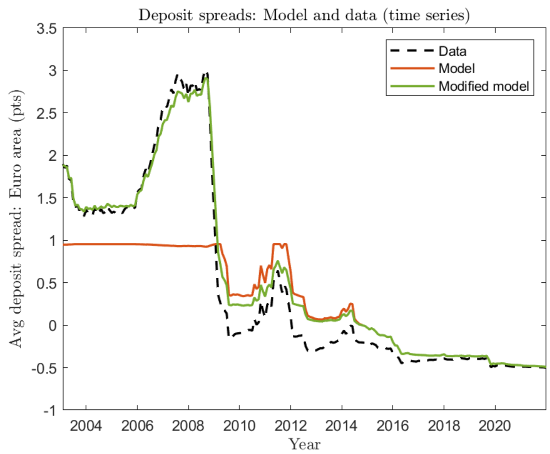

Figure C.1: The deposit spread in the data (dashed black line) is plotted along with the deposit spread in the benchmark model (red line) and the modified model in which deposit rates are set according to (C.1) (green line). Model predictions are based on the EONIA rate.

deposit rate in the data, respectively, we predict the deposit spread $i_t - i_t^D$ in the model (taking $i_t = i_t^{Data}$ and using (C.1)), and then we choose ω to minimize

$$\frac{1}{T}\sum((i_t^{Data} - i_t^{D,Data}) - (i_t - i_t^D))^2.$$

The calibration $\omega = 0.39$ coincides closely with the pass-through estimated by Ulate (2021a) using micro-level data: that paper finds a pass-through close to zero when $i_t < 50$bp and a pass-through close to 0.5 when $i_t > 50$bp.

Figure C.1 displays the deposit spread in the data as well as the predicted deposit spread (given the policy rate i_t) in our benchmark model and in the modified model where deposit rates are set according to (C.1). Our benchmark deposit rate co-moves with the data well in the period of low rates, but not in the pre-2008 period when rates routinely exceeded 2%. The modified deposit rate, on the other hand, co-moves closely with the actual spread throughout the sample. Importantly, this modification of our model does not substantially change the main predictions: in the modified model, we find a reversal rate for investment of -0.4% and a reversal rate for bank lending of around -1.1%. Intuitively, these reversal rates are higher because when pass-through is imperfect, interest rate cuts reduce banks' interest income even in positive territory via the reduction in the deposit spread.

Returns on equity. In Section IV.A, we discuss our model's predictions for the impulse response of banks' accounting return on equity. In our model, ROE over a one-year period is defined as

$$ROE_{t,t+4} = \frac{DIV_{t+1} + DIV_{t+2} + DIV_{t+3} + DIV_{t+4} + N_{t+4}}{N_t}.$$

Figure C.2 plots the change in $ROE_{t,t+4}$ at the impact of a -10bp Taylor rule innovation for various initial levels of the policy rate.

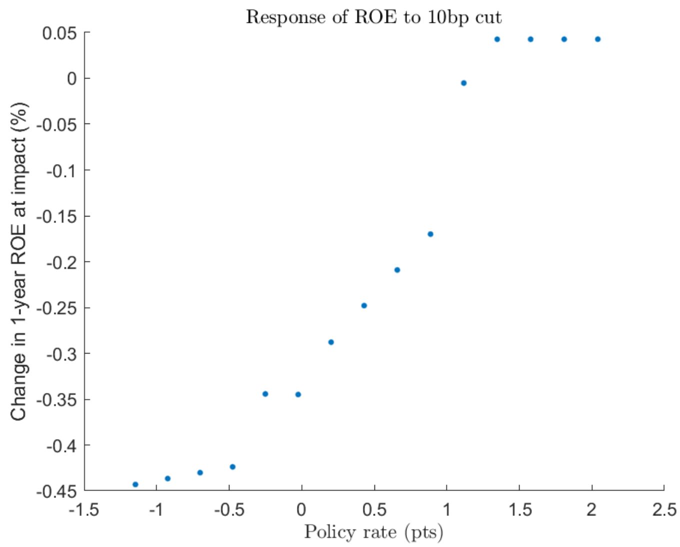

Figure C.2: The response at the impact of a -10bp Taylor rule innovation of returns on equity over one year, $ROE_{t,t+4}$, is plotted for various initial levels of the policy rate.

C.2 Sensitivity results

This section discusses sources for some parameters in the model and the remainder of our sensitivity analysis.

Our model has a set of standard parameters that were simply calibrated to match values previously used in the literature (rather than to match some target in the particular data we analyze). In particular, we calibrate each parameter to the exact value used in the ECB's New Area-Wide Model II (Coenen et al. 2019). Sources that use similar values for each of the parameters in Table 1 are listed below in Table C.1.

Figures C.3 illustrates the sensitivity of our results to these parameters, with the exception of the monetary policy parameters ρ^{mp} and ϕ^{π}. Fortunately, our estimate of the reversal rate at the impact of a monetary shock is not very sensitive to the value of any of them.

Figure C.4 shows how the reversal rate depends on the parameters of the Taylor rule. We plot results for the persistence ρ^{mp} of the nominal rate, the coefficient ϕ^{π} on inflation, and also add a coefficient ϕ^{y} on output (which is equal to zero by assumption in our benchmark calibration).

[7]Several papers use a value of $h \approx 0.8$ instead. See, for example, Christiano, Motto, and Rostagno (2014) or Gertler and Karadi (2011).

[8]These authors report a macro Frisch elasticity of 0.5 in Table 1. Smets and Wouters (2007) provide a similar estimate of $\frac{1}{1.52}$.

[9]The authors estimate ε using a retail markup estimate of 35%, from Martins, Scarpetta, and Pilat (1996) and Jean and Nicoletti (2002).

[10]Most papers in the literature use the Calvo (1983) formulation of price stickiness. A frequency of price adjustment ω is equivalent to a Rotemberg cost θ if $\frac{(1-\omega)(1-\beta\omega)}{\omega} = \frac{\varepsilon-1}{\theta}$. The parameter value we use is roughly in line with the evidence provided by Nakamura and Steinsson (2008), which suggests firms adjust prices once per year.

[11]Another conventional parametrization for the Taylor rule is $\rho^{mp} = 0.8; \phi^{\pi} = 1.5$, and a coefficient on the output gap $\phi^{y} = \frac{0.5}{4}$. See Del Negro et al. (2017); Gertler and Karadi (2011), or Eggertsson et al. (2019).

Table C.1: Sources for conventional parameters

Parameter	**Description**	**Source(s)**
σ	IES parameter	NAWM II; Christiano, Eichenbaum, and Evans (2005); Christiano, Motto, and Rostagno (2014); Del Negro et al. (2017); Gerali et al. (2010)
h	Habit formation[7]	NAWM II; Christiano, Eichenbaum, and Evans (2005); Smets and Wouters (2007)
φ	Frisch elasticity	NAWM II; Chetty et al. (2011)[8]
δ	Capital depreciation	NAWM II; Christiano, Eichenbaum, and Evans (2005); Del Negro et al. (2017); Gerali et al. (2010); Gertler and Karadi (2011); Justiniano, Primiceri, and Tambalotti (2010); Smets and Wouters (2007)
α	Capital share	NAWM II; Christiano, Eichenbaum, and Evans (2005); Del Negro et al. (2017); Gertler and Karadi (2011)
ε	Retail elasticity	NAWM II[9]; Gertler and Karadi (2011)
θ	Rotemberg cost[10]	NAWM II; Christiano, Motto, and Rostagno (2014); Del Negro et al. (2017); Gertler and Karadi (2011)
ϕ^{π}	Taylor rule inflation coefficient	NAWM II
ρ^{mp}	Taylor rule inertia[11]	NAWM II

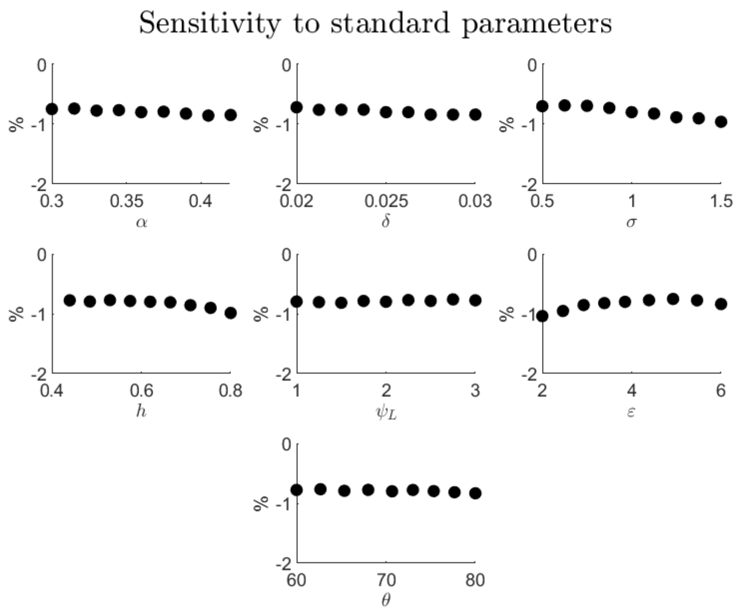

Figure C.3: Sensitivity of the reversal rate estimate to the parameters calibrated in Table 1, with the exception of the monetary policy parameters ρ^{mp} and ϕ^{π}. The reversal rate of aggregate investment on impact of a monetary shock is plotted against a range of values for each parameter. All other parameters are held fixed at their values in our benchmark calibration.

Again, the reversal rate does not seem to depend much on the value of those parameters.

Next, we address the sensitivity of our results to the calibration of parameters in Table 2. Two of these parameters are uniquely identified by bank balance sheet data: the payout rate γ, which is identified by banks' Tier-1 capitalization ratio, the liquid asset demand parameter ζ, which is identified from the loan-to-bond ratio on bank balance sheets. We plot sensitivity results for parameters κ^I, τ, ε^L, and ε^D in Figures C.5 and C.6, and Section IV.B plots results for parameters κ^L and $\frac{A^b}{A^{nb}}$.

Figure C.5 illustrates the sensitivity of the reversal rate to two key parameters: the elasticity of investment to the price of capital κ^I and the maturity τ of bonds on bank balance sheets. The reversal rate is decreasing in κ^I because, when investment is more sensitive to capital prices, a reduction in investment by bank-dependent firms generates a greater countervailing investment response by non-bank-dependent firms. As expected, the reversal rate is decreasing in τ as well: when banks' bonds have a longer maturity, they experience greater capital gains, so interest rate cuts are not as detrimental to their net worth. In both cases, however, the reversal rate remains in negative territory below -40bp.

The composition of banks' profits also has important implications for the reversal rate. Figure C.6 depicts the dependence of the reversal rate on the elasticity of loan demand ε^L and the elasticity of deposit demand $|\varepsilon^D|$. Since we re-calibrate the model to keep banks' capitalization ratio $\frac{N^*}{L^*}$ constant, these comparative statics should be interpreted as changing the *shares* of bank profits

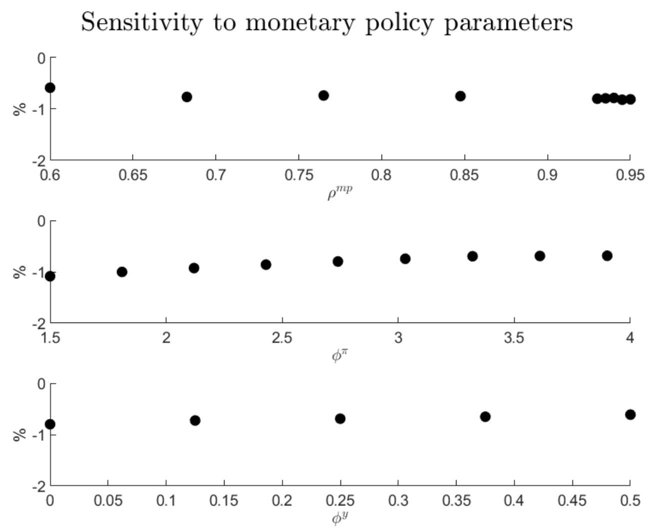

Figure C.4: Sensitivity of the reversal rate estimate (of aggregate investment on impact) to the parameters of the Taylor rule, $(\rho^{mp}, \phi^{\pi}, \phi^{y})$. The reversal rate of aggregate investment on impact of a monetary shock is plotted against a range of values for each parameter. All other parameters are held fixed at their values in our benchmark calibration.

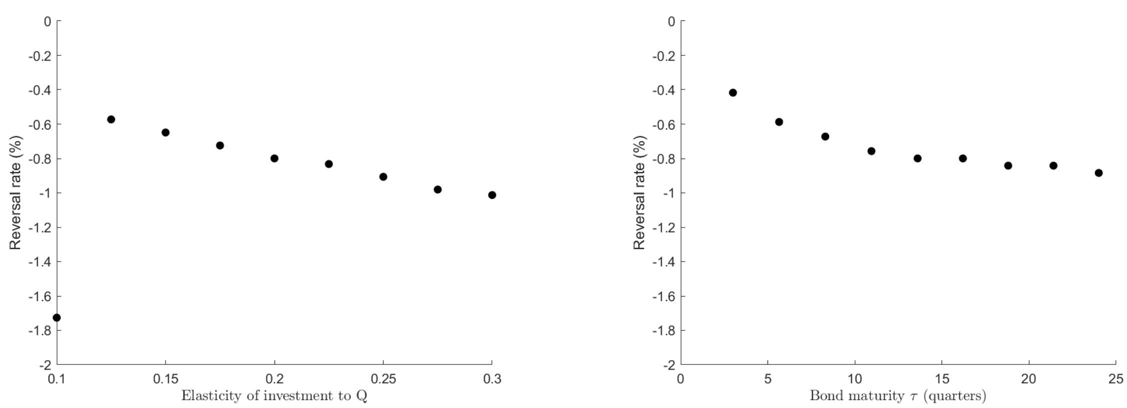

Figure C.5: Sensitivity of the reversal rate (of aggregate investment on impact) to the investment adjustment cost parameter κ^I (reported as the elasticity of investment to Q, left panel) and the expected maturity of banks' bonds (reported in quarters, right panel).

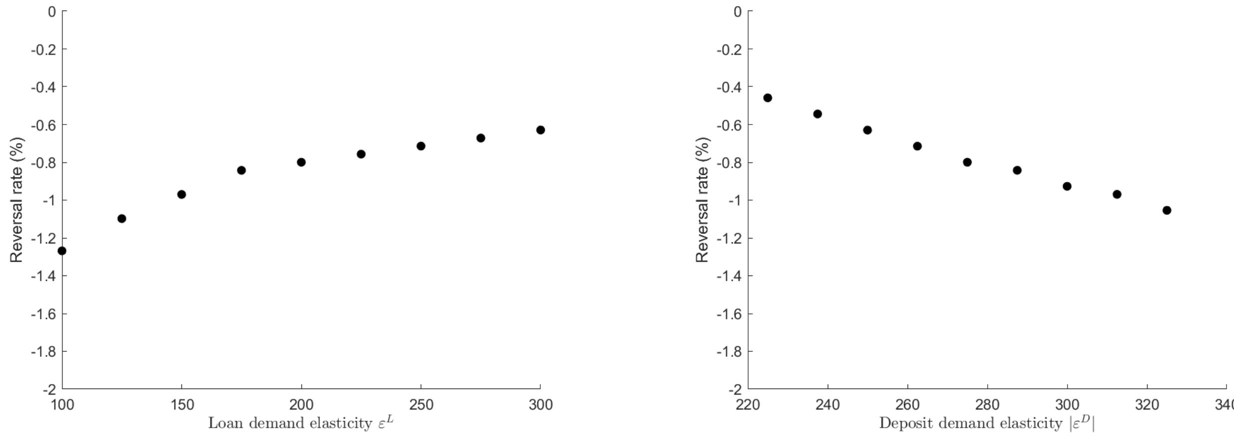

Figure C.6: Dependence of the reversal rate on the values of the loan demand elasticity parameter ε^L (left panel) and the absolute value of the deposit demand elasticity parameter ε^D (right panel). In each panel, all other parameter values are set as in our benchmark calibration.

accounted for by loans and deposits, respectively. The reversal rate is *increasing* in the elasticity of loan demand but *decreasing* in the elasticity of deposit demand. The key mechanism underlying this result is that when loan markets are relatively competitive, or when deposit markets are relatively uncompetitive, then bank profitability is highly reliant on deposit spreads. Interest rate cuts into negative territory are then significantly detrimental to bank net worth, leading to a higher reversal rate. One would expect a relationship between the reversal rate and the share of banks' profits attributable to deposit market power, but there is not necessarily a relationship between the overall concentration of the banking sector and the reversal rate.

Therefore, it remains to analyze the sensitivity of the results to the remaining parameters: (1) the time rate of preference β (calibrated to the level of interest rates), (2) the returns-to-scale parameter ν in the production function (calibrated to the consumption/investment ratio), (3) the fraction of bank-dependent firms ξ (calibrated to SME data), (4) the benefit μ^D banks receive from issuing deposits, (5) the satiation point $\mathcal{L}^*$ of liquid asset balances (calibrated to the change in the deposit/GDP ratio from 2000 to 2014), and (6) the equity injection $\hat{N}$ banks receive each period (calibrated to the equity issuance/asset ratio for the banking sector).

Figure C.7 displays the results for these parameters. Again, for most parameters the estimated reversal rate on impact remains mostly within a reasonable range of our estimate of -0.8%, even though we consider a wide range for each individual parameter. In no case is the reversal rate far below our preferred estimate. Two cases in which the reversal rate is somewhat sensitive to parameter values is when the benefit μ^D of issuing deposits is implausibly large (on the order of a percentage point annually) and when the time rate of preference β is far from its calibrated value. This is not surprising because μ^D affects the level of deposit rates and β affects the level of both deposit rates and the policy rate, and of course the reversal rate depends on steady-state interest rates. The results would not be as sensitive to μ^D, for instance, if deposit demand elasticity ε^D were simultaneously changed to keep the steady-state deposit spread constant.

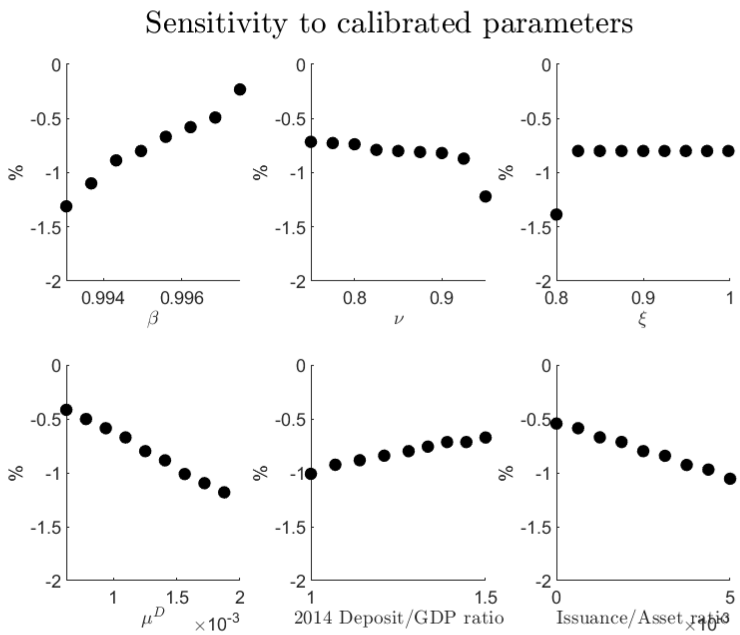

Figure C.7: Sensitivity of the reversal rate estimate to selected parameters calibrated in Table 2. The reversal rate on impact of a monetary shock is plotted against a range of values for each parameter. All other parameters are held fixed at their values in our benchmark calibration.

C.3 Analysis of alternative initial shocks

The goal of our calibration is to estimate the *level* of interest rates at which additional interest rate cuts become contractionary for bank lending and investment. In the benchmark model, we answer this question by first hitting the economy with a large monetary shock that reduces the policy rate to a low level and then adding an additional small monetary shock that reduces the policy rate by an additional 10 basis points. We then compute the *marginal* response of bank lending and investment to this 10-basis point shock. There is no particular reason that the initial large shock has to be a monetary shock, however – it suffices to consider *any* shock that would reduce interest rates.

Therefore, we examine two additional types of large shocks in this section: a shock to the discount factor β that makes agents more patient (as in Eggertsson and Woodford, 2003), and a shock to firm productivity (A_t^b, A_t^{nb}) that reduces the natural rate. For the discount factor shock, we assume that agents' subjective discount factor $\beta_t = e^{\epsilon_t^\beta}\beta$, where ϵ_t^β is the shock to the discount factor. We assume an "MIT" shock: agents learn ϵ_0^β at $t = 0$, and there are no additional shocks to the discount factor thereafter, $\epsilon_t^\beta = 0$ for $t > 0$. When we consider shocks to productivity, we impose an exactly analogous process for (A_t^b, A_t^{nb}). In each case, the shock does not persist after the first period. Nevertheless, since the Taylor rule followed by the central bank has inertia, *interest rates* are just as persistent as in our benchmark model.

Figure C.8 plots the response of investment at the impact of the 10-basis point Taylor rule

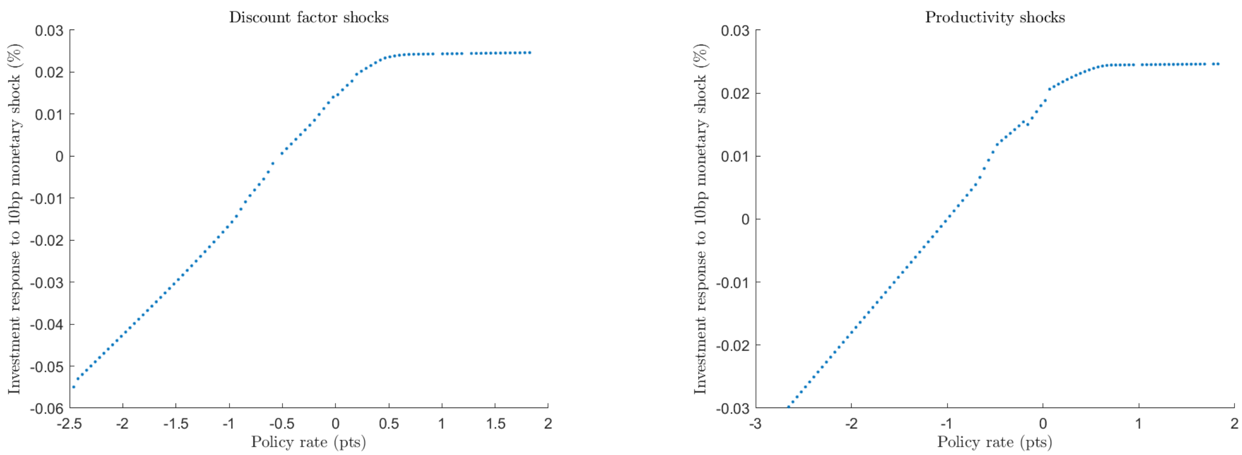

Figure C.8: Initial interest rate plotted against the marginal response of investment to a 10-basis point Taylor rule innovation. The left panel depicts the case in which the level of the initial interest rate is determined by a discount factor shock occurring at $t = 0$. The right panel depicts the case in which the level of the initial interest rate is determined by a productivity shock occurring at $t = 0$. In both cases, the model is solved under perfect foresight.

shock against the interest rate after the initial large shock. In both cases, the estimated reversal rate is reasonably close to our benchmark estimate of -0.8%: in the case of a productivity shock it is -1.1%, whereas in the case of a discount factor shock it is -0.6%.

It is sensible that in these cases, the reversal rate does not depend too heavily on the source of the initial interest rate displacement. The theoretical model showed that in partial equilibrium, banks' initial net worth, the tightness of the leverage constraint, and the dependence of banks' profits on the interest rate were the main determinants of the reversal rate. The qualitative dependence of bank profits and initial capital gains on the sequence of interest rates is relatively unchanged by the type of shock considered. Therefore, the reversal rate differs across these two cases mostly via general equilibrium effects that affect loan demand and deposit demand through changes in prices.

Note, however, that the reversal rate would be significantly *higher* than -0.8% following any shock that both reduces interest rates and directly decreases bank net worth on impact. For example, suppose bank net worth were to receive a substantial negative shock at the same time as a negative shock to demand (captured by a discount factor shock). This situation could be interpreted as a financial crisis coupled with a demand recession. In this scenario, banks are initially much more constrained than they would be following a demand shock only. Hence, further interest rate cuts are particularly detrimental to bank lending in this type of recession, so a reversal in aggregate investment should be expected to occur at a higher interest rate.

C.4 The accounting treatment of bonds

In this section, we extend the model to incorporate bonds that are held-to-maturity, as discussed in Section IV.E. We begin with a general setup. Then, we demonstrate that our main theoretical results continue to hold in this setup. Finally, we solve the model quantitatively and show that the reversal rate is not highly sensitive to the fraction of banks' bonds that are held-to-maturity.

C.4.1 General setup

We extend the model as follows. The asset side of a bank's balance sheet now consists of loans L_t, a real quantity of held-to-maturity (H) bonds $S_t^H = \frac{Q^{B*}B_t^H}{P_t}$, and a real quantity of market-to-market (M) bonds $S_t^M = Q_t^B B_t^M$. Just like marked-to-market bonds, held-to-maturity bonds mature stochastically with probability $\frac{1}{\tau}$. On the liabilities side, banks have deposits D_t and regulatory (accounting) capital N_t (the counterpart of net worth in our benchmark model), so the balance sheet identity is

$$L_t + S_t^H + S_t^M = N_t + D_t. \tag{C.2}$$

A bank's regulatory capital evolves according to

$$\begin{aligned} N_{t+1} = \frac{1-\gamma}{1+\pi_{t+1}} \Big(&(1+i_t)S_t^M + (1+i^*)S_t^H + (1+i_t^L)L_t - (1+i_t^D)D_t \\ &- \Psi^L(N_t, L_t) - \Psi^D(S_t^M, D_t) \Big) \quad \forall \;\; t \geq 0. \end{aligned} \tag{C.3}$$

Note that banks pay out a fraction γ of their regulatory capital each period and that their liquidity constraints are based on their holdings of *trading* (marked-to-market) bonds, since held-to-maturity bonds cannot be traded when the bank needs liquidity. Here, we use the facts that in equilibrium, the return on marked-to-market bonds must be equal to the risk-free rate after the initial unanticipated shock,

$$\frac{(1-\frac{1}{\tau})Q_{t+1}^B + \frac{1}{\tau}\frac{1}{1+\pi_{t+1}}}{Q_t^B} = \frac{1+i_t}{1+\pi_{t+1}},$$

and that the accounting income from held-to-maturity bonds equals the steady-state risk-free rate times the accounting value of those bonds,

$$\frac{(1-\frac{1}{\tau})Q^{B*} + \frac{1}{\tau}}{P_{t+1}} B_t^H = \frac{1+i^*}{1+\pi_{t+1}} \frac{Q^{B*}B_t^H}{P_t},$$

since $Q^{B*} = \frac{1}{1+\tau i^*}$.

A fraction ϕ of banks' steady-state bond holdings are marked as held-to-maturity, so

$$B_0^H = \phi B^*, \tag{C.4}$$

and a fraction $\frac{1}{\tau}$ mature each period, meaning the real value of these bonds evolves according to

$$\frac{Q^{B*}B_t^H}{P_t} = \frac{1-\frac{1}{\tau}}{1+\pi_t} \frac{Q^{B*}B_{t-1}^H}{P_{t-1}} \quad \forall \;\; t \geq 1. \tag{C.5}$$

Banks' initial regulatory capital in period 0 is equal to its steady-state value plus the capital gains on marked-to-market bonds,

$$N_0 = N^* + (1-\gamma)(1-\frac{1}{\tau})(1-\xi)(Q_0^B - Q^{B*})B^*. \tag{C.6}$$

Banks maximize the present value of their dividends γN_t,

$$\max_{i_t^L, i_t^D, S_t^M} \sum_{t=0}^{\infty} \beta \Lambda_t \gamma N_t \text{ s.t. } (C.2),\ (C.3). \tag{C.7}$$

C.4.2 Theoretical results

Now we specialize the model to demonstrate that analogues of our theoretical results hold. As before, we consider the bank's problem in partial equilibrium with no inflation. The bank faces deposit and loan demand schedules $D(i^D, i)$ and $L(i^L)$, and its net worth and liquidity constraints are specified by constraints

$$\frac{L_t}{N_t} \leq \psi^L,\ \frac{S_t^M}{D_t} \geq \psi^D \tag{C.8}$$

as in the theoretical section of the paper.

In each period, banks maximize their net worth in the next period, which as before reduces to solving the problem

$$NII_t(N_t, i_t) = \max_{i_t^L, i_t^D, S_t^M} i_t S_t^M + i^* S_t^H + i_t^L L_t - i_t^D D_t \text{ s.t. } (C.2),\ (C.8). \tag{C.9}$$

Here the net interest income function can be written as a function of t because S_t^H is a function of t only, per (C.5). We can write the evolution of regulatory capital concisely as

$$N_{t+1} = (1-\gamma)(N_t + NII_t(N_t, i_t)). \tag{C.10}$$

Hence, we have reduced this problem to a more general version of the bank's problem in our benchmark theoretical model: the only difference is that now the net interest income function depends on t directly as well as the state variables N and i. Applying the envelope theorem to (C.9), we have

$$\frac{\partial NII_t}{\partial i} = S_t^M + (i_t - i_t^D)\frac{\partial D(i_t^D, i_t)}{\partial i_t}. \tag{C.11}$$

As before, the partial derivative in the second term is equal to zero when $i_t < 0$, since households do not hold bonds when rates are negative. Furthermore, S_t^M must be positive in light of the liquidity constraint. Therefore, net interest income is increasing in i_t for all sufficiently low i_t. Moreover, we have assumed a non-trivial lending constraint ($\psi^L < \infty$), and net interest income must be an increasing function of N_t (since it is the objective function in an optimization problem). These are the properties required for Lemma B.3, our key lemma. A final point is that $NII_t(N, i)$ is decreasing in t, since the interest payments of held-to-maturity bonds decay geometrically. Thus, for $i < \underline{i}$, the analogue of Lemma B.2 will imply that N_t decreases monotonically to some long-run

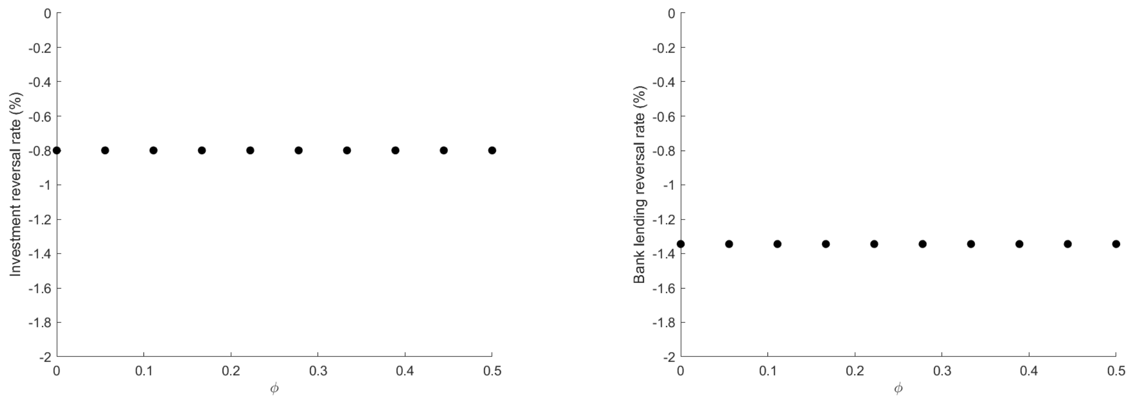

Figure C.9: The reversal rate for investment (left panel) and bank lending (right panel) in the extended model with bonds held-to-maturity (HTM) is plotted against the fraction ϕ of bonds on banks' balance sheet that are marked as HTM.

level $\tilde{N}(i)$. Then it is easy to check that our theoretical results continue to hold in this extended model. They can be proved in an exactly analogous way to our benchmark results.

C.4.3 Quantitative results

We now briefly describe the quantitative results that we obtain in this extension of the model. We use the equations derived at the beginning of this section to replace the corresponding equations in the benchmark model. We solve the model for various values of the new parameter ϕ and find the reversal rate (for both investment and bank lending) for each value. Specifically, in Dynare we introduce bonds that are held-to-maturity as an exogenous unanticipated shock at $t = 0$, with ϕ modulating the value of that shock.

Figure C.9 plots the reversal rate for investment (left panel) and bank lending (right panel) as a function of ϕ. Clearly, the reversal rate does not vary much with this parameter. The benchmark model has $\phi = 0$ (i.e., no bonds are held to maturity). We use the range $\xi \in [0, 0.5]$ because it seems plausible given the data – at least in the US, banks tend to mark 15-30% of their bonds as held-to-maturity. Intuitively, this result arises because quantitatively, the two countervailing effects of held-to-maturity bonds approximately offset one another: banks do not benefit from capital gains on bonds that are held-to-maturity, but the interest payments generated by those bonds are also shielded from the decrease in interest rates.

C.5 Comparison to a standard DSGE model

In addition to this sensitivity analysis, we perform checks to make sure our model behaves reasonably near its steady state. We compare its performance to that of a workhorse medium-scale DSGE model. Our model, in fact, embeds all of the elements present in Christiano, Eichenbaum, and Evans (2005, henceforth referred to as CEE) with the exception of variable capital utilization and wage stickiness. Hence, we can compare our results to those of a version of CEE with fixed

Table C.2: CEE (2005) parameters

Parameter	Description	Value
β	Time rate of preference	0.9925
σ	IES parameter	1.0
h	Habit formation	0.62
α	Capital share	0.36
ν	Scale parameter	1.0
$\frac{A^b}{A^{nb}}$	Relative prod. bank-dependent firms	1.0
κ^I	Capital adjustment cost	2.48
ε	Retail elasticity	6.0
θ	Rotemberg cost	18.5[12]
ρ^{mp}	Taylor rule persistence	0.8
ϕ^π	Taylor rule inflation coefficient	1.5
ϕ^y	Taylor rule output coefficient	0.125

capital utilization and flexible wages. To do so, we shut down all of the financial frictions in our model: the loan market is competitive ($\varepsilon^L \to \infty$), banks do not face leverage costs in lending ($\kappa^L = 0$), and bank-dependent firms are just as productive as non-bank-dependent firms ($\frac{A^b}{A^{nb}} = 1$). Hence, both types of firms are identical and issue debt at the policy rate. Then, we set the remaining (relevant) parameters to the values reported by CEE. Those authors include a coefficient ϕ^y on output in the Taylor rule, so for the purposes of this experiment, we include one as well. The relevant parameter values are listed in Table C.2.

We then compute two impulse response functions. First, we compute the impulse response of our benchmark model to a 10-basis point Taylor rule innovation near the steady state. Next, we compute the impulse response to the same shock in an economy where (1) financial frictions have been shut down, and (2) the remaining relevant parameters are set to the values in Table C.2, as in CEE. Figure C.10 plots the impulse response of macroeconomic aggregates.

Our model produces responses that are largely similar to those in the CEE model without variable capital utilization or wage stickiness. In particular, the impulse responses for the policy rate i, output Y, consumption C, hours H, and investment I are almost identical. Inflation is somewhat less responsive in our model than in CEE. The reason is straightforward: in CEE, prices are more flexible, $\theta = 18.5$ in CEE versus $\theta = 70.7$ in our model.

As a final check, we also compare our model's impulse responses to a "frictionless" model in which all parameters are set to their benchmark values except for the leverage cost parameter κ^L, which is set to zero. In this frictionless model, there are no net worth constraints on lending, so the transmission of monetary policy should be similar to that in a standard New Keynesian model. Figure C.11 confirms that this is indeed the case.

[12] CEE (2005) use Calvo pricing, but in their log-linear solution, Calvo and Rotemberg frictions are equivalent. Here, we report the equivalent value of the Rotemberg cost parameter for consistency with our model.

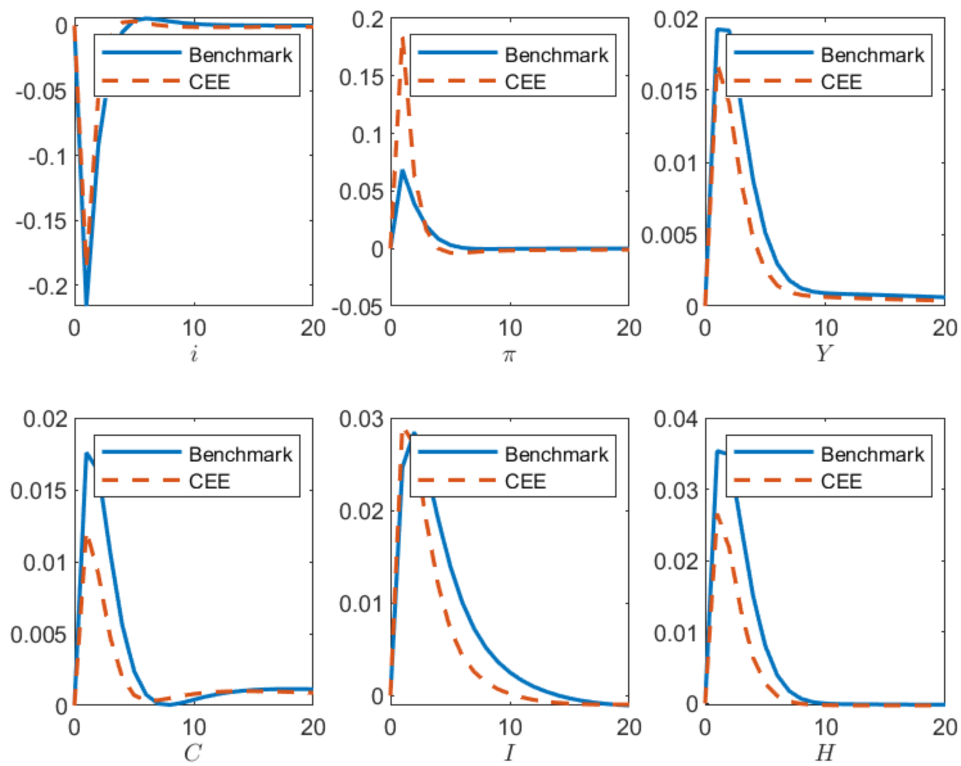

Figure C.10: A comparison of impulse responses in our model (blue lines) to those in a version of Christiano, Eichenbaum, and Evans (2005) without wage stickiness or variable capital utilization (dashed red lines). Both economies are hit with a ten-basis point Taylor rule innovation at their steady states. The parameters used in the CEE model are listed in Table C.2. All financial frictions are shut down, and loan markets are assumed to be perfectly competitive.

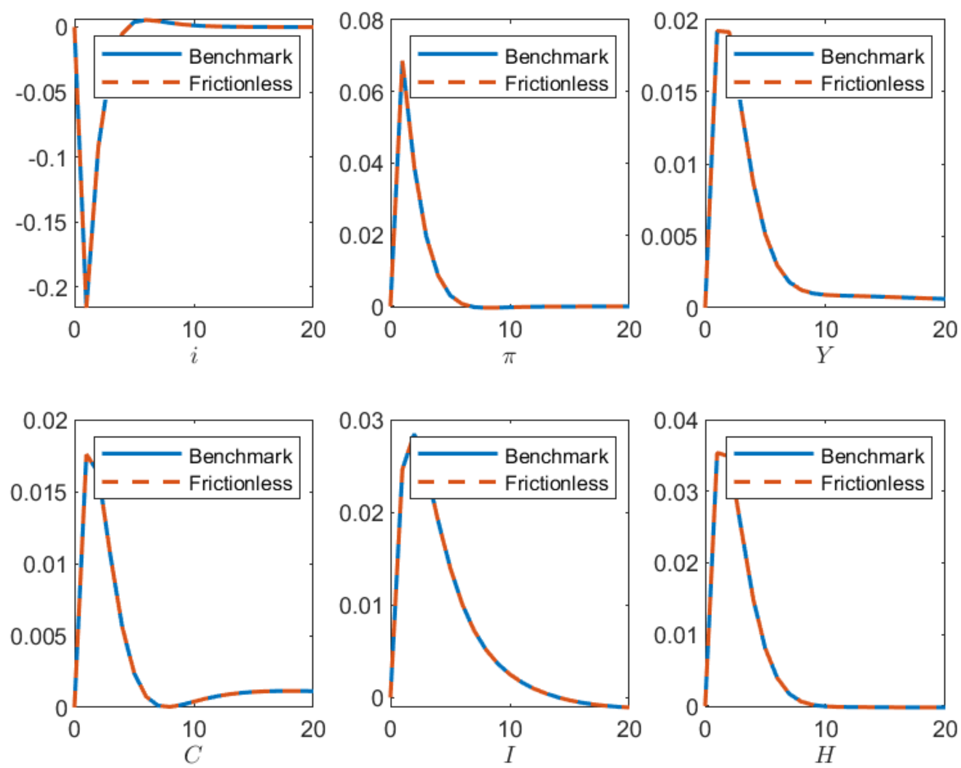

Figure C.11: A comparison of impulse responses in our model (blue lines) to those in a frictionless version of the model with κ^L set to zero (dashed red lines). All other parameters in the frictionless model are set to their values in Tables 1 and 2. Both economies are hit with a ten-basis point Taylor rule innovation at their steady states.

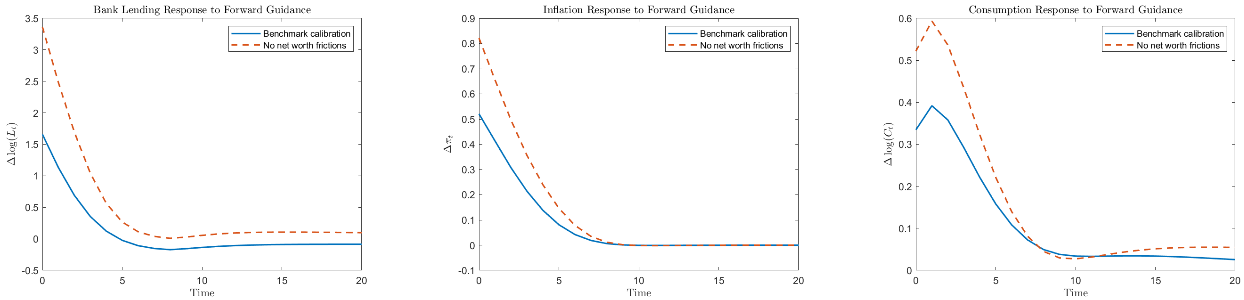

Figure C.12: Responses of bank lending (left panel), inflation (center panel), and aggregate consumption (right panel) to a forward guidance shock in two economies: our benchmark economy (blue lines) and a frictionless economy in which the cost of leverage κ^L is set to zero (dashed red lines), with all other parameters held fixed at their original values. At $t = 0$, the central bank announces that it will hold the interest rate at -1% for eight periods and then return to a Taylor rule.

C.6 Additional forward guidance results

Finally, we report additional results on the effectiveness of forward guidance in our model versus a "frictionless" model in which banks' cost of leverage, parameterized by κ^L, is set to zero (as in Section III.E). Previously, we showed that in our model aggregate investment and output do not respond nearly as strongly to a forward guidance shock as they would in the frictionless model. Here, we report analogous results for inflation, consumption, and bank lending.

Figure C.12 illustrates the results. As with aggregate investment and output, the responses of bank lending, inflation, and consumption in our model are roughly half as large as in the benchmark model. The intuition is similar: given that agents foresee a future decline in investment, demand and asset prices do not initially respond as strongly as they would in a frictionless model, where there is no eventual decline in investment. Hence, inflation does not increase nearly as much either.

Importantly, our model also produces conventional responses to forward guidance when the initial shock to interest rates is small, i.e., when rates are kept below the natural rate but they remain in positive territory. Figure C.13 illustrates the results of a forward guidance policy in our model and in the "frictionless" model when interest rates are held at 1.5% (rather than -1%) for eight quarters. The variables plotted are the same as those in Figure C.12. Given that the model's response to forward guidance in positive territory is largely the same as that of a standard model, it is clear that our results are driven by the reversal rate mechanism. When interest rates are not cut low enough, banks' net interest income does not decline much, so their net worth constraints play essentially no role. On the other hand, when net worth constraints are expected to be costly for banks in the future, households anticipate a future recession, so demand and inflation do not respond as strongly to monetary stimulus in the present.

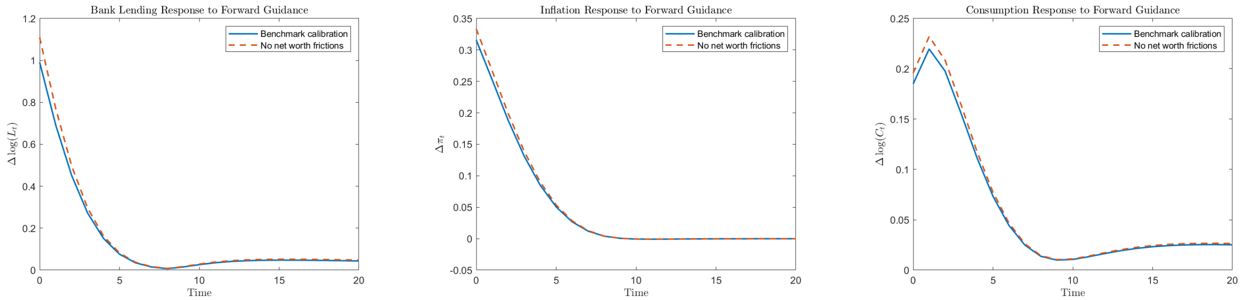

Figure C.13: Responses of bank lending (left panel), inflation (center panel), and aggregate consumption (right panel) to a forward guidance shock in two economies: our benchmark economy (blue lines) and a frictionless economy in which the cost of leverage κ^L is set to zero (dashed red lines), with all other parameters held fixed at their original values. At $t = 0$, the central bank announces that it will hold the interest rate at 1.5% for eight periods and then return to a Taylor rule.

D Monopolistic competition in loan and deposit markets

In this section, we elaborate on the monopolistic loan and deposit market competition introduced in Section III.B. Our treatment follows that originally developed by Gerali et al. (2010). We begin by providing a formulation in which households (firms) have CES preferences over the deposits (loans) issued by different banks and describe agents' optimization problems. Then, we examine the case of symmetric equilibrium in which banks choose to issue deposits, which is the relevant one for our calibration. Finally, for completeness, we discuss the case in which banks shut down deposit issuance, despite the fact that in our calibration this never occurs in equilibrium. At the end of this section, we additionally show how the CES form of loan demand can be derived from a discrete choice problem, leaving the corresponding derivation for deposit demand to Ulate (2021b).

D.1 Firm optimization problem

A bank-dependent firm can borrow from each bank $j \in [0, 1]$, and loans offered by different banks are imperfect substitutes. The total quantity of resources raised by a bank that takes loans K_{jt} from each bank j is $K_t = \left(\int_0^1 K_{jt}^{\frac{\varepsilon^L - 1}{\varepsilon^L}} dj \right)^{\frac{\varepsilon^L}{\varepsilon^L - 1}}$. Thus, the firm's problem is

$$\max_{K_{jt},H_t} \; P_t^I A^b (K_t^\alpha H_t^{1-\alpha})^\nu + (1-\delta) P_t Q_t K_t - P_{t-1} Q_{t-1} \int_0^1 (1+i^L_{j,t-1}) K_{jt} dj - W_t H_t \;\text{ s.t. }\; K_t = \left(\int_0^1 K_{jt}^{\frac{\varepsilon^L - 1}{\varepsilon^L}} dj \right)^{\frac{\varepsilon^L}{\varepsilon^L - 1}}.$$

Conditional on the firm's total demand for capital K_t, the first-order condition implies that the firm's loan demand from bank j is

$$K_{jt} = \left(\frac{1+i^L_{j,t-1}}{1+i^L_{t-1}}\right)^{-\varepsilon^L} K_t,$$

where the aggregate loan rate is

$$1+i^L_{t-1} = \left(\int_0^1 (1+i^L_{j,t-1})^{1-\varepsilon^L} dj\right)^{\frac{1}{1-\varepsilon^L}}.$$

D.2 Household optimization problem

A household is permitted to deposit at any bank $j \in [0,1]$, and its total deposit holdings satisfy

$$D_t = \left(\int_0^1 D_{jt}^{\frac{\varepsilon^D-1}{\varepsilon^D}} dj\right)^{\frac{\varepsilon^D}{\varepsilon^D-1}},$$

where D_{jt} is the quantity of deposits held at bank j. The household then solves

$$\max_{C_t,H_t,B_t,D_{jt},M_t} \sum_{t=0}^{\infty} \beta^t \big(\log C_t + \zeta\Phi(D_t+M_t)\big)$$

$$\text{s.t. } C_t + B_t + \int_0^1 D_{jt} dj + M_t \le \frac{W_t}{P_t} H_t \frac{1+i_{t-1}}{1+\pi_t} B_{t-1} + \int_0^1 \frac{1+i^D_{j,t-1}}{1+\pi_t} D_{j,t-1} dj + \frac{M_{t-1}}{1+\pi_t} + \Pi_t + T_t,$$

$$D_t = \left(\int_0^1 D_{jt}^{\frac{\varepsilon^D-1}{\varepsilon^D}} dj\right)^{\frac{\varepsilon^D}{\varepsilon^D-1}}, B_t, D_t, M_t \ge 0.$$

If the household demands a positive quantity of deposits, its demand for deposits satisfies $D_{jt} = \left(\frac{1+i^D_{jt}}{1+i^D_t}\right)^{-\varepsilon^D} D_t$ for all j, except for possibly a set of measure zero.[13] The aggregate deposit rate $1+i^D_t$ is given by

$$1+i^D_t = \left(\int_0^1 (1+i^D_{jt})^{1-\varepsilon^D} dj\right)^{\frac{1}{1-\varepsilon^D}}.$$

The household always demands a positive quantity of deposits if i^D_t is positive. If i^D_t is negative, it does not demand deposits from any bank. Households are indifferent between deposits and cash when $i^D_t = 0$, but we assume the household breaks the tie by investing only in deposits.

We additionally make an assumption to rule out the possibility that individual banks set negative deposit rates in equilibrium, so that the aggregate deposit rate remains at zero even when the

[13]The household can arbitrarily change its demand for some measure-zero set of banks' deposits without changing the value it achieves in its objective function.

policy rate goes negative. Namely, we assume that when a measure-zero set of banks set negative deposit rates $i_{jt}^D < 0$, then the household invests nothing in those banks. However, if a positive measure set negative deposit rates, the household's deposit demand satisfies $D_{jt} = \left(\frac{1+i_{jt}^D}{1+i_t^D}\right)^{-\varepsilon^D} D_t$ for all j.

Therefore, the deposit demand faced by a bank takes the form

$$D_{jt} = \begin{cases} \left(\frac{1+i_{jt}^D}{1+i_t^D}\right)^{-\varepsilon^D} D_t & i_{jt}^D \geq 0 \quad \text{or} \quad \{j' : i_{j't}^D < 0\} \text{ has positive measure} \\ 0 & i_{jt}^D < 0 \quad \text{and} \quad \{j' : i_{j't}^D < 0\} \text{ has measure zero} \end{cases}. \tag{D.1}$$

D.3 Symmetric equilibrium

Then, the problem of bank j is

$$NII(N_{jt}, i_t) = \max_{i_{jt}^L, i_{jt}^D, B_{jt}^L} i_t B_{jt}^L + i_{jt}^L \left(\frac{1+i_{jt}^L}{1+i_t^L}\right)^{-\varepsilon^L} L_t - (i_{jt}^D - \mu^D) \left(\frac{1+i_{jt}^D}{1+i_t^D}\right)^{-\varepsilon^D} \mathbf{1}\{i_{jt}^D \geq 0\} D_t - \Psi^L(N_{jt}, L_{jt})$$

$$\text{s.t. } L_{jt} + B_{jt}^L = N_{jt} + D_{jt}.$$

As long as it is optimal for the bank to issue deposits and deposit demand is large enough, the first-order conditions imply

$$1 + i_{jt}^L = \frac{\varepsilon^L}{\varepsilon^L - 1}(1 + i_t + \frac{\partial \Psi^L}{\partial L_t}),$$

$$1 + i_{jt}^D = \begin{cases} \frac{\varepsilon^D}{\varepsilon^D - 1}(1 + i_t + \mu^D) & i_t \geq -\frac{1}{\varepsilon^D} - \mu^D \\ 0 & i_t < -\frac{1}{\varepsilon^D} - \mu_D \end{cases}.$$

The loan rate is set as a constant markup $\frac{\varepsilon^L}{\varepsilon^L - 1}$ over the bank's opportunity cost of lending, which is i (the payoff of instead investing in a bond) plus $\frac{\partial \Psi^L}{\partial L}$ (the marginal cost of taking additional leverage). Similarly, the deposit rate is a mark-down below the nominal rate i_t plus the additional marginal benefit μ^D of issuing deposits.

D.4 Bank shutdown

In this section, we describe the conditions under which banks may voluntarily decide to stop issuing deposits. If a bank does not issue deposits, it simply lends out its net worth. It must set an interest rate of

$$1 + \tilde{i}^L = \left(\frac{L}{N}\right)^{\frac{1}{\varepsilon^L}} \times \frac{\varepsilon^L}{\varepsilon^L - 1}(1 + i), \tag{D.2}$$

which is also its (nominal) return on net equity.

On the other hand, a bank that issues deposits sets a loan rate of

$$1 + i^{L*} = \frac{\varepsilon^L}{\varepsilon^L - 1}(1 + i + \frac{\partial \Psi^L}{\partial L}). \tag{D.3}$$

The return on equity of a bank that follows this strategy is

$$
\begin{aligned}
ROE^* &= \frac{1+i^{L*}}{1+\pi}\frac{L}{N} + \frac{1+i}{1+\pi}\frac{N+D-L}{N} - \frac{1-\mu^D}{1+\pi}\frac{D}{N} - \Psi^L \\
&= \frac{1+i}{1+\pi}\left(\frac{\varepsilon^L}{\varepsilon^L-1}\frac{L}{N} + \frac{N+D-L}{N}\right) - \frac{1-\mu^D}{1+\pi}\frac{D}{N} + \frac{\varepsilon^L}{\varepsilon^L-1}\frac{L}{N}\frac{\partial\Psi^L}{\partial L} - \frac{\Psi^L}{N} \\
&= \frac{1+i}{1+\pi}\left(1 + \frac{1}{\varepsilon^L-1}\frac{L}{N}\right) + \frac{i-\mu^D}{1+\pi}\frac{D}{N} + \frac{\varepsilon^L}{\varepsilon^L-1}\frac{L}{N}\frac{\partial\Psi^L}{\partial L} - \frac{\Psi^L}{N} \\
&= \frac{1}{1+\pi}\left(1 + \frac{1}{\varepsilon^L-1}\frac{L}{N}\right) + \frac{i}{1+\pi}\left(1 + \frac{1}{\varepsilon^L-1}\frac{L}{N} + \frac{D}{N}\right) + \frac{\mu^D}{1+\pi}\frac{D}{N} \\
&+ \frac{\varepsilon^L}{\varepsilon^L-1}\frac{L}{N}\frac{\partial\Psi^L}{\partial L} - \frac{\Psi^L}{N}
\end{aligned}
$$

If banks prefer not to shut down, it must be that $ROE^* \geq \frac{1+\tilde{i}^L}{1+\pi}$, or

$$
i \geq \hat{i} \equiv \frac{\left(\frac{L}{N}\right)^{\frac{1}{\varepsilon^L}}\frac{\varepsilon^L}{\varepsilon^L-1} - \frac{1}{\varepsilon^L-1}\frac{L}{N} - 1 - \mu^D\frac{D}{N} + (1+\pi)\left(\frac{\Psi^L}{N} - \frac{\varepsilon^L}{\varepsilon^L-1}\frac{L}{N}\frac{\partial\Psi^L}{\partial L}\right)}{1 + \frac{1}{\varepsilon^L-1}\frac{L}{N} + \frac{D}{N} - \left(\frac{L}{N}\right)^{\frac{1}{\varepsilon^L}}\frac{\varepsilon^L}{\varepsilon^L-1}}. \tag{D.4}
$$

In our benchmark calibration, we find $\hat{i} \approx -1.5\%$ in steady state, so this inequality is verified for the interest rate cuts we consider.

If (D.4) does not hold, however, the equilibrium conditions differ from those stated in the main analysis. In this regime, a fraction μ of banks will shut down deposit issuance. Those banks set their loan rates according to (D.2), issuing an aggregate quantity of loans $\tilde{L} = \mu N$. The remaining banks set their loan rates according to (D.3), issuing an aggregate quantity of loans L^*. Total lending L satisfies

$$
L = \left(\mu\tilde{L}^{\frac{\varepsilon^L-1}{\varepsilon^L}} + (1-\mu)L^{*\frac{\varepsilon^L-1}{\varepsilon^L}}\right)^{\frac{\varepsilon^L}{\varepsilon^L-1}}.
$$

The final equilibrium condition is that the returns on equity of both types of banks must be equal so that banks are indifferent between issuing deposits and not issuing, $ROE^* = 1 + \tilde{i}^L$.

E Micro-founding banks' net worth process

In this section, we outline how to micro-found the process followed by bank net worth in our calibrated model. Recall that in the model, banks pay out a fraction γ of net worth each period and receive an additional equity injection of $\hat{N}$.

As in the benchmark model, there is a continuum of households. We index households by $j \in [0,1]$. Each household owns the corresponding bank j. A household is comprised of two types of members: a fraction ω of "workers" and a complementary fraction $1-\omega$ of "bankers." In each period, each worker is given an equal share of the household's financial wealth. Workers then choose how much labor to supply and how much to save in bonds, deposits, and cash. At the end of the period, any resources not saved by workers are consumed by the household jointly, and its members enjoy utility $u(C_t, C_{t-1}, H_t)$, where H_t denotes aggregate hours supplied by workers.

The household's bankers enter a period with net worth N_{jt} and jointly solve (6). They choose a deposit rate, a loan rate, and bond holdings for bank j in order to maximize net interest income each period. Importantly, no worker in household j is matched with bank j, so the bankers borrow only from other households.

At the end of a period, a fraction γ of bankers return to the household and become workers. When a banker dies, she returns to the household with a fraction γ of the bank's net worth N_{jt}. In order to keep the fraction of bankers in the household constant, a fraction $\frac{1-\omega}{\omega}\gamma$ of workers become bankers in each period as well, bringing with them an amount of startup capital $\tilde{N}$.

We can now write down the net worth accumulation processes for the household and for the bank. The household's net worth N_t^H evolves according to

$$N_{t+1}^H = (1+i_t)B_t + (1+i_t^D)D_t + M_t + W_tH_t - P_tC_t + \Pi_t + T_t + \gamma N_{t+1} - \frac{1-\omega}{\omega}\gamma\tilde{N}.$$

(We drop the j subscripts because all households behave identically in equilibrium.) The last two terms reflect the dividends brought home by bankers and the net worth injected into the bank in each period, which is just the net bank dividend term Π_t^D in the budget constraint (2). The bank's net worth follows

$$\begin{aligned} N_{t+1} =& \frac{1+i_t}{1+\pi_{t+1}}Q_t^B B_t^L + \frac{1+i_t^L}{1+\pi_{t+1}}L_t - \frac{1+i_t^D}{1+\pi_{t+1}}D_t \\ & - \Psi^L(N_t, L_t) - \Psi^D(Q_t^B B_t^L, D_t) - \gamma N_{t+1} + \frac{1-\omega}{\omega}\gamma\tilde{N}, \end{aligned}$$

where the last two terms reflect net worth paid out by returning bankers and net worth brought in by entrant bankers. Note that the net worth $\frac{1-\omega}{\omega}\gamma\tilde{N}$ brought into the bank is independent of time (as well as household and bank net worth), so it can be written simply as $\hat{N}$.

For completeness, we state workers' problem, which is:

$$\max_{C_t, H_t, B_t, D_t, M_t} \; \Lambda_t C_t - \Lambda_t^H H_t + \Lambda_t^{\mathcal{L}}\big(D_t + M_t\big) + \beta\Lambda_{t+1}\Big(\frac{(1+i_t)B_t + (1+i_t^D)D_t + M_t}{1+\pi_{t+1}}\Big)$$

subject to

$$C_t + B_t + D_t + M_t \le \frac{W_t}{P_t}H_t + \frac{1+i_{t-1}}{1+\pi_t}B_{t-1} + \frac{1+i_{t-1}^D}{1+\pi_t}D_{t-1} + \frac{1}{1+\pi_t}M_{t-1} + \Pi_t + T_t + \gamma N_t - \hat{N},$$

where $\Lambda_t = u_1(C_t, C_{t-1}, H_t)$ is the household's marginal utility of consumption at t, $\Lambda_t^H = u_3(C_t, C_{t-1}, H_t)$ is the household's marginal disutility of labor, and $\Lambda_t^{\mathcal{L}} = \zeta\Phi'(\mathcal{L}_t)$ is the household's marginal utility of liquidity.

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