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APPENDIX

"Effective Demand Failures and the Limits of Monetary Stabilization Policy"

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Here we present additional details of the arguments in the main text, and proofs of the stated lemmas and propositions.

A Welfare Analysis: Preliminary Results

This section provides proofs of the main results in section I.

A.1 Proof of Lemma 1

As explained in the main text, in the case of any rotationally-invariant allocation of resources, if we let $U^j(t;\xi,\phi)$ be the flow utility (2) in the case of disturbances (ξ,ϕ), we must have

$$U^j(t;\xi,R\phi) \;=\; U^{j-1}(t;\xi,\phi) \tag{A.1}$$

for each sector j. Then consider the contribution to the ex-ante expected value of (1) from the terms corresponding to the different possible outcomes in a particular rotation family (6). Because the N different outcomes must each have the same ex-ante probability, the contribution from these terms must be proportional to their equally-weighted average,

$$\bar{U}^j \;\equiv\; \frac{1}{N}\sum_{h=0}^{N-1}\sum_{t=0}^{\infty}\beta^t U^j(t;\xi,R^h\phi).$$

But it follows from (A.1) that

$$U^j(t;\xi,R^h\phi) \;=\; U^{j-1}(t;\xi,R^{h-1}\phi) \;=\; U^{j-2}(t;\xi,R^{h-2}\phi) \;=\; U^{j-h}(t;\xi,\phi)$$

for any integer h. Hence we can alternatively write

$$\bar{U}^j \;=\; \frac{1}{N}\sum_{h=0}^{N-1}\sum_{t=0}^{\infty}\beta^t U^{j-h}(t;\xi,\phi) \;=\; \frac{1}{N}\sum_{t=0}^{\infty}\beta^t\sum_{i=1}^{N}U^i(t;\xi,\phi). \tag{A.2}$$

The final expression on the right in (A.2) is independent of the sector j for which we compute ex-ante expected utility. Hence units in all sectors agree about the ex-ante ranking of alternative feasible rotationally-invariant allocations of resources, as stated in the lemma. The welfare criterion stated in the lemma is just N times the final expression on the right in (A.2).

A.2 Proof of Lemma 2

For given disturbance sequences (ξ, ϕ), we wish to find the consumption allocation $\{c_k^j(t)\}$ and production plan $\{y_k(t)\}$ for all $t \geq 0$ that maximizes (7), subject to the constraints that

$$\sum_j c_k^j(t) \ = \ y_k(t) \tag{A.3}$$

for each sector k at each date t. Note that the welfare measure (7) consists of a sum of separate terms for each good k at each date t. Since the constraints (A.3) each also involve only the production and consumption of a single good k at a single date t, we can separate the optimization problem into a set of independent problems, one for each good k and each date t.

If k is a good for which $\phi_k(t) = 0$, then the static optimization is simply the choice of $y_k(t)$ to minimize the disutility of supply $v(y_k(t); \xi_t)$. The solution is obviously $y_k(t) = 0$, which then implies that the only feasible consumption allocation involves $c_k^j(t) = 0$ for all j as well. (Thus in the example of a "pandemic shock" discussed in the main text, where $\phi_1(0) = 0$, it is optimal for there to be no production or consumption of good 1 in period 0.)

If instead $\phi_k(t) > 0$, the static optimization problem requires that we choose $y_k(t)$ and the $\{c_k^j(t)\}$ for $j = 1, \ldots, N$ to maximize

$$\sum_{h \in H} \phi_k(t) \alpha_h u(c_k^{k-h}(t)/(\alpha_h \phi_k(t)); \, \xi_t) \ - \ v(y_k(t); \, \xi_t), \tag{A.4}$$

subject to the constraint (A.3). Substituting the left-hand side of (A.3) for $y_k(t)$ in (A.4), we obtain an objective that is purely a function of the consumption allocation. We further note that this function is monotonically decreasing in $c_k^{k-h}(t)$, for any $h \notin H$ (if such sectors exist). Hence the optimum must involve $c_k^{k-h}(t) = 0$ for any $h \notin H$. With this substitution, we are left with an objective that is a function of the quantities $\{c_k^{k-h}(t)\}$ for $h \in H$,

$$\sum_{h \in H} \phi_k(t) \alpha_h u(c_k^{k-h}(t)/(\alpha_h \phi_k(t)); \, \xi_t) \ - \ v(\sum_{h \in H} c_k^{k-h}(t); \, \xi_t).$$

Because this last objective is a strictly concave function of its arguments, it has a unique optimum characterized by the first-order conditions. Moreover, the Inada conditions on the functions u, v imply that there cannot be a corner solution; hence we need consider only the first-order conditions for an interior maximum. These require that $u'(c^{k-h}(t)/\alpha_h \phi_k(t)); \, \xi_t)$ be the same quantity for each $h \in H$. Since $u'(c; \, \xi_t)$ is a monotonically decreasing function of c, this in turn implies that $c^{k-h}(t)/\alpha_h \phi_k(t)$ must be the same for each $h \in H$. This in turn is only consistent with (A.3) if $c_k^{k-h}(t) = \alpha_{k-h} y_k(t)$ for each $h \in H$. Since the same expression holds for $h \notin H$ as well (where it simply states that $c_k^{k-h}(t) = 0$), the optimal consumption allocation must satisfy (9) for all j, as stated in the lemma.

Finally, using this result, we can rewrite the objective (A.4) as a function of $y_k(t)$ alone, obtaining

$$\phi_k(t) u(y_k(t)/\phi_k(t); \, \xi_t) \ - \ v(y_k(t); \, \xi_t). \tag{A.5}$$

The optimal output level is the $y_k(t)$ that maximizes (A.5). Since the objective involves only $\phi_k(t)$ and the disturbances ξ_t, the solution is of the form $y_k(t) = y^*(\phi_k(t); \, \xi_t)$, where the function $y^*(\phi;\xi)$ is the same for any sector k. Moreover (A.5) is again a strictly concave function, and the Inada conditions imply that we must have an interior optimum. There is thus a unique solution $y^*(\phi;\xi)$, implicitly defined by the first-order condition (8) stated in the lemma.

This establishes Lemma 2. In addition, differentiation of (8) implies that

$$\frac{\partial y^*}{\partial \phi} \;=\; \frac{y^*}{\phi}\frac{-u''/\phi}{v''-u''/\phi} \;>\; 0,$$

so that y^* is a monotonically increasing function of ϕ, as stated in the main text. In the limit as $\phi \to 0$, the first term in (A.5) approaches zero regardless of the value of $y_k(t)$; hence the objective approaches a monotonically decreasing function of $y_k(t)$ for all $y_k(t) > 0$, which is maximized when $y_k(t) = 0$. Hence $y^* \to 0$ as $\phi \to 0$, for any disturbances ξ.

A.3 Proof of Proposition 1

The first-best optimal allocation of resources in this case has already been established in Lemma 2:

$$y_k(t) = y_t^*, \qquad c_k^j(t) = \alpha_{k-j}\cdot y_k(t) \tag{A.6}$$

for all j, k, and all $t \geq 0$. We wish to prove that this resource allocation, together with prices

$$p_k(t) = P^*(t), \qquad 1+i(t) \;=\; (1+r_t^*)\frac{P^*(t+1)}{P^*(t)} \tag{A.7}$$

for all k and all $t \geq 0$, represents an equilibrium.

We first observe the prices and quantities specified in (A.6)–(A.7) imply that sector j's end-of-period asset position each period will satisfy $b^j(t) = a^j(t)$, as a consequence of (11). Substitution of this conclusion into (13) implies that

$$a^j(t+1) \;=\; (1+i(t))a^j(t) \;-\; \tau(t+1)$$

for each $t \geq 0$. Then under the hypothesis that $a^j(0) = a(0)/N$ and that $\tau(t+1)$ each period is consistent with the fiscal authority's target path for the public debt, the assumed behavior implies that $a^j(t) = a(t)/N$ for all $t \geq 0$. This in turn implies that $b^j(t) = a(t)/N > 0$ each period, so that the hypothesized spending plan is consistent with the borrowing constraint (12) each period. Hence the hypothesized spending plan represents a feasible plan for units in sector j. We next show that it is their optimal plan, i.e., that it is the feasible plan with the highest value for the objective (1).

Conditions (11) and (13) imply a law of motion for the nominal asset position of the form

$$\frac{a^j(t+1)}{P(t+1)} \;=\; (1+i(t))\frac{P(t)}{P(t+1)}\left[\frac{a^j(t)}{P(t)} + \frac{p_j(t)y_j(t) - \sum_k p_k(t)c_k^j(t)}{P(t)}\right] \;-\; \frac{\tau(t+1)}{P(t+1)}$$

for each $t \geq 0$. The prices and interest rates assumed in (A.7) imply that in the conjectured equilibrium, this can be written in the form

$$\beta\rho_{t+1}\left[\frac{a^j(t+1)}{P^*(t+1)} + \frac{\tau(t+1)}{P^*(t+1)}\right] \;=\; \rho_t\left[\frac{a^j(t)}{P^*(t)} + y_j(t) - \sum_k c_k^j(t)\right],$$

where

$$\rho_t \equiv u'(y_t^*; \xi_t), \tag{A.8}$$

using the definition of the natural rate of interest in (17). Multiplying both sides of this equation by β^t and summing for values of t from 0 to $T-1$ yields the condition

$$\beta^T \rho_T \frac{a^j(T)}{P^*(T)} + \sum_{t=1}^{T} \beta^t \rho_t \frac{\tau(t)}{P^*(t)} = \rho_0 \frac{a^j(0)}{\bar{p}} + \sum_{t=0}^{T-1} \beta^t \rho_t [y_j(t) - \sum_k c_k^j(t)], \tag{A.9}$$

which any feasible spending plan must satisfy for any $T \geq 1$.

We further note that because $\underline{b}^j(T-1) = 0$, (12) and (13) imply that $a^j(T) \geq -\tau(T)/N$ under any feasible plan. In addition, (15) together with (19) implies that under any policy of the kind hypothesized, we must have

$$\lim_{T\to\infty} \beta^T \rho_T \frac{\tau(T)}{P^*(T)} = 0, \tag{A.10}$$

so that any feasible plan must satisfy

$$\lim_{T\to\infty} \beta^T \rho_T \frac{a^j(T)}{P^*(T)} \geq 0. \tag{A.11}$$

Together with the fact that (A.9) must hold for arbitrary T, this implies that any feasible spending plan must satisfy the integrated intertemporal budget constraint

$$\sum_{t=0}^{\infty} \beta^t \rho_t \sum_k c_k^j(t) \leq \rho_0 \frac{a^j(0)}{\bar{p}} + \sum_{t=0}^{\infty} \beta^t \rho_t y_j(t) - \sum_{t=1}^{\infty} \beta^t \rho_t \frac{\tau(t)}{P^*(t)}. \tag{A.12}$$

Moreover, under the hypothesized spending plan, $a^j(t) = a(t)/N$ each period; (19) then implies that under this plan, condition (A.11) holds with equality, and hence (A.12) is satisfied with equality.

We next show that the hypothesized spending plan is the optimal one for a unit in sector j, among all plans consistent with the constraint (A.12). This is a problem of maximizing a concave objective (1) subject to a single linear inequality constraint. Moreover, the fact that goods $k \notin K_j(t)$ have positive prices, but have no effect on the objective (1) makes it obvious that the optimal solution requires that $c_k^j(t) = 0$ for any $k \notin K_j(t)$, as is true of the hypothesized spending plan (A.6). It then remains only to show that the hypothesized plan is optimal among all those satisfying the constraint that $c_k^j(t) = 0$ for all $k \notin K_j(t)$.

We then seek to maximize a function that is increasing in each of its arguments $c_k^j(t)$ (for $k \in K_j(t)$), and strictly concave. The fact that the objective is increasing requires that the optimal plan must satisfy the constraint (A.12) with equality, but this is true of the hypothesized plan for each sector j, as just shown. In addition, the strict concavity of the objective implies that there must be a unique optimum, characterized by the first-order conditions. The Inada conditions on the utility function $u(c;\xi)$ further imply that the optimal plan must involve positive consumption of each of the goods $k \in K_j(t)$ in each period $t \geq 0$. It follows that a consumption plan $\{c_k^j(t)\}$ is optimal if and only if, in addition to satisfying the budget constraint (A.12) with equality, the marginal rate of substitution between any two goods is equal to their relative price.

The conjectured prices and quantities (A.6)–(A.7) imply that for any two goods $k, k' \in K_j(t)$ in some period t, we have

$$\frac{u'(c_k^j(t)/\alpha_{k-j};\, \xi_t)}{u'(c_{k'}^j(t)/\alpha_{k'-j});\, \xi_t)} \;=\; \frac{\rho_t}{\rho_t} \;=\; 1 \;=\; \frac{P^*(t)}{P^*(t)} \;=\; \frac{p_k(t)}{p_{k'}(t)},$$

so that the intra-temporal optimality condition is satisfied. The marginal utility of an additional unit of expenditure in period t is then equal to $u'(y_t^*; \xi_t)/P^*(t)$, regardless of which good $k \in K_j(t)$ it is spent on. Furthermore, the conjectured prices and quantities (A.6)–(A.7) imply that for any two successive periods $t, t+1$,

$$\beta \frac{\rho_{t+1}}{\rho_t} \;=\; \frac{1}{1+r_t^*} \;=\; \frac{1}{1+i(t)} \frac{P(t+1)}{P(t)},$$

so that the inter-temporal optimality condition is satisfied as well.

Thus all necessary and sufficient conditions are satisfied to establish that given the conjectured prices, the hypothesized spending plan is optimal among all plans consistent with the budget (A.12). Of course, this last constraint is a weaker condition than the full set of requirements for a spending plan to be feasible (in particular, it neglects the borrowing constraints), so the set of feasible plans is a proper subset of the set of plans consistent with (A.12). However, we have already verified that the hypothesized spending plan satisfies all of the conditions for feasibility (in particular, it also satisfies the borrowing constraint (12) each period); hence optimality among all plans in the larger set is a sufficient condition to establish optimality among those plans in the smaller set of feasible plans. We have therefore established that the hypothesized spending plan for each sector j is optimal, given the conjectured prices.

The conjectured paths $\{y_k(t)\}$ for the production levels also satisfy (14) for each sector k at each date $t \geq 0$. We must further show that the conjectured goods prices for dates $t \geq 1$ are market-clearing prices. This follows from (14), if for each sector k, $y_k(t)$ is the quantity that units in sector k wish to supply, taking the price at which they can sell as given. Since we have already shown that the marginal utility of additional nominal income in period t is $u'(y_t^*; \xi_t)/P^*(t)$ in any sector, the first-order condition for optimal supply in period t is satisfied if and only if

$$\frac{v'(y_k(t);\, \xi_t)}{p_k(t)} \;=\; \frac{u'(y_t^*; \xi_t)}{P^*(t)}. \tag{A.13}$$

But for any sector k, the conjectured values for $y_k(t)$ and $p_k(t)$ satisfy (A.13), because of (10).

We must also show that the conjectured path of interest rates $\{i(t)\}$ clears the market for liquid assets in each period $t \geq 0$. But we have already shown above that the hypothesized plan for units in any sector j implies that $b^j(t) = a^j(t) = a(t)/N$ in each period, from which it follows that $\sum_j b^j(t) = a(t)$, and the asset market clears.

It thus remains only to verify that the conjectured paths are consistent with the government policies specified in the proposition. We have already discussed the consistency of these paths with the specified target path for the public debt and the specified borrowing limits. The conjectured prices (A.7) imply that in any period $t \geq 0$, the right-hand side of (16) is equal to

$$\log(1 + r^*) \; + \; \pi^*(t+1) \; \geq \; 0,$$

where the sign is guaranteed by (18). Hence the specified monetary policy rule requires that $\log(1 + i(t))$ equal the above expression, or alternatively, that

$$1 + i(t) \;=\; (1 + r_t^*)\frac{P^*(t+1)}{P^*(t)}.$$

But this is exactly the path of interest rates specified in (A.7). Hence the prices and quantities specified in (A.6)–(A.7) constitute an equilibrium, under the policies specified in the proposition.

Note that the proposition asserts only that an equilibrium of this kind exists under these policies, and does not address the question whether this is the only possible equilibrium consistent with the policies. We could go further, and establish local determinacy of the proposed equilibrium, using the methods discussed in Woodford (2003) for the case of a single-sector model. We do not pursue such issues here, noting only that the issues connected to uniqueness of equilibrium in this model are similar to those that arise in the single-sector model.

A.4 Proof of Proposition 2

In this case, equilibrium prices are the same as in the one described in Proposition 1, and equilibrium interest rates are the same, except for $i(0)$. Equilibrium quantities are the same for all $t \geq 1$. In period $t = 0$, quantities are instead given by

$$y_k(0) \;=\; y(0), \qquad c_k^j(0) \;=\; \alpha_{k-j}\, y(0), \tag{A.14}$$

where $y(0)$ is the quantity implicitly defined by

$$u'(y(0);\, \xi_0) \;=\; \beta(1 + i(0))\frac{\bar{p}}{P^*(1)}u'(y_1^*;\, \xi_1), \tag{A.15}$$

given the interest rate $i(0)$.

The proof that these prices and quantities represent a perfect foresight equilibrium under the assumed policy follows the same lines as in the proof of Proposition 1. First, we show that the proposed plan for each unit is feasible, given the assumed prices. The plans (A.14) imply that the circular flow of payments is again perfectly balanced in period zero, so that $b^j(0) = a^j(0) = a(0)/N$ for each j, just as in the case considered in Proposition 1. This implies that the borrowing constraint (12) is satisfied in period zero. And because $b^j(0)$ is the same for all j, one must have $a^j(1) = a(1)/N$ for each j, which are the same initial asset positions at the beginning of period 1 as in Proposition 1. It then follows that the proposed plan satisfies the feasibility constraints in all periods $t \geq 1$, just as in the proof of Proposition 1.

Next, we show that a feasible plan for each unit must satisfy an integrated intertemporal budget constraint of the form (A.12), where ρ_t continues to be defined by (A.8) for all $t \geq 1$, but in period $t = 0$ we instead define

$$\rho_0 \;=\; \beta(1 + i(0))\frac{\bar{p}}{P^*(1)}u'(y_1^*;\, \xi_1),$$

which is only equal to the value defined in (A.8) if the interest rate is chosen so as to ensure that $y_0 = y_0^*$. With this modification, the demonstration that a feasible plan satisfies the intertemporal budget constraint proceeds as in the proof of Proposition 1. Moreover, we can again show that the hypothesized plan for each unit satisfies this constraint with equality.

After this, in order to show that the proposed plan for each unit is the optimal plan among those consistent with the budget constraint (A.12), we need only to show that the marginal rates of substitution are all equal to the corresponding relative prices. In the case of marginal rates of substitution between goods in periods $t \geq 1$, all quantities and prices are the same as in Proposition 1, so that the equality of marginal rates of substitution and relative prices has already been shown. Condition (A.14) implies that the marginal rate of substitution between any two goods that units in sector j consume in period zero is equal to their relative price (which is 1); and condition (A.15) implies that the marginal rate of substitution between real expenditure in period zero and real expenditure in period one is equal to the real interest rate between those two periods. Hence all of the necessary and sufficient conditions for optimality are satisfied. Then the fact that the plan is optimal among those consistent with the weaker constraint (A.12), while it also satisfies all of the additional conditions required for feasibility, together with the fact that all feasible plans must satisfy (A.12), implies that the proposed plan for each unit is optimal among all feasible plans.

Next, we have already shown in the proof of Proposition 1 that the conjectured paths $\{y_k(t)\}$ for the production levels also satisfy (14) for each sector k at each date $t \geq 1$. Condition (A.14) implies that this is true in period $t = 0$ as well. We have also already shown in the proof of Proposition 1 that the prices in all periods $t \geq 1$ are market-clearing prices. And once again, the hypothesized plan for units in any sector j implies that $b^j(t) = a^j(t) = a(t)/N$ in each period, from which it follows that $\sum_j b^j(t) = a(t)$, and the asset market clears in every period $t \geq 0$.

Finally, we have already shown in the proof of Proposition 1 that the conjectured prices and quantities in all periods $t \geq 1$ are consistent with the specification of monetary and fiscal policy in those periods. The prices and quantities assumed here for period $t = 0$ are also consistent with our alternative specification of policy in period $t = 0$. Hence all of the conditions for a perfect foresight equilibrium, under the assumed policy, have been shown to be satisfied.

In this equilibrium, the level of output y_0 is determined by (A.15). Since all quantities in this formula are taken as given, except the values of $y(0)$ and $i(0)$, the equation establishes a structural relationship between these two quantities. The left-hand side of (A.15) is a decreasing function of y_0, while the right-hand side is an increasing function of $i(0)$; it follows that $y(0)$ is a decreasing function of $i(0)$. When $i(0)$ takes the value specified by (16), the equation implies that $y_0 = y_0^*$. Hence y_0 is less than or greater than y_0^* according to whether $i(0)$ is greater or less than the value specified by (16), as asserted in the proposition.

B Equilibrium with Asymmetric Disturbances

Here we present proofs of the main results in section II of the main text.

B.1 Proof of Lemma 3

The proof that the prices and quantities described in the proposition constitute a flexible-price perfect foresight equilibrium for all periods $t \geq 1$, taking as given the real financial wealth of each sector at the beginning of period 1, follows the same kind of reasoning as in the proof of Proposition 1.

First we verify that given the conjectured prices, the quantities described in the proposition represent a feasible plan for each sector j. The vector $\tilde{\boldsymbol{a}}(1)$, together with the target $a(1)$ for the public debt after taxes are collected at the beginning of period 1, implies that

$$a^j(1) \;=\; \frac{a(1)}{N} \;+\; [\tilde{a}^j(1) - (1/N)\sum_\ell \tilde{a}^\ell(1)]P^*(1) \;=\; \frac{a(1)}{N} \;+\; \frac{f^j}{1-\beta}P^*(1).$$

Then the fact that the stationary quantities $\{c_k^j\}$ and y_j satisfy (23) implies that (11) is satisfied each period, if we further specify that

$$a^j(t) \;=\; \frac{a(t)}{N} \;+\; \frac{f^j}{1-\beta}P^*(t), \qquad b^j(t) \;=\; \frac{a(t)}{N} \;+\; \frac{\beta}{1-\beta}f^j P^*(t) \tag{B.1}$$

for all $t \geq 1$. This path for $\{b^j(t)\}$ must also satisfy the borrowing constraint (12) each period, if the borrowing limit satisfies (22). Hence the hypothesized plan for sector j is feasible, given the conjectured prices and the specified policy.

Next we show that any feasible plan for periods $t \geq 1$ must satisfy an integrated intertemporal budget constraint. We can proceed as in the proof of Proposition 1 to show that under any feasible plan for a unit in sector j, we must have

$$\beta^{T-1}\frac{a^j(T)}{P^*(T)} \;+\; \sum_{t=1}^{T}\beta^{t-1}\frac{\tau(t)}{NP^*(t)} \;=\; \tilde{a}^j(1) \;+\; \sum_{t=1}^{T-1}\beta^{t-1}[q_j^* y_j(t) - \sum_k q_k^* c_k^j(t)], \tag{B.2}$$

for any $T \geq 2$. (The factors $\{\rho_t\}$ are now omitted, because the absence of aggregate disturbances implies that now $\rho_t = \rho_1$ for all $t \geq 1$; but we must now take account of intra-period relative prices, assumed to equal 1 in (A.9).)

The argument after this step is slightly more complicated than in the proof of Proposition 1, because we no longer assume that $\underline{b}^j(t) = 0$ each period. However, (12) and (13) in the more general case imply that

$$\frac{a^j(T)}{P^*(T)} \;\geq\; \frac{(1+i(T-1))\underline{b}^j(T-1)}{P^*(T)} \;-\; \frac{\tau(T)}{NP^*(T)} \;=\; \beta^{-1}\frac{\underline{b}^j(T-1)}{P^*(T-1)}.$$

Then the assumption (20) together with (A.10) guarantees that any feasible plan must satisfy (A.11), even under our more general assumption about the borrowing limits.[47] This together

[47]Because of this, the result in Proposition 1 could easily be generalized to allow the more flexible kind of borrowing limits assumed here.

with (B.2) then implies that any feasible plan must satisfy the integrated intertemporal budget constraint

$$\sum_{t=1}^{\infty} \beta^{t-1} \sum_k q_k^* c_k^j(t) \;\leq\; \tilde{a}^j(1) + \sum_{t=1}^{\infty} \beta^{t-1} \left[q_j^* y_j(t) - \frac{\tau(t)}{NP^*(t)} \right]. \tag{B.3}$$

In addition, (15) and (19) imply that

$$\sum_{t=1}^{\infty} \beta^{t-1} \frac{\tau(t)}{P^*(t)} \;=\; \frac{(1+i(0))a(0)}{P^*(1)} \;=\; \sum_{\ell=1}^{N} \tilde{a}^{\ell}(1).$$

Substituting this into the right-hand side of (B.3) and using the definition (24), the intertemporal budget constraint can alternatively be written

$$\sum_{t=1}^{\infty} \beta^{t-1} \sum_k q_k^* c_k^j(t) \;\leq\; \sum_{t=1}^{\infty} \beta^{t-1} q_j^* y_j(t) + \frac{f^j}{1-\beta}. \tag{B.4}$$

Thus under the assumptions of the proposition, any feasible plan for units in sector j must satisfy (B.4). Moreover, it follows from (23) that the hypothesized stationary plan satisfies (B.4) with equality.

Next we show that the hypothesized plan for units in sector j maximizes the terms in (1) for periods $t \geq 1$, among all plans consistent with (B.4). We have already shown that the hypothesized plan satisfies (B.4) with equality, so it remains only to show that all marginal rates of substitution are equal to the corresponding relative prices. The fact that the stationary allocation is a competitive equilibrium of the static model requires that

$$\frac{u'(c_{j+h}^j/\alpha_h; \bar{\xi})}{q_{j+h}^*} \;=\; \frac{v'(y_j; \bar{\xi})}{q_j^*} \tag{B.5}$$

for each $h \in H$ (i.e., for each of the goods that units in sector j consume in periods $t \geq 1$). This implies that all intra-temporal marginal rates of substitution are equal to the corresponding relative prices. We also note that the marginal utility of additional expenditure on the composite good in any period, for units in sector j, is given by (any of) the expressions in (B.5). Thus under the hypothesized plan, this marginal utility is constant over time, and the marginal rate of substitution between expenditure on the composite good in periods t and $t+1$ (for any $t \geq 1$) is equal to β^{-1}. This is exactly the real interest rate between these periods, under the hypothesized prices, because of (25). Thus we verify all of the necessary and sufficient conditions for the hypothesized plan to maximize (1) among all plans consistent with (B.4). Since we have also shown that all feasible plans must satisfy (B.4), and that the hypothesized plan is feasible, it follows that the hypothesized plan maximizes (1) among all feasible plans.

Next we show that markets clear under the hypothesized plans for all sectors. The fact that goods markets clear each period follows from the fact that the intra-temporal resource allocation each period corresponds to a Walrasian competitive equilibrium of the static model. And condition (B.1) implies that

$$\sum_{j=1}^{N} a^j(t) \;=\; a(t) + \frac{\sum_{j=1}^{N} f^j}{1-\beta} P^*(t) \;=\; a(t)$$

each period, so that the asset market clears as well.

Finally, we show that the conjectured paths are consistent with the specification of policy. The only non-trivial condition to check is consistency with the Taylor rule (16) for interest-rate policy. Because the inflation is consistent with the central bank's target in each of the periods $t \geq 1$ under the conjectured paths, the right-hand side of (16) corresponds simply to $P^*(t+1)/(\beta P^*(t))$, which is greater than 1 under the assumption (18). Hence the assumed monetary policy rule requires that $1+i(t)$ equal this quantity each period, which is precisely what (25) assumes. Hence the conjectured paths are consistent with policy, and constitute a perfect foresight equilibrium under the assumed policy.

Moreover, in this equilibrium, the borrowing constraint (12) is not a binding constraint in any period $t \geq 1$, as we have shown that each unit's intertemporal plan would also be optimal if the unit were subject to only an integrated intertemporal budget constraint. Note also that our analysis implies that it would be possible to relax the borrowing limit $\underline{b}^j(t)$ in any period $t \geq 1$ without this implying any change in the equilibrium paths of prices or quantities.

B.2 Proof of Lemma 4

The function $V^j(\tilde{a};\, \tilde{\boldsymbol{a}}(1))$ is the maximum achievable value of the discounted utility in periods $t \geq 1$ for a unit in sector j that carries real pre-tax wealth $\tilde{a}$ into period $t = 1$, if the prices and interest rates are the ones associated with the stationary equilibrium determined by the aggregate vector of pre-tax wealths $\tilde{\boldsymbol{a}}(1)$. The envelope theorem then implies that the partial derivative of V^j with respect to $\tilde{a}$, evaluated at $\tilde{a} = \tilde{a}^j(1)$, will equal the marginal utility of real expenditure in period 1 in the stationary equilibrium. Thus we must have

$$\Lambda^j(\tilde{a}^j(1);\, \tilde{\boldsymbol{a}}(1)) \;=\; \frac{u'(c^j_k/\alpha_{k-j};\, \bar{\xi})}{q^*_k} \;=\; \frac{v'(y_j;\, \bar{\xi})}{q^*_j}, \tag{B.6}$$

where we note that in the stationary equilibrium, the middle expression must have the same value for each k such that $\alpha_{k-j} > 0$, and the final expression must also have this same value, as a consequence of the first-order conditions for optimization by units in sector j.

Moreover, it follows from Lemma 3 that the allocation of resources in the stationary equilibrium depends only on the value of the vector $\boldsymbol{f}$ implied by the vector $\tilde{\boldsymbol{a}}(1)$. Thus both the second and third expressions in (B.6) are functions of $\boldsymbol{f}$ that are independent of the value of β, and we can define

$$\Lambda^{*j}(\boldsymbol{f}) \;\equiv\; \frac{v'(y_j(\boldsymbol{f});\, \bar{\xi})}{q^*_j(\boldsymbol{f})}.$$

Here $y_j(\boldsymbol{f})$ means the output supply by sector j in the static competitive equilibrium associated with the vector of net transfers $\boldsymbol{f}$, and $q^*_j(\boldsymbol{f})$ is the relative price of the sector j good in that same equilibrium. Note that we have assumed that $\tilde{\boldsymbol{a}}(1)$ is such that $\boldsymbol{f} \in U$, so that we have a uniquely defined static competitive equilibrium with the vector $\boldsymbol{f}$. It then follows not only that $\Lambda^{*j}(\boldsymbol{f})$ is uniquely defined, but that it varies continuously with variation in the elements of $\boldsymbol{f}$.

Our definition of the equilibrium selection has also assumed that when $\boldsymbol{f} = \boldsymbol{0}$, the static competitive equilibrium is the one in which $y_j = \bar{y}, q_j^* = 1$ for each sector. Thus we have

$$\Lambda^{*j}(\boldsymbol{0}) \;=\; v'(\bar{y};\, \bar{\xi}) \;=\; u'(\bar{y};\, \bar{\xi}).$$

And since $\Lambda^{*j}(\boldsymbol{f})$ must be a continuous function near $\boldsymbol{f} = \boldsymbol{0}$, we must have

$$\lim_{\boldsymbol{f}\to\boldsymbol{0}} \Lambda^{*j}(\boldsymbol{f}) \;=\; u'(\bar{y};\bar{\xi})$$

for each j, as asserted in the lemma.

B.3 Proof of Proposition 3

Here we consider equilibrium determination when $a(0)$ is arbitrarily close to zero (though we assume that $a(0) > 0$), and $\underline{b}^j(0) = 0$ for all j. In this case, satisfaction of the borrowing constraint (12) by all sectors requires that $0 \leq b^j(0) \leq a(0)$ for each sector, and hence that each element of $\boldsymbol{f}$ must satisfy the bounds

$$-\frac{1}{N}(1-\beta)\frac{(1+i(0))}{P^*(1)}a(0) \;\;\leq\;\; f^j \;\;\leq\;\; \frac{N-1}{N}(1-\beta)\frac{(1+i(0))}{P^*(1)}a(0). \qquad \text{(B.7)}$$

As $a(0) \to 0$, for fixed values of the other model parameters, both the upper and lower bounds converge to 0. Hence we can assure that in equilibrium, all elements of $\boldsymbol{f}$ must be as close as may be desired to zero, by fixing a sufficiently small value for $a(0)$.

Lemma 4 then implies that the value of $\Lambda^j(\tilde{a}^j(1);\, \tilde{\boldsymbol{a}}(1))$ must approach $u'(\bar{y};\bar{\xi})$, so that under the assumption about $i(0)$ maintained in the proposition, the Euler condition (27) takes the simpler form (29).

In addition, given that $\beta < 1$, the bounds (B.7) can alternatively be written

$$-\frac{1}{N}a(0) \;\;\leq\;\; b^j(0) \;\;\leq\;\; \frac{N-1}{N}a(0).$$

Thus we observe that in the limit as $a(0) \to 0$, the equilibrium value of each of the $\{b^j(0)\}$ must approach zero. We thus calculate the equilibrium in period zero for the limiting case in which we must have $b^j(0) = 0$ for all j. It then follows, as discussed in the main text, that the vector $\boldsymbol{c}(0)$ of sectoral expenditure levels must satisfy $\boldsymbol{c}(0) = \boldsymbol{A}\boldsymbol{c}(0)$.

We now show that under our assumptions, the matrix $\boldsymbol{A}$ must have a unique right eigenvector $\boldsymbol{\pi}$ with an associated eigenvalue of 1. We first note that the definition of the matrix $\boldsymbol{A}$ in (5) implies that $A_{kj} \geq 0$ for all k, j, and that $\sum_{k=1}^N A_{kj} = 1$ for every j, or in vector notation, that

$$\boldsymbol{e}'\boldsymbol{A} \;=\; \boldsymbol{e}'.$$

This indicates that 1 must be an eigenvalue of the matrix $\boldsymbol{A}$, with $\boldsymbol{e}'$ the associated left eigenvector. Any eigenvalue must also have at least one associated right eigenvector; thus it remains only to establish that the right eigenvector $\boldsymbol{\pi}$ is unique (up to normalization).

We observe from the properties noted in the previous paragraph that $\boldsymbol{A}$ is a non-negative matrix (Gantmacher, 1959, chap. XIII, Definition 1) that is furthermore a stochastic matrix

(Definition 4).[48] Any non-negative matrix necessarily has a maximal (Frobenius-Perron) eigenvalue $\bar{\lambda}$ with the properties that (i) $\bar{\lambda}$ is a non-negative real number, and (ii) $|\lambda| \leq \bar{\lambda}$ for all eigenvalues λ of the matrix (where $|\lambda|$ denotes the modulus of an eigenvalue that may be complex); moreover, (iii) the left and right eigenvectors associated with the maximal eigenvalue are real-valued and non-negative in all elements (Gantmacher, Theorem 3). In the more specific case of a stochastic matrix, the maximal eigenvalue is 1 (Gantmacher, p. 83). The associated left eigenvector is $\boldsymbol{e}'$, which is obviously non-negative in all elements; but there must be a (non-zero) right eigenvector $\boldsymbol{\pi}$ that is also non-negative in all of its elements. Because $\boldsymbol{\pi} \geq \boldsymbol{0}$, we can normalize the right eigenvector to satisfy $\boldsymbol{e}'\boldsymbol{\pi} = 1$.

To go further it is useful to write the matrix $\boldsymbol{A}$ in the normal form defined in Gantmacher (sec. XIII.4).[49] This involves partitioning the N sectors (the rows and columns of the matrix) into disjoint subsystems $\{S_1, \ldots, S_s\}$, each of which is irreducible, in the sense that any two sectors $j \neq z$ within the same subsystem can be linked by a sequence of sectors $(j, k, l, \ldots, y, z)$ all within the same subsystem, with the property that j buys goods produced in k, k buys goods produced in l, $\ldots$, and y buys goods produced in z. We further define a subsystem as "isolated" if each of the sectors $j \in S_i$ spend only on products of sectors in subset S_i. Then Gantmacher shows that one can order the subsystems so that the first $g \geq 1$ of them are the (only) isolated subsystems; and the subsystems S_i for $g+1 \leq i \leq s$ each have the property that each of the sectors $j \in S_i$ spends only on products produced in subsystems S_k with $k \leq i$. Thus if one re-orders the sectors in accordance with this ordering of the subsystems, the matrix $\boldsymbol{A}$ has a normal form representation that is upper block-triangular, with all off-diagonal blocks being blocks of zeroes in the first g block columns.

In the case that $\phi_k(0) > 0$ for all k, our assumption that $\alpha_1 > 0$ implies that there is only one isolated subsystem, which is the complete system of all N sectors (1 buys from 2, which buys from 3, $\ldots$, which buys from N, which buys from 1). Hence in this case, we must have $g = 1, s = 1$. The situation is only slightly more complicated if there exists a sector for which $\phi_k(0) = 0$. We have assumed that there can be at most one such sector; let it be sector 1 (as in the numerical examples shown in Figure 2). In this case, we must have $g = 1, s \geq 2$. The only isolated subsystem must be the one that contains sector N; the assumption that $\alpha_1 > 0$ implies that 1 buys from 2 which buys from $\ldots$ which buys from $N-1$ which buys from N, so that any isolated subsystem must contain sector N. In addition, subsystem S_s must consist solely of sector 1, since no other sector buys anything from sector 1 (as a result of the pandemic); thus there must be at least two subsystems.

Gantmacher (Theorem 12) shows that a stochastic matrix $\boldsymbol{A}$ has a unique right eigenvector $\boldsymbol{\pi}$ with an associated eigenvalue of 1 if and only if $g = 1$ in the normal form representation (i.e., there is a unique isolated subsystem). In this case the Frobenius-Perron eigenvector $\boldsymbol{\pi}$ corresponds to the uniquely defined stationary long-run probabilities of occupying the N different states, if $\boldsymbol{A}$ is interpreted as the matrix of transition probabilities defining a homogeneous N-state Markov chain. The elements of this eigenvector satisfy $\pi_j > 0$ for all $j \in S_1$, and $\pi_j = 0$ for all other j. We have shown that this result necessarily applies to our model.

[48]More precisely, the transpose $\boldsymbol{A}'$ is a stochastic matrix as defined in Gantmacher. Below we translate the properties of stochastic matrices established in Gantmacher into statements about the matrix $\boldsymbol{A}$.

[49]More precisely, we put $\boldsymbol{A}'$ in the form shown in Gantmacher (p. 75).

This unique solution for $\boldsymbol{\pi}$ allows us then to solve uniquely for the vector $\boldsymbol{c}(0) = \Omega\boldsymbol{\pi}$, where the value of $\Omega > 0$ is given by (31), as explained in the main text, in order to satisfy (29). This establishes the proposition.

Because $\boldsymbol{c}(0)$ is a multiple of $\boldsymbol{\pi}$, it has the property that $c^j(0) > 0$ for all j in S_1, while $c^j(0) = 0$ for all other sectors. For example, in the case of the uniform network structure shown in the left panel of Figure 1, S_1 consists of sectors $\{2, 3, 4, 5\}$, while S_2 consists of $\{1\}$. In the case of the chain structure shown in the right panel of Figure 1, instead (and regardless of the value of the parameter λ), the irreducible subsystems are $S_1 = \{5\}$, $S_2 = \{4\}$, $S_3 = \{3\}$, $S_4 = \{2\}$, and $S_5 = \{1\}$, among which only S_1 is an isolated subsystem. This explains why, in Figure 2, we have $c^j(0) = 0$ only for $j = 1$ in the left panel, while instead $c^j(0) = 0$ for all $j \leq 4$ in the right panel.

B.4 Proof of Corollary 1

We know from Lemma 2 that the first-best consumption level $c_k^j(0)$ of any good k that is consumed by sector j in period zero is given by (9), where $y_k(0) = y^*(\phi_k(0); \bar{\xi})$, and the function $y^*(\phi; \xi)$ is implicitly defined by (8). We have also shown in section A.2 above that $y^*(\phi; \xi)$ is an increasing function of ϕ, for a given disturbance vector ξ. Because we have assumed that v is at least weakly convex in y, $v'(y^*; \bar{\xi})$ must be non-decreasing when ϕ increases. Thus the right-hand side of (8) must be non-decreasing, and so the left-hand side cannot decrease either. Because $u'(c; \bar{\xi})$ is a decreasing function of c, it follows that y^*/ϕ must be non-increasing when ϕ increases.

Then since the optimal level of production for sector k satisfies $y^*/\phi = \bar{y}$ when $\phi_k(0) = 1$, we must have $y^*/\phi \geq \bar{y}$ for all $0 < \phi_k(0) \leq 1$. Thus the optimal level of production in sector k is necessarily no smaller than $\phi_k(0)\bar{y}$. Combining (9) with this result, we conclude that the optimal consumption of good k by sector j must satisfy

$$c_k^{j,opt}(0) \;\geq\; \alpha_{k-j}\phi_k(0)\bar{y}. \tag{B.8}$$

But the quantity on the right-hand side here is the equilibrium consumption if the borrowing constraint does not bind for units in sector j, from (32). This in turn is an upper bound on equilibrium consumption, since a binding borrowing constraint can only reduce $c^j(0)$ and hence (because of (4) reduce sector j consumption of all goods of which there is positive consumption. Thus we must have

$$c_k^j(0) \;=\; A_{kj}c^j(0) \;\leq \alpha_{k-j}\phi_k(0)\bar{y} \;\leq\; c_k^{j,opt}(0) \tag{B.9}$$

for each good k. Thus no good can be consumed in an amount greater than the amount required for the first-best optimal allocation, as stated in the corollary.

Furthermore, if sector j is borrowing-constrained in period zero, the first inequality in (B.9) must be a strict inequality. Thus in this case we must have $c_k^j(0) < c_k^{j,opt}(0)$, as stated in the corollary (condition (i)). Finally, if v is a strictly convex function of y, the function $v'(y^*; \bar{\xi})$ must strictly increasing (and not just non-decreasing) when ϕ increases. From this we can conclude (by an argument parallel to the one made above, under the weaker assumption of weak convexity) that y^*/ϕ must be a decreasing function of ϕ (not just non-increasing). This implies that $y^*/\phi \geq \bar{y}$ for all $0 < \phi_k(0) < 1$, from which we conclude that

(B.8 must be a strict inequality for any $0 < \phi_k(0) < 1$.[50] This then implies that the last inequality in (B.9) must be a strict inequality, so that again we must have $c_k^j(0) < c_k^{j,opt}(0)$, as stated in the corollary (condition (ii)).

B.5 Algebra of the examples in Figure 2

In the case of a uniform network structure (the left panel of Figure 2), $A_{kj} = 1/(N-1)$ for all j and any $k \neq 1$; the Frobenius-Perron maximal right eigenvector is then easily seen to be

$$\boldsymbol{\pi} \;=\; (0 \quad 1/(N-1) \;\; \ldots \;\; 1/(N-1))'.$$

(Because every sector spends the same amount in each sector $k \neq 1$, but nothing in sector 1, the eigenvector must have this property as well.) Furthermore, $\omega_j = (N-1)/N$ for all j. Hence the maximal value in the problem on the right-hand side of (31) is achieved by all sectors $j \neq 1$, and the equilibrium expenditure vector is given by

$$\boldsymbol{c}(0) \;=\; (0 \quad (N-1)/N \;\ldots\; (N-1)/N)' \cdot \bar{y},$$

as shown in the left panel of Figure 2 for the case $N = 5$. In this example, expenditure collapses completely in sector 1 (which no longer receives any income), but it is reduced in sectors $j \neq 1$ only to the extent that it is efficient for these sectors to reduce their spending (given that they no longer can or should buy sector-1 goods).

The collapse of effective demand is much more severe (and the inefficiency much greater) in the case of a "chain" network. In this case, one can show that the Frobenius-Perron maximal right eigenvector is given by

$$\boldsymbol{\pi} \;=\; (0 \;\ldots\; 0 \quad 1)'.$$

Sector 1 cannot spend at all, because it receives no income. Given that sector 1 cannot spend, sector 2 receives no income other than its own within-sector spending. But because sector 2 does not spend all of its income within-sector, an eigenvalue with eigenvector 1 must involve zero spending by this sector as well. Continuing iteratively in this way, one can show that every sector but sector N must have zero expenditure.

The argument no longer goes through in the case of sector N, because — given that they can no longer buy sector 1 goods — units in sector N spend all of their income within-sector. Hence all elements of $\boldsymbol{\pi}$ but the final one must equal zero. In the problem on the right-hand side of (31), sector N achieves the maximum. Then given that $c^{*N} = (1-\alpha_1)\bar{y} = (1-\lambda)\bar{y}$, the equilibrium expenditure vector is given by

$$\boldsymbol{c}(0) \;=\; (0 \;\ldots\; 0 \quad 1-\lambda)' \cdot \bar{y},$$

as shown in the right panel of Figure 2 for the case $N = 5, \lambda = 0.8$.

[50] Note that (B.8) still holds with equality if $\phi_k(0)$ is equal to either 0 or 1. In the former case, both sides are equal to zero; in the latter case, both sides are equal to $\alpha_{k-j}\bar{y}$. Note also that (B.8) also holds with equality for any good k such that $\alpha_{k-j} = 0$, so that sector j does not wish to consume it, regardless of the value of $\phi_k(0)$.

These two cases illustrate the two extremes with regard to the degree of collapse of aggregate expenditure and output in the limiting case in which $a(0) \to 0$. For a general network structure with a fraction $\alpha_0 = 1/N$ of within-sector spending by all sectors (as in these examples), we can show that aggregate spending $c^{agg}(0) \equiv \sum_{j=1}^{N} c^j(0)$ must fall within the bounds

$$(1/N)\bar{y} \;\leq\; c^{agg}(0) \;\leq\; (N-2+(1/N))\bar{y} \;<\; (N-1)\bar{y} \;=\; y^* \;\equiv\; \sum_{j=1}^{N} c^{*j}.$$

Here the lower bound is established by the fact that at least one sector must not be borrowing-constrained, and since that sector cannot be sector 1, its first-best level of spending c^{*j} must at least equal $\alpha_0\bar{y} = \bar{y}/N$. The upper bound is established by the fact that spending by sector 1 must be zero, and that spending in every other sector must be bounded above by c^{*j}, which cannot exceed $\bar{y}$ for any sector. Both of these bounds are achievable, since (as just shown) the chain network achieves the lower bound while the uniform network achieves the upper bound.

B.6 Proof of Proposition 4

We derive the solution to (35) and demonstrate its uniqueness in several steps. We begin by observing that there must be a unique solution, before deriving a closed-form expression for that solution.

B.6.1 Existence of a unique solution to the "Keynesian cross"

Let $\boldsymbol{\delta} >> \mathbf{0}$ be fixed, and consider the set of vectors $\boldsymbol{c}(0)$ that satisfy (35). We can show that there must be a unique solution using properties of positive concave mappings that are reviewed in Cavalcante *et al.* (2016). For any vector $\boldsymbol{c}(0)$, let $\boldsymbol{F}(\boldsymbol{c}(0))$ be the vector defined by the right-hand side of (35); thus $\boldsymbol{F}(\cdot)$ maps N-vectors into N-vectors. If $\boldsymbol{\delta}(0) >> \mathbf{0}$, we can further show that $\boldsymbol{F}(\cdot)$ is a positive mapping, in the sense that for any $\boldsymbol{c}(0) \geq \mathbf{0}$, we have $\boldsymbol{F}(\boldsymbol{c}(0)) >> \mathbf{0}$. Let $F_j(\cdot)$ be the jth element of $\boldsymbol{F}(\cdot)$, that is, the implied value for $c^j(0)$. Then for each j, we need to show that for any $\boldsymbol{c}(0) \geq \mathbf{0}$, $F_j(\boldsymbol{c}(0)) > 0$. Since $\boldsymbol{A} \geq \mathbf{0}$, $\boldsymbol{c}(0) \geq \mathbf{0}$ implies that $\boldsymbol{A}\boldsymbol{c}(0) \geq \mathbf{0}$. Then under the hypothesis that $\boldsymbol{\delta}(0) >> \mathbf{0}$, we must have

$$\frac{1}{\bar{p}}\boldsymbol{\delta}(0) \;+\; \boldsymbol{A}\boldsymbol{c}(0) \;>>\; \mathbf{0}.$$

Thus the jth element of this vector must be positive, for any j. Since $c^{*j} > 0$ as well, the minimum of the two quantities must be positive. Thus $F_j(\boldsymbol{c}(0))$ is necessarily positive, as required.

We can further show that each of the functions $F_j(\cdot)$ is concave. This requires that for any vectors $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, and any scalar $0 \leq \alpha \leq 1$,

$$F_j(\alpha\boldsymbol{c}_1 + (1-\alpha)\boldsymbol{c}_2) \;\geq\; \alpha F_j(\boldsymbol{c}_1) + (1-\alpha)F_j(\boldsymbol{c}_2). \qquad \text{(B.10)}$$

Given the definition of $F_k(\cdot)$ in (35) as the minimum of two functions, this holds if and only if *both* of the inequalities

$$\frac{\delta^j}{\bar{p}} + \sum_k A_{jk}[\alpha c_1^k + (1-\alpha)c_2^k] \;\geq\; \alpha F_j(\boldsymbol{c}_1) + (1-\alpha)F_j(\boldsymbol{c}_2), \tag{B.11}$$

$$c^{*j} \;\geq\; \alpha F_j(\boldsymbol{c}_1) + (1-\alpha)F_j(\boldsymbol{c}_2) \tag{B.12}$$

are necessarily satisfied. But inequality (B.11) follows from the fact that

$$F_j(\boldsymbol{c}_i) \;\leq\; \frac{\delta^j}{\bar{p}} + \sum_k A_{jk} c_i^k$$

for each of the cases $i = 1, 2$; and inequality (B.12) follows from the fact that

$$F_j(\boldsymbol{c}_i) \;\leq\; c^{*j}$$

for each of the cases $i = 1, 2$. Hence (B.10) is satisfied, and $F_j(\cdot)$ is a concave function for each j. This in turn means that $\boldsymbol{F}(\cdot)$ is a concave mapping.

Thus $\boldsymbol{F}(\cdot)$ is a positive concave mapping. Moreover, there exists a finite upper bound $\bar{\boldsymbol{c}}$ with the property that $\boldsymbol{F}(\boldsymbol{c}(0)) \leq \bar{\boldsymbol{c}}$ for all $\boldsymbol{c}(0) \leq \bar{\boldsymbol{c}}$; this is the bound $\bar{\boldsymbol{c}} = \boldsymbol{c}^*$, where the elements of the vector $\boldsymbol{c}^*$ are defined in (30). It then follows from Cavalcante *et al.* (2016, Proposition 1 and Facts 4.1 and 4.2) that the mapping $\boldsymbol{F}(\cdot)$ has a unique fixed point. This means that the system of equations (35) has a unique solution $\boldsymbol{c}(0)$.

B.6.2 Properties of the solution: monotonicity

For any vector $\boldsymbol{\delta} >> \boldsymbol{0}$, let this unique fixed point be denoted $\bar{\boldsymbol{c}}(\boldsymbol{\delta})$. (Note that the mapping $\boldsymbol{F}(\cdot)$ depends on the vector $\boldsymbol{\delta}$.) One can easily establish several features of the functional dependence of the fixed point on $\boldsymbol{\delta}$. In addition to being useful in the proof of Proposition 4, these will be used in our later analysis of the effects of fiscal policy and credit policy.

First, because $\boldsymbol{0} << \boldsymbol{F}(\boldsymbol{c}) \leq \boldsymbol{c}^*$ for all $\boldsymbol{c}$, it is clear that the fixed point must satisfy $0 << \bar{\boldsymbol{c}}(\boldsymbol{\delta}) \leq \boldsymbol{c}^*$ for all $\boldsymbol{\delta} \geq \boldsymbol{0}$.

We can also show that each of the component functions $\bar{\boldsymbol{c}}_j(\boldsymbol{\delta})$ must be at least weakly increasing in each of the elements of $\boldsymbol{\delta}$. Consider any two vectors $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2$ such that $\boldsymbol{\delta}_2 \geq \boldsymbol{\delta}_1 >> \boldsymbol{0}$. Then we can show that we must have $\bar{\boldsymbol{c}}(\boldsymbol{\delta}_2) \geq \bar{\boldsymbol{c}}(\boldsymbol{\delta}_1)$ for each j. Let $\boldsymbol{F}_i(\cdot)$ be the mapping defined by the right-hand side of (35) when $\boldsymbol{\delta} = \boldsymbol{\delta}_i$, for $i = 1, 2$, and further define the mapping

$$\tilde{\boldsymbol{F}}(\boldsymbol{\zeta}) \;\equiv\; \boldsymbol{F}_2(\bar{\boldsymbol{c}}(\boldsymbol{\delta}_1) + \boldsymbol{\zeta}) \;-\; \bar{\boldsymbol{c}}(\boldsymbol{\delta}_1),$$

defined for an arbitrary vector $\boldsymbol{\zeta} \geq \boldsymbol{0}$. Then $\boldsymbol{c}$ will be a fixed point of $\boldsymbol{F}_2$ if and only if $\boldsymbol{\zeta} = \boldsymbol{c} - \bar{\boldsymbol{c}}(\boldsymbol{\delta}_1)$ is a fixed point of $\tilde{\boldsymbol{F}}$.

It is evident that $\tilde{\boldsymbol{F}}(\cdot)$ is a continuous mapping, with the upper bound $\tilde{\boldsymbol{F}}(\boldsymbol{\zeta}) \leq \boldsymbol{c}^* - \bar{\boldsymbol{c}}(\boldsymbol{\delta}_1)$ for all $\boldsymbol{\zeta}$. Moreover, for any $\boldsymbol{\zeta} \geq \boldsymbol{0}$, we must have

$$\begin{aligned} \tilde{\boldsymbol{F}}(\boldsymbol{\zeta}) \;&\geq\; \boldsymbol{F}_2(\bar{\boldsymbol{c}}(\boldsymbol{\delta}_1)) \;-\; \bar{\boldsymbol{c}}(\boldsymbol{\delta}_1) \\ &\geq\; \boldsymbol{F}_1(\bar{\boldsymbol{c}}(\boldsymbol{\delta}_1)) \;-\; \bar{\boldsymbol{c}}(\boldsymbol{\delta}_1) \;=\; \boldsymbol{0}. \end{aligned}$$

Thus $\tilde{\boldsymbol{F}}(\cdot)$ maps the set of vectors satisfying the bounds

$$\mathbf{0} \;\leq\; \boldsymbol{\zeta} \;\leq\; \boldsymbol{c}^* \,-\, \bar{\boldsymbol{c}}(\boldsymbol{\delta}_1)$$

into itself. These bounds define a compact, convex subset of $\mathbb{R}^N$. Hence by Brouwer's fixed-point theorem, there must be a vector $\boldsymbol{\zeta}^*$ in this set that is a fixed point of the mapping $\tilde{\boldsymbol{F}}(\cdot)$. It follows that $\boldsymbol{c} = \bar{\boldsymbol{c}}(\boldsymbol{\delta}_1) + \boldsymbol{\zeta}^*$ is a fixed point of $\boldsymbol{F}_2(\cdot)$, and since (as shown above) the latter mapping must have a unique fixed point, it follows that we must have

$$\bar{\boldsymbol{c}}(\boldsymbol{\delta}_2) \;=\; \bar{\boldsymbol{c}}(\boldsymbol{\delta}_1) \,+\, \boldsymbol{\zeta}^* \;\geq\; \bar{\boldsymbol{c}}(\boldsymbol{\delta}_1).$$

Hence each of the functions $\bar{c}_j(\boldsymbol{\delta})$ must be weakly increasing in each of the elements of $\boldsymbol{\delta}$, as asserted.

This means that along any continuous expansion path $\boldsymbol{\delta}(s)$ for the vector $\boldsymbol{\delta}$ specifying period-zero liquidity, where the real variable s indexes distance along the expansion path [not time], if each element of $\boldsymbol{\delta}(s)$ is at least weakly increasing in s, then each element of $\bar{\boldsymbol{c}}(\boldsymbol{\delta}(s))$ will be weakly increasing in s as well. Among other things, this means that along any such expansion path (representing situations that can be reached through progressively larger lump-sum transfers in period zero), if the borrowing constraint no longer constrains some sector j for the level of transfers parameterized by s, then sector j will not be borrowing-constrained for any vector of transfers corresponding to a point $s' > s$ along the expansion path.

Thus for each sector, there will be a single point along the expansion path at which that sector shifts from being borrowing-constrained (for all levels of transfers below that point) to being unconstrained (for all levels of transfers beyond that point). This is illustrated for two different network structures in Figure 3 (where the labeled points $\{\hat{a}_i\}$ on the horizontal axis are levels of initial liquid assets at which another sector ceases to be borrowing-constrained). If we let C be the subset of the sectors that are borrowing-constrained in the case of a particular vector of initial asset positions, then as one proceeds along any monotonic expansion path, the set C remains the same except at a finite number of points, and at any point where C changes, increasing s can only result in the subtraction of elements from C.

The set of possible vectors $\boldsymbol{\delta}$ can thus be partitioned into regions corresponding to different subsets C of borrowing-constrained sectors. We have already shown (in the proof of Proposition 3) that for all $\boldsymbol{\delta}(0)$ close enough to $\mathbf{0}$, the set of unconstrained sectors will be U_0, the set of sectors j for which the maximum value is achieved in the problem on the right-hand side of (31); hence at such points the set of constrained sectors will be C_0, the complement of U_0. Because the set of constrained sectors can only shrink as a result of additional initial transfers (or further relaxations of borrowing constraints), it follows that for all $\boldsymbol{\delta}(0) >> \mathbf{0}$, C must be an element of $\mathcal{C}$, the set of all subsets of C_0 (including the empty set $\emptyset$ as well as C_0 itself).

B.6.3 The local solution for a given subset C of borrowing-constrained sectors

If we know what the set C is for a given vector $\boldsymbol{\delta}$, it is straightforward to compute the equilibrium expenditure vector $\bar{\boldsymbol{c}}(\boldsymbol{\delta})$ at that point.[51] The solution vector $\boldsymbol{c}(0)$ must satisfy

$$c^j(0) \;=\; \frac{\delta^j}{\bar{p}} + \sum_k A_{jk} c^k(0)$$

for all $j \in C$, and

$$c^j(0) \;=\; c^{*j}$$

for all $j \notin C$. This is a system of linear equations to solve for $\boldsymbol{c}(0)$.

The first of these sets of equations can be written in vector form as

$$\hat{\boldsymbol{c}} \;=\; \hat{\boldsymbol{\delta}} + \boldsymbol{A}_{CC}\hat{\boldsymbol{c}} + \boldsymbol{A}_{CU}\check{\boldsymbol{c}}^*,$$

where $\hat{\boldsymbol{c}}$ is the vector of elements of the solution $\boldsymbol{c}(0)$ corresponding to sectors $j \in C$,; $\hat{\boldsymbol{\delta}}$ is the vector collecting the values of $\delta^j/\bar{p}$ for the sectors $j \in C$; $\check{\boldsymbol{c}}^*$ is the vector of elements of $\boldsymbol{c}^*$ corresponding to sectors $k \notin C$; and the matrix $\boldsymbol{A}$ has been partitioned as in (43) in the main text. This system of linear equations has a unique solution (and hence the complete system has a unique solution) if and only if the matrix $\boldsymbol{I} - \boldsymbol{A}_{CC}$ is non-singular.

This is necessarily the case for any $C \in \mathcal{C}$. Note that in order for $\boldsymbol{I} - \boldsymbol{A}_{CC}$ to be singular, there would have to exist a vector $\boldsymbol{u} \neq \boldsymbol{0}$ such that $\boldsymbol{A}_{CC}\boldsymbol{u} = \boldsymbol{u}$. This would require that the set of sectors C be an isolated subsystem (that is, sectors $j \in C$ spend only on the products of sectors $k \in C$). But we have shown in the proof of Proposition 3 that under our assumptions, the only isolated subsystem can be S_1, the one containing sector N. We have further shown that the eigenvector $\boldsymbol{\pi}$ has non-zero elements only for sectors $j \in S_1$. Hence the elements of U_0 (of which there must be at least one) belong to S_1; it follows that not all of S_1 can belong to C_0, and thus that not all of S_1 can belong to any $C \in \mathcal{C}$. We can therefore conclude that C cannot be an isolated subsystem, from which it follows that $\boldsymbol{I} - \boldsymbol{A}_{CC}$ must be non-singular.

We can show something stronger, which is that all eigenvalues of $\boldsymbol{A}_{CC}$ must be inside the unit circle (i.e., have modulus less than 1). We note that $\boldsymbol{A}_{CC}$ is a non-negative matrix, though no longer a stochastic matrix (because the set C cannot be an isolated subsystem, as just discussed). It follows from Gantmacher (1959, chap. XIII, Theorem 3) that $\boldsymbol{A}_{CC}$ has a non-negative real eigenvalue r, such that $|\lambda| \leq r$ for all of the other eigenvalues of the matrix. This maximal eigenvalue is bounded above by

$$r \;\leq\; \max_{j \in C} \sum_{k \in C} A_{kj} \;\leq\; 1.$$

[51] Note that it may be ambiguous whether to include a particular sector j in the set C or not, as in equilibrium the sector's income may be just enough to allow it to spend the optimal quantity c^{*j}, with end-of-period assets $b^j(0) = 0$. In this case, it does not matter whether we consider the set C to include the sector j or not; the solution obtained for $\bar{\boldsymbol{c}}(\boldsymbol{\delta})$ will be the same in either case. In such a case, the value of $\boldsymbol{\delta}$ lies on the boundary between two regions corresponding to different sets of constrained sectors; but since the function $\bar{\boldsymbol{c}}(\boldsymbol{\delta})$ is continuous at such boundaries, it does not matter to which region the boundary case is assigned. Note that if instead we wish to compute the effect of a *change* in $\boldsymbol{\delta}$, it will matter how we define the set of constrained sectors C; but in that case, the right answer will depend on the direction in which $\boldsymbol{\delta}$ is to be changed.

Here the first inequality follows from Gantmacher (p. 68), and the second from the definition of the matrix $\boldsymbol{A}$.

However, we have just shown that 1 cannot be an eigenvector of $\boldsymbol{A}_{CC}$. Thus the maximal eigenvalue must satisfy $r < 1$, from which it follows that $|\lambda| < 1$ for every eigenvalue of $\boldsymbol{A}_{CC}$. This allows us to write

$$(\boldsymbol{I} - \boldsymbol{A}_{CC})^{-1} \;=\; \boldsymbol{I} \,+\, \boldsymbol{A}_{CC} \,+\, (\boldsymbol{A}_{CC})^2 \,+\, (\boldsymbol{A}_{CC})^3 \,+\, \ldots$$

where the infinite sum must converge because the eigenvalues of $\boldsymbol{A}_{CC}$ have modulus less than 1. Since each of the terms on the right-hand side is a non-negative matrix, it follows that

$$(\boldsymbol{I} - \boldsymbol{A}_{CC})^{-1} \;\geq\; \boldsymbol{0}. \tag{B.13}$$

The system of linear local equations therefore has a unique solution

$$\boldsymbol{c}^{loc}(\boldsymbol{\delta}; C) \;=\; \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{A}_{CC})^{-1}(\hat{\boldsymbol{\delta}} + \boldsymbol{A}_{CU}\check{\boldsymbol{c}}^*) \\ \check{\boldsymbol{c}}^* \end{bmatrix},$$

where the solution is partitioned as in (43); this is the solution (37 referred to in the proposition. (See the proof of Lemma 6, below, for further details.) Here we have written the set C as an argument of the function, because there is a separate function of this kind for each possible choice of $C \in \mathcal{C}$. We further see that for any C, the solution is a linear function of $\boldsymbol{\delta}$ and $\boldsymbol{c}^*$, and that the matrices of coefficients denoted $\boldsymbol{M}$ and $\boldsymbol{N}$ in (37) contain only non-negative elements, because of (B.13) and the fact that all elements of $\boldsymbol{A}$ are non-negative.

B.6.4 The unique global solution

We see then that if we can determine which set of sectors C is the borrowing-constrained set in the case of any given vector of initial asset balances, we can determine the value of $\bar{\boldsymbol{c}}(\boldsymbol{\delta})$ at that point. We next show how to do this. Fixing the vector $\boldsymbol{\delta}$, let $\bar{C}$ be the set of constrained sectors in the solution to the "Keynesian cross" system (35), and let C instead be any other element of $\mathcal{C}$. Then let

$$\boldsymbol{\zeta} \;\equiv\; \boldsymbol{c}^{loc}(\boldsymbol{\delta}; \bar{C}) \,-\, \boldsymbol{c}^{loc}(\boldsymbol{\delta}; C)$$

measure the difference between the linear solution under the assumption that sectors $\bar{C}$ are constrained and the linear solution under the assumption instead that C is the set of constrained sectors. (Also, in what follows, let us write $\boldsymbol{c}^{loc}(\boldsymbol{\delta}; C)$ simply as $\boldsymbol{c}$, and $\boldsymbol{c}^{loc}(\boldsymbol{\delta}; \bar{C})$ as $\bar{\boldsymbol{c}}$.)

In the case of any sector $j \in C$, we must have

$$c^j \;=\; \hat{\delta}^j \,+\, \sum_k A_{jk} c^k,$$

$$\bar{c}^j \;\leq \hat{\delta}^j \,+\, \sum_k A_{jk} \bar{c}^k,$$

where the second condition holds for all j given that $\bar{\boldsymbol{c}}$ is a solution to (35). Subtracting the first equation from the second yields the implication

$$\zeta^j \;\leq\; \sum_k A_{jk}\zeta^k$$

for all $j \in C$. Instead, in the case of any sector $j \notin C$, we must have

$$c^j \;=\; c^{*j},$$

$$\bar{c}^j \;\leq\; c^{*j},$$

where again the second condition holds for all j given that $\bar{\boldsymbol{c}}$ is a solution to (35). Subtracting the first equation from the second yields the implication

$$\zeta^j \;\leq\; 0$$

for all $j \notin C$.

Then if we let $\hat{\boldsymbol{\zeta}}$ be the vector of elements of $\boldsymbol{\zeta}$ corresponding to sectors $j \in C$, and $\check{\boldsymbol{\zeta}}$ the vector of elements corresponding to sectors $j \notin C$, we must have

$$\hat{\boldsymbol{\zeta}} \;\leq\; \boldsymbol{A}_{CC}\hat{\boldsymbol{\zeta}} + \boldsymbol{A}_{CU}\check{\boldsymbol{\zeta}}, \qquad \check{\boldsymbol{\zeta}} \leq \boldsymbol{0}.$$

If we let $\boldsymbol{u} \equiv (\boldsymbol{I} - \boldsymbol{A}_{CC})\hat{\boldsymbol{\zeta}}$, then the first inequality implies that $\boldsymbol{u} \leq \boldsymbol{0}$, and hence that

$$\hat{\boldsymbol{\zeta}} \;=\; (\boldsymbol{I} - \boldsymbol{A}_{CC})^{-1}\boldsymbol{u} \;\leq\; \boldsymbol{0},$$

using (B.13). This together with the second inequality implies that $\boldsymbol{\zeta} \leq 0$, and hence that

$$\boldsymbol{c}^{loc}(\boldsymbol{\delta}; \bar{C}) \;\leq\; \boldsymbol{c}^{loc}(\boldsymbol{\delta}; C). \tag{B.14}$$

The fact that (B.14) must hold for any $C \in \mathcal{C}$ then implies that $\bar{C}$ must be the selection of borrowing-constrained sectors that implies that

$$\bar{\boldsymbol{c}}(\boldsymbol{\delta}) \;=\; \min_{C \in \mathcal{C}} \boldsymbol{c}^{loc}(\boldsymbol{\delta}; C). \tag{B.15}$$

That is, for any $\boldsymbol{\delta} >> \boldsymbol{0}$, $\bar{C}$ must be one of the elements of $\mathcal{C}$ that solve the minimization problem on the right-hand side of (B.15). Since (B.15) must hold for arbitrary $\boldsymbol{\delta}$, this gives us a closed-form solution for the function $\bar{\boldsymbol{c}}(\boldsymbol{\delta})$ for all $\boldsymbol{\delta} >> \boldsymbol{0}$.

If for values of $\boldsymbol{\delta}$ on the boundary of the positive orthant we select as the relevant solution to (35) the vector $\boldsymbol{c}(0)$ that can be reached as the limit of a sequence of solutions $\boldsymbol{c}_n \to \boldsymbol{c}(0)$ corresponding to a non-increasing sequence of vectors $\boldsymbol{\delta}_n \to \boldsymbol{\delta}$ with $\boldsymbol{\delta}_n >> \boldsymbol{0}$ for each n, then also for these values of $\boldsymbol{\delta}$ the solution for $\boldsymbol{c}(0)$ will be the one given by (B.15). (This follows immediately from the fact that the functions defined in (B.15) are all continuous functions of $\boldsymbol{\delta}$.) Hence (B.15) is the desired solution for all $\boldsymbol{\delta} \geq \boldsymbol{0}$, as stated in equation (38) of the proposition.

B.7 Proof of Corollary 2

The proof follows exactly the same lines as in the proof of Corollary 1 above. Even when we allow non-negligible values for the $\{a^j(0)\}$, in the limit as $\beta \to 1$, the Euler condition again reduces to the form (29), which implies the upper bound (30) on spending by each sector. This was the crucial result needed to establish Corollary 1. In the case that the $\{a^j(0)\}$ are non-negligible, the equation for the level of spending if a sector is borrowing-constrained changes, but the equation for its level of spending if it is not borrowing-constrained is unchanged, and the previous conclusions continue to hold.

B.8 Proof of Corollary 3

The solution to the system (35) involves no binding borrowing constraint for any sector if and only if $c^j(0) = c^{*j}$ for each j is a solution to this system. Since we have established that the solution must be unique, if this is a solution it must be the unique solution; so it suffices that we check whether this vector of expenditure levels satisfies all of the equations in the system (35).

Substituting the candidate solution for $\boldsymbol{c}(0)$ into (34), we see that this equation holds if and only if

$$\frac{\delta^j}{\bar{p}} + \sum_k A_{jk} c^{*k} \;\geq\; c^{*j}.$$

Using (30) to substitute for the elements of $\boldsymbol{c}^*$ in this inequality, and recalling the definition (5), we obtain the requirement

$$\frac{\delta^j}{\bar{p}} + \phi_j(0)\bar{y} \;\geq\; \omega_j \bar{y}.$$

This implies a lower bound for δ^j that is equivalent to condition (39). Hence the candidate solution satisfies (34) for all j if and only if δ^j satisfies the lower bound (39) for all j. Hence (39) holding for all j is necessary and sufficient for the solution to the system (35) to involve no binding borrowing constraints, as asserted in the corollary.

We have already noted that if borrowing constraints do not bind, consumption demands must be given by (32) for each sector. Then (28) implies that equilibrium production in each sector k must equal

$$y_k(0) \;=\; \sum_j c_k^j(0) \;\sum_j \alpha_{k-j}\phi_k(0)\bar{y} \;=\; \phi_k(0)\cdot\bar{y}.$$

This establishes that the resource allocation must be (40).

Finally, in the special case that $a^j(0) = a(0)/N$ and $\underline{b}^j(0) = 0$ for all j, we have $\delta^j = a(0)/N$ for all j. In this case, (39) reduces to

$$a(0) \;\geq\; N\bar{p}\bar{y}\cdot[\sum_k \alpha_{k-j}\phi_k(0) - \phi_j(0)].$$

This condition holds for all j if and only if $a(0)$ satisfies the lower bound (41), as stated in the corollary.

C Equilibrium with Fiscal Transfers

Here we present proofs of the main results in section III of the main text.

C.1 Proof of Lemma 5

In any equilibrium, regardless of the nature of policy, the consumption allocation in period zero must satisfy (4). We wish to consider the conditions under which this can be consistent with achievement of the first-best allocation of resources, characterized in Lemma 2.

Consistency of (4) with (9) requires that for each pair of sectors j, k such that $\alpha_{k-j}\phi_k(0) > 0$ (so that j wishes to consume good k in period zero), one must have

$$\frac{c^j(0)}{\omega_j} \;=\; \frac{y_k(0)}{\phi_k(0)}. \qquad \text{(C.1)}$$

(Note that $c_k^j(0)/(\alpha_{k-j}\phi_k(0))$ must equal both of these quantities, so that they must be equal.) This means that for each of the goods k consumed by sector j, the value of $y_k(0)/\phi_k(0)$ must be the same (and positive). But condition (8) implies that for each sector k,

$$u'(y_k(0)/\phi_k(0);\, \xi_0) \;=\; v'(y_k(0);\, \xi_0). \qquad \text{(C.2)}$$

Hence $v'(y_k(0); \xi_0)$ must be the same positive quantity for each of the sectors k consumed by sector j.

Since this conclusion is independent of the identity of sector j, we can show more generally that $v'(y_k(0); \xi_0)$ must be the same positive quantity for any two sectors that both sell a positive amount to some single sector. Our assumptions about the network structure further imply that the sector of sectors with $\phi_k(0) > 0$ (and that are therefore consumed by someone) form an indecomposable system. It follows that $v'(y_k(0); \xi_0)$ must be the same positive quantity for all k such that $\phi_k(0) > 0$.[52]

If $v(y; \xi)$ is strictly convex, as assumed in the lemma, then $v'(y; \xi_0)$ is an increasing function of y, and the conclusion of the previous paragraph is only possible if $y_k(0)$ is the same quantity for all k such that $\phi_k(0) > 0$. Moreover, the Inada condition on the function $u(c; \xi_0)$ implies that (C.2) cannot have $y_k(0) = 0$ as a solution, for any $\phi_k(0) > 0$. Hence the common value for $y_k(0)$ for all of the sectors that produce must be positive.

Condition (C.2) further implies that if $y_k(0)$ is the same for every sector that produces, $y_k(0)/\phi_k(0)$ must also be the same for each of these sectors. And since the common value of $y_k(0)$ is positive, this is only possible if $\phi_k(0)$ is the same for each of the sectors with $\phi_k(0) > 0$. This establishes the lemma.

Note that if instead $v(y; \xi_0)$ is a linear function of y (one of the cases considered in Proposition 5), $v'(y; \xi_0)$ will be the same for all $y > 0$. Hence in this case, the conclusion reached above does not follow: it is possible to have different levels of production $y_k(0)$ in different sectors that all produce in period zero, and still achieve the first-best allocation of resources. In this case, we do not need to assume that $\phi_k(0)$ is the same for each of

[52] Note however that the value of $v'(y_k(0); \xi_0)$ can be different for a sector with $\phi_k(0) = 0$, if one exists, since in that case the ratio $c_k^j(0)/(\alpha_{k-j}\phi_k(0))$ is undefined, and (C.1) need not be satisfied for this value of k.

the sectors with $\phi_k(0) > 0$ in order for the first-best allocation to be achievable under an appropriate policy, as Proposition 5 shows.

C.2 Proof of Proposition 5

We shall show that the equilibrium allocation of resources under this policy is given by

$$c_k^j(t) \;=\; \alpha_{k-j}\phi_k(t)\bar{y}, \qquad y_k(t) = \phi_k(0)\bar{y}$$

for any j, k, and any period $t \geq 0$. The associated equilibrium prices are $p_k(t) = P^*(t)$ for each k in any period $t \geq 0$, and the nominal interest rate $i(t)$ is given by (25) in each period $t \geq 0$. The associated path of beginning-of-period nominal asset balances is given by $a^j(t) = a(t)/N$ for each sector j in any period $t \geq 1$, while end-of-period asset balances are given by $b^j(t) = a(t)/N$ for each sector in any period $t \geq 0$.

Note that if this is indeed an equilibrium under the proposed policy, (40) holds in period zero, as asserted in the proposition; equilibrium prices and quantities in all periods $t \geq 1$ are as specified in Proposition 1; and borrowing constraints do not bind in any sector in any period, since

$$b^j(t) \;=\; a(t)/N \;>\; 0 \;\geq\; \underline{b}^j(t)$$

for any sector j in any period $t \geq 0$.

Thus if the conjectured allocation is indeed an equilibrium, it remains only to show that this allocation of resources is the first-best optimal allocation defined in Lemma 2. It is easily verified that the proposed allocation satisfies (9) for all j, k, and t. In addition, the proposed output levels in any period $t \geq 1$ are $y_k(t) = \bar{y}$ for all k, and these satisfy (8) for each k and any $t \geq 1$. When $t = 0$, the proposed allocation specifies that $y_k(0) = 0$ for any sector with $\phi_k(0) = 0$, as is required by Lemma 2. Thus we need only verify that the proposed output levels in each of the sectors with $\phi_k(0) > 0$ also satisfy (8) in period $t = 0$.

Under case (i) of the hypothesis, this condition becomes

$$u'(y_k(0)/\phi_k(0);\, \bar{\xi}) \;=\; \nu,$$

which is satisfied by the proposed allocation for each k such that $\phi_k(0) > 0$, since in the case of these preferences, $u'(\bar{y}; \bar{\xi}) = \nu$. Under case (ii) of the hypothesis, the condition becomes

$$u'(\bar{y}; \bar{\xi}) \;=\; v'(\bar{y}; \bar{\xi})$$

for every sector k with $\phi_k(0) = 1$, and this condition is satisfied by the definition of $\bar{\xi}$. Thus the proposed allocation satisfies all of the conditions stated in Lemma 2 for the first-best optimal allocation.

We then need only to show that the quantities and prices proposed in the first paragraph above do indeed constitute a perfect foresight equilibrium, under the policy specified in the proposition. This can be established using the same method as in the proof of Proposition 1. Because the proof is straightforward (following the proof strategy already illustrated in the earlier proof), we omit the details. We confine ourselves here to a sketch of the intuition for the result. First, the proposed end-of-period balances $\{b^j(0)\}$ imply that $\tilde{a}^j(1)$ is the same for each sector j, so that (24) implies that $\boldsymbol{f} = \mathbf{0}$ under the conjectured paths. And we have already shown in Lemma 3 that if $\boldsymbol{f} = \boldsymbol{0}$, the prices and quantities specified above represent a perfect foresight equilibrium from period $t = 1$ onward. Second, an equilibrium in which $\boldsymbol{f} = \boldsymbol{0}$ going into period $t = 1$ must have the equilibrium allocations described in Proposition 4, even if $\beta < 1$; the assumption that $\beta \to 1$ was only used in the proof of that proposition to guarantee that $\boldsymbol{f} = \boldsymbol{0}$. Third, the initial asset balances $\{a^j(0)\}$ specified in (42) imply that conditions (39) are satisfied for all sectors, in the case of any borrowing limits $\underline{b}^j(0) \leq 0$. Hence the same argument as is used in the proof of Corollary 3 can again be used to show that no sectors are borrowing-constrained in period zero, and that the equilibrium allocation of resources in period zero must be given by (32). And finally, the initial balances together with this pattern of spending and production in period zero imply the specified end-of-period balances $\{b^j(0)\}$, and hence that $\boldsymbol{f} = \boldsymbol{0}$. Thus we obtain an equilibrium of the conjectured form.

C.3 Proof of Lemma 6

Let $C \in \mathcal{C}$ be the set of constrained sectors in the case of initial liquidity $\boldsymbol{\delta}$. (Because our definition specifies that $c^j(0)$ is strictly less than c^{*j} for each of the constrained sectors, these will continue to be the constrained sectors in the case of any small enough increase in the vector of initial assets.) Then we know, for each sector j, which of the two terms on the right-hand side of (34) $c^j(0)$ is equal to; this allows us to replace the nonlinear equation system (35) by a system of linear equations, that must hold locally at $\boldsymbol{\delta}$ and for any alternative vector $\boldsymbol{\delta'} \geq \boldsymbol{\delta}$ close enough to it.

If we order the sectors so that all of the sectors in C (if any) come first, and partition the matrix $\boldsymbol{A}$ as in (43), then the local version of (35) can be written as

$$\begin{bmatrix} \hat{\boldsymbol{c}} \\ \check{\boldsymbol{c}} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\delta}} \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{A}_{CC} & \boldsymbol{A}_{CU} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{c}} \\ \check{\boldsymbol{c}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{c}}^* \\ \check{\boldsymbol{c}}^* \end{bmatrix}.$$

Here $\hat{\boldsymbol{c}}$ is the vector of expenditures $c^j(0)$ for the sectors $j \in C$, $\check{\boldsymbol{c}}$ is the vector of expenditures for sectors $j \notin C$; the vectors $\hat{\boldsymbol{c}}^*$ and $\check{\boldsymbol{c}}^*$ similarly collect the values of c^{*j} for the two groups of sectors; and $\hat{\boldsymbol{\delta}}$ similarly collects the values of $\delta^j/\bar{p}$ for the sectors $j \in C$.

For any $C \in \mathcal{C}$, we have already shown in the proof of Proposition 4 that $\boldsymbol{A}_{CC}$ has all of its eigenvalues inside the unit circle. It then follows that the matrix $\boldsymbol{I} - \boldsymbol{A}_{CC}$ must be invertible, and that its inverse can be expressed as the infinite sum on the right-hand side of (44). Hence the linear local system of equations has a unique solution of the form (37), where the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ are the ones given in the statement of the lemma. Moreover, the sub-matrix $\boldsymbol{M}_{CC}$ can alternatively be written as in (44).

Since $\boldsymbol{A}_{CC} \geq \boldsymbol{0}$, it follows from (44) that $\boldsymbol{M}_{CC} \geq \boldsymbol{0}$ as well. This together with the fact that $\boldsymbol{A}_{CU} \geq \boldsymbol{0}$ implies that all elements of the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ must be non-negative.

C.4 Fiscal transfer multipliers: An example

Our model implies that the multiplier effects of fiscal transfers can be quite different, depending both on the sectors receiving the transfer and which sectors' expenditure we are concerned with. As an example, consider again the chain network with fraction λ of out-of-sector purchases (right panel of Figure 1), and consider the case of a pandemic shock which makes it impossible to consume the output of sector 1, while other sectors' products are unaffected. Suppose that we compute the multipliers for small additional transfers, starting from liquid asset balances that satisfy the inequality

$$\sum_{j \neq N} a^j(0) \ < \ \lambda \bar{p} \bar{y}. \tag{C.3}$$

When this inequality is satisfied, sector N is the only unconstrained sector (as already established in section B.5, for the limiting case in which $a(0) \to 0$), so that the set C consists of sectors $\{1, 2, \ldots, N-1\}$. The matrix of transfer multipliers is in this case equal to

$$\boldsymbol{M} \ = \ \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ \lambda^{-1} & \lambda^{-1} & 0 & \ldots & 0 & 0 \\ \lambda^{-1} & \lambda^{-1} & \lambda^{-1} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda^{-1} & \lambda^{-1} & \lambda^{-1} & \ldots & \lambda^{-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

The aggregate expenditure multiplier for transfers to sector k can be obtained by summing column k of the matrix $\boldsymbol{M}$; this is largest (equal to $1 + (N-2)/\lambda$) for transfers to sector 1, and smallest (zero) for transfers to sector N. In the numerical example discussed in the paper, $N = 5$ and $\lambda = 4/5$. In this case, the aggregate expenditure multiplier for a transfer to sector 1 is

$$1 \, + \, \lambda^{-1} \, + \, \lambda^{-1} \, + \, \lambda^{-1} + 0 \ = \ 4.75,$$

as reported in the main text. The multiplier effect of uniformly distributed transfers on sector j spending can be obtained by averaging the elements of row j of the matrix; these are largest (equal to $(N-1)/(\lambda N)$) for sector $N-1$, and smallest (again zero) for sector N. In the numerical example discussed in the paper, the aggregate expenditure multiplier for uniformly distributed transfers is the average over the five columns of the sums of all five rows, or

$$\frac{1}{5}[4.75 + 3.75 + 2.50 + 1.25 + 0.00] \ = \ 2.45, \tag{C.4}$$

as reported in the main text.

In the event that fewer sectors are borrowing-constrained, additional rows and columns of $\boldsymbol{M}$ must be set to zero. We continue to have a matrix of the form

$$\boldsymbol{M} \ = \ \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ \lambda^{-1} & \lambda^{-1} & 0 & \ldots & 0 & 0 \\ \lambda^{-1} & \lambda^{-1} & \lambda^{-1} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

but if $\hat{a}_{i-1} \leq a(0) < \hat{a}_i$, for any $i \geq 1$, the last i rows of the matrix are all zeroes. Thus when $i = 1$, as assumed above, the aggregate expenditure multiplier for uniformly distributed transfers is given by (C.4). But if $i = 2$, it is instead equal to only

$$\frac{1}{5}[3.50 + 2.50 + 1.25 + 0.00 + 0.00] \ = \ 1.45;$$

if $i = 3$, it is equal to only

$$\frac{1}{5}[2.25 + 1.25 + 0.00 + 0.00 + 0.00] \;=\; 0.70;$$

and if $i = 4$, it is equal to only

$$\frac{1}{5}[1.00 + 0.00 + 0.00 + 0.00 + 0.00] \;=\; 0.20.$$

And of course, once $a(0) \geq \hat{a}_4$, the entire matrix of multipliers is equal to zero.

The differentiation of alternative multipliers according to the sector affected (the separate rows of $\boldsymbol{M}$) is relevant for calculation of the welfare effects of transfer policies, since the marginal utility of additional spending varies across sectors. For example, when (C.3) holds, the marginal utility of additional real expenditure by any sector $j \neq N$ is given by

$$\mu^j \;\equiv\; u'(c^j(0)/(1-\alpha_{1-j}) \;=\; u'(\lambda^{-1}\sum_{k=1}^{j} a^k(0)),$$

while for sector N it is

$$\mu^N \;\equiv\; u'(c^N(0)/(1-\alpha_1)) \;=\; u'(\bar{y}).$$

It follows that in the case of any $\boldsymbol{a}(0) >> \mathbf{0}$ satisfying (C.3),

$$\mu^1 \;>\; \mu^2 \ldots \mu^{N-1} \;>\; \mu^N.$$

The welfare effect of transfers to each of the different sectors is then given not by $\boldsymbol{e}'\boldsymbol{M}$ (the vector of column sums), but by $\boldsymbol{\mu}'\boldsymbol{M}$, where $\boldsymbol{\mu}$ is the N-vector with jth element equal to μ^j.

C.5 Proof of Corollary 4

Given a one-parameter family of transfer policies (45), let $c^j(s)$ be the solution for $c^j(0)$ in the case of the transfer policy indexed by s. It follows from (38) that

$$c^j(s) \;=\; \min_{C \in \mathcal{C}} c^{j,loc}(\boldsymbol{\alpha} + \boldsymbol{\gamma} \cdot s;\, C).$$

It follows from Lemma 6 that each of the functions $c^{j,loc}(\boldsymbol{\alpha} + \boldsymbol{\gamma} \cdot s;\, C)$ is a non-increasing linear function of s. Since $c^j(s)$ is the minimum of a finite collection of such functions, it must be a non-increasing, piecewise-linear, concave function. This then implies that its right derivative, $m_j(s)$, must a piecewise constant function, non-increasing in s, and everywhere non-negative in value.

It remains only to show that $m_j(s)$ is eventually equal to zero for all s above some finite bound. But since $c^j(s)$ can have only a finite number of segments with different slopes, there must be some finite $\bar{s}$ such that $m_j(s)$ is constant for all $s \geq \bar{s}$. Suppose that this terminal value is positive. It would follow that $c^j(s)$ would be an increasing linear function of s for all $s \geq \bar{s}$. But it follows from (30) that $c^j(s)$ must be bounded. Hence we obtain a contradiction, and can conclude that instead $m_j(s)$ must equal zero for all $s \geq \bar{s}$, as asserted in the corollary.

C.6 Proof of Lemma 7

For any choice of the parameters specifying policy in period 1, and any value of β close enough to 1, there must be an equilibrium of the following kind. First, the equilibrium for all $t \geq 1$ is a stationary equilibrium of the kind characterized in Lemma 3, for some vector $\boldsymbol{f} \in U$. (The stipulation that β be close enough to 1 is in order to ensure that $\boldsymbol{f} \in U$.) Second, for each sector j, the consumption plan in period $t = 0$ must be of the form (4) for some choice of $c^j(0)$. This allows us to compute end-of-period balances $b^j(0)$ as a function of $c^j(0)$, so that constraint (12) implies an upper bound for $c^j(0)$, that depends on the policy specification (both on $a^j(0)$ and on $\underline{b}^j(0)$). Total expenditure $c^j(0)$ must also satisfy

$$u'(c^j(0)/\omega_j; \bar{\xi}) \;\geq\; \psi\Lambda^{*j}(\boldsymbol{f}), \tag{C.5}$$

as a consequence of (27). This also implies an upper bound for $c^j(0)$ that depends on policy (in particular, that depends on the monetary policy parameter ψ). Since at least one of the constraints (12) and (C.5) must hold with equality, the optimal choice of $c^j(0)$ must be the minimum of these two upper bounds, for each sector j. Third, the vector $\boldsymbol{f}$ must satisfy (24), where

$$\tilde{a}^j(1) \;=\; \psi\frac{b^j(0)}{\bar{p}}$$

for each sector j, and we can compute $b^j(0)$ from the sector's choice of $c^j(0)$, as just indicated. This gives us a fixed-point relationship, $\boldsymbol{f} = \boldsymbol{\Psi}(\boldsymbol{f})$, that the vector $\boldsymbol{f}$ must satisfy. The mapping $\boldsymbol{\Psi}$ is defined for any $\boldsymbol{f} \in U$.

In the limiting case $\beta = 1$, this fixed-point relationship becomes a mapping $\boldsymbol{\Psi}(\boldsymbol{f}) = \mathbf{0}$ for all $\boldsymbol{f}$, and there exists a unique fixed point, $\boldsymbol{f}^* = \mathbf{0}$. For any values of β close enough to 1, the mapping will still be a contraction, and will have a unique fixed point $\boldsymbol{f}^*$ near $\mathbf{0}$. Hence there is a perfect foresight equilibrium of the kind proposed above. Moreover, in the limit as $\beta \to 1$, the fixed point $\boldsymbol{f}^* \to \mathbf{0}$. Lemma 4 then implies that the stationary allocation in periods $t \geq 1$ approaches the one characterized in Proposition 1 (for the case in which $\xi_t = \bar{\xi}$ each period), which is to say, the stationary allocation that represents the first-best optimum when $\xi_t = \bar{\xi}$.

It then follows that $\bar{U}^j \to U^*$ in this limit, so that (47) must hold, as asserted in the lemma. This in turn implies that (48) provides an equivalent welfare ranking of alternative policies in this limit.

We also note that Lemma 4 implies that $\Lambda^{*j}(\boldsymbol{f}) \to u'(\bar{y}; \bar{\xi})$ in the limit as $\beta \to 1$, so that the upper bound implied by (C.5) approaches

$$c^j(0) \;\leq\; c^{*j}(\psi) \;\equiv\; \psi\omega_j\bar{y}, \tag{C.6}$$

a generalization of (30). The vector $\boldsymbol{c}(0)$ of equilibrium spending levels in period zero is then the solution to the fixed-point system (35), given by (38), in which we now substitute the more general definition of c^{*j} given in (C.6). This result is useful for characterizing the effects of interest-rate policy, discussed further below.

C.7 Proof of Proposition 6

Substitution of (2) into the welfare measure (48), and simplification using the fact that in any equilibrium the consumption plan for each sector must be of the form (4), allows us to write the welfare measure in the form

$$W_0 \;=\; \sum_{j=1}^{N} \omega_j G(c^j(0)/\omega_j) \;+\; \sum_{k=1}^{N} H(y_k(0)), \tag{C.7}$$

where

$$G(c) \;\equiv\; u(c;\bar{\xi})) \;-\; u'(\bar{y};\bar{\xi}) \cdot c, \qquad H(y) \;\equiv\; v'(\bar{y};\bar{\xi}) \cdot y \;-\; v(y;\bar{\xi}).$$

We can then use our solutions for the effects of transfers on $c^j(0)$ and $y_k(0)$ to calculate the welfare gradient.

Differentiating (C.7) with respect to each element of the vector of transfers, we obtain equation (49) given in the main text, where $\boldsymbol{g}$ is the vector with elements

$$g_j \;=\; G'(c^j(0)/\omega_j)$$

for each j, and $\boldsymbol{h}$ is the vector with elements

$$h_k \;=\; H'(y_k(0))$$

for each k. (We use that $\boldsymbol{M}$ is the matrix of expenditure multipliers and $\boldsymbol{M}^Y = \boldsymbol{AM}$ is the matrix of output multipliers.) Differentiating the functions G and H, we obtain the expressions for g_j and h_k given in the main text.

We further observe that $G(c)$ is a strictly concave function, that reaches its (unique) maximum at $c = \bar{y}$, and that $H(y)$ is another (at least weakly) concave function, which achieves its maximum value at $y = \bar{y}$ (though the maximum need not be unique). It follows that g_j must be positive for all $c^j(0)/\omega_j < \bar{y}$ (or for all $c^j(0) < c^{*j} \equiv \omega_j \bar{y}$, as stated in the text), and similarly that h_k must be non-negative for all $y_k(0) \leq \bar{y}$. Since the elements of $\boldsymbol{M}$ and $\boldsymbol{M}^Y$ are all non-negative, we obtain the result that the elements of the welfare gradient must all be non-negative if we consider additional transfers at a point where $c^j(0) \leq c^{*j}$ for all j and $y_k(0) \leq \bar{y}$ for all k.

When $\psi = 1$, the Euler condition (29) must hold. Then because of (30), we must have $c^j(0) \leq c^{*j}$ for all j in all cases. This in turn implies that $c^j_k(0) \leq \alpha_{k-j}\phi_k(0)\bar{y}$ for all j, k, using (4), and hence that $y_k(0) \leq \phi_k(0)\bar{y}$ for all k, as a consequence of (14). (The reasoning is the same as in the derivation of (B.9). Under the further assumption that $0 \leq \phi_k(0) \leq 1$ for all sectors, we must have $y_k(0) \leq \bar{y}$ for all k, and all elements of the welfare gradient must be non-negative.

Now consider the effects of a transfer to sector j only, meaning that the vector of transfers is proportional to $\boldsymbol{e}_j$, the vector with 1 as its jth element, and all other elements equal to 0. If $j \notin C$ (sector j is not borrowing-constrained), it follows from Lemma 6 that $\boldsymbol{Me}_j = \boldsymbol{0}$, as a consequence of which $w_j \equiv \boldsymbol{w}'\boldsymbol{e}_j = 0$. If instead $j \in C$, it follows from (44) that $M_{jj} > 0$, and hence that

$$w_j \;\equiv\; \boldsymbol{w}'\boldsymbol{e}_j \;\geq\; \boldsymbol{g}'\boldsymbol{Me}_j \;\geq\; g_j M_{jj} \;>\; 0.$$

C.8 Proof of Proposition 7

The proof proceeds in the same way as in the proof of Proposition 5. In the earlier proof, the key to establishing existence of an equilibrium with the first-best optimal allocation of resources was demonstrating that the assumed policies were consistent with (i) an equilibrium for periods $t \geq 1$ of the kind characterized in Lemma 3 with $\boldsymbol{f} = \mathbf{0}$; and (ii) an equilibrium in period $t = 0$ in which no sectors are borrowing-constrained, so that the equilibrium allocation of resources is given by (40). In the limit as $\beta \to 1$, we must have $\boldsymbol{f} \to \mathbf{0}$ regardless of the nature of equilibrium in period $t = 0$, so that condition (i) is now more easily established.

Given an equilibrium for periods $t \geq 1$ with $\boldsymbol{f} = \mathbf{0}$, the Euler condition takes the form (29), which implies that the equilibrium in period zero must satisfy (30). It then follows, as in the proof of Corollary 3, that (39) is a sufficient condition for the existence of an equilibrium in period zero in which borrowing constraints do not bind in any sector. One can then establish that the equilibrium must involve the first-best optimal allocation of resources using the same argument as in the proof of Proposition 5.

D Equilibrium with Interest-Rate Policy

Here we present proofs of the main results in section IV of the main text.

D.1 Proof of Proposition 8

The proof follows exactly the same lines as the proof of Proposition 3, but taking into account the fact that (29) now takes the more general form (C.5). As before, the assumption that initial asset balances are negligible and that borrowing is impossible implies that the equilibrium for periods $t \geq 1$ must be of the kind characterized in Lemma 3, for the case in which $\boldsymbol{f} = \mathbf{0}$. The equilibrium in period $t = 0$ must be one in which $b^j(0) \to 0$ for each sector j, with the consequence that $\boldsymbol{c}(0)$ must be a multiple of the maximal eigenvector $\boldsymbol{\pi}$, as in Proposition 3. The fact that the inequality (51) must hold for each sector, and with equality for at least one sector, then implies that the multiplicative factor Ω must be given by (53).

D.2 Proof of Corollary 5

It follows from Proposition 8 that the allocation of resources in periods $t \geq 1$ is independent of the choice of ψ; hence (7) is maximized by the policy that maximizes the single-period welfare criterion (48). Using (2) and (4), we can write (48) in the form

$$W_0 \;=\; \sum_j \left[\omega_j u \left(\frac{c^j(0)}{\omega_j}; \bar{\xi} \right) \;-\; v(y_j(0); \bar{\xi}) \right]. \tag{D.1}$$

Here both the quantities $\{c^j(0)\}$ and $\{y_j(0)\}$ depend on ψ purely through the effect of ψ on the value of $\hat{y}(\psi)$. Hence we can reduce the problem of choosing ψ to maximize (7) to the problem of choosing $\hat{y}$ to maximize (D.1). (The optimal ψ will then be whatever value is required in order for $\hat{y}(\psi)$ to equal the optimal value of $\hat{y}$.)

Proposition 8 implies that each of the quantities $\{c^j(0)\}$ and $\{y_j(0)\}$ is a non-negative multiple of $\hat{y}(\psi)$; in addition, at least one of the $\{c^j(0)\}$ and at least one of the $\{y_j(0)\}$ are positive, and hence strictly increasing function of $\hat{y}$. It then follows from (D.1) that W_0

must be a strictly concave function of ψ, with a continuous derivative. Hence $W_0(\psi)$ must have a unique maximum, which must furthermore be the unique point consistent with the first-order condition that $\partial W_0/\partial\hat{y}$ be equal to zero.

Differentiating (D.1) with respect to $\hat{y}$, we obtain

$$\begin{aligned}
\frac{\partial W_0}{\partial \hat{y}} &= \sum_j \left[u'\left(\frac{c^j(0)}{\omega_j};\, \bar{\xi}\right)\frac{\partial c^j(0)}{\partial \hat{y}} \;-\; v'(y_j(0);\, \bar{\xi})\frac{\partial y_j(0)}{\partial \hat{y}}\right] \\
&= \sum_j \left[u'\left(\frac{c^j(0)}{\omega_j};\, \bar{\xi}\right)\frac{\partial c^j(0)}{\partial \hat{y}}\right] \;-\; \nu(\bar{\xi})\cdot\sum_k \frac{\partial y_k(0)}{\partial \hat{y}} \\
&= \sum_j \left\{\left[u'\left(\frac{c^j(0)}{\omega_j};\, \bar{\xi}\right) - \nu(\bar{\xi})\cdot\sum_k A_{kj}\right]\frac{\partial c^j(0)}{\partial \hat{y}}\right\} \\
&= \sum_j \left\{\left[u'\left(\frac{c^j(0)}{\omega_j};\, \bar{\xi}\right) - \nu(\bar{\xi})\right]\frac{\partial c^j(0)}{\partial \hat{y}}\right\}, \qquad \text{(D.2)}
\end{aligned}$$

using the assumed form for $v(y;\xi)$.

The corollary assumes that each sector j will belong either to a set of borrowing-constrained sectors C with $c^j(0) = 0$ (which requires that $\pi_j = 0$), or to the complementary set of sectors U, for which (51) holds with equality. (Note that under the assumptions of this corollary, the set of borrowing-constrained sectors is the same for all values of ψ.) For any $j \in C$, the fact that $\pi_j = 0$ means that $\partial c^j(0)/\partial\hat{y} = 0$, using the solution for $c^j(0)$ in Proposition 8. Hence these terms contribute nothing to the sum in (D.2). Moreover, the fact that (51) holds with equality for every sector $j \in U$ allows us to write the remaining terms in the form

$$\frac{\partial W_0}{\partial \hat{y}} = \left[u'\left(\hat{y};\, \bar{\xi}\right) - \nu(\bar{\xi})\right]\sum_{j\in U}\frac{\partial c^j(0)}{\partial \hat{y}}. \qquad \text{(D.3)}$$

The solution in Proposition 8 implies that $c^j(0)$ is a linearly increasing function of $\hat{y}$ for each of the sectors $j \in U$ (and there must be at least one such sector); hence the sum of the $\partial c^j(0)/\partial\hat{y}$ terms must be positive. Moreover, the fact that $u(c;\bar{\xi})$ is strictly concave implies that $u'\left(\hat{y};\,\bar{\xi}\right)$ must be a decreasing function of $\hat{y}$. Therefore the expression in square brackets is positive if $\hat{y} < \bar{y}$ and negative if $\hat{y} > \bar{y}$. It follows that $\partial W_0 \partial\hat{y}$ is positive for all $\hat{y} < \bar{y}$ and negative for all $\hat{y} > \bar{y}$.

From this we can conclude that W_0 is (uniquely) maximized when ψ is chosen so that $\hat{y}(\psi) = \bar{y}$, which holds if and only if $\psi = 1$ (the policy assumed in Proposition 3).

D.3 Proof of Proposition 9

The result that the equilibrium for periods $t \geq 1$ is the same as in Proposition 1 can be established in the same way as in the proof of Proposition 4, since it is once again the case that in the limit as $\beta \to 1$, we must have $\boldsymbol{f} \to \boldsymbol{0}$, regardless of the nature of the equilibrium allocation in period $t = 0$.

The demonstration that the system of equations (54) has a unique solution for $\boldsymbol{c}(0)$ also proceeds in the same way as in the proof of Proposition 4. In fact, for any given specification of policy, the equations (and hence their solution) are exactly the same, except that the vector of parameters $\boldsymbol{c}^* >> \boldsymbol{0}$ in the previous discussion is now replaced by the vector $\boldsymbol{\omega} \cdot \hat{y}(\psi)$, which indicates how the elements of the vector depend on the choice of ψ. (The more specific assumption that $\boldsymbol{c}^* = \boldsymbol{\omega} \cdot \bar{y}$ was never used in the derivation of (38), only the fact that the elements of the vector were all positive.) Hence we obtain a unique solution $\boldsymbol{c}(\boldsymbol{\delta}; \psi)$, given by our previous solution (38), but with the above substitution for the vector $\boldsymbol{c}^*$.

Finally, the fact that the right-hand side of (54) is homogeneous of degree one in $(\boldsymbol{c}(0), \boldsymbol{\delta}, \hat{y})$ implies that if some values $(\boldsymbol{c}(0), \boldsymbol{\delta}, \hat{y})$ satisfy the equation, the values $(\lambda \boldsymbol{c}(0), \lambda \boldsymbol{\delta}, \lambda \hat{y})$ must satisfy it as well, for any multiplicative factor $\lambda > 0$. This means that if $\boldsymbol{c}(0) = \boldsymbol{c}$ is a solution to (54) in the case of a policy that implies parameters $(\boldsymbol{\delta}, \hat{y})$, then $\boldsymbol{c}(0) = \lambda \boldsymbol{c}$ will be a solution in the case of a policy that implies parameters $(\lambda \boldsymbol{\delta}, \lambda \hat{y})$. Thus the function $\boldsymbol{c}(\boldsymbol{\delta}; \psi)$ must be a homogeneous degree one function of $(\boldsymbol{\delta}, \hat{y}(\psi))$.

From this it follows that the function can be written in the form

$$\boldsymbol{c}(\boldsymbol{\delta}; \psi) \;=\; \frac{\hat{y}(\psi)}{\bar{y}} \cdot \bar{\boldsymbol{c}} \left(\frac{\bar{y}}{\hat{y}(\psi)} \boldsymbol{\delta} \right),$$

where

$$\bar{\boldsymbol{c}}(\boldsymbol{\delta}) \;\equiv\; \boldsymbol{c}(\boldsymbol{\delta};\, 1).$$

Moreover, when $\psi = 1$, the system (54) reduces to (35), the system for which we have already determined the solution. Hence the function $\bar{\boldsymbol{c}}(\boldsymbol{\delta})$ must be the one defined in (38).

Given this solution for the sectoral expenditure levels, the fact that prices are predetermined the level $p_k(0) = \bar{p}$ for all k implies that the complete allocation of resources in period zero is given by (4) and (14).

D.4 Proof of Corollary 6

It follows from the form of the solution for $\boldsymbol{c}(0)$ in Proposition 9, together with (38), that we can write

$$\begin{aligned} \boldsymbol{c}(0) &= \frac{\hat{y}(\psi)}{\bar{y}} \cdot \min_{C \in \mathcal{C}} \boldsymbol{c}^{loc} \left(\frac{\bar{y}}{\hat{y}(\psi)} \boldsymbol{\delta};\, C \right) \\ &= \frac{\hat{y}(\psi)}{\bar{y}} \cdot \min_{C \in \mathcal{C}} \left\{ \frac{\bar{y}}{\hat{y}(\psi)} \boldsymbol{M}(C) \boldsymbol{\delta} \,+\, \boldsymbol{N}(C) \boldsymbol{c}^* \right\} \\ &= \min_{C \in \mathcal{C}} \Big\{ \boldsymbol{M}(C) \boldsymbol{\delta} \,+\, \boldsymbol{N}(C) \boldsymbol{\omega} \cdot \hat{y}(\psi) \Big\}, \end{aligned} \tag{D.4}$$

where the notation $\boldsymbol{M}(C)$, $\boldsymbol{N}(C)$ indicates that the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ in (37) depend on the choice of the set C. The expression on the final line indicates that each of the solutions $c^j(0)$ is the minimum of a finite collection of non-decreasing linear functions of $\hat{y}(\psi)$. It follows that $c^j(0)$ must be a non-decreasing, piecewise linear, concave function of $\hat{y}(\psi)$.

Since $y^{agg}(0) \equiv \sum_k y_k(0) = \sum_j c^j(0)$, aggregate output is the sum of a finite collection of non-decreasing, piecewise linear, concave functions of $\hat{y}(\psi)$, and hence must itself be a non-decreasing, piecewise linear, concave function of $\hat{y}(\psi)$. Moreover, for every partition $C \in \mathcal{C}$, the sectors $j \in U_0$ (which must include at least one sector) are among the unconstrained sectors. It then follows from Lemma 6 that for any $j \in U_0$, the coefficient

$$\boldsymbol{e}_j' \boldsymbol{N}(C)\boldsymbol{\omega} \ = \ \omega_j \ > \ 0$$

for any $C \in \mathcal{C}$, so that $c^j(0)$ must be increasing as $\hat{y}$ increases, no matter how large $\hat{y}$ may be. Since at least one such sector exists, the sum $y^{agg}(0) = \sum_j c^j(0)$ must be strictly increasing as well. Thus $y^{agg}(0)$ is an increasing, piecewise linear, concave function of $\hat{y}(\psi)$, as asserted in the corollary.

Each of the quantities $c_k^j(0)$, $y_k(0)$, for arbitrary j, k, is similarly a linear combination of the different elements of $\boldsymbol{c}(0)$ with non-negative weights; and so by the same kind of argument, we can show that each of these quantities is a non-decreasing, piecewise linear, concave function of $\hat{y}(\psi)$, as asserted in the corollary.

Let us next consider how the set C of borrowing-constrained sectors changes as we lower ψ, increasing $\hat{y}(\psi)$. The system (54) implies that for each sector j,

$$c^j(0) \ = \ \min\{L^j(\hat{y}(\psi);\, C(\psi)),\, \omega_j \hat{y}(\psi)\}$$

where $C(\psi)$ is the set of borrowing-constrained sectors in the case of interest-rate policy ψ, and for each possible choice of C, $L^j(\hat{y}; C)$ is the function

$$L^j(\hat{y}; C) \ = \ \frac{\delta^j}{\bar{p}} \ + \ \boldsymbol{e}_j' \boldsymbol{A} \boldsymbol{c}^{loc}(\hat{y}; C), \tag{D.5}$$

where we use the notation

$$\boldsymbol{c}^{loc}(\hat{y}; C) \ \equiv \ \boldsymbol{M}(C)\boldsymbol{\delta} \ + \ \boldsymbol{N}(C)\boldsymbol{\omega} \cdot \hat{y}(\psi)$$

for the functions of $\hat{y}$ on the right-hand side of (D.4) associated with the different possible choices of C.

Consider a value of ψ at which a particular set of sectors $C(\psi) = C_1$ are borrowing-constrained. This means that for sectors $j \in C_1$, $L^j(\hat{y}(\psi); C_1) < \omega_j \hat{y}(\psi)$, while for the unconstrained sectors, $L^j \geq \omega_j \hat{y}$. And now consider whether the set $C(\psi)$ remains equal to C_1 if ψ is reduced relative to this initial value. In order for this to be the case, it must continue to be true that $L^j(\hat{y}(\psi); C_1) < \omega_j \hat{y}(\psi)$ for sectors $j \in C_1$, while $L_j \geq \omega_j \hat{y}(\psi)$ continues to hold for all of the other sectors.

We see from (D.5) that for each j, $L^j(\hat{y}(\psi); C_1)$ is the sum of a non-negative constant plus a non-negative term proportional to $\hat{y}(\psi)$; it follows that $L^j(\hat{y}(\psi); C_1$ either grows in proportion to the growth of $\hat{y}(\psi)$ (in the case that $\delta^j = 0$), or less than proportionally to the growth of $\hat{y}(\psi)$ (in the case that $\delta^j > 0$), when ψ is reduced. In either case, the fact that $L^j(\hat{y}(\psi); C_1) < \omega_j \hat{y}(\psi)$ initially for sectors $j \in C_1$ means that this inequality must continue to hold for any smaller value of ψ, given that $\omega_j \hat{y}(\psi)$ grows in proportion to $\hat{y}(\psi)$. Hence it cannot be the case that any of the borrowing-constrained sectors ceases to be borrowing-constrained as a result of a reduction in ψ.

On the other hand, it is possible that a reduction in ψ will eventually increase $\hat{y}(\psi)$ sufficiently for one of the initially unconstrained sectors to cease to be unconstrained. If so, at that point the set $C(\psi)$ must change; but the change must always be an increase in the set of borrowing-constrained sectors. The sectors $j \in C_1$ that were previously borrowing-constrained must continue to be constrained. Thus as ψ is reduced, the set $C(\psi)$ can only be increasing, as asserted in the corollary.

Finally, we consider the behavior of the solution to the system (54) when the real interest rate is very low. We begin by noting that one of the functions $\boldsymbol{c}^{loc}(\hat{y}; C)$ on the right-hand side of (D.4) is the one associated with the partition $C = C_0$. In the case of this choice of the set of constrained sectors, the definition of the eigenvector $\boldsymbol{\pi}$ implies that

$$(\boldsymbol{I} - \boldsymbol{A}_{CC})\boldsymbol{\pi}_C \;=\; \boldsymbol{A}_{CU}\boldsymbol{\pi}_U,$$

and hence that

$$\boldsymbol{\pi} \;=\; \left[\begin{array}{c} \boldsymbol{N}_{CU} \\ \boldsymbol{N}_{UU} \end{array}\right] \boldsymbol{\pi}_U. \tag{D.6}$$

Moreover, for each of the sectors $j \in U_0$, it follows from (31) that

$$\pi_j \;=\; [\max_{\ell}(\pi_\ell/\omega_\ell)] \cdot \omega_j;$$

hence we have

$$\boldsymbol{\pi}_U \;=\; [\max_{\ell}(\pi_\ell/\omega_\ell)] \cdot \boldsymbol{\omega}_U.$$

Substitution of this into (D.6) yields

$$\boldsymbol{\pi} \;=\; [\max_{\ell}(\pi_\ell/\omega_\ell)] \cdot \left[\begin{array}{c} \boldsymbol{N}_{CU} \\ \boldsymbol{N}_{UU} \end{array}\right] \boldsymbol{\omega}_U. \tag{D.7}$$

Hence

$$\boldsymbol{N}(C_0)\boldsymbol{\omega} \;=\; \left[\begin{array}{c} \boldsymbol{N}_{CU} \\ \boldsymbol{N}_{UU} \end{array}\right] \boldsymbol{\omega}_U \;=\; \frac{\boldsymbol{\pi}}{\max_\ell(\pi_\ell/\omega_\ell)},$$

and the local solution $\boldsymbol{c}^{loc}(\hat{y}(\psi);\, C_0)$ is given by the formula on the right-hand side of (56). It remains to show that this choice of C is the one that yields the lowest values for all elements of $\boldsymbol{c}(0)$, in the case of any large enough value of $\hat{y}(\psi)$.

But we know from Proposition 4 that the solution $\bar{\boldsymbol{c}}(\boldsymbol{\delta})$ is a continuous function of $\boldsymbol{\delta}$; thus there must exist a neighborhood $\mathcal{N}$ of $\mathbf{0}$ such that for any $\boldsymbol{\delta} \in \mathcal{N}$, the solution $\bar{\boldsymbol{c}}(\boldsymbol{\delta})$ has the same set of constrained sectors as the solution for $\boldsymbol{\delta} = \mathbf{0}$. And when $\boldsymbol{\delta} = \mathbf{0}$, the solution to (35) is the expenditure vector characterized in Proposition 3, for which the set of constrained sectors is C_0. Thus for any $\boldsymbol{\delta} \in \mathcal{N}$, the choice of C that results in the lowest value for all of the elements of $\boldsymbol{c}(0)$ must be C_0.

It follows that in the solution (55), $\boldsymbol{c}(0)$ must be given by the local solution corresponding to the choice $C = C_0$ (i.e., by the right-hand side of (56)) for all values of ψ such that $(\bar{y}/\hat{y}(\psi))\boldsymbol{\delta} \in \mathcal{N}$. We have already shown that $\hat{y}(\psi)$ can be made arbitrarily large by choosing a small enough value of ψ. Hence we can choose a $\underline{\psi} > 0$ such that $\psi < \underline{\psi}$ implies that $(\bar{y}/\hat{y}(\psi))\boldsymbol{\delta} \in \mathcal{N}$. This is then a bound such that the solution for $\boldsymbol{c}(0)$ must be the one given in (56) for any $\psi < \underline{\psi}$, as asserted in the corollary.

D.5 Proof of Corollary 7

The proof follows the same lines as the proof of Corollary 5. It follows from Proposition 9 that the allocation of resources in periods $t \geq 1$ is independent of the choice of ψ. Hence (7) is again maximized by the policy that maximizes the single-period welfare criterion (D.1), and we can again reduce the problem of choosing ψ to maximize (7) to the problem of choosing $\hat{y}$ to maximize (D.1). The main new complication that arises here comes from the fact that now the set of borrowing-constrained sectors can change as we vary ψ, for a given specification of $\boldsymbol{\delta}$.

Corollary 6 implies that each of the quantities $\{c^j(0)\}$ and $\{y_j(0)\}$ is a non-decreasing, concave function of $\hat{y}$; and whenever ψ increases, at least one of the $\{c^j(0)\}$ and at least one of the $\{y_j(0)\}$ must increase. It then follows from (D.1) that W_0 must be a strictly concave function of ψ. It is no longer the case that the derivative $\partial W_0/\partial \hat{y}$ must be continuous at all points; but W_0 must have well-defined left and right derivatives at all points, and these must be equal at all but some finite number of values of ψ (the ones at which the set of borrowing-constrained sectors changes). The strict concavity of $W_0(\psi)$ continues to imply that the function must have a unique maximum. This must furthermore be the unique point consistent with the first-order condition: W_0 is maximized at ψ if (a) either the left or right derivative $\partial W_0/\partial \hat{y}$ is equal to zero at ψ, or (b) the left derivative is greater than zero at ψ while the right derivative is less than zero.

Differentiating (D.1) with respect to $\hat{y}$, we again obtain (D.2). This expression is correct for either the left or right derivative of W_0, as long as one understands each of the to be correspondingly left or right derivatives. We wish to evaluate the sign of both the left and right derivative at the value $\psi = 1$.

Let C and U be the sets of constrained and unconstrained sectors respectively, in the solution (38) when $\psi = 1$ and $\boldsymbol{\delta}$ has the specified value. For any $j \in C$, we must have

$$c^j(0) \;=\; \frac{\delta^j}{\bar{p}} + \sum_k A_{jk} c^k(0) \;<\; \omega_j \hat{y}(\psi)$$

when $\psi = 1$, and since the solution is continuous in ψ, this will continue to be true for all ψ in a neighborhood of 1. Under the hypothesis that $\boldsymbol{A}_{CU} = \mathbf{0}$, the equality reduces to

$$c^j(0) \;=\; \frac{\delta^j}{\bar{p}} + \sum_{k \in C} A_{jk} c^k(0)$$

for each $j \in C$. As discussed in the proof of Lemma 6, this system of equations can be uniquely solved for values

$$c^j(0) \;=\; \frac{1}{\bar{p}} \boldsymbol{e}_j' \boldsymbol{M}_{CC} \hat{\boldsymbol{\delta}}.$$

This will be the solution for all values of ψ in the neighborhood of 1 where these sectors continue to be borrowing-constrained.

Hence we have $\partial c^j(0)/\partial \hat{y} = 0$ for all $j \in C$, as both the left and right derivative at any ψ in the neighborhood of 1. It follows that near 1 we can express (D.2) more simply as

$$\frac{\partial W_0}{\partial \hat{y}} \;=\; \sum_{j \in U} \left\{ \left[u'\left(\frac{c^j(0)}{\omega_j}; \bar{\xi} \right) - \nu(\bar{\xi}) \right] \frac{\partial c^j(0)}{\partial \hat{y}} \right\}.$$

Again, the fact that (51) holds with equality for every sector $j \in U$ allows us to write this in the simpler form (D.3).

The derivatives $\partial c^j(0)/\partial \hat{y}$ are all non-negative, and there must be at least one sector $j \in U$ for which both the left and right derivative are positive; hence the sum of the $\partial c^j(0)/\partial \hat{y}$ terms must be positive, whether a left or right derivative is considered. Thus regardless of whether we consider the left or right derivative, the sign of $\partial W_0/\partial \hat{y}$ must be the same as the sign of $u'\left(\hat{y}; \bar{\xi}\right) - \nu(\bar{\xi})$. This is positive if $\hat{y} < \bar{y}$, negative if $\hat{y} > \bar{y}$, and exactly equal to zero if and only if $\hat{y} = \bar{y}$ exactly.

From this we again conclude that W_0 is (uniquely) maximized when ψ is chosen so that $\hat{y}(\psi) = \bar{y}$, which holds if and only if $\psi = 1$ (the policy assumed in Proposition 4).