Online Appendix

Bargaining over a Divisible Good in the Market for Lemons

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A. Appendix for online publication

A.1 Proof of Lemma 1

Proof. We show that (a) holds for the weaker solution concept of PBE. For any $K \in \{1, \ldots, m\}$ and for any PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$, let $H_K(\sigma_B, (\sigma_L, \sigma_H), \beta)$ denote the set of histories h^t with $K(h^t) = K$ and $\beta(h^t) = 0$.

We show first that (a) holds when only one unit remains. Let $\bar{u}_L$ denote the supremum, over all PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$, of the low-type seller's continuation payoff at histories $h^t \in H_1(\sigma_B, (\sigma_L, \sigma_H), \beta)$. Assume towards a contradiction that $\bar{u}_L > 0$ and take $\varepsilon = \left(\frac{1-\delta}{2}\right) \bar{u}_L$. There must exist a PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$ and a history $\bar{h}^t \in H_1(\sigma_B, (\sigma_L, \sigma_H), \beta)$ at which the buyer offers $\varphi_t = (1, p)$ for some $p \in [\bar{u}_L - \varepsilon, \bar{u}_L]$. The low-type seller must accept this offer with probability one. To see why, notice that if the low-type seller rejects this offer with positive probability, then $(\bar{h}^t, (\varphi_t, R)) \in H_1(\sigma_B, (\sigma_L, \sigma_H), \beta)$ and therefore the low-type seller's continuation payoff is at most $\bar{u}_L$. But then, since $\bar{u}_L - \varepsilon > \delta \bar{u}_L$, it is not optimal for the low-type seller to reject φ_t. For the same reason, the low-type seller must accept the offer $\varphi'_t = \left(1, \bar{u}_L - \frac{3}{2}\varepsilon\right)$ with probability one. Thus, the buyer has a profitable deviation at $\bar{h}^t$ since he strictly prefers the offer φ'_t to φ_t.

We show next that (a) holds for any number of remaining units K. We proceed by induction. Fix $K \in \{2, \ldots, m\}$ and assume that for any PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$ and for any $h^t \in H_1(\sigma_B, (\sigma_L, \sigma_H), \beta) \cup \ldots \cup H_{K-1}(\sigma_B, (\sigma_L, \sigma_H), \beta)$, the low-type seller"s continuation payoff is zero. Again, let $\bar{u}_L$ denote the supremum, over all PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$, of the low-type seller's continuation payoff at histories $h^t \in H_K(\sigma_B, (\sigma_L, \sigma_H), \beta)$. Towards a contradiction, assume that $\bar{u}_L > 0$ and take $\varepsilon = \left(\frac{1-\delta}{2}\right)\bar{u}_L$. There must exist a PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$ and a history $\bar{h}^t \in H_K(\sigma_B, (\sigma_L, \sigma_H), \beta)$ at which the buyer offers $\varphi_t = (k, p)$ for some $p \in [\bar{u}_L - \varepsilon, \bar{u}_L]$ and some $k \leq K$. Using the induction hypothesis and an argument similar to the one presented in the previous paragraph, we conclude that the low-type seller must accept this offer with probability one. However, the same is true for the offer $\varphi'_t = \left(k, \bar{u}_L - \frac{3}{2}\varepsilon\right)$ which is, therefore, strictly preferred to φ_t. Again, this shows that the buyer has a profitable deviation at $\bar{h}^t$ and concludes the proof of part (*a*) of Lemma 1.

We show (*b*) by contradiction. Assume that there exist two histories h^t and $\tilde{h}^{t'}$ with the same state variables but with $V_B(h^t) < V_B\left(\tilde{h}^{t'}\right)$. The buyer then has a profitable deviation after history h^t. He can choose the same actions as he chooses after history $\tilde{h}^{t'}$. Since the seller's strategy depends only on state variables, then he reacts as he does after history $\tilde{h}^{t'}$, and so the buyer's continuation payoff increases.

We show (*c*) by contradiction. Assume instead that there is a history h^t where the high-type seller obtains a positive continuation payoff: $V_H(h^t) > 0$. Over all histories with positive continuation payoffs, pick those with the smallest number of remaining units $\underline{K} = \min\left\{K(h^t) : V_H(h^t) > 0\right\}$. Let $\alpha = \sup\left\{V_H(h^t) : K(h^t) = \underline{K}\right\}$ denote an upper bound for the high-type seller's continuation payoff when only $\underline{K}$ units remain. Finally, let $\varepsilon \equiv (1-\delta)\alpha/3$.

There must exist a history h^t with $K(h^t) = \underline{K}$ at which the buyer makes an offer (k, p) that the high-type seller accepts, and the offer satisfies $1 \leq k \leq \underline{K}$ and $p > \frac{c}{m}k + \alpha - \varepsilon$. This in turn implies that the low-type seller also accepts this offer (otherwise, by Lemma 1(*a*), he gets a total payoff of zero). Consider instead the following deviation by the buyer; he offers $\left(k, \frac{c}{m}k + \alpha - \varepsilon\right)$. If the high-type seller rejects this offer, he obtains a continuation payoff of at most $\delta\alpha < \alpha - \varepsilon$, so he accepts it. For the same reason as above, the low-type seller also accepts this offer. Both the original offer and the deviation lead to the same state variables, and therefore to the same continuation payoff to the buyer, as shown in Lemma 1(*b*). This implies that the deviation is profitable. This shows part (*c*) of Lemma 1.

Consider next part (*d*) of Lemma 1. Whenever $\beta(h^t) = 0$, the result follows immediately from Lemma 1(*a*). Otherwise, the zero bound on the continuation payoff for the high type seller directly implies a $\frac{c}{m}K(h^t)$ upper bound for the continuation payoff for the low-type seller. ■

A.2 Proof of Lemma 2

Proof. In the case $\beta(h^t) = 0$ all units are traded in the first period (this follows immediately from Lemma 1(a)). Assume instead that $\beta(h^t) > 0$ and consider an offer $\varphi_t = (k, p)$ with $k < K(h^t)$ and $p < \frac{c}{m}k$. We show that such an offer is not accepted with positive probability. By contradiction, assume that this offer is accepted with positive probability. A high-type seller would never accept such an offer, so it must be the low-type seller who accepts this offer with probability $\sigma_L^t(h^t, \varphi_t) > 0$.

A rejection then leads to a posterior $\beta' \in \big(\beta(h^t), 1\big)$. Whenever the low-type seller accepts, the buyer immediately learns that the seller is of low type. Then, in the following period all remaining units are traded, at zero cost. The buyer obtains the following payoff from this offer:

$$[1 - \beta(h^t)]\sigma_L^t(h^t, \varphi_t) \left[\sum_{s=K(h^t)-k+1}^{K(h^t)} \Lambda_s^m v_L - p + \delta \sum_{s=1}^{K(h^t)-k} \Lambda_s^m v_L \right] \\ + \left[1 - \beta(h^t) \left(1 - \sigma_L^t(h^t, \varphi_t)\right)\right] V_L(\beta', K)$$

Consider instead an offer to pay p in exchange for *all* remaining units. If the low-type seller accepts, he obtains the same payoff as from accepting the previous offer. Moreover, because of stationarity, a rejection leads to the same belief β' as before. Then, the low-type seller accepts this offer with the same probability as the previous offer. The buyer, however, obtains the following *higher* payoff from this offer:

$$[1 - \beta(h^t)]\sigma_L^t(h^t, \varphi_t) \left[\sum_{s=K(h^t)-k+1}^{K(h^t)} \Lambda_s^m v_L - p + \sum_{s=1}^{K(h^t)-k} \Lambda_s^m v_L \right] \\ + \left[1 - \beta(h^t) \left(1 - \sigma_L^t(h^t, \varphi_t)\right)\right] V_L(\beta', K)$$

Then, if an offer for $k < K(h^t)$ remaining units was accepted with positive probability, the buyer would rather make an offer for all remaining units, so there would be a profitable deviation. ■

A.3 Proof of Proposition 2

In this proof we consider a good divided into a fixed number of units equal to m a fixed period length equal to Δ. We thus suppress the dependence of all variables on m and Δ.

The proof is divided into two parts. In Part A we define the notion of a consistent quadruplet $(\mathcal{V}_L, P, W, y)$ of intertwined functions. We show that whenever a consistent quadruplet $(\mathcal{V}_L, P, W, y)$ exists, then a stationary PBE must exist. Our proof is constructive: we derive equilibrium strategies and beliefs from the consistent quadruplet. In Part B we construct a consistent quadruplet $(\mathcal{V}_L, P, W, y)$.

Part A. The consistent quadruplet $(\mathcal{V}_L, P, W, y)$

We first describe the components of the quadruplet $(\mathcal{V}_L, P, W, y)$. The function $\mathcal{V}_L(K, q)$: $\{1, \ldots, m\} \times [0, \hat{q}] \to \mathbb{R}$ determines the strategy of the low-type seller, as described in the definition of stationary PBE. The function $P(K, q) : \{1, \ldots, m\} \times [0, \hat{q}] \to \mathbb{R}$ pins down the screening offer $(K, P(K, q))$ that induces (transformed) posterior belief q if rejected. The function $W(K, q) : \{1, \ldots, m\} \times [0, \hat{q}] \to \mathbb{R}$ represents the buyer's (normalized) continuation payoff. Finally, the function $y(K, q) : \{1, \ldots, m\} \times [0, \hat{q}] \to \{1, \ldots, m\} \cup [0, \hat{q}]$ specifies the offers that the buyer makes on the equilibrium path.

Part A contains four steps. The first three define the notion of a consistent quadruplet $(\mathcal{V}_L, P, W, y)$. In step 1 we derive the function P from the function $\mathcal{V}_L$. In step 2 we turn to the buyer's optimization problem. We take as given the behavior of the low-type seller, which is summarized by P. We define the buyer's value function W and his best response correspondence. From this best response correspondence, in step 3 we select the offer $y(K, q)$ that the buyer makes in state (K, q). We construct a candidate value function $\mathcal{V}_L'$ for the low-type seller from the functions y and P. Finally, we say that the quadruplet $(\mathcal{V}_L, P, W, y)$ is *consistent* if $\mathcal{V}_L' = \mathcal{V}_L$.

In step 4 we construct strategies from the consistent quadruplet $(\mathcal{V}_L, P, W, y)$ and show that these strategies (together with appropriate beliefs) form a stationary PBE.

Step 1. From $\mathcal{V}_L$ to P. Consider a (left-continuous) candidate function $\mathcal{V}_L$ with $0 \leq \mathcal{V}_L(K,q) \leq \frac{c}{m}K$ for all (K,q). This function determines the low-type seller's behavior, following the definition of stationary PBE.[1] This same definition also pins down the high-type seller behavior: he accepts any offer for k units if and only if he receives in exchange a payment greater or equal than $\frac{c}{m}k$.

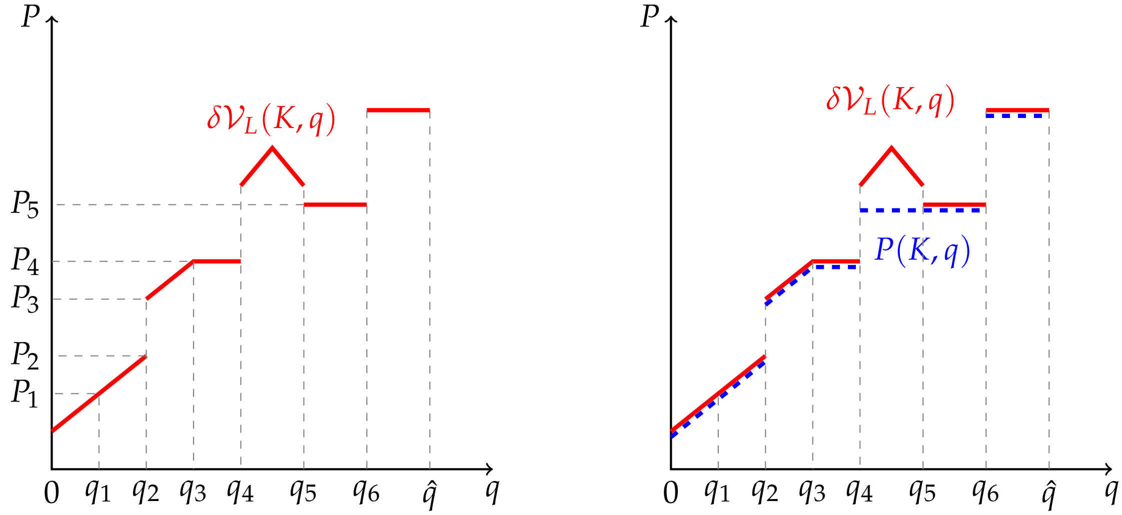
We study the buyer's best response to the seller's behavior implied by $\mathcal{V}_L(K,q)$. We can restrict attention to two types of offers: universal and screening. Universal offers are simple: the buyer offers a payment $\frac{c}{m}k$ for some (or all) remaining units $k \leq K$, both sellers accept and beliefs do not change.

Screening offers involve both a price and a transformed posterior belief. A price induces a probability of acceptance, which in turn leads to a transformed posterior belief after the offer is rejected. As we show below, different prices may induce the same posterior. Moreover, there may be some posteriors that no price can induce. We define a modified problem where the buyer who starts a period with a (transformed) belief $q \in [0, \hat{q}]$ can induce any (transformed) posterior belief $q' \in [q, \hat{q}]$ by choosing a unique price $P(K, q')$. We show in step 4 that solutions to the modified problem coincide with those of the original one.

We first illustrate how we derive $P(K,q)$ from $\mathcal{V}_L(K,q)$ and then provide the formal definition of $P(K,q)$. Consider the function $\delta\mathcal{V}_L(K,q)$ shown in Figure 1(a). It is simple to see that the price $P_1 = \delta\mathcal{V}_L(K,q_1)$ induces posterior belief q_1. This is because the function $\delta\mathcal{V}_L(K,q)$ lies above P_1 for posteriors greater than q_1. In fact, obtaining $P(K,q)$ would be straightforward if $\mathcal{V}_L(K,q)$ was continuous and strictly increasing. However, consider for example posterior belief q_2, which is induced by all prices in the range $[P_2, P_3]$. The buyer's preferred price in that range is the lowest: P_2; and thus we set $P(K,q_2) = P_2$.

The set of induced beliefs may be non-convex. The price P_4 induces posterior belief q_4, but no price induces posterior beliefs on the range $[q_3, q_4)$. To restore convexity, in the

[1]The function $\mathcal{V}_L(K,q)$ maps one-to-one to a function $\mathcal{V}_L(K,\beta) : m \times [\hat{\beta}, 1] \to \mathbb{R}$. The definition of stationary PBE pins down the behavior of the low-type seller through the function $\mathcal{V}_L(K,\beta)$.



(a) Multiplicity and non-convexity

(b) The function $P(K,q)$

Figure 1: Derivation of $P(K,q)$ from $\mathcal{V}(K,q)$

modified problem we allow the buyer to induce any belief $q \in [q_3, q_4)$ by paying the price $P(K,q) = P_4$. Similarly, the buyer cannot induce posterior beliefs in the range (q_4, q_6). We allow the buyer to induce any belief $q \in (q_4, q_6)$ by paying the price $P(K,q) = P_5$. Differently than before, $P(K,q) < \delta\mathcal{V}_L(K,q)$ for the interval $q \in (q_4, q_5]$.

Formally, we let $P(K,q)$ be the largest weakly increasing function below $\delta\mathcal{V}_L(K,q)$. As an example, the dashed line in Figure 1(b) depicts the function $P(K,q)$ derived from $\delta\mathcal{V}_L(K,q)$ in Figure 1(a). Whenever the buyer can induce a posterior q but cannot induce posteriors in some range $(q-\eta, q)$, our definition implies that $P(k,q') = \delta\mathcal{V}_L(K,q)$ for all $q' \in (q-\eta, q)$. By doing so, the function $P(K,q)$ becomes flat in some region. Claim 1 in step 4 shows that the buyer never chooses interior points in flat regions, which guarantees that the solutions to the modified problem coincide with those of the original one.

Step 2. From P to W. The buyer's modified problem. We now formalize the buyer's (modified) dynamic optimization problem. With a slight abuse of notation, let $V_B(K,q)$ denote the buyer's continuation payoff when the state is (K,q). For convenience, we work directly with the buyer's *normalized* continuation payoff

$$W(K,q) \equiv (1-q)V_B(K,q).$$

We set $W(0,q)=0$ and

$$W(K,\hat{q}) = (1-\hat{q})\left[\left(\sum_{s=1}^{K}\Lambda_s^m\right)v_H - \frac{c}{m}K\right].$$

For all other cases, we define $W(K,q)$ recursively by:

$$\begin{aligned} W(K,q) = \max\Bigg\{ & \max_{q'\in[q,\hat{q}]} \overbrace{(q'-q)\left[\left(\sum_{s=1}^{K}\Lambda_s^m\right)v_L - P(K,q')\right] + \delta W(K,q')}^{(*)\text{ Screening. Offer } P(K,q') \text{ for } K \text{ units. If rejected, induced belief is } q'}, \\ & \max_{0\le k\le K-1} \underbrace{\left\{\left(\sum_{s=k+1}^{K}\Lambda_s^m\right)[(\hat{q}-q)v_L + (1-\hat{q})v_H] - (1-q)\frac{c}{m}(K-k) + \delta W(k,q)\right\}}_{(**)\text{ Universal offer. Request next } K-k \text{ units in exchange for payment } \frac{c}{m}(K-k)} \Bigg\} \end{aligned} \quad (1)$$

The first component $(*)$ of equation (1) provides the continuation payoff when the buyer induces belief q' through a screening offer. The second component $(**)$ of equation (1) provides the continuation payoff when the buyer makes a universal offer for $K-k$ units. The buyer compares the value of the best screening offer (optimal q') with the value of the best universal offer (optimal k) to choose which kind of offer to make.[2]

Equation (1) defines the buyer's *modified* problem. When the state is (K,q) with $q\in[0,\hat{q})$ we allow the buyer to induce any state (K,q') with $q'\ge q$ by making the screening offer $(K, P(K,q'))$. This includes states that cannot be reached in the original game, like (K,q_5) in Figure 1.

Let $Y(K,q)$ denote the set of solutions to the problem in equation (1). A screening offer that induces posterior q' is of the form $(K, P(K,q'))$. When such offer is optimal, we let $q'\in Y(K,q)$. A universal offer for $K-k$ units is of the form $\left(K-k, \frac{c}{m}(K-k)\right)$. When such offer is optimal, we let $k\in Y(K,q)$.

Step 3. From P and W to y and $\mathcal{V}_L'$. The notion of consistent quadruplet. We combine the low-type seller's behavior, implicit in P, with the buyer's optimal behavior to construct a *candidate* value function $\mathcal{V}_L'(K,q)$ for the low-type seller. Let $\mathcal{V}_L'(K,q)$ be de-

[2] The buyer's continuation payoff is always positive, so his individual rationality constraint is satisfied. To see this, note that the buyer can always choose $q'=q$ in equation (1).

fined recursively by:

$$\mathcal{V}_L'(K,q) = \min\left\{\min_{q'\in Y(K,q)} P\left(K,q'\right), \min_{k\in Y(K,q)} \frac{c}{m}(K-k) + \delta\mathcal{V}_L'(k,q)\right\} \tag{2}$$

As equation (2) shows, we construct $\mathcal{V}_L'$ by always selecting the offer that minimizes the low-type seller's continuation payoff from all of the buyer's optimal choices $Y(K,q)$. Let $y(K,q) \in Y(K,q)$ denote the buyer's choice that solves (2). There may be many solutions to (2), but if so, one of them is universal.[3] In such case, we let $y(K,q)$ be the universal offer associated to the lowest k.

Finally, we say that a quadruplet $(\mathcal{V}_L, P, W, y)$ is *consistent* if its components are linked as described in steps 1 to 3 and if the derived $\mathcal{V}_L'$ satisfies $\mathcal{V}_L' = \mathcal{V}_L$.

Step 4. From the consistent quadruplet $(\mathcal{V}_L, P, W, y)$ to a stationary PBE.

a. Definition of strategies and beliefs. Fix a consistent quadruplet $(\mathcal{V}_L, P, W, y)$. Our definition of stationary PBE, together with $\mathcal{V}_L$, fully pins down the seller's strategy. Both types accept with probability one any offer (k,p) with $p \geq \frac{c}{m}k$. The high-type seller rejects offers (k,p) with $p < \frac{c}{m}k$ with probability one, while the low-type seller accepts them with probability pinned down by $\mathcal{V}_L$.

We next specify the buyer's strategy and beliefs. We first define for each t a set of histories $\widehat{H}^t$ that is not reached on the equilibrium path. We say that $h^t \in \widehat{H}^t$ whenever h^t contains either 1) a rejected offer (k,p) with $p \geq \frac{c}{m}k$, or 2) an accepted partial offer. Whenever $h^t \in \widehat{H}^t$, we let the buyer assign probability zero to the seller being of high type. Also, we let the buyer offer a payment of zero for all remaining units after any history $h^t \in \widehat{H}^t$.[4]

If instead $h^t \notin \widehat{H}^t$, the buyer's offer depends on the state $\left(K(h^t), q(h^t)\right)$ and on the actions (φ_{t-1}, a_{t-1}) in $t-1$. The buyer's strategy and beliefs are as follows:

1. If $(\varphi_{t-1}, a_{t-1}) = ((k,p), A)$ with $p \geq \frac{c}{m}k$, then the belief is unchanged: $q(h^t) =$

[3] To see why, assume that $P(K,q') = P(K,\tilde{q}')$ for $q' \in Y(K,q)$ and $\tilde{q}' \in Y(K,q)$. Since $P(K,q)$ is weakly increasing, then $P(K,q)$ is constant between q' and $\tilde{q}'$. But this cannot happen; Claim 1 shows that the buyer never chooses interior points in flat regions of $P(k,q)$.

[4] The set $\widehat{H}^t$ contains some but not all off-path histories. Below we specify the buyer's strategy and beliefs for all histories on path, and also for the remaining off-path histories.

$q(h^{t-1})$. The buyer makes the offer implied by $y(K(h^t), q(h^t))$.

2. If $(\varphi_{t-1}, a_{t-1}) = ((k, p), R)$ with $p < \frac{c}{m}k$, then

   a. If $p \leq P\left(K(h^{t-1}), q(h^{t-1})\right)$, then the belief is unchanged: $q(h^t) = q(h^{t-1})$. The buyer makes the offer implied by $y(K(h^t), q(h^t))$.

   b. If $p > P\left(K(h^{t-1}), q(h^{t-1})\right)$ and $p = P\left(K(h^{t-1}), q\right)$ for some $q > q(h^{t-1})$, then the belief $q(h^t)$ is given by the probability of acceptance implied in the definition of stationary PBE. The buyer makes the offer implied by $y(K(h^t), q(h^t))$.

   c. If $p > P\left(K(h^{t-1}), q(h^{t-1})\right)$ and $p \neq P\left(K(h^{t-1}), q\right)$ for all $q > q(h^{t-1})$, then the belief $q(h^t)$ is given by the probability of acceptance implied in the definition of stationary PBE. The buyer randomizes among the elements of $Y(K(h^t), q(h^t))$ to rationalize the probability of acceptance of the low-type seller in $t-1$.[5]

b. Verification that strategies and beliefs form a stationary PBE. The strategy of the high-type seller is optimal. On-path, the buyer never pays more than $\frac{c}{m}k$ for any k. Then, it is optimal to accept any offer greater or equal than $\frac{c}{m}k$ for any k and to reject otherwise.

The optimality of the low-type seller's strategy follows from $\mathcal{V}_L = \mathcal{V}'_L$. Assume that the buyer and the seller follow the equilibrium strategies specified above. Then, in any on-path history h^t with state $(K, q) = \left(K(h^t), q(h^t)\right)$ the function $\mathcal{V}_L(K, q)$ satisfies:

$$\mathcal{V}_L(K, q) = \begin{cases} \frac{c}{m}(K-k) + \delta\mathcal{V}_L(k, q) & \text{if } y(K, q) = k \\ P\left(K, q'\right) = \delta\mathcal{V}_L\left(K, q'\right) & \text{if } y(K, q) = q' \end{cases} \tag{3}$$

Equation (3) follows from the definition of $\mathcal{V}'_L$ in equation (2), the equality $\mathcal{V}'_L = \mathcal{V}_L$, the definition of $P(K, q)$ and the fact that the buyer never chooses an induced posterior in a

[5]Supoose that $p > P\left(K(h^{t-1}), q(h^{t-1})\right)$, $p \neq P\left(K(h^{t-1}), q\right)$ for all $q > q(h^{t-1})$ and that the new belief is $q(h^t)$. Then, $\delta\mathcal{V}_L\left(K(h^t), q(h^t)\right) < p < \delta \lim_{q \downarrow q(h^t)} \mathcal{V}_L\left(K(h^t), q\right)$. One element of $Y\left(K(h^t), q(h^t)\right)$ yields a continuation payoff of $\mathcal{V}_L\left(K(h^t), q(h^t)\right)$ to the low-type seller, while another one yields a continuation payoff of $\lim_{q \downarrow q(h^t)} \mathcal{V}_L\left(K(h^t), q\right)$ to the low-type seller. In period t the buyer randomizes between these two elements of $Y\left(K(h^t), q(h^t)\right)$ so that the low-type seller's continuation payoff in period $t-1$ (if he rejects the screening offer) is exactly p. Note that this implies that off-the-equilibrium path the low-type seller's continuation payoff may depend not only on the state but also on the offer in the previous period.

flat region of $P(K,q)$. Therefore, $\mathcal{V}_L(K,q)$ is the on-path continuation payoff of the low-type seller.

The low-type seller obtains a continuation payoff of zero if he rejects a universal offer. The first line of equation (3) shows that he obtains a strictly positive payoff if he instead accepts it. Then, it is optimal for the low-type seller to accept a universal offer.[6] The second line of equation (3) shows that the low-type seller is indifferent between accepting and rejecting the screening offers that the buyer makes on path. Consider instead a buyer who deviates and makes a partial offer $(k, P(K,q'))$ with $k < K$. If the low-type seller accepts, he obtains $P(K,q')$ in the current period and zero from then on. If he instead rejects, his continuation payoff is $\delta\mathcal{V}_L\left(K,q'\right)$. Thus, the low-type seller is also willing to randomize in this case.[7]

We construct the strategy of the buyer by choosing for every history h^t elements from the set $Y\left(K(h^t), q(h^t)\right)$ of best responses in the modified problem. The difference between the original and modified problem lies in the set of posteriors that screening offers can induce. While in the modified problem the buyer can induce the whole set of posteriors $[q,\hat{q}]$ at any state (K,q), the set of posteriors that he can induce in the original game may be limited. Claim 1 shows that the best response correspondence $Y\left(K,q\right)$ in the modified problem only induces posteriors that are feasible in the original game.

CLAIM 1. THE BUYER NEVER CHOOSES A POSTERIOR IN A FLAT REGION OF $P(K,\cdot)$. *If $q' \in Y(K,q)$, then $P(K,q'') > P(K,q')$ for every $q'' > q'$.*

See Section T.2 of the Technical Addendum for the proof.

This proves that the strategy of the buyer is optimal.

Part B. Construction of the consistent quadruplet $(\mathcal{V}_L, P, W, y)$

We construct a consistent quadruplet $(\mathcal{V}_L, P, W, y)$ through two processes of induction (and a fixed point argument). In the base step of the first process of induction we construct the quadruplet $(\mathcal{V}_L(1,\cdot), P(1,\cdot), W(1,\cdot), y(1,\cdot))$, which deals with the case when only one

[6] For this same reason it is optimal for the low-type seller to accept any offer (k,p) with $p > \frac{c}{m}k$.

[7] The buyer could also deviate by making an offer (k,p) with $k \leq K$ and $p \neq P(K,q')$. The equilibrium strategies that we define also guarantee that the low-type seller behaves optimally. We omit the details.

unit remains. In the inductive step there are K units left, with $1 < K \leq m$. We assume that the quadruplet $(\mathcal{V}_L(k,\cdot), P(k,\cdot), W(k,\cdot), y(k,\cdot))$ has already been constructed for all $k \in \{1,\ldots,K-1\}$ and construct the quadruplet $(\mathcal{V}_L(K,\cdot), P(K,\cdot), W(K,\cdot), y(K,\cdot))$.

The second process of induction is nested within the first one. We explain this process in detail in steps 1 to 3 below. Let K be the number of remaining units and assume that the quadruplet $(\mathcal{V}_L(k,\cdot), P(k,\cdot), W(k,\cdot), y(k,\cdot))$ has already been constructed for all $k \in \{1,\ldots,K-1\}$. In the base step, we construct $(\mathcal{V}_L(K,\cdot), P(K,\cdot), W(K,\cdot), y(K,\cdot))$ for $q \in [\overline{q}, \hat{q}]$ for some $\overline{q} < \hat{q}$ (see step 1 below). In the inductive step (indexed by n), we assume that the quadruplet $(\mathcal{V}_L(K,\cdot), P(K,\cdot), W(K,\cdot), y(K,\cdot))$ has already been constructed for $q \in [q_n, \hat{q}]$. We extend $(\mathcal{V}_L(K,\cdot), P(K,\cdot), W(K,\cdot), y(K,\cdot))$ to $q \in [q_{n+1}, \hat{q}]$ with $q_{n+1} < q_n$ (we explain this in step 2a below). This inductive step involves a fixed point argument that we describe in detail in step 2b. Finally, we show that in a finite number ($\tilde{n}$) of steps $q_{\tilde{n}} = 0$ (step 3 below).

Step 1. Quadruplet in interval $q \in [\overline{q}, \hat{q}]$. Claim 2 describes the simple form that the quadruplet $(\mathcal{V}_L, P, W, y)$ takes when transformed beliefs are sufficiently close to $\hat{q}$. The intuition behind Claim 2 is simple. If the buyer is sufficiently convinced that the seller is of high type, he is better off trading right away. He offers to pay the high type's cost in exchange for all remaining units. Both types accept and the game ends. This leads directly to the quadruplet's form in Claim 2.

CLAIM 2. *There exists $\overline{q} < \hat{q}$, such that any consistent quadruplet $(\mathcal{V}_L, P, W, y)$ must satisfy*

$$\begin{aligned}
\mathcal{V}_L(K,q) &= \frac{c}{m}K, \\
P(K,q) &= \delta\frac{c}{m}K, \\
W(K,q) &= \sum_{s=1}^{K} \Lambda_s^m \left[(\hat{q}-q)\,v_L + (1-\hat{q})\,v_H\right] - (1-q)\frac{c}{m}K > 0 \quad \textit{and} \\
y(K,q) &= K
\end{aligned}$$

for every $q \in [\overline{q}, \hat{q}]$ and for every $K \in \{1,\ldots,m\}$.

Proof. Assume that there are K remaining units. A buyer who makes a screening offer obtains a (normalized) continuation payoff bounded above by

$$(\hat{q}-q)\sum_{s=1}^{K}\Lambda_s^m v_L + (1-\hat{q})\,\delta\left(\sum_{s=1}^{K}\Lambda_s^m v_H - \frac{c}{m}K\right).$$

Moreover, for a sufficiently high $q < \hat{q}$, the expression above is strictly smaller than

$$\sum_{s=1}^{K}\Lambda_s^m\left[(\hat{q}-q)\,v_L + (1-\hat{q})\,v_H\right] - (1-q)\frac{c}{m}K$$

which represents the continuation payoff for the buyer when he makes a universal offer for all remaining units. This continuation payoff is strictly positive for sufficiently high $q < \hat{q}$. This, in turn, implies that there exists $\bar{q} < \hat{q}$ such that for any $q \in [\bar{q}, \hat{q}]$ and for any $K \in \{1, \ldots, m\}$, screening offers are strictly dominated by a universal offer for all remaining units, and this universal offer leads to strictly positive payoffs. Therefore, the best universal offer is to buy all units immediately, which leads to the expressions for W and y outlined above. These expressions, in turn, imply that $\mathcal{V}_L$ and P are as above. ■

Step 2. Extension of quadruplet from interval $[q_n, \hat{q}]$ to interval $q \in [q_{n+1}, \hat{q}]$. The extension of the quadruplet consists of two sub-steps. In the first one (a), we only allow the buyer to make screening offers. We find an interval $[q_{n+1}, q_n]$ where the optimal screening offer induces posterior belief above q_n. If universal offers were not allowed (i.e., if there were only one unit left, as in DL), this would conclude the extension to $[q_{n+1}, q_n]$. In the second sub-step (b), we give the buyer the possibility of making universal offers. This modifies the low-type seller's continuation payoff – and therefore the function $P(K, \cdot)$ – in the interval $[q_{n+1}, q_n]$. We allow the buyer to re-optimize, given the modified function $P(K, \cdot)$, which in turn changes the low-type seller's continuation payoff. We continue this process until we reach a fixed point.

a. Only screening offers. Fix the number of remaining units K. Assume that the quadruplet $(\mathcal{V}_L(k, \cdot), P(k, \cdot), W(k, \cdot), y(k, \cdot))$ is already defined for all $1 \leq k \leq K-1$ and that the quadruplet $(\mathcal{V}_L(K, \cdot), P(K, \cdot), W(K, \cdot), y(K, \cdot))$ is defined for $q \in [q_n, \hat{q}]$.

We define two auxiliary value functions for the buyer that represent continuation payoffs from making screening offers. First, for $q \in [0, q_n]$ we let $W^I(K, q)$ represent the buyer's payoff from making a screening offer that leads to posterior $q' \geq q_n$:

$$W^I(K,q) = \max_{q' \geq q_n} \left(q' - q\right) \left(\sum_{s=1}^{K} \Lambda_s^m v_L - P\left(K,q'\right)\right) + \delta W\left(K,q'\right) \tag{4}$$

Let $X(K,q) \in [q_n, \hat{q}]$ denote the set of solutions to the above maximization problem, and let $\underline{x}(K,q)$ and $\overline{x}(K,q)$ denote respectively the smallest and largest elements of $X(K,q)$.

Second, let $P^I(K,q) = \delta P\left(K, \underline{x}(K,q)\right)$ denote an auxiliary pricing function for $q \in [0, q_n]$. The function $W^{II}(K,q)$ represents the buyer's payoff from making a screening offer $\left(K, P^I(K,q)\right)$ that leads to posterior $q' \in [q, q_n]$ (and to a continuation payoff W^I afterwards):

$$W^{II}\left(K,q\right) = \max_{q' \in [q,q_n]} \left(q' - q\right) \left(\sum_{s=1}^{K} \Lambda_s^m v_L - P^I\left(K,q'\right)\right) + \delta W^I\left(K,q'\right) \quad \text{for } q \in [0, q_n]$$

Let the endpoint q_{n+1} be defined by $q_{n+1} = \max\left\{q \in [0, q_n] : W^I(K,q) \leq W^{II}(K,q)\right\}$ if the set is non-empty and $q_{n+1} = 0$ otherwise.

Claim 3. *Endpoints are strictly decreasing: $q_{n+1} < q_n$. Moreover, the continuation payoff $W^I(K,q)$ is continuous and satisfies $W^I(K,q) > 0$ for all $q \in [q_{n+1}, q_n]$.*

Proof. The continuation payoff $W^I(K,q_n)$ is strictly positive because it is bounded below by $\delta W(K,q_n) > 0$. By definition, $W^{II}(K,q_n) = \delta W^I(K,q_n)$, and so $W^{II}(K,q_n) < W^I(K,q_n)$. Finally, the theorem of the maximum guarantees that the functions $W^I(K,\cdot)$ and $W^{II}(K,\cdot)$ are continuous. Therefore, $q_{n+1} < q_n$. Next, by definition, for any $q \in (q_{n+1}, q_n]$, we have $W^I(K,q) > W^{II}(K,q) \geq \delta W^I(K,q)$. Thus, for any $q \in (q_{n+1}, q_n]$, we have $W^I(K,q) > 0$. It only remains to be shown that $W^I(K,q_{n+1}) > 0$, which we do in Section T.3 of the Technical Addendum. ■

b. Fixed Point. We define a sequence of quadruplets

$$\left\{\left(\mathcal{V}_L^\ell(K,\cdot), P^\ell(K,\cdot), W^\ell(K,\cdot), y^\ell(K,\cdot)\right)\right\}_{\ell=1,2,\ldots} \quad \text{for the interval } [q_{n+1}, \hat{q}].$$

The first element of the sequence is as follows. For $q \in (q_n, \hat{q}]$, we set

$$\left(\mathcal{V}_L^1(K,q), P^1(K,q), W^1(K,q), y^1(K,q)\right) = \left(\mathcal{V}_L(K,q), P(K,q), W(K,q), y(K,q)\right).$$

For $q \in [q_{n+1}, q_n]$ we instead set

$$W^1(K,q) = \max\Bigg\{ W^I(K,q), \\ \max_{0 \le k \le K-1} \left\{ \sum_{s=k+1}^{K} \Lambda_s^m \left[(\hat{q} - q)\, v_L + (1-\hat{q})\, v_H \right] - (1-q)\frac{c}{m}(K-k) + \delta W(k,q) \right\} \Bigg\}$$

and we let $y^1(K,q)$ be the solution that gives the lowest continuation payoff to the low-type seller.[8] The screening offer in W^I leads to a state (K, q') with $q' \geq q_n$. The continuation payoff $\mathcal{V}_L(K,q')$ is already defined for this state. Similarly, a universal offer leads to a state (k,q) with $k < K$, for which the continuation payoff $\mathcal{V}_L(k,q)$ is already defined. Thus, we extend $\mathcal{V}_L^1(K,\cdot)$ to the interval $[q_{n+1}, q_n]$ as follows:

$$\mathcal{V}_L^1(K,q) = \begin{cases} \delta \mathcal{V}_L(K,q') & \text{if } y^1(K,q) = q' \\ \frac{c}{m}(K-k) + \delta \mathcal{V}_L(k,q) & \text{if } y^1(K,q) = k \end{cases}$$

Finally, in the interval $[q_{n+1}, q_n]$, we define $P^1(K,\cdot)$ to be the largest weakly increasing function below $\delta \mathcal{V}_L^1(K,\cdot)$.

We define the remaining elements of the sequence of quadruplets recursively. For any $\ell \geq 1$, we define the $\ell + 1$'th element of the sequence as follows. First, we set

$$W^{\ell+1}(K,q) = \max\Bigg\{ \max_{q' \in [q,\hat{q}]} (q'-q) \left[\left(\sum_{s=1}^{K} \Lambda_s^m \right) v_L - P^\ell(K,q') \right] + \delta W^\ell(K,q'), \\ \max_{0 \le k \le K-1} \left\{ \left(\sum_{s=k+1}^{K} \Lambda_s^m \right) \left[(\hat{q} - q)\, v_L + (1-\hat{q})\, v_H \right] - (1-q)\frac{c}{m}(K-k) + \delta W(k,q) \right\} \Bigg\}.$$

[8] As in Step 3 of Part A, whenever there are many solutions with the same continuation payoff, then there must exist at least one that implies a universal offer $\left(K-k, \frac{c}{m}(K-k)\right)$. Of all such universal offers, we pick the one with the lowest k.

Next, we let $y^{\ell+1}(K,q)$ be the solution to the above problem that gives the lowest continuation payoff to the low-type seller. Denote that continuation payoff by $\mathcal{V}_L^{\ell+1}(K,q)$. Finally, let $P^{\ell+1}(k,\cdot)$ be the largest weakly increasing function below $\delta V_L^{\ell+1}(K,\cdot)$.

CLAIM 4. *There exists ℓ^* such that*

$$\begin{aligned}\left(\mathcal{V}_L^{\ell^*}(K,\cdot),P^{\ell^*}(K,\cdot),W^{\ell^*}(K,\cdot),y^{\ell^*}(K,\cdot)\right)\\ =\left(\mathcal{V}_L^{\ell^*+1}(K,\cdot),P^{\ell^*+1}(K,\cdot),W^{\ell^*+1}(K,\cdot),y^{\ell^*+1}(K,\cdot)\right).\end{aligned}$$

Proof. For every $q \geq q_{n+1}$ and for every $\ell > 1$, $W^\ell(k,q) \geq W^1(k,q) > 0$. Then, there exists $\eta > 0$ such that for $q \in [q_{n+1}, q_n]$ and for every $\ell > 1$, $W^\ell(k,q) > \eta$.

If the claim fails, for any positive integer T there exist ℓ, $q \in [q_{n+1}, q_n)$, and a sequence $\{q^\tau\}_{\tau=0}^T$ with $q^0 = q$, $q^T < q + \frac{1}{T}$ and $y^\ell(K, q^{\tau-1}) = q^\tau$ for all $\tau \in \{1, \ldots, T\}$. The buyer's continuation payoff $W^\ell(K,q)$ is bounded above:

$$W^\ell(K,q) < \left(\frac{1}{T}+\delta^T\right)\sum_{s=1}^{K}\Lambda_s^m v_H$$

Finally, pick T so that

$$\left(\frac{1}{T}+\delta^T\right)\sum_{s=1}^{K}\Lambda_s^m v_H < \eta.$$

But $W^\ell(K,q) > \eta$, so we have reached a contradiction. ■

At the end of the n'th inductive step, the quadruplet is already defined for $q \geq q_n$. We extend the quadruplet to $q \in [q_{n+1}, q_n)$ by setting it equal to the fixed point defined above:

$$(\mathcal{V}_L(K,q),P(K,q),W(K,q),y(K,q)) = \left(\mathcal{V}_L^{\ell^*}(K,q),P^{\ell^*}(K,q),W^{\ell^*}(K,q),y^{\ell^*}(K,q)\right).$$

Step 3. Extension to interval $[0,\hat{q}]$ takes finitely many steps. In the last step of the construction, we show that it takes finitely many steps to extend the quadruplet to the whole interval $[0,\hat{q}]$.

CLAIM 5. *There exists $\tilde{n}$ so that $q_{\tilde{n}} = 0$.*

See Section T.4 of the Technical Addendum for the proof.

Finally, note that $W(K,q) > 0$ for every (K,q). Thus it is never optimal for the buyer to make two consecutive universal offers. Formally, if $k \in Y(K,q)$ for some (K,q), then $k' \notin Y(k,q)$. Assume towards a contradiction that $k \in Y(K,q)$ and $k' \in Y(k,q)$. Then,

$$\begin{aligned} W(K,q) &= \left(\sum_{s=k+1}^{K} \Lambda_s^m\right) [(\hat{q}-q)\, v_L + (1-\hat{q})\, v_H] - (1-q)\frac{c}{m}(K-k) + \delta W(k,q) \\ &< \left(\sum_{s=k+1}^{K} \Lambda_s^m\right) [(\hat{q}-q)\, v_L + (1-\hat{q})\, v_H] - (1-q)\frac{c}{m}(K-k) + W(k,q) \\ &= \left(\sum_{s=k'+1}^{K} \Lambda_s^m\right) [(\hat{q}-q)\, v_L + (1-\hat{q})\, v_H] - (1-q)\frac{c}{m}(K-k') + \delta W(k',q) \end{aligned}$$

This shows that, at state (K,q), the buyer strictly prefers to make a universal offer for $K-k'$ units, instead of making one for $K-k$ units. Thus, $k \notin Y(K,q)$. ■

A.4 Convergence as bargaining frictions vanish

Lemma 1. Convergence as bargaining frictions vanish. *Fix m.*

(a) Consider an arbitrary sequence of vanishing frictions $\{\Delta_n\}_{n=1}^{\infty} \to 0$. *The associated sequences* $\left\{K_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{q_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ *and* $\left\{\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ *have subsequences that converge pointwise.*

(b) There exist functions $K_m(\cdot)$, $q_m(\cdot)$, $\{P_m(K,\cdot)\}_{K=1}^{m}$ *and* $\{W_m(K,\cdot)\}_{K=1}^{m}$ *such that for any sequence of vanishing frictions* $\{\Delta_n\}_{n=1}^{\infty} \to 0$, *the associated sequences* $\left\{K_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{q_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ *and* $\left\{\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ *converge pointwise to* $K_m(\cdot)$, $q_m(\cdot)$, $\{P_m(K,\cdot)\}_{K=1}^{m}$ *and* $\{W_m(K,\cdot)\}_{K=1}^{m}$, *respectively, except for finitely many points.*[9]

Proof of part (a). For any $\Delta > 0$, the functions $K_m^{\Delta}(\cdot)$ and $q_m^{\Delta}(\cdot)$ are monotonic in time elapsed τ and the function $P_m^{\Delta}(K,\cdot)$ is monotonic in q for all $K \in \{1,\ldots,m\}$. Therefore,

[9] The finitely many points where pointwise convergence may not occur correspond to impasses. At any impasse at state (K,q), $P_m^{-}(K,q)$ and $P_m^{+}(K,q)$ are well defined. We set $P_m(K,q) = P_m^{+}(K,q)$. This is without loss of generality, as the limit equilibrium outcome as bargaining frictions vanish does not depend on this choice.

they all have bounded variation. Moreover, all these functions are bounded above and below by bounds that do not depend on Δ. By Helly's First Theorem (Theorem 6.1.18 in Kannan and Krueger [1996]), $\left\{K_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{q_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$ and $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ all have subsequences that converge pointwise.

Fix $K \in \{1,\ldots,m\}$. The functions $\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{n=1}^{\infty}$ are uniformly equicontinuous since they all have the same Lipschitz constant $v_H \sum_{s=1}^{K} \Lambda_s^m$. They are also uniformly bounded. Then, the Arzelà-Ascoli Theorem guarantees that $\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{n=1}^{\infty}$ has a subsequence that converges uniformly. ■

Proof of part (b). In Proposition 3 we show that all convergent sequences $\left\{K_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{q_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ and $\left\{\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ have the same limit. ■

A.5 Proof of Proposition 3

In this proof we introduce an algorithm that characterizes the limit equilibrium outcome as bargaining frictions vanish.[10] Proposition 3 follows immediately from this characterization.

We consider a sequence of vanishing bargaining frictions $\{\Delta_n\}_{n=1}^{\infty} \to 0$ with associated sequences $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$, $\left\{\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ and $\left\{\left(K_m^{\Delta_n}(\cdot), q_m^{\Delta_n}(\cdot)\right)\right\}_{n=1}^{\infty}$ that converge pointwise, by Lemma 1(a). We characterize the limits of these associated sequences, which we denote by $\{P_m(K,\cdot)\}_{K=1}^{m}$, $\{W_m(K,\cdot)\}_{K=1}^{m}$ and $(K_m(\cdot), q_m(\cdot))$.

We describe both on-path and off-path behavior: we specify how quantities and beliefs evolve starting from any state (K,q). We let $K_m(\tau;(K,q))$ and $q_m(\tau;(K,q))$ denote respectively the number of remaining units and the belief at time elapsed τ if the starting state at time elapsed zero is (K,q).[11] The on-path limit equilibrium outcome as bargaining frictions vanish $(K_m(\tau), q_m(\tau))$ then corresponds to $(K_m(\tau;(m,0)), q_m(\tau;(m,0)))$.

Our algorithm proceeds by induction. In each step we characterize the limit functions $\{P_m(K,\cdot)\}_{K=1}^{m}$, $\{W_m(K,\cdot)\}_{K=1}^{m}$ and $(K_m(\cdot), q_m(\cdot))$ for different subsets of the state space

[10] We do this for generic values of the parameters (for details, see Remark 1 on page 20 of this Appendix).

[11] As in the main body of the paper, these functions are left-continuous in τ. These functions are uniquely identified at all states, except at finitely many states, which correspond to (on- and off-path) impasses. For these states, the functions $K_m(\tau;(K,q))$ and $q_m(\tau;(K,q))$ reflect the evolution after the impasse is resolved.

$\{1, \ldots, m\} \times [0, \hat{q}]$. In the base step ($j = 0$), we identify a candidate impasse $(k_1, q_1) = (1, \bar{q}_m(1))$. We characterize the limit functions for all states $(1, q)$ with $q < q_1$ (Claim 6) and for all states (K, q) with $q \geq q_1$ (Claim 7). At each (non-final) step $j \geq 1$ of the inductive process we identify a candidate impasse (k_{j+1}, q_{j+1}) with $k_{j+1} > k_j$ and $q_{j+1} < q_j$. Claims 8, 9 and 10 characterize the limit functions for all states (K, q) with either 1) $K \in \{k_j + 1, \ldots, k_{j+1}\}$ and $q \in [0, q_j)$, or 2) $K \in \{k_{j+1} + 1, \ldots, m\}$ and $q \in [q_{j+1}, q_j)$. In particular, these claims show that the candidate impasse (k_j, q_j) is reached from the candidate impasse (k_{j+1}, q_{j+1}).

The algorithm ends after finitely many steps with a characterization of the limit functions for the whole state space $\{1, \ldots, m\} \times [0, \hat{q}]$ and with a collection $\left\{(k_j, q_j)\right\}_{j=1}^{J}$ of J candidate impasses. All candidate impasses are on-path: the limit equilibrium outcome as bargaining frictions vanish consists of a sequence of phases of fast trade and impasses summarized by $\left\{(k_j, q_j)\right\}_{j=1}^{J}$.

The base step (j = 0)

In the base step we obtain the first candidate impasse $(k_1, q_1) = (1, \bar{q}_m(1))$. Claim 6 shows that the candidate impasse $(1, \bar{q}_m(1))$ is reached without delay starting from any state $(1, q)$ with $q < \bar{q}_m(1)$.

CLAIM 6. *For all $q < \bar{q}_m(1)$, we have*

$$
\begin{aligned}
P_m(1, q) &= \frac{\left(\Lambda_1^m v_L\right)^2}{c/m}, \\
W_m(1, q) &= \left(\bar{q}_m(1) - q\right)\left(\Lambda_1^m v_L\right)\left(1 - \frac{\Lambda_1^m v_L}{c/m}\right) \quad \textit{and}
\end{aligned}
$$

$$
\left(K_m\left(\tau; (1, q)\right), q_m\left(\tau; (1, q)\right)\right) = \begin{cases} (1, \bar{q}_m(1)) & \textit{if } \tau \leq \tau_1 \\ (0, \hat{q}) & \textit{if } \tau > \tau_1 \end{cases} \quad \textit{with} \;\; \tau_1 = \frac{2}{r} \ln\left(\frac{c/m}{\Lambda_1^m v_L}\right).
$$

The proof of Claim 6 is in DL, so we omit it.

Claim 7 shows that starting at any state (K, q) with $K \in \{1, \ldots, m\}$ and $q \in [\bar{q}_m(1), \hat{q}]$, the game ends without delay.

Claim 7. *For all (K, q) with $K \in \{1, \ldots, m\}$ and $q \in [\bar{q}_m(1), \hat{q}]$ we have*

$$
\begin{aligned}
&P_m(K,q) = K\frac{c}{m},\\
&W_m(K,q) = (\hat{q}-q)\left(\sum_{s=1}^{K}\Lambda_s^m v_L - K\frac{c}{m}\right) + (1-\hat{q})\left(\sum_{s=1}^{K}\Lambda_s^m v_H - K\frac{c}{m}\right) \quad \textit{and}\\
&\left(K_m\left(\tau;(K,q)\right), q_m\left(\tau;(K,q)\right)\right) = (0,\hat{q}) \quad \forall \tau \geq 0.
\end{aligned}
$$

Proof. At all states (K, q) with $K \in \{1, \ldots, m\}$ and $q \in [\bar{q}_m(1), \hat{q}]$, except for $(1, \bar{q}_m(1))$, the buyer can guarantee a strictly positive continuation payoff by making a universal offer for K units. Thus, the game ends without delay. The low-type seller can always mimic the high-type seller's behavior. Therefore, as bargaining frictions vanish, the price that the low-type seller is willing to accept for K units must converge to $K\frac{c}{m}$. Then, the function $P_m(1, \cdot)$ is discontinuous at $(1, \bar{q}_m(1))$. We assign $P_m(1, \bar{q}_m(1)) = P_m^+(1, \bar{q}_m(1))$. We do the same with the outcome $\left(K_m\left(\tau;(1,\bar{q}_m(1))\right), q_m\left(\tau;(1,\bar{q}_m(1))\right)\right)$, i.e. we take the limit from the right. In this way, these functions evaluated at $(1, \bar{q}_m(1))$ reflect what happens right after the impasse $(1, \bar{q}_m(1))$ is resolved. We follow this convention also for the next impasses. ■

The algorithm then continues to the first inductive step ($j = 1$).

The inductive step (j $\geq$ 1)

The previous step $j - 1$ provides a (candidate) impasse (k_j, q_j) of length τ_j. The impasse (k_j, q_j) satisfies $\bar{q}_m(k_j + 1) < q_j$ and $k_j < m$. All previous steps together provide a characterization of the limit functions for all states (K, q) with either $K \leq k_j$, or $q \geq q_j$, or both.

As we do in the main body of the paper, throughout this proof we focus on the "limit game" in the sense that the low-type seller's behavior is summarized by the limit function $P_m(\cdot, \cdot)$. We consider a simple course of action that brings the buyer from any state (K, q) with $K \in \{k_j + 1, \ldots, m\}$ and $q \in [0, q_j]$ to the impasse (k_j, q_j). The buyer first makes the universal offer $\left(K - k_j, \frac{c}{m}(K - k_j)\right)$ and then the screening offer $\left(K, P_m^-(k_j, q_j)\right)$. The function $\mathcal{W}(K,q) : \{k_j+1,\ldots,m\} \times [0,q_j] \to \mathbb{R}$, defined in equation (5), denotes the buyer's (normalized) payoff from following this simple course of action.

$$\mathcal{W}(K,q) \equiv (\hat{q}-q)\left[\sum_{s=k_j+1}^{K} \Lambda_s^m v_L - (K-k_j)\frac{c}{m}\right] + (1-\hat{q})\left[\sum_{s=k_j+1}^{K} \Lambda_s^m v_H - (K-k_j)\frac{c}{m}\right] + \left(q_j - q\right)\left[\sum_{s=1}^{k_j} \Lambda_s^m v_L - P_m^-\left(k_j,q_j\right)\right] \tag{5}$$

Remark 1. *The following two conditions hold for generic values of the parameters:*

$$\mathcal{W}\left(K,0\right) \neq 0 \quad \textit{for all } K \in \{k_j+1,\ldots,m\} \tag{6a}$$

$$\mathcal{W}\left(K,\bar{q}_m(K)\right) \neq 0 \quad \textit{for all } K \in \{k_j+1,\ldots,m\} \tag{6b}$$

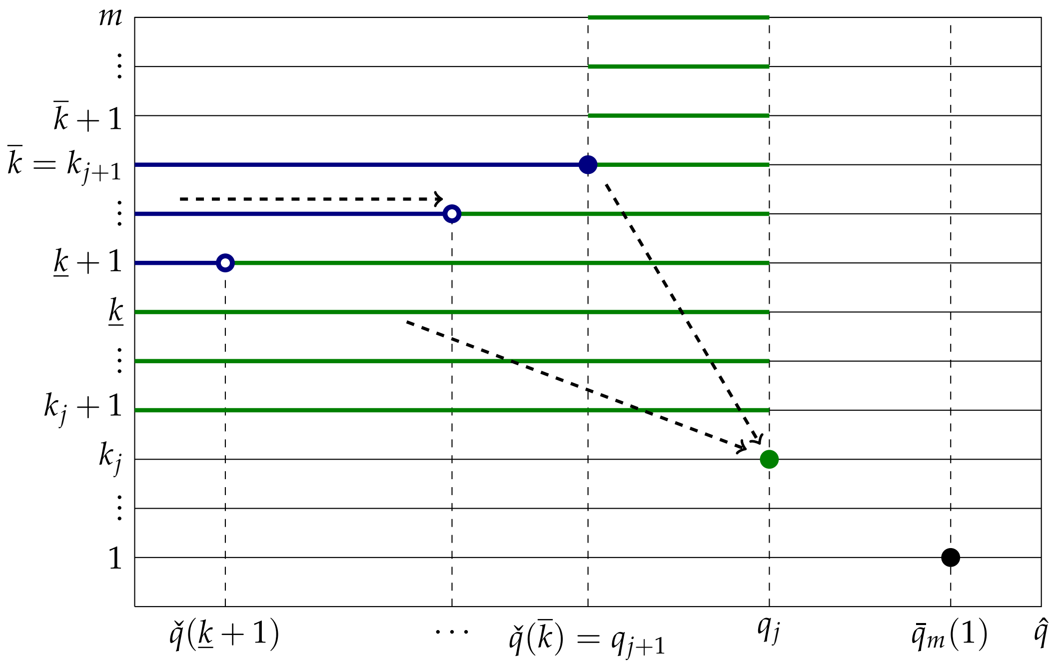
Throughout this proof we restrict attention to parameters that satisfy these two conditions.

The function $\mathcal{W}(\cdot,\cdot)$ satisfies $\mathcal{W}(K,q_j) > 0$ because $\bar{q}_m(K) \leq \bar{q}_m(k_j+1) < q_j$. Moreover, $\mathcal{W}(\cdot,0)$ is strictly decreasing in K. Given the genericity condition (6a), we next let

$$\underline{k} = \begin{cases} \max\left\{K \in \{k_j+1,\ldots,m\} : \mathcal{W}(K,0) > 0\right\} & \text{if } \mathcal{W}(k_j+1,0) > 0 \\ k_j & \text{if } \mathcal{W}(k_j+1,0) < 0 \end{cases}$$

We split the remainder of the inductive step into two parts, a and b. If $\underline{k} = m$, the algorithm proceeds with part a and then ends. If $k_j < \underline{k} < m$, the algorithm proceeds first with part a and then with part b. If $\underline{k} = k_j$, the algorithm skips part a and moves directly to part b. Throughout the description of these two parts, we refer to Figure 2 to facilitate their exposition.

Part a. In this part we characterize the equilibrium outcome for all states (K,q) with $K \in \left\{k_j+1,\ldots,\underline{k}\right\}$ and $q \in [0,q_j)$. At any such state, the buyer can guarantee a positive continuation payoff by following the simple course of action described above. We represent this area of the state space with thick green lines in Figure 2. We show in Claim 8 how starting an any state (K,q) with $K \in \left\{k_j+1,\ldots,\underline{k}\right\}$ and $q \in [0,q_j)$, the state (k_j,q_j) is



Notes: The green circle at state (k_j, q_j) denotes the candidate impasse from the previous step $j-1$. Thick green lines represent states (K, q) with $\mathcal{W}(K, q) > 0$, while thick blue lines represent states (K, q) with $\mathcal{W}(K, q) < 0$. Dashed black arrows illustrate transitions without delay. Filled circles represent on-path impasses, while empty circles represent off-path impasses.

Figure 2: The inductive step ($j \geq 1$) of the algorithm

reached without delay. The state remains there for time elapsed τ_j, i.e. there is an impasse of length τ_j at state (k_j, q_j). After the impasse is resolved, the evolution of the number of remaining units and of beliefs is as specified in the previous step of the induction process.

CLAIM 8. *For all $K \in \{k_j+1, \ldots, \underline{k}\}$ and for all $q \in [0, q_j)$ we have:*

$$P_m(K,q) = (K-k_j)\frac{c}{m} + P_m^-\left(k_j, q_j\right)$$

$$W_m(K,q) = \mathcal{W}(K,q)$$

$$\left(K_m\left(\tau;(K,q)\right), q_m\left(\tau;(K,q)\right)\right) = \begin{cases} \left(k_j, q_j\right) & \text{if } \tau \leq \tau_j \\ \left(K_m\left(\tau-\tau_j;(k_j,q_j)\right), q_m\left(\tau-\tau_j;(k_j,q_j)\right)\right) & \text{if } \tau > \tau_j \end{cases}$$

See Section T.5 of the Technical Addendum for the proof.

If $\underline{k} = m$, then (k_j, q_j) is the first impasse and the algorithm ends. Otherwise, the algorithm proceeds to part *b*.

Part b. We first let

$$\bar{k} = \max\left\{K \in \{\underline{k}+1,\ldots,m\} : \mathcal{W}\left(K, \bar{q}_m(K)\right) > 0\right\}.$$

Furthermore, for all $K \geq \underline{k}+1$ we let $\check{q}(K) \in (0, q_j)$ be defined by $\mathcal{W}(K, \check{q}(K)) = 0$. In this part we derive the functions of interest for all states (K, q) with either 1) $K \in \{\underline{k}+1,\ldots,\bar{k}\}$ and $q < q_j$ or 2) $K > \bar{k}$ and $q \in [\check{q}(\bar{k}), q_j)$. To do so, we first prove the following fact.

FACT 1. *The following inequalities hold:*

$$\frac{\partial \mathcal{W}(K,q)}{\partial q} = (K - k_j)\frac{c}{m} + P_m^-\left(k_j, q_j\right) - \sum_{s=1}^{K} \Lambda_s^m v_L > 0 \quad \forall\, K > \underline{k} \tag{7a}$$

$$\bar{q}_m(\bar{k}+1) < \check{q}(\bar{k}) < \bar{q}_m(\bar{k}) \tag{7b}$$

$$\check{q}(\underline{k}+1) < \check{q}(\underline{k}+2) < \cdots < \check{q}(\bar{k}-1) < \check{q}(\bar{k}) \tag{7c}$$

where if $\bar{k} = m$, replace (7b) by $\check{q}(\bar{k}) < \bar{q}_m(\bar{k})$.

Proof. First, for (7a), note that $\mathcal{W}(K, 0) < 0$ and $\mathcal{W}(K, q_j) > 0$ for all $K > \underline{k}$. Moreover, $\mathcal{W}(K, q)$ is linear in q. Thus, $\mathcal{W}(K, q)$ is strictly increasing in q for all $K > \underline{k}$.[12] Second, for (7b), note that by the definition of $\bar{k}$, $\mathcal{W}\left(\bar{k}, \bar{q}_m(\bar{k})\right) > 0$. Since $\mathcal{W}(K, q)$ is strictly increasing, then $\check{q}(\bar{k}) < \bar{q}_m(\bar{k})$. If $\bar{k} = m$, this finishes the proof of (7b). Otherwise, note that the definition of $\bar{k}$ (and the genericity condition (6b)) imply that $\mathcal{W}\left(\bar{k}+1, \bar{q}_m(\bar{k}+1)\right) < 0$. Since $\mathcal{W}\left(\bar{k}+1, \bar{q}_m(\bar{k}+1)\right) = \mathcal{W}\left(\bar{k}, \bar{q}_m(\bar{k}+1)\right)$, then $\bar{q}_m(\bar{k}+1) < \check{q}(\bar{k})$. Finally, regarding equation (7c), note that:

$$\mathcal{W}(K, q) = \mathcal{W}(K-1, q) + (\hat{q} - q)\left[\Lambda_K^m v_L - \frac{c}{m}\right] + (1 - \hat{q})\left[\Lambda_K^m v_H - \frac{c}{m}\right]$$

Then, $\mathcal{W}(K, q) \geq \mathcal{W}(K-1, q) \Leftrightarrow q \geq \bar{q}_m(K)$. Suppose that $\check{q}(K) < \bar{q}_m(K)$. Then, $0 = \mathcal{W}(K, \check{q}(K)) < \mathcal{W}(K-1, \check{q}(K))$ and so $\check{q}(K-1) < \check{q}(K)$. Since, $\check{q}(\bar{k}) < \bar{q}_m(\bar{k})$, an inductive

[12] The strict monotonicity of $\mathcal{W}(K,q)$ together with the equality $\mathcal{W}\left(K, \bar{q}_m(K)\right) = \mathcal{W}\left(K-1, \bar{q}_m(K)\right)$ implies that $\mathcal{W}\left(K, \bar{q}_m(K)\right) > 0$ for all $K \in \{\underline{k}+1,\ldots,\bar{k}\}$. Furthermore, $\check{q}(K) < \bar{q}_m(K+1)$ for all $K \in \{\underline{k}+1,\ldots,\bar{k}-1\}$.

argument shows equation (7c).[13] ■

The buyer can guarantee a positive continuation payoff at any state (K, q) with $K \in \{\underline{k}+1, \ldots, \overline{k}\}$ and $q \in (\check{q}(K), q_j)$. This follows directly from the definition of $\check{q}(\cdot)$. The buyer can also guarantee a positive continuation payoff at any state (K, q) with $K \in \{\overline{k}+1, \ldots, m\}$ and $q \in [\check{q}(\overline{k}), q_j)$. This follows from the first inequality in equation (7b) and the fact that $\bar{q}_m(\cdot)$ is strictly decreasing in K. We represent these areas of the state space with thick green lines in Figure 2. As in Claim 8, starting from any state (K, q) with $\mathcal{W}(K, q) > 0$, the state (k_j, q_j) is reached without delay and an impasse of length τ_j occurs. Claim 9 summarizes these findings.[14] We omit the proof of Claim 9 since it is analogous to that of Claim 8.

CLAIM 9. *For all (K, q) with either 1) $K \in \left\{\underline{k}+1, \ldots, \overline{k}\right\}$ and $q \in \left[\check{q}(K), q_j\right)$ or 2) $K \in \left\{\overline{k}+1, \ldots, m\right\}$ and $q \in \left[\check{q}(\overline{k}), q_j\right)$ we have*

$$
\begin{aligned}
P_m(K, q) &= (K - k_j)\frac{c}{m} + P_m^-\left(k_j, q_j\right), \\
W_m(K, q) &= \mathcal{W}(K, q) \quad \textit{and} \\
\left(K_m\left(\tau; (K, q)\right), q_m\left(\tau; (K, q)\right)\right) &= \begin{cases} \left(k_j, q_j\right) & \textit{if } \tau \leq \tau_j \\ \left(K_m\left(\tau - \tau_j; (k_j, q_j)\right), q_m\left(\tau - \tau_j; (k_j, q_j)\right)\right) & \textit{if } \tau > \tau_j. \end{cases}
\end{aligned}
$$

Claim 10 completes the description of the limit functions in the inductive step. States (K, q) with $K \in \{\underline{k}+1, \ldots, \overline{k}\}$ and $q < \check{q}(K)$ have $\mathcal{W}(K, q) < 0$. We represent these states with thick blue lines in Figure 2. Claim 10 shows that starting from any such (K, q), the state shifts without delay to $(K, \check{q}(K))$, where an impasse of length $\rho(K)$ occurs. The reason behind this impasse is that the function $P_m(K, \cdot)$ must be discontinuous at $\check{q}(K)$ for any $K \in \{\underline{k}+1, \ldots, \overline{k}\}$. If it were continuous, the buyer's continuation payoff would be negative at states (K, q) with q close (and to the left) of $\check{q}(K)$. This impasse makes the

[13] It is easy to show in a similar way that $\check{q}(\overline{k}) > \check{q}(\overline{k}+1) > \ldots > \check{q}(m)$. Thus $\overline{k} = \arg\max_{K \in \left\{k_j+1, \ldots, m\right\}} \{\check{q}(K)\}$, which is consistent with the definition of k_2 in section III.B.

[14] In Claim 10 we show that there is a (potentially off-path) impasse at every state $(K, \check{q}(K))$ with $K \in \left\{\underline{k}+1, \ldots, \overline{k}\right\}$. Following the convention established in Claim 7, the limit functions evaluated at $(K, \check{q}(K))$ reflect the outcome after the impasse is resolved.

price $P_m^-\left(K,\check{q}(K)\right)$ low enough so that the buyer finds it optimal to move to state $(K,\check{q}(K))$ without delay.

CLAIM 10. *For all (K,q) with $K\in\{\underline{k}+1,\ldots,\overline{k}\}$ and $q\in[0,\check{q}(K))$ we have:*

$$P_m(K,q)=\frac{\left(\sum_{s=1}^{K}\Lambda_s^m v_L\right)^2}{(K-k_j)\frac{c}{m}+P_m^-\left(k_j,q_j\right)},$$
$$W_m(K,q)=\left(\check{q}(K)-q\right)\left(\sum_{s=1}^{K}\Lambda_s^m v_L\right)\left(1-\frac{\sum_{s=1}^{K}\Lambda_s^m v_L}{(K-k_j)\frac{c}{m}+P_m^-\left(k_j,q_j\right)}\right)$$

and

$$\left(K_m\left(\tau;(K,q)\right),q_m\left(\tau;(K,q)\right)\right)=\begin{cases}(K,\check{q}(K)) & \text{if } \tau\le\rho(K)\\ \Big(K_m\left(\tau-\rho(K);(K,\check{q}(K))\right), \\ \qquad q_m\left(\tau-\rho(K);(K,\check{q}(K))\right)\Big) & \text{if } \tau>\rho(K)\end{cases}$$
$$\text{with } \rho(K)=\frac{2}{r}\log\left(\frac{(K-k_j)\frac{c}{m}+P_m^-(k_j,q_j)}{\left(\sum_{s=1}^{K}\Lambda_s^m\right)v_L}\right).$$

See Section T.5 of the Technical Addendum for the proof.

We finally describe how the inductive step concludes. We let $\left(k_{j+1},q_{j+1}\right)=\left(\overline{k},\check{q}\left(\overline{k}\right)\right)$ and $\tau_{j+1}=\rho(\overline{k})$. If $\overline{k}<m$, then the algorithm proceeds to the next inductive step. If $\overline{k}=m$, then the algorithm ends. Since m is finite, the algorithm ends in finitely many steps.

When the algorithm ends, it provides a collection $\left\{(k_j,q_j)\right\}_{j=1}^{J}$ of candidate impasses and a complete characterization of the limit functions. The last inductive step shows that starting at the initial state $(m,0)$, the state (k_J,q_J) is reached without delay and an impasse of length τ_J ensues. Each inductive step shows how after the impasse in state (k_j,q_j) is resolved, the state shifts without delay to (k_{j-1},q_{j-1}), where an additional impasse of length τ_{j-1} occurs. The base step shows that the game ends after the last impasse $(1,\bar{q}_m(1))$ is reached.

To sum up, all impasses in $\left\{(k_j,q_j)\right\}_{j=1}^{J}$ occur on-path.[15] Thus, the limit equilibrium

[15] All other impasses identified in Claim 10 in each inductive step are off-path.

outcome as bargaining frictions vanish consists of a sequence of phases of fast trade an impasses characterized by $\left\{(k_j, q_j)\right\}_{j=1}^{J}$. ■

A.6 Proof of Proposition 4

We first show equation (8d). We then proceed with the proof of equation (8b), which is the most involved part of the proof of Proposition 4 and includes several steps. We finally show how the remaining equations in Proposition 4 follow from equations (8b) and (8d).

Proof of equation (8d). Any impasse (k_j^m, q_j^m) must satisfy $\bar{q}_m(k_j^m + 1) < q_j^m < \bar{q}_m(k_j^m)$ (see Proposition 3). Together with the definitions of $\bar{q}(\cdot)$ and $\bar{q}_m(\cdot)$, and replacing $z_j^m = k_j^m/m$ when needed, this implies

$$\bar{q}\left(z_j^m + \frac{1}{m}\right) = \bar{q}\left(\frac{k_j^m + 1}{m}\right) < \bar{q}_m(k_j^m + 1) < q_j^m < \bar{q}_m(k_j^m) < \bar{q}\left(\frac{k_j^m - 1}{m}\right) = \bar{q}\left(z_j^m - \frac{1}{m}\right)$$

Notice that $\left|\frac{d\bar{q}(z)}{dz}\right|$ is bounded by some constant $\check{\rho} < \infty$ (because $\frac{d\lambda(\cdot)}{dz}$ is continuous). Thus,

$$\left|q_j^m - \bar{q}(z_j^m)\right| < \max\left\{\left|\bar{q}\left(z_j^m - 1/m\right) - \bar{q}(z_j^m)\right|; \left|\bar{q}\left(z_j^m + 1/m\right) - \bar{q}(z_j^m)\right|\right\} < \check{\rho}/m.$$

The bound $\check{\rho}$ is independent of j, so $\max\left\{\left|q_j^m - \bar{q}(z_j^m)\right|\right\}_{j=1}^{J_m} < \check{\rho}/m$, which leads to equation (8d):

$$\lim_{m\to\infty} \max\left\{\left|q_j^m - \bar{q}(z_j^m)\right|\right\}_{j=1}^{J_m} = 0$$

Proof of equation (8b). We split this proof in two parts. In the first one we construct a sequence of limits of consecutive impasses and show how to link these limits. In the second one we use this construction to show that limits of consecutive impasses must be arbitrarily close.

Construction of the sequence of limits of consecutive impasses. Assume towards a contradiction that

$$\limsup_{m\to\infty}\left(\max\left\{q_{j-1}^m - q_j^m\right\}_{j=2}^{J_m}\right) > 0.$$

Then, by taking a subsequence if necessary, we may assume that a sequence of consecutive impasses $\left\{\left(z^m_{j_m}, q^m_{j_m}\right), \left(z^m_{j_m-1}, q^m_{j_m-1}\right)\right\}_{m=1}^{\infty}$ that converges to $((z_0, q_0), (z_{-1}, q_{-1}))$ with $q_0 > q_{-1}$ exists. Equation (8d) guarantees that $q_0 = \bar{q}(z_0)$ and $q_{-1} = \bar{q}(z_{-1})$

The buyer obtains a zero continuation payoff at every impasse. Thus, the difference $W_m(mz^m_{j_m}, q^m_{j_m}) - W_m(mz^m_{j_m-1}, q^m_{j_m-1})$, which we express in equation (8), is also zero:[16]

$$\begin{aligned}\left(q^m_{j_m-1} - q^m_{j_m}\right)\left[\int_0^{z^m_{j_m}} \lambda(z) v_L dz - P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)\right] \\ + \left(\hat{q} - q^m_{j_m-1}\right)\int_{z^m_{j_m-1}}^{z^m_{j_m}} \left[\lambda(z) v_L - c\right] dz + (1-\hat{q})\int_{z^m_{j_m-1}}^{z^m_{j_m}} \left[\lambda(z) v_H - c\right] dz = 0\end{aligned} \tag{8}$$

The left hand side of equation (8) is continuous in $\left(z^m_{j_m}, q^m_{j_m}\right)$, $\left(z^m_{j_m-1}, q^m_{j_m-1}\right)$ and in $P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)$. Moreover it strictly decreases in $P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)$, with derivative bounded away from zero. Hence, since $\left\{\left(z^m_{j_m}, q^m_{j_m}\right)\right\}_{m=1}^{\infty}$ and $\left\{\left(z^m_{j_m-1}, q^m_{j_m-1}\right)\right\}_{m=1}^{\infty}$ converge, then $\left\{P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)\right\}_{m=1}^{\infty}$ must also converge. We let P^+_0 denote its limit. Equation (9) expresses equation (8) in the limit:

$$\begin{aligned}(q_{-1} - q_0)\left[\int_0^{\psi(q_0)} \lambda(z) v_L dz - P^+_0\right] \\ + (\hat{q} - q_{-1})\int_{\psi(q_{-1})}^{\psi(q_0)} \left[\lambda(z) v_L - c\right] dz + (1-\hat{q})\int_{\psi(q_{-1})}^{\psi(q_0)} \left[\lambda(z) v_H - c\right] dz = 0\end{aligned} \tag{9}$$

with a change of variables taking advantage of $z_\ell = \psi(q_\ell)$ for $\ell \in \{0, -1\}$, where $\psi(\cdot)$ is the inverse of $\bar{q}(\cdot)$. Equation (9) links the limits (z_0, q_0) and (z_{-1}, q_{-1}).

We show next that $q_{-1} < \bar{q}(0)$ (and so $z_{-1} > 0$). Assume towards a contradiction that $q_{-1} = \bar{q}(0)$ and $z_{-1} = 0$. This implies that $P^+_0 = z_0 c$.[17] Using this, we rewrite the left hand side of equation (9) as

$$(\hat{q} - q_0)\left[\int_0^{\psi(q_0)} \left[\lambda(z) v_L - c\right] dz\right] + (1-\hat{q})\int_0^{\psi(q_0)} \left[\lambda(z) v_H - c\right] dz < 0$$

where the inequality follows from the definition of $\psi(\cdot)$. This leads to a contradiction.

[16] We use equation (4c) to obtain equation (8).

[17] Equation (4b) implies that $P^-_m\left(mz^m_{j_m-1}, q^m_{j_m-1}\right) < v_L \int_0^{z^m_{j_m-1}} \lambda(z) dz$, which converges to zero as $m \to \infty$. This and equation (4c) imply that $P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)$ becomes arbitrarily close to $(z^m_{j_m} - z^m_{j_m-1})c$ as $m \to \infty$.

For every (large enough) m there exists an impasse $\left(z^m_{j_m-2}, q^m_{j_m-2}\right)$ that occurs after $\left(z^m_{j_m-1}, q^m_{j_m-1}\right)$ is resolved. This is because the last impasse occurs at $z = \frac{1}{m}$ and $z_{-1} > 0$. Assume, by taking a subsequence if necessary, that the sequence $\left\{\left(z^m_{j_m-2}, q^m_{j_m-2}\right)\right\}_{m=1}^{\infty}$ converges to (z_{-2}, q_{-2}). By an argument like the one for q_{-1}, then also $q_{-2} < \bar{q}(0)$.

We show next that $q_{-1} < q_{-2}$. Assume instead that $q_{-1} = q_{-2}$ (so $z_{-1} = z_{-2}$). Equation (4c) then implies $\lim_{m\to\infty} P^+_m\left(mz^m_{j_m-1}, q^m_{j_m-1}\right) - P^-_m\left(mz^m_{j_m-2}, q^m_{j_m-2}\right) = 0$. Proposition 3 guarantees that in general

$$P^-_m\left(mz^m_{j_m-2}, q^m_{j_m-2}\right) < v_L \int_0^{z^m_{j_m-2}} \lambda(z)dz < v_L \int_0^{z^m_{j_m-1}} \lambda(z)dz < P^+_m\left(mz^m_{j_m-1}, q^m_{j_m-1}\right).$$

Thus, $q_{-1} = q_{-2}$ implies $\lim_{m\to\infty} P^+_m\left(mz^m_{j_m-1}, q^m_{j_m-1}\right) = \lim_{m\to\infty} P^-_m\left(mz^m_{j_m-2}, q^m_{j_m-2}\right) = v_L \int_0^{z_{-1}} \lambda(z)dz$. Finally, we link $P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)$ and $P^+_m\left(mz^m_{j_m-1}, q^m_{j_m-1}\right)$ using equations (4b) and (4c) and take limits to obtain

$$P^+_0 = (z_0 - z_{-1})\, c + v_L \int_0^{z_{-1}} \lambda(z)dz.$$

We plug this expression for P^+_0 in the left hand side of equation (9) and obtain the following contradiction:

$$(\hat{q} - q_0) \int_{\psi(q_{-1})}^{\psi(q_0)} [\lambda(z)v_L - c]\, dz + (1 - \hat{q}) \int_{\psi(q_{-1})}^{\psi(q_0)} [\lambda(z)v_H - c]\, dz < 0$$

The same argument that shows that the sequence $\left\{P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)\right\}_{m=1}^{\infty}$ must converge to P^+_0 also guarantees that the sequence $\left\{P^+_m\left(mz^m_{j_m-1}, q^m_{j_m-1}\right)\right\}_{m=1}^{\infty}$ must converge, and its limit, which we denote by P^+_{-1} must satisfy an equation like (9):

$$\begin{aligned} &(q_{-2} - q_{-1}) \left[\int_0^{\psi(q_{-1})} \lambda(z)v_L dz - P^+_{-1}\right] \\ &+ (\hat{q} - q_{-2}) \int_{\psi(q_{-2})}^{\psi(q_{-1})} [\lambda(z)v_L - c]\, dz + (1 - \hat{q}) \int_{\psi(q_{-2})}^{\psi(q_{-1})} [\lambda(z)v_H - c]\, dz = 0 \end{aligned}$$

The previous equation links the limits (z_{-1}, q_{-1}) and (z_{-2}, q_{-2}) of the sequences of consecutive impasses $\left\{\left(z^m_{j_m-1}, q^m_{j_m-1}\right)\right\}_{m=1}^{\infty}$ and $\left\{\left(z^m_{j_m-2}, q^m_{j_m-2}\right)\right\}_{m=1}^{\infty}$.

We next link the limit prices P_0^+ and P_{-1}^+ using equations (4b) and (4c). Equation (4c) links $P_m^+\left(mz_{j_m}^m, q_{j_m}^m\right)$ and $P_m^-\left(mz_{j_m-1}^m, q_{j_m-1}^m\right)$. Equation (4b) links $P_m^-\left(mz_{j_m-1}^m, q_{j_m-1}^m\right)$ and $P_m^+\left(mz_{j_m-1}^m, q_{j_m-1}^m\right)$. Using these equations together, and taking limits, we obtain

$$P_0^+ = \left[\psi(q_0) - \psi(q_{-1})\right] c + \frac{\left(v_L \int_0^{\psi(q_{-1})} \lambda z dz\right)^2}{P_{-1}^+}. \tag{10}$$

We proceed recursively and construct, taking subsequences if necessary, a collection of sequences of impasses $\left\{\left\{\left(z_{j_m-\ell}^m, q_{j_m-\ell}^m\right)\right\}_{m=1}^{\infty}\right\}_{\ell=0}^{\infty}$, where, for every ℓ, the sequence $\left\{\left(z_{j_m-\ell}^m, q_{j_m-\ell}^m\right)\right\}_{m=1}^{\infty}$ converges to $(z_{-\ell}, q_{-\ell})$ as m grows to infinity. Furthermore, for every ℓ, the sequence $\left\{P_m^+\left(mz_{j_m-\ell}^m, q_{j_m-\ell}^m\right)\right\}_{m=1}^{\infty}$ converges to $P_{-\ell}^+$.

For every $\ell = 0, 1, \ldots$ the limits of consecutive impasses must satisfy equations (11) and (12).

$$\begin{aligned}
&\left(q_{-(\ell+1)} - q_{-\ell}\right)\left[\int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz - P_{-\ell}^+\right] \qquad (11)\\
&\quad + \left(\hat{q} - q_{-(\ell+1)}\right)\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})} \left[\lambda(z) v_L - c\right] dz + (1-\hat{q})\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})} \left[\lambda(z) v_H - c\right] dz = 0
\end{aligned}$$

$$P_{-\ell}^+ = \left[\psi(q_{-\ell}) - \psi(q_{-(\ell+1)})\right] c + \frac{\left(v_L \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) dz\right)^2}{P_{-(\ell+1)}^+} \tag{12}$$

These conditions mirror equations (9) and (10). Finally, limit beliefs satisfy

$$q_0 < q_{-1} < \ldots < q_{-\ell} < \ldots < \bar{q}(0). \tag{13}$$

Bounding the distance between limits of consecutive impasses. In the remainder of the proof we focus on the collection $\left\{\left(q_{-\ell}, P_{-\ell}^+\right)\right\}_{\ell=0}^{\infty}$ which satisfies equations (11), (12), and (13). We show that the limit beliefs $\{q_{-\ell}\}_{\ell=0}^{\infty}$ are arbitrarily close to each other. To do this, we obtain explicit bounds that link successive limit impasses by using equations (11) and (12). These bounds link differences between consecutive beliefs and also differences between prices and valuations. Facts 2 and 3 state the first bounds (see Section T.6 of the Technical Addendum for their proof).

FACT 2. *There exists $\eta^* > 0$ such that for every $\ell \geq 1$, if $q_{-(\ell+1)} - q_{-\ell} < \eta^*$, then $q_{-\ell} - q_{-(\ell-1)} < \frac{4}{3}\left(q_{-(\ell+1)} - q_{-\ell}\right)$.*

FACT 3. *There exists constants $b_1 > 0$ and $b_2 > 0$ such that for every $\ell = 0, 1, \ldots,$ we have:*

$$\frac{\left[P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz\right] - \left[P^+_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz\right]}{q_{-(\ell+1)} - q_{-\ell}} \leq b_1 \left(q_{-(\ell+1)} - q_{-\ell}\right) \quad (14)$$

$$q_{-(\ell+1)} - q_{-\ell} \leq b_2 \left[P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz\right] \quad (15)$$

Using Facts 2 and 3 we prove Claims 11 and 12, which provide further bounds. Claim 11 links successive differences between prices and valuations and Claim 12 links differences between successive beliefs.

CLAIM 11. *Consider ℓ' and ℓ'' with $0 \leq \ell' < \ell''$. Let $\varepsilon > 0$ and $\eta > 0$ be such that $q_{-(\ell+1)} - q_{-\ell} < \varepsilon$ for all $\ell \in \{\ell', \ldots, \ell'' - 1\}$ and $q_{-\ell''} - q_{-\ell'} < \eta$. Then, for every $\ell \in \{\ell', \ldots, \ell'' - 1\}$, we have:*

$$P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz < P^+_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz + \varepsilon \eta b_1$$

Proof. For every $\ell \in \{\ell', \ldots, \ell'' - 1\}$ we have

$$\begin{aligned} P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz &= P^+_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz \\ &\quad + \left(q_{-(\ell+1)} - q_{-\ell}\right) \frac{P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz - \left(P^+_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz\right)}{q_{-(\ell+1)} - q_{-\ell}} \\ &< P^+_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz + \varepsilon b_1 \left(q_{-(\ell+1)} - q_{-\ell}\right) \end{aligned}$$

where the inequality follows from $q_{-(\ell+1)} - q_{-\ell} < \varepsilon$ and equation (14) in Fact 3. Applying the same argument recursively leads to

$$\begin{aligned} P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz &< P^+_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz + \varepsilon b_1 \sum_{\tilde{\ell}=\ell}^{\ell''-1} \left(q_{-(\tilde{\ell}+1)} - q_{-\tilde{\ell}}\right) \\ &< P^+_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz + \varepsilon \eta b_1 \qquad \blacksquare \end{aligned}$$

Claim 12. *Consider ℓ' and ℓ'' with $1 \leq \ell' < \ell''$. Let $0 < \varepsilon < \eta^*$ and $0 < \eta < (3b_1b_2)^{-1}$ be such that $q_{-(\ell+1)} - q_{-\ell} < \varepsilon$ for all $\ell \in \{\ell', \ldots, \ell''-1\}$, $q_{-\ell''} - q_{-\ell'} < \eta$ and $P^+_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz < (3b_2)^{-1}\varepsilon$. Then, $q_{-\ell'} - q_{-(\ell'-1)} < \varepsilon$.*

Proof. We have

$$q_{-(\ell'+1)} - q_{-\ell'} \leq b_2 \left[P^+_{-\ell'} - \int_0^{\psi(q_{-\ell'})} \lambda(z) v_L dz \right] < b_2 \left(P^+_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz + \varepsilon \eta b_1 \right)$$
$$< b_2 \left((3b_2)^{-1}\varepsilon + \varepsilon (3b_1b_2)^{-1} b_1 \right) < \frac{2}{3}\varepsilon$$

where the first inequality follows from equation (15) in Fact 3 and the second one from Claim 11. This, together with Fact 2, implies that

$$q_{-\ell'} - q_{-(\ell'-1)} < \frac{4}{3} \left(q_{-(\ell'+1)} - q_{-\ell'} \right) < \left(\frac{4}{3}\right) \left(\frac{2}{3}\right) \varepsilon < \varepsilon \qquad \blacksquare$$

Claim 12 provides the last intermediate result to complete the proof of equation (8b). The sequence $\{q_{-\ell}\}_{\ell=0}^{\infty}$ is strictly increasing and bounded above by $\bar{q}(0)$. Then, it has a limit, which we denote by $q_{-\infty}$. With this, applying L'Hôpital's rule to equation (11) we obtain

$$\lim_{\ell \to \infty} P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz = 0.$$

We focus on elements of the sequence $\{q_{-\ell}\}_{\ell=0}^{\infty}$ which are sufficiently close to $q_{-\infty}$. Let $\ell' = \min\left\{\ell : q_{-\ell} \geq q_{-\infty} - (6b_1b_2)^{-1}\right\}$. Fix $\varepsilon = \frac{1}{2}\min\{q_{-\ell'} - q_{-\ell'+1}; \eta^*\} > 0$ and pick ℓ'' such that:

$$\max\left\{ q_{-(\ell''+1)} - q_{-\ell''}; P^+_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz \right\} < \min\left\{\varepsilon, (3b_2)^{-1}\varepsilon\right\}$$

Then, applying Claim 12 recursively, we obtain $q_{-\ell'} - q_{-\ell'+1} < \varepsilon$, which is a contradiction and completes the proof of equation (8b).

Proof of equations (8a), (8c), (8e) and (8f). Equations (8b) and (8d) together imply equation (8a). Equation (8) links *any* sequence of consecutive impasses. We take the limit of equation (8) as m grows large, use equations (8b) and (8d) and apply L'Hôpital's rule to obtain equation (8f). Equation (8e) follows from equation (8f) and equation (4b) in

Proposition 3. Finally, we show equation (8c) by contradiction. Assume instead that, taking subsequences if necessary, $\lim_{m\to\infty} z^m_{J_m} = \bar{z} < 1$. This, together with equation (8e), implies that, in the limit, the buyer's continuation payoff at the beginning of the game is negative:

$$\lim_{m\to\infty} W_m(m,0) = \hat{q}\left[\int_{\bar{z}}^{1} [\lambda(z)v_L - c]\, dz\right] + (1-\hat{q})\left[\int_{\bar{z}}^{1} [\lambda(z)v_H - c]\, dz\right] < 0$$

This can never happen, so we have reached a contradiction. ■

A.7 Proof of Proposition 1

Proof. We present here the proof for (a). The cases (b), (c) and (d) follow the same argument. Assume towards a contradiction that the result does not hold for (a). Equation (3b) implies that

$$\tilde{z}^{*\prime}(0) = \frac{rv_L \int_0^1 \tilde{\lambda}(z)dz}{v_L\tilde{\lambda}(1) - c} < \frac{rv_L \int_0^1 \lambda(z)dz}{v_L\lambda(1) - c} = z^{*\prime}(0).$$

Let $\underline{\tau} = \min\{\tau > 0 : \tilde{z}^*(\tau) = z^*(\tau)\}$. It follows again from equation (3b) that

$$\tilde{z}^{*\prime}(\underline{\tau}) = \frac{rv_L \int_0^{\tilde{z}^*(\underline{\tau})} \tilde{\lambda}(z)dz}{v_L\tilde{\lambda}(\tilde{z}^*(\underline{\tau})) - c} < \frac{rv_L \int_0^{z^*(\underline{\tau})} \lambda(z)dz}{v_L\lambda(z^*(\underline{\tau})) - c} = z^{*\prime}(\underline{\tau}).$$

But then there exists $\tau' \in (0,\underline{\tau})$ with $\tilde{z}^{*\prime}(\tau') = z^{*\prime}(\tau')$, reaching a contradiction. Finally, notice that $z^*(0) = 1$ and that $z^{*\prime}(\cdot)$ does not depend on v_H or $\hat{\beta}$. ■

A.8 Proof of Proposition 5

Proof. Let $\lambda(z) = 1$ for every $z \in [0,1]$. Fix the number of units m and the period length Δ. Let $W(1,\cdot)$ and $P(1,\cdot)$ be respectively the buyer's normalized payoff and the price function when one unit remains. These functions are as in DL, so $W(1,q) > 0$ for every $q \in [0,\hat{q}]$. Suppose that for every $K \in \{1,\ldots,m\}$ and for every $q \in [0,\hat{q}]$, $W(K,q) = KW(1,q)$ and $P(K,q) = KP(1,q)$. Finally consider a belief $q' \in [0,\hat{q}]$ such that the buyer makes a screening offer at state $(1,q')$. The following argument shows that it is not optimal for the buyer to make a universal offer at any state (K, q') with $K \in \{2, \dots, m\}$. Assume towards a contradiction that it is optimal to make a universal offer for $K - k$ units. Then,

$$\begin{aligned} W(K, q') = KW(1, q') &\leq \frac{K-k}{m} \left[\left(\hat{q} - q' \right) v_L + (1 - \hat{q})\, v_H - (1 - q')c \right] + \delta k W(1, q') \\ &< \frac{K-k}{m} \left[\left(\hat{q} - q' \right) v_L + (1 - \hat{q})\, v_H - (1 - q')c \right] + k W(1, q') \end{aligned}$$

This in turn, implies that

$$W(1, q') < \frac{1}{m} \left[\left(\hat{q} - q' \right) v_L + (1 - \hat{q})\, v_H - (1 - q')c \right]$$

which violates the assumption that a screening offer is optimal at state $(1, q')$. This argument directly implies Proposition 5 when gains from trade are constant.

An argument analogous to the one in the previous paragraphs extends the result to the case of increasing gains from trade. We omit the proof here.[18] ■

References

KANNAN, R. AND C. KRUEGER (1996): *Advanced analysis on the real line*, Springer Verlage Publications.

[18] An earlier version of our paper contains further details on the cases of constant and increasing returns.