Online Appendix for
"Rank Uncertainty in Organizations"

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B. Proofs for Section 5

B.1. Proof of Proposition 1

For the present proof and those of later propositions, we define the following convenient notation for reasoning about the principal's problem.

Notation 1. For each $i, i', j, j' \in N$, let $D(i, i', j, j')$ be the matrix with its (i, j) and (i', j') entries taking value 1, its (i, j') and (i', j) entries taking value -1, and all other entries taking value zero.

Notation 2. Given a ranking matrix μ and for each $i \in N$, let $I_i(\mu)$ denote agent i's **incentive effect** defined as follows:

$$I_i(\mu) := \sum_{j=1}^{N} \mu_{ij} \left[P(j) - P(j-1) \right].$$

When not confusing, we will omit the dependence on μ and simply write I_i.

To begin our proof of Proposition 1, we fix an optimal ranking matrix μ^* with the minimum number of zero entries. As convex combinations of optimal ranking matrices are themselves optimal, this minimality property in fact implies that, for any optimal ranking matrix μ, every zero entry of μ^* corresponds to a zero entry of μ, so that μ^* in fact has the minimum set of zero entries. Toward constructing our order, we define $J^*(i) := \{j \in N : \ \mu^*_{ij} > 0\}$ for each $i \in N$.

We first establish, via a perturbation argument, the following structural claim that we will use several times.

Claim 1. *Suppose* $i_1, i_2, j_1, j_2, j_3 \in N$ *are such that* $j_1 < j_2 < j_3$*, that* $j_1, j_3 \in J^*(i_1)$*, and that* $j_2 \in J^*(i_2)$*. Then* $j_1, j_2, j_3 \in J^*(i_1) \cap J^*(i_2)$*.*

Proof. Given $\varepsilon, \varepsilon' > 0$, define the $N \times N$ matrix

$$\mu := \mu^* + \varepsilon D(i_1, i_2, j_2, j_1) + \varepsilon' D(i_1, i_2, j_2, j_3).$$

As $\mu^*_{i_1 j_1}, \mu^*_{i_1 j_3}, \mu^*_{i_2 j_2} > 0$, the matrix μ is doubly stochastic with strictly positive entries wherever μ^* has strictly positive entries, as long as $\max\{\varepsilon, \varepsilon'\}$ is small enough. Define now the ratio

$$\rho := \frac{[P(j_2) - P(j_2 - 1)] - [P(j_1) - P(j_1 - 1)]}{[P(j_3) - P(j_3 - 1)] - [P(j_2) - P(j_2 - 1)]},$$

which is strictly positive because P is strictly supermodular. Upon choosing $\varepsilon, \varepsilon'$ to further satisfy $\frac{\varepsilon'}{\varepsilon} = \rho$, direct computation shows that μ generates the exact same incentive effects $(I_i)_{i \in N}$ as μ^* does;[29] μ is therefore also optimal. That $\mu_{i_1 j_2}, \mu_{i_2 j_1}, \mu_{i_2 j_3} > 0$ then implies $\mu^*_{i_1 j_2}, \mu^*_{i_2 j_1}, \mu^*_{i_2 j_3} > 0$ as well by definition of μ^*. *Q.E.D.*

Next, using the above structural claim, we derive more detail on the set of nonzero entries over the next three claims. First, in Claim 2, we show that the set of nonzero entries of each row is an interval. Then, in Claim 3, we show that, if two rows have distinct sets of columns in which they are nonzero, then these two column sets can overlap at most once. Finally, in Claim 4, we strengthen the latter to show that any two rows must have nonzero entries in either the exact same set of columns or in disjoint sets of columns.

Claim 2. *Suppose* $i_1, j_1, j_2, j_3 \in N$ *are such that* $j_1 < j_2 < j_3$ *and* $j_1, j_3 \in J^*(i_1)$*. Then* $j_2 \in J^*(i_1)$*.*

[29] For example, for i_1, we obtain

$$I_{i_1}(\mu) = I_{i_1}(\mu^*) + \varepsilon \left\{ \begin{array}{c} [P(j_2) - P(j_2 - 1)] \\ -[P(j_1) - P(j_1 - 1)] \end{array} \right\} + \rho\varepsilon \left\{ \begin{array}{c} [P(j_2) - P(j_2 - 1)] \\ -[P(j_3) - P(j_3 - 1)] \end{array} \right\} = I_{i_1}(\mu^*).$$

Proof. Assume for a contradiction that the claim is false. So there is some $j_2 \in N$ such that $j_1 < j_2 < j_3$ and $\mu^*_{i_1 j_2} = 0$. As the j_2 column of μ^* sums to 1, there is some $i_2 \in N$ such that $\mu^*_{i_2 j_2} > 0$. But then Claim 1 implies $\mu^*_{i_1 j_2} > 0$, a contradiction. *Q.E.D.*

Claim 3. *Suppose $i_1, i_2 \in N$ have $J^*(i_1) \neq J^*(i_2)$. Then $|J^*(i_1) \cap J^*(i_2)| \leq 1$.*

Proof. Assume for a contradiction that $J^*(i_1) \neq J^*(i_2)$ and $|J^*(i_1) \cap J^*(i_2)| > 1$. Without loss, say $J^*(i_1) \nsubseteq J^*(i_2)$. Then let $j_1 := \min J^*(i_1)$ and $j_3 := \max J^*(i_1)$. That $|J^*(i_1) \cap J^*(i_2)| > 1$ implies that $j_3 \neq \min J^*(i_2)$ and $j_1 \neq \max J^*(i_2)$. There therefore exists $j_2 \in J^*(i_2)$ such that $j_1 < j_2 < j_3$. But Claim 1 then says $j_1, j_3 \in J^*(i_2)$, so that Claim 2 implies $J^*(i_1) \subseteq J^*(i_2)$, a contradiction. *Q.E.D.*

Claim 4. *Suppose $i_1, i_2 \in N$ have $J^*(i_1) \neq J^*(i_2)$. Then $J^*(i_1) \cap J^*(i_2) = \emptyset$.*

Proof. Assume for a contradiction that $j^* \in J^*(i_1) \cap J^*(i_2)$. By Claim 2 and Claim 3, $S_+ := \{i \in N : \max J^*(i) > j^*\}$ cannot contain both i_1 and i_2; nor can $S_- := \{i \in N : \min J^*(i) < j^*\}$ contain both i_1 and i_2. But $J^*(i_1)$ and $J^*(i_2)$ cannot both be $\{j^*\}$, so that at least one of S_+, S_- contains one of i_1, i_2. We will derive a contradiction from $i_1 \in S_-$, the other three cases being completely analogous.

In this case, $i_2 \notin S_-$, so that $\min J^*(i_2) = j^*$. Moreover, by Claim 3, every $i \in S_-$ has $|J^*(i) \cap J^*(i_2)| \leq 1$, so that (since both $J^*(i), J^*(i_2)$ are intervals by Claim 2) $\max J^*(i) \leq j^*$.

Finally, observe that

$$\begin{aligned}
|S_-| - (j^* - 1) &= \left(\sum_{i \in S_-} \sum_{j \in N} \mu^*_{ij} \right) - \left(\sum_{j=1}^{j^*-1} \sum_{i \in N} \mu^*_{ij} \right) \\
&= \left(\sum_{i \in S_-} \sum_{j=1}^{j^*} \mu^*_{ij} \right) - \left(\sum_{j=1}^{j^*-1} \sum_{i \in S_-} \mu^*_{ij} \right) \\
&= \sum_{i \in S_-} \mu^*_{ij^*} \in [\mu^*_{i_1 j^*},\ 1 - \mu^*_{i_2 j^*}],
\end{aligned}$$

where the second equality follows from the dropped entries of μ^* being zero. Hence, it follows that $|S_-| - (j^* - 1) \subseteq (0, 1)$, which contradicts $|S_-| - (j^* - 1)$ being an integer. *Q.E.D.*

Now, we can define a weak order $\succsim$ on N by saying $i \sim i'$ if and only if $J^*(i) = J^*(i')$, and $i \succ i'$ if and only if $\max J^*(i) < \min J^*(i')$. The relation is obviously transitive, and it is complete by Claim 2 and Claim 4. By construction of $\succsim$, under μ^* there is complete rank uncertainty over any $\sim$ equivalence class (because for any $i \sim j$ we have $\mu^*_{ij} > 0$), and i is ranked above i' in any optimal ranking matrix if $i \succ i'$.

All that remains is to check that $c_i < c_{i'}$ whenever $i \succ i'$. Assume otherwise for a contradiction: so $i \succ i'$ but $c_i \geq c_{i'}$. Observe that the incentive effects generated by μ^* satisfy $I_i < I_{i'}$. As $c_i \geq c_{i'}$, switching the i and i' rows from μ^* would weakly improve the principal's objective (given that $f_i(\mu)$ is a submodular function of I_i and c_i), preserving optimality. But the new ranking matrix would have a nonzero entry where μ^* does not, a contradiction.

B.2. Proof of Proposition 2

In what follows, let $\succsim$ be as given by Proposition 1. The proof of Proposition 2 proceeds in three steps. First, we provide two preliminary results that we will use in our arguments. Second, we explicitly characterize the form of the order $\succsim$. Finally, we specialize this characterization to understand when it is perfectly coarse or perfectly fine.

Step 1. We provide two preliminary results. First, in the next claim, we show that the ordering induced by $\left\{\frac{I_i}{\sqrt{c_i}}\right\}_{i\in N}$ at an optimal ranking matrix respects the order $\succsim$; this is an expression of the principal's first-order conditions.

Claim 5. *If $i \succsim i'$, then any optimal ranking matrix induces $\frac{I_i}{\sqrt{c_i}} \geq \frac{I_{i'}}{\sqrt{c_{i'}}}$.*

Proof. Let μ be an optimal ranking matrix, and take $i \succsim i'$ with $i \neq i'$. Given the unique-bonuses result in Theorem 2, the incentive effects $\{I_{\tilde{i}}\}_{\tilde{i}\in N}$ remain unchanged if we replace μ with any other optimal ranking matrix. Therefore, by Proposition 1, we may assume without loss that μ exhibits complete rank uncertainty over every $\sim$ equivalence class. A consequence is that $\mu_{ii}, \mu_{i'i'} > 0$, implying that $\mu + \varepsilon D(i, i', i', i)$ is a ranking matrix for small enough $\varepsilon > 0$. Optimality of μ then requires that the directional derivative of the principal's objective in direction $D(i, i', i', i)$ be nonnegative. By direct computation, this derivative is equal to

$$P(N)\left\{[P(i') - P(i'-1)] - [P(i) - P(i-1)]\right\}\left[\frac{c_{i'}}{I_{i'}(\mu)^2} - \frac{c_i}{I_i(\mu)^2}\right],$$

which (by strict supermodularity of P and because $i < i'$) has the same sign as $\frac{c_{i'}}{I_{i'}(\mu)^2} - \frac{c_i}{I_i(\mu)^2}$. Therefore, $\frac{c_{i'}}{I_{i'}(\mu)^2} \geq \frac{c_i}{I_i(\mu)^2}$, as desired. *Q.E.D.*

Second, we establish the following result comparing the incentive effects given by two ranking matrices.

Claim 6. *Take two ranking matrices μ', μ'' and $k_1, k_2, k_3 \in N$ with $k_1 \leq k_2 < k_3$. Suppose that:*

1. *$\forall i \in \{k_1, \ldots, k_3\}$, $j \notin \{k_1, \ldots, k_3\}$ we have $\mu''_{ij} = 0$;*
2. *$\forall i \in \{k_1, \ldots, k_2\}$, $j \notin \{k_1, \ldots, k_2\}$ we have $\mu'_{ij} = 0$;*
3. *$\exists i \in \{k_1, \ldots, k_2\}$, $j \in \{k_2, \ldots, k_3\}$ such that $\mu''_{ij} > 0$.*

Then, $\displaystyle\sum_{i=k_1}^{k_2} I_i(\mu') < \sum_{i=k_1}^{k_2} I_i(\mu'')$.

Proof. It is straightforward to construct a new ranking matrix $\tilde{\mu}$, also with property 2, such that $\tilde{\mu}_{ij} \geq \mu''_{ij}$ $\forall i, j \in \{k_1, \ldots, k_2\}$. Property 2, and the fact that both $\tilde{\mu}$ and μ' are doubly stochastic, implies $\displaystyle\sum_{i=k_1}^{k_2} I_i(\mu') = \sum_{i=k_1}^{k_2} I_i(\tilde{\mu})$. Notice that $\tilde{\mu}_i$ is weakly first-order-stochastically dominated by μ''_i for each $i \in \{k_1, \ldots, k_2\}$ by construction and property 1. Also notice that at least one such dominance relationship holds strictly because of property 3. Since $P(j) - P(j-1)$ is strictly increasing in $j \in N$, we have that $I_i(\tilde{\mu}) \leq I_i(\mu'')$ $\forall i \in \{k_1, \ldots, k_2\}$ with at least one strict inequality. This means that $\displaystyle\sum_{i=k_1}^{k_2} I_i(\tilde{\mu}) < \sum_{i=k_1}^{k_2} I_i(\mu'')$. *Q.E.D.*

Step 2. Having shown that the principal's first-order conditions take the simple form in Claim 5, we can convert this result into a complete characterization of the order $\succsim$. To achieve this characterization, Claim 7 and Claim 8 below derive concrete algebraic conditions on $\succsim$ that follow from these first-order conditions, and then Claim 9 shows that no two distinct orders can satisfy the same concrete conditions. We provide the complete characterization of $\succsim$ in Claim 10.

In what follows, the following function on sets of agents will be of use.

Notation 3. Given any nonempty set $S \subseteq N$, let

$$\varphi(S) := \frac{\sum_{j\in S}[P(j) - P(j-1)]}{\sum_{i\in S}\sqrt{c_i}}.$$

Remark 2. An important property φ satisfies, which is easy to establish given its "fractional sum" form, is a (strict) betweenness property. Specifically, any collection $\mathcal{S} \subseteq 2^N$ of pairwise disjoint, nonempty sets has

$$\min_{S\in\mathcal{S}} \varphi(S) \leq \varphi\left(\bigcup \mathcal{S}\right) \leq \max_{S\in\mathcal{S}} \varphi(S),$$

with both inequalities strict if $\{\varphi(S)\}_{S\in\mathcal{S}}$ are not all the same. We take this property for granted throughout the proof.

Claim 7. *Let S be some $\sim$ equivalence class. Any optimal ranking matrix μ has $I_i(\mu) = \varphi(S)\sqrt{c_i}$ for every $i \in S$.*

Proof. By Claim 5, there is some $\overline{I} \in \mathbb{R}$ such that $I_i(\mu) = \overline{I}\sqrt{c_i}$ for every $i \in S$. But, by Proposition 1, we know that $\mu_{ii'} = \mu_{i'i} = 0$ for every $i \in S$ and $i' \in N \setminus S$. Therefore,

$$\overline{I}\sum_{i\in S}\sqrt{c_i} = \sum_{i\in S} I_i(\mu) = \sum_{j\in S}[P(j) - P(j-1)] = \varphi(S)\sum_{i\in S}\sqrt{c_i},$$

implying $\overline{I} = \varphi(S)$. *Q.E.D.*

Claim 8. *Let S be some $\sim$ equivalence class, and $S' := [1, i'] \cap S$ for some $i' \in S$ with $i' < \max S$. Then $\varphi(S') < \varphi(S)$.*

Proof. Proposition 1 yields some optimal ranking matrix μ such that $\mu_{ij} > 0$ for every $i, j \in S$. Claim 7 says each $i \in S$ has $\varphi(S)\sqrt{c_i} = I_i(\mu)$, so that

$$\frac{\varphi(S')}{\varphi(S)} = \frac{\sum_{j\in S'}[P(j) - P(j-1)]}{\varphi(S)\sum_{i\in S'}\sqrt{c_i}} = \frac{\sum_{i\in S'} I_i(\delta)}{\sum_{i\in S'} I_i(\mu)},$$

where δ is the identity matrix. By Claim 6, this ratio is strictly below 1. *Q.E.D.*

Claim 9. *Suppose weak orders $\succsim_1, \succsim_2$ are such that, for both $k \in \{1, 2\}$:*

1. $1 \succsim_k \cdots \succsim_k N$*;*
2. *Every pair S, S' of $\sim_k$ equivalence classes with $S \ll S'$ have $\varphi(S) \geq \varphi(S')$;*
3. *Every $\sim_k$ equivalence class S and $i' \in S \setminus \{\max S\}$ have $\varphi(S \cap [1, i']) < \varphi(S)$.*

Then $\succsim_1$ and $\succsim_2$ are identical.

Proof. Assume for a contradiction that the claim fails. Given the first condition above, both $\succsim_1, \succsim_2$ are fully determined by their equivalence classes. So let $i_0 \in N$ be the lowest-labeled agent such that $S_1 := \{i \in N : \ i \sim_1 i_0\} \neq \{i \in N : i \sim_2 i_0\} =: S_2$. By construction, $i_0 = \min S_1 = \min S_2$, and the first condition implies that both S_1 and S_2 are intervals in N. Without loss, say $S_2 \not\subseteq S_1$, so that $\max S_2 > \max S_1$. Now, define $\mathcal{S}_1$ to be the set of all $\sim_1$ equivalence classes contained in $[i_0, \max S_2)$, and let $S_1' := \{i \in N : \ i \sim_1 \max S_2\}$.

Note that as S_2 is the disjoint union of $\mathcal{S}_1 \cup \{S_1' \cap S_2\}$, the betweenness property says

$$\begin{aligned} \varphi(S_2) &\leq \max\{\varphi(S) : \ S \in \mathcal{S}_1 \text{ or } S = S_1' \cap S_2\} \\ &\leq \max\{\varphi(S) : \ S \in \mathcal{S}_1 \text{ or } S = S_1'\} \\ &= \varphi(S_1). \end{aligned}$$

The second inequality follows from applying the third property to $\succsim_1$ because $S_1' \cap S_2$ is an initial segment of S_1'. Moreover, the equality follows from noting that $S \ll S_1'$ for every $S \in \mathcal{S}_1$ and applying the second property to $\succsim_1$. But applying the third property to $\succsim_2$ delivers $\varphi(S_1) < \varphi(S_2)$, a contradiction. *Q.E.D.*

Claim 10. *The weak order $\succsim$ is the unique transitive complete relation satisfying:*

1. $1 \succsim \cdots \succsim N$*;*
2. *Every pair S, S' of $\sim$ equivalence classes with $S \ll S'$ have $\varphi(S) \geq \varphi(S')$;*
3. *Every $\sim$ equivalence class S and $i' \in S \setminus \{\max S\}$ have $\varphi(S \cap [1, i']) < \varphi(S)$.*

Proof. By definition, $\succsim$ satisfies the first property. It satisfies the second property by Claim 5 and Claim 7. It satisfies the third property by Claim 8. But then, no other order can satisfy these three properties by Claim 9. *Q.E.D.*

Step 3. With the above claims in hand, the proposition is easy to establish. To see the second part of the proposition, observe that there is an optimal ranking matrix exhibiting complete rank uncertainty over the whole set N of agents if and only if $1 \sim \cdots \sim N$, which Claim 10 shows holds if and only if $\varphi(\{1, \ldots, n\}) < \varphi(N)$ for every $n \in \{1, \ldots, N-1\}$.

To see the first part of the proposition, observe that the identity (ranking) matrix $\delta = [\delta_{ij}]_{i,j \in N}$ is the unique optimal ranking matrix if and only if $1 \succ \cdots \succ N$, which Claim 10 shows holds if and only if $\varphi(\{1\}) \geq \cdots \geq \varphi(\{N\})$. All that remains, then, is to establish that δ is an optimal ranking matrix if and only if it is the uniquely optimal ranking matrix. But this result follows directly from the following claim (taking $\succsim'$ to satisfy $1 \succ' \cdots \succ' N$).

Claim 11. *Suppose $\succsim'$ is a weak order on N such that some optimal ranking matrix has i ranked higher than i' for every $i \succ' i'$. Then $\succsim$ is a (weak) refinement of $\succsim'$, i.e., $i \succ i'$ for any $i \succ' i'$. Therefore, $\succsim$ is the finest order on N with this property.*[30]

Proof. Let μ' be an optimal ranking matrix that has i ranked higher than i' for every $i \succ' i'$, and (appealing to Proposition 1) let μ be an optimal ranking matrix whose nonzero entries are exactly $\{(i,j) \in N^2 : \ i \sim j\}$. Fix $i^* \in N$ and let $S := \{i \in N : \ i \succsim' i^*\}$. It suffices to show that $i \succ i'$ for any $i \in S$ and $i' \in N \setminus S$. Assume otherwise, for a contradiction.

[30] That $\succsim$ as delivered by Proposition 1 is the finest order with this property implies it can be inferred directly from the set of nonzero entries of any optimal ranking matrix. This observation plays no role in the proof of the current proposition.

Observe that all nonzero $S \times N$ entries of μ' are in $S \times \{1, \ldots, |S|\}$, but that μ has at least one strictly positive entry in $S \times \{|S|+1, \ldots, N\}$. Claim 6 then implies $\sum_{i \in S} I_i(\mu') < \sum_{i \in S} I_i(\mu)$, contradicting the unique-bonuses result in Theorem 2. *Q.E.D.*

B.3. Proof of Proposition 3

In what follows, define $\varphi^* : N \to \mathbb{R}$ by letting $\varphi^*(i) := \varphi\left(\{i' \in N : \ i' \sim i\}\right)$, where $\succsim$ is as given by Proposition 1. By Claim 7, the optimal bonus paid to each agent $i \in N$ is exactly $b_i^* = \frac{\sqrt{c_i}}{\varphi^*(i)}$, and so his markup is exactly $\frac{b_i^*}{c_i} = \frac{1}{\varphi^*(i)\sqrt{c_i}}$. It therefore suffices to show that $\varphi^*(i)$ weakly decreases as any agent's marginal cost increases, and that $\varphi^*(i)\sqrt{c_i}$ weakly increases as c_i increases.

To see the above, first observe that any set $S \subseteq N$ which contains agent i has

$$\varphi(S) = \frac{1}{\sum_{i' \in S} \sqrt{c_{i'}}} \sum_{j \in S} [P(j) - P(j-1)],$$

which weakly decreases with the vector of marginal costs, and

$$\varphi(S)\sqrt{c_i} = \frac{\sqrt{c_i}}{\sum_{i' \in S} \sqrt{c_{i'}}} \sum_{j \in S} [P(j) - P(j-1)],$$

which weakly increases with agent i's marginal cost.

Therefore, the proposition will follow directly if $\varphi^*(i)$ is an increasing function of the vector $(\varphi(S))_{S \subseteq N : i \in S}$. But this fact follows directly from the following claim.

Claim 12. *Each agent* $i \in N$ *has* $\varphi^*(i) = \max_{i_1 \in \{i, \ldots, N\}} \min_{i_0 \in \{1, \ldots, i\}} \varphi\left(\{i_0, \ldots, i_1\}\right)$.

Proof. As $\max_{i_1 \in \{i, \ldots, N\}} \min_{i_0 \in \{1, \ldots, i\}} \varphi\left(\{i_0, \ldots, i_1\}\right)$ is always weakly below $\min_{i_0 \in \{1, \ldots, i\}} \max_{i_1 \in \{i, \ldots, N\}} \varphi\left(\{i_0, \ldots, i_1\}\right)$, it suffices to show that

$$\min_{i_0 \in \{1, \ldots, i\}} \max_{i_1 \in \{i, \ldots, N\}} \varphi\left(\{i_0, \ldots, i_1\}\right) \leq \varphi^*(i) \leq \max_{i_1 \in \{i, \ldots, N\}} \min_{i_0 \in \{1, \ldots, i\}} \varphi\left(\{i_0, \ldots, i_1\}\right).$$

Let us establish that $\min_{i_0 \in \{1, \ldots, i\}} \max_{i_1 \in \{i, \ldots, N\}} \varphi\left(\{i_0, \ldots, i_1\}\right) \leq \varphi^*(i)$, the other inequality following by a symmetric argument.[31] Toward establishing the inequality, let $i_0^* := \min\{i_0 \in N :\ i_0 \sim i\}$ and $i_1^* := \max\{i_1 \in N :\ i_1 \sim i\}$, and take an arbitrary $i_1 \in \{i, \ldots, N\}$. We aim to show that $\varphi^*(i) \geq \varphi\left(\{i_0^*, \ldots, i_1\}\right)$, i.e., that $\varphi\left(\{i_0^*, \ldots, i_1^*\}\right) \geq \varphi\left(\{i_0^*, \ldots, i_1\}\right)$.

To accomplish this, we provide an alternative characterization of $\succsim$. Let $\ell_0 := 0$ and, working recursively for $k \in N$, define

$$\ell_k := \begin{cases} N & : \ \ell_{k-1} = N, \\ \min \arg\max_{i' \in \{\ell_{k-1}+1, \ldots, N\}} \varphi\left(\{\ell_{k-1}+1, \ldots, i'\}\right) & : \ \ell_{k-1} < N. \end{cases}$$

Letting $S^* := \{\ell_k\}_{k \in N}$, we can then define the order $\succsim'$ on N by letting $i' \succsim' i''$ if and only if $\min\left(\{i', \ldots, N\} \cap S^*\right) \leq \min\left(\{i'', \ldots, N\} \cap S^*\right)$. It is easy to see that this order satisfies the three properties listed in Claim 10: the first and third are immediate, and the second follows from applying betweenness of φ to the union of any two adjacent $\sim'$ equivalence classes. Claim 10 then implies that $\succsim'$ is exactly $\succsim$. But then, $i_0^* - 1 = \ell_{k-1}$ for some $k \in N$. It follows by construction that $i_1^* = \ell_k$ maximizes $\varphi(\{i_0^*, \ldots, i'\})$ over $i' \in \{i_0^*, \ldots, N\}$, delivering the desired inequality. *Q.E.D.*

[31] To see that symmetry obtains, note that the only conditions on which we base our arguments below—betweenness of φ, and the conditions established in Claim 10—would apply directly if we were to replace $\succsim$ with $\precsim$ and φ with $-\varphi$. In particular, by strict betweenness, the third condition in Claim 10 is equivalent to requiring that every $\sim$ equivalence class S and $i' \in S \setminus \{\min S\}$ have $\varphi(S \cap [i', N]) > \varphi(S)$.