ONLINE APPENDICES FOR AER PUBLICATION "DYNAMIC PRICE COMPETITION, LEARNING-BY-DOING AND STRATEGIC BUYERS" BY SWEETING, JIA, HUI AND YAO

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A Methods for Finding Equilibria

In this Appendix we describe the two methods that we use to find equilibria for our analyses in Sections 3, 4 and 5 (excepting Section 5.1).

A.1 Equation Solving.

One method for finding an equilibrium for a fixed set of parameters is to numerically solve the 4,743 value, continuation probability and first-order condition equations in Section 2 using the `fsolve` tool in MATLAB.[32] We specify tolerances of 1e-14 on the variables and on the objective function.

We use equation solving to identify equilibria from which we can start homotopies, and also, as we describe below, to fill in any gaps in a homotopy path that results from the homotopy algorithm stalling.

A.2 Homotopies

This Appendix details of our implementation of the homotopy algorithm, using the example of the b^p-homotopies that we use in Section 3. The methods used for other homotopies are similar. Our description of the homotopy algorithm follows the description in Besanko, Doraszelski, Kryukov, and Satterthwaite (2010) closely, and our implementation is based on the code of Besanko, Doraszelski, and Kryukov (2014) (BDK), and we use their numerical tolerances.

[32] We have used this tool with both numerical and analytic derivatives, and using different algorithms.

A.2.1 Overview.

An equilibrium for a given set of parameters is defined as the solution to the 4,743 equations presented in Section 2 . We can write these equations collectively as

$$F(\mathbf{x}; b^p, \rho, \sigma) = \mathbf{0}, \tag{A.1}$$

where $\mathbf{x} = (\mathbf{V}^*, \mathbf{V}^{INT*}, \mathbf{p}^*, \lambda^*)$ (i.e., values, for buyers and sellers, and strategies) and we are implicitly conditioning on other parameters that we hold fixed such as the discount factor and the entry cost and scrap value distribution parameters. The objective of a b^p-homotopy is to explore the correspondence

$$F^{-1} = \{\mathbf{x} | F(\mathbf{x}; b^p, \rho, \sigma) = \mathbf{0}, b^p \in [0, 1]\}. \tag{A.2}$$

To follow the correspondence, the homotopy method introduces an ancillary parameter s, so that equation (A.2) becomes,

$$F^{-1} = \{\mathbf{x}(s) | F(\mathbf{x}(s); b^p(s), \rho, \sigma) = \mathbf{0}, b^p \in [0, 1]\}. \tag{A.3}$$

Assuming that a vector $\mathbf{x}$ satisfies the equations, the following conditions must be satisfied for the homotopy to remain on the correspondence

$$\frac{\partial F(\mathbf{x}(s); b^p(s), \rho, \sigma)}{\partial \mathbf{x}} \mathbf{x}'(s) + \frac{\partial F(\mathbf{x}(s); b^p(s), \rho, \sigma)}{\partial b^p} b^{p\prime}(s) = \mathbf{0} \tag{A.4}$$

where $\frac{\partial F(\mathbf{x}(s); b^p(s), \rho, \sigma)}{\partial \mathbf{x}}$ is a (4,743 x 4,743) matrix, $\mathbf{x}'(s)$ and $\frac{\partial F(\mathbf{x}(s); b^p(s), \rho, \sigma)}{\partial b^p}$ are both (4,743 x 1) vectors and $b^{p\prime}(s)$ is a scalar. The solution to these differential equations will have the following form, where $y_i'(s)$ is the derivative of the ith element of $\mathbf{y}(s) = (\mathbf{x}(s), b^p(s))$,

$$y_i'(s) = (-1)^{i+1} \det \left(\left(\frac{\partial F(\mathbf{y}(s); \rho, \sigma)}{\partial \mathbf{y}} \right)_{-i} \right) \tag{A.5}$$

where $_{-i}$ means that the ith column is removed from the (4,743 x 4,744) matrix $\frac{\partial F(\mathbf{y}(s); \rho, \sigma)}{\partial \mathbf{y}}$.

A.2.2 Implementation.

The homotopy procedure is implemented using the FORTRAN routines FIXPNS and STEPNS from HOMPACK90. Jacobians are computed numerically, although we specify which elements of the Jacobian are non-zero.[33,34] The algorithm keeps track of the values of $\mathbf{x}$ and b^p at each step on the path, which we can then use to compute associated outcomes, such as HHI^∞ and P^∞, which we do using the same code as BDK.

Restarting. A practical problem that arises is that a homotopy can stall or start taking an apparently endless sequence of increasingly small steps. We use a few different approaches to try to complete a path. One approach involves running homotopies in the opposite direction (e.g., decreasing b^p, rather than increasing b^p) from equilibria that have already been found. This often connects up sections of a path that have been found using different homotopy runs. If this does not work, we try to identify an adjacent equilibrium by solving the equilibrium equations for a close value of b^p, and then use this value to start a new homotopy path. If this path also does not progress, we solve the equations for additional small changes of b^p.

Computational Burden. The time taken to run a homotopy is usually between one hour and seven hours, when it is run on the University of Maryland's Department of Economics cluster. The servers on this cluster have the configurations of Dell PowerEdge R620 2x Intel Xeon E5-2680 v2 384GB.

[33]STEPNS is a predictor-corrector algorithm where hermetic cubic interpolation is used to guess the next point, and an iterative procedure is then used to return to the path.

[34]For details of the HOMPACK subroutines, please consult manual of the algorithm at `https://users.wpi.edu/~walker/Papers/hompack90,ACM-TOMS_23,1997,514-549.pdf`.

B Definition of SELPM Equilibria.

As explained in Section 2, we pay particular attention to one type of non-accommodative equilibria which we call SELPM equilibria.

Definition *A symmetric equilibrium has the "**S**ome **E**xit **L**eads to **P**ermanent **M**onopoly" (SELPM) property if there is some state $e_1^* > 1$, where (i) $\lambda_1(e_1, e_2) = 1$ for all $e_1 \geq e_1^*$*[35] *and $\forall e_2$, including $e_2 = 0$; (ii) $\lambda_2(e_1^*, e_2) < 1$ for some e_2 where $0 < e_2 < e_1^*$, and $\lambda_2(e_1, 0) = 0$ for all $e_1 \geq e_1^*$.*

After some additional discussion of this definition, Appendix C provides a classification of the equilibria identified by the σ- and ρ-homotopies in Sections 4 into accommodative, SELPM and two alternative types of non-accommodative equilibria. Appendix D details the algorithm that we use to identify whether at least one SELPM equilibrium exists.

Discussion. The High and Mid-HHI baseline ($b^p = 0$) equilibria in Table 1 (illustrative parameters) are both SELPM: $e_1^* = 30$ satisfies the definition in both cases. In fact, it is usually the case that $e_1^* = M = 30$ satisfies the definition if an equilibrium is SELPM.[36] Note that our algorithm that tests whether a SELPM equilibrium exists will stop when at the highest e_1 that satisfies the criteria for e_1^*.

Figures B.1 and B.2 provide examples of how play may move through the state space in SELPM equilibria. The first figure shows two paths where we assume that the sellers use the baseline High-HHI equilibrium strategies. The red line shows a path where both sellers make a sale in the first two periods of the game, and the game then evolves to (M, M). The black line shows a path where seller 1 makes the first $M - 4$ sales, and seller 2 then exits. Once seller 1 has made a sale, there is no possibility of entry by a potential entrant seller 2, and the games moves to $(M, 0)$.

Figure B.2 provides a second (hypothetical) example where a potential entrant seller 2 could enter in some states. However, $e_1 = 30$ satisfies the definition of e_1^*, so the equilibrium is SELPM.

[35] Note, that as we are only looking at symmetric equilibria, this condition implies that $\lambda_2(e_1, e_2) = 1$ for all $e_2 \geq e_1^*$.

[36] Intuitively, a laggard will have the strongest incentive to exit, and a potential entrant the least incentive to enter, when it is as far behind the leader as possible.

Figure B.1: Baseline High-HHI Equilibrium: Examples of Possible Paths Through the State Space For a Game Starting at (1,1). The numbers in each cell are seller 2's continuation probabilities. The circular arrows indicate no sale being made, due to the buyer choosing the "outside option".

$\mathbf{e_1}$

$\mathbf{e_2}$

	out	1	2	...	$M-4$	$M-3$	$M-2$	$M-1$	M
out	0.9583	0	0		0	0	0	0	0
1	1	0.9996	0.7799		0.7777	0.7777	0.7777	0.7777	0.7777
2	1	1	1		1	1	1	1	1
...									
$M-4$	1	1	1		1	1	1	1	1
$M-3$	1	1	1		1	1	1	1	1
$M-2$	1	1	1		1	1	1	1	1
$M-1$	1	1	1		1	1	1	1	1
M	1	1	1		1	1	1	1	1

The following are examples of strategies where the equilibrium would not be SELPM:

1. accommodative equilibria (i.e., $\lambda_2(e_1, e_2) = 1$ for all $e_1, e_2 \geq 1$);

2. an equilibrium where $\lambda_2(M, 0) > 0$ (for example, due to a low lower bound on entry costs);

3. an equilibrium where $\lambda_1(M, M) < 1$ (for example, due to a high upper bound on scrap values and/or intense duopoly competition); or,

4. an equilibrium where $\lambda_2(M, 0) = 0$, $\lambda_1(M, e_2) = 1$ for $e_2 \geq 2$ or $e_2 = 0$, but $\lambda_1(M, 1) < 1$. If the state reaches $(30, 2)$ the game will either proceed to (M, M) (permanent duopoly) with no exit, or seller 2 may exit and seller 1 will be a permanent monopolist. However, it fails to meet our definition because seller 1 may exit in state $(M, 1)$. We require the condition that $\lambda_1(e_1, e_2)$ in all $e_1 \geq e_1^*$ because it allows us to construct

Figure B.2: Alternative SELPM Example with Possible Re-entry: Examples of Possible Paths Through the State Space For a Game Starting at (1,1). The numbers in each cell are seller 2's continuation probabilities. The circular arrows indicate no sale being made, due to the buyer choosing the "outside option".

$\mathbf{e_1}$

$\mathbf{e_2}$	out	1	2	...	$M-4$	$M-3$	$M-2$	$M-1$	M
out	0.958	0.758	0.643		0.123	0.042	0	0	0
1	1	1	0.945		0.689	0.675	0.672	0.669	0.666
2	1	1	1		0.695	0.689	0.686	0.683	0.681
...									
$M-4$	1	1	1		1	1	1	1	1
$M-3$	1	1	1		1	1	1	1	1
$M-2$	1	1	1		1	1	1	1	1
$M-1$	1	1	1		1	1	1	1	1
M	1	1	1		1	1	1	1	1

an algorithm that can check whether a SELPM equilibria exists or not under weaker assumptions on the scale of problem for which we can find all equilibria. Of course, if this equilibrium exists, it is also possible that a different equilibrium, where $\lambda_1(30, 1) = 1$, will satisfy the SELPM definition.

C Classification of Equilibria.

We now classify the equilibria identified in Section 4 into different types. Two mutually exclusive types are accommodative (see definition in Section 2) and SELPM. For a complete discussion of what we find, it is also useful to define two other types.

Definition *An equilibrium has the "**A**ny **E**xit **L**eads to **P**ermanent **M**onopoly" (AELPM) property if (i)* $\lambda_1(\mathbf{e}) = 1$ *for all* $\mathbf{e} = (e_1, e_2)$ *where* $e_1 \geq e_2$*; (ii) there is some* $\mathbf{e} = (e_1, e_2)$ *where* $e_1 > e_2 > 0$ *and* $\lambda_2(\mathbf{e}) < 1$*, and (iii) for any* $\mathbf{e} = (e_1, e_2)$ *where* $e_1 > e_2 > 0$ *and* $\lambda_2(\mathbf{e}) < 1$*,* $\lambda_2(e_1', 0) = 0$ *for* $e_1' \geq e_1$*.*

In an AELPM equilibrium, the only exit from duopoly will be by a strict laggard and there will be no re-entry once a laggard exits. Any AELPM equilibrium will be SELPM.[37] But, SELPM equilibria may not be AELPM. For example, the High-HHI baseline equilibrium in Table 1 is not AELPM because there is a chance that sellers exit in state (1,1), so that there is a small probability that both sellers exit, in which case there may be re-entry.

We also consider equilibria that satisfy BDK's definition of "aggressive" equilibria.

Definition *An equilibrium is "aggressive" if* $p_1(\mathbf{e}) < p_1(e_1, e_2+1), p_2(\mathbf{e}) < p_2(e_1, e_2+1)$*, and* $\lambda_2(\mathbf{e}) < \lambda_2(e_1, e_2 + 1)$ *for some state* $\mathbf{e} = (e_1, e_2)$ $e_1 > e_2 > 0$*.*

This definition depends on both prices and continuation strategies. Aggressive equilibria are not accommodative, and they may or may not be AELPM or SELPM.

C.1 Classification for σ- and ρ-Homtopies for the Illustrative Parameters.

Figure C.1(a) and (b) shows a classification of the equilibria found by σ- and ρ-homotopies for different values of b^p. The other parameters are held at their baseline values. The HHI^∞ is shown on the y-axis. The different line styles indicate the different type of equilibria. Recall that all AELPM equilibria are SELPM. For these parameters we find that:

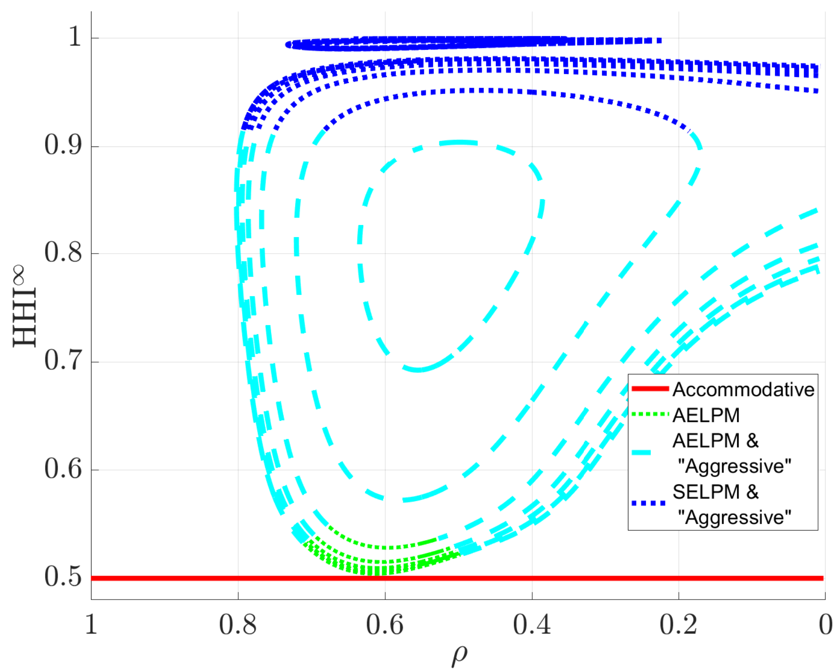
[37] In particular, the fact that a leader will never exit rules out the fourth type of non-SELPM equilibrium above. Any value of e_1 where there is some possibility that a laggard seller 2 exits will meet the e_1^* definition.

- all identified equilibria are either accommodative or SELPM (i.e., this classification is exhaustive for the equilibria that the homotopies identify for these parameters); and
- all identified aggressive equilibria are SELPM, although many SELPM equilibria are not aggressive.

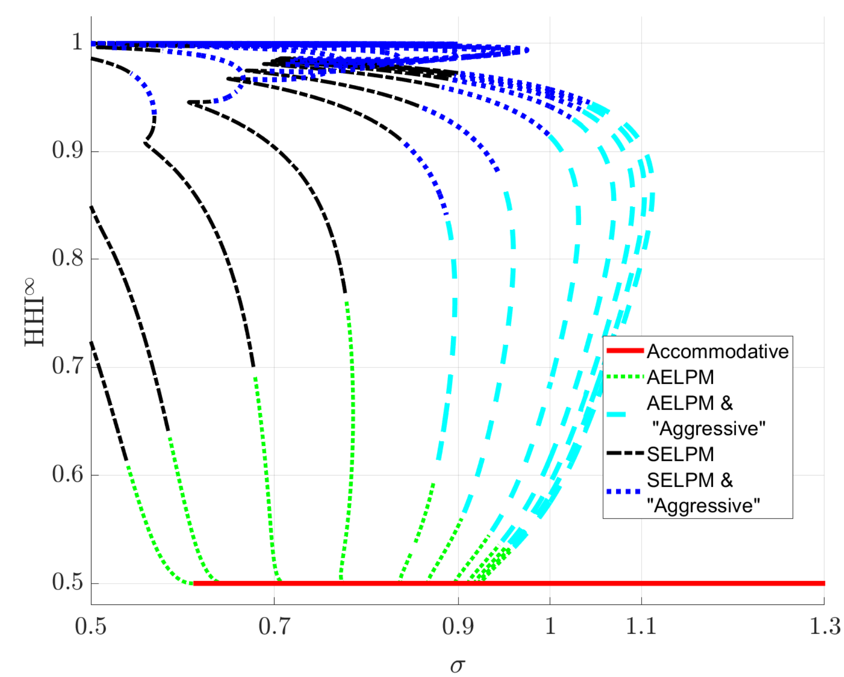
There is also an interesting pattern where the AELPM equilibria tend not to be the equilibria with the highest implied values of HHI^{∞}. Even though high HHI^{∞} equilibria tend to have low duopoly prices, and it is not attractive for a potential entrant to enter against a monopolist, there is usually some probability of exit in symmetric duopoly states, particularly (1,1), so these equilibria do not meet the AELPM criteria.

Figure C.1: Classification of Equilibria Identified by ρ- and σ-Homotopies for Various b^p. Other parameters at their illustrative values. See text Figures 4(a) and (c) for which line corresponds to which b^p. Equilibria indicated as "SELPM" or "AELPM" only do not satisfy the definition of aggressive equilibria.

(a) ρ-Homotopy Paths ($\sigma = 1$)



(b) σ-Homotopy Paths ($\rho = 0.75$)



D The Algorithm for Identifying if SELPM Equilibria Exist.

A property of SELPM equilibria is that once the state e_1^* has been reached, state transitions have the directional property that the state will evolve to (M, M) or $(M, 0)$ without returning to a previously visited state. As discussed by Iskhakov, Rust, and Schjerning (2016), recursive algorithms, which solve for equilibria in a sequence of individual states, can be used when states evolve directionally. However, the way that we use this idea is novel in at least two ways. First, we consider a directional property that applies to a certain type of equilibrium in part of the state space, rather than a property which has to apply to all equilibria given primitives of the model. Second, we apply a recursive algorithm to find whether this type of equilibrium exists, rather than trying to find all equilibria.

We proceed as follows. First, we describe how the algorithm proceeds through the state space, and how it terminates in success or failure, without providing details of how we solve for equilibrium strategies in any particular state. Instead, we make assumptions about our ability to solve for all equilibria in a particular state given continuation values if the state changes.[38] Second, we provide the proof that, under these assumptions, our algorithm will terminate in success if and only if a SELPM equilibrium exists. Finally, we detail the mechanics of how we solve for equilibria in different types of states.

D.1 Overview of the Algorithm

The algorithm recursively solves for equilibrium strategies in each state until we either (i) find an equilibrium path where there is an e_1 state that meets the SELPM definition of e_1^* ("success"), or (ii) find that all paths are inconsistent with SELPM ("failure"). Notably, either outcome may be achieved by going only through a small part of the state space.

Figure D.1 describes the recursive path that the algorithm takes through the state space. The key feature is that we only construct and follow paths that are consistent with SELPM in states $e_1 \geq e_1^*$. For example, this implies that if the industry becomes a monopoly then

[38] As noted by Iskhakov, Rust, and Schjerning (2016), assumptions are needed as no algorithms are guaranteed to find all equilibria in particular states, outside of some special cases that do not apply here. However, we explain why we are confident that, in practice, we are able to find all equilibria.

Figure D.1: Outline of the Recursive Algorithm.

Main Program:

Create matrix E that will contain strategies and values on the equilibrium path.

Solve for equilibrium strategies and values in all states where $(e_1,0)$ for all $e_1>0$ assuming that $\lambda_1(e_1,0)=1$ and $\lambda_2(e_1,0)=0$ for all such states. Store these strategies and values in E.

Solve for equilibrium prices (which will equal static Nash prices) and values in state (30,30) assuming that $\lambda_1(30,30)=\lambda_2(30,30)=1$. If implied $\beta V_1^S(30,30) < (\overline{X}+\Delta X)$, i.e., a seller may exit, then there are no SELPM equilibria, and the program terminates. Otherwise, add these strategies to E.

Set e1==30, e2==29, call [fail,success] = recursion_function(e1,e2,E)

If success==1, there is a SELPM equilibrium.

If fail==1, there is no SELPM equilibrium.

Program terminates.

Recursive Function

function [fail,success] = recursion_function(e1start,e2start,E)

% Initialize variables
Set success=0
Set fail=0
Set e1=e1start
Set e2=e2start

% Outer loop, decreasing over the states of the leader
While success==0 && fail==0 && e1>=1,

   % Initialize
   e2=e2start

   % Inner loop, decreasing over the states of the laggard
   While success==0 && fail==0 && e2>=0,

      % In a state where firm 2 is a potential entrant, determine whether we have identified an e1 state satisfying the SELPM definition, or whether the equilibrium path is inconsistent with SELPM

      If e2==0,
            Given the continuation values in E, check whether firm 2 would want to enter in state $(e_1,0)$ if $\lambda_1(e_1,0)=1$.

            If yes, this is not a SELPM path, set fail=1.

            If no, check whether firm 1 would want to continue with probability 1 if $\lambda_2(e_1,0)=0$.
                  If no, this is not a SELPM path, set fail=1.
                  If yes, check whether this path involves some positive probability of laggard exit when the leader state is e_1.
                        If yes, a SELPM path is identified, set success=1.

      Else % states where both firms are active
            If e1==e2, % symmetric state

Figure D.1: Outline of the Recursive Algorithm cont.

```
                Find assumed-to-be unique state-specific symmetric
                equilibrium prices and values in (e₁, e₂), using
                continuation values from E, assuming λ₂(e₁,e₁)=λ₁(e₁,e₂)=1.

                Check that these values imply that the leader will want to
                continue with probability 1.
                      If no, this is not a SELPM path, set fail=1.
                      If yes, record the prices and values for (e₁,e₂) in E.

                If e1==1 && no exit from any e1 states, an accommodative
                equilibrium has been identified, set fail=1.
          else, % e1>e2

                Find all pricing and seller 2 continuation probability
                state-specific equilibria for (e₁,e₂), using continuation
                values from E, when assume λ₁(e₁,e₂)=1.

                Identify state-specific equilibria where λ₂(e₁,e₂)<1-(e-10)
                as equilibria with laggard exit.

                Remove any state-specific equilibria that imply the leader
                might want to exit from consideration.

                For the remaining equilibria:
                      If there are none, set fail=1.
                      If there is exactly one, record the prices and values
                      for (e₁,e₂) in E.
                      If there are multiple, set local_fail==1, and then
                      while success==0 and local_fail==1,
                            for each state-specific equilibrium in turn
                            add the state-specific equilibrium to E to
                                  create E',
                            set e2'=e2-1 and call
                            [local_fail,success]=
                            recursion_function(e1,e2',E')

                      end (while)
                      Set fail=local_fail.
          end (the if e1==e2 condition)
    end (the e2==0 condition)
    Set e2=e2-1.
  end (inner loop)
  Set e1=e1-1.
  Set e2start=e1.
end (outer loop)

end (function)
```

it will remain so. We therefore solve for equilibrium prices and values in duopoly states assuming that this will be what happens in monopoly states, before verifying that, in fact, potential entrants would not want to enter. The SELPM-consistent equilibrium strategies and values on the current path (including for monopoly states that the algorithm has not yet reached) are stored in a set of matrices, that, for ease of description, we collective label as E.[39]

To understand the process, consider the illustrative parameters with $b^p = 0$. The algorithm solves for equilibria in monopoly states when seller 1 and the buyer assumes that the potential entrant will not enter. Consistent with Table 1, this implies the incumbent will set prices of 8.54 in state (30,0), and for example, 8.72 in state (2,0), although this price will only be relevant if the search for an e_1^* continues back to $e_1 = 2$. It then solves for the SELPM-consistent equilibrium in state (30,30), where neither seller will exit, before progressing through the states (30,29), (30,28),...,(30,2), using the continuation values in the states that the game could move to in a SELPM equilibrium (including (30,0)) in order to solve the game in a particular state. In these states, we find that the only SELPM-consistent equilibria have $\lambda_2 = 1$. In state (30,1) we find three equilibria with $\lambda_2 = 0.7777$, 0.9577 and 1. The algorithm selects the 0.9577 (Mid-HHI) equilibrium to try first. In this case, it only needs to check if $\lambda_1(30,0) = 1$ and $\lambda_2(30,0) = 0$ given the implied $V_1^S(30,1)$ and $V_2^S(30,1)$. Both checks are passed so the criteria for e_1^* are satisfied by $e_1 = 30$ and the algorithm terminates in success. If, counterfactually, we had found multiple equilibria in state (30,2), then algorithm would have selected one path, extended that path to find an equilibrium in state (30,1) and then performed the check on continuation probabilities in state (30,0). If the SELPM conditions are rejected on one path, the next path, if one is available, is chosen.

D.2 Properties of the Algorithm

We make two claims about the property of the algorithm.

Claim 1 *If the algorithm terminates in success, then a SELPM equilibrium exists.*

[39]Our code also assumes that seller 2 is the leader, rather than seller 1. We present our description with seller 1 as the leader as it is easier to follow.

Proof. Inspection reveals that if the algorithm terminates in success for $e_1 = e_1'$ then (i) $\lambda_1(e_1, e_2) = 1$ for all $e_1 \geq e_1'$ and all e_2, including $e_2 = 0$[40], and (ii) $\lambda_2(e_1', e_2) < 1$ for some $0 < e_2 < e_1'$ and $\lambda_2(e_1, 0) = 0$ for all $e_1 \geq e_1'$.

Therefore, the path that terminates in success has equilibrium strategies and values consistent with SELPM for all states where $e_1 \geq e_1'$, and e_1' satisfies the criteria for e_1^* in the definition.[41]

It remains to show that a set of equilibrium strategies and values in earlier states must exist that, when combined with these strategies and values, would form an equilibrium in the whole game. In a SELPM equilibrium, once state e_1^* has been reached, play will only move through states where equilibrium strategies and values have been calculated by the algorithm. Therefore, we only require that an equilibrium exists in a reduced game where the states are $e_1 = 1, ..., e_1^* - 1$ and the terminal payoffs of players if a buyer purchases from seller 1 in a state $(e_1^* - 1, e_2)$ are $V_i^{S,INT}(e_1^*, e_2)$ and $V^{B,INT}(e_1^*, e_2)$. Existence of an equilibrium in this reduced game follows from the arguments in Doraszelski and Satterthwaite (2010). ■

To prove that the algorithm will terminate in success if a SELPM equilibrium exists, we make three additional assumptions.

Assumption 1 *There is a unique state-specific equilibrium (i.e., values of* p_1, V_1^S, $V_1^{S,INT}$, V^B, $V^{B,INT}$ *satisfying the monopoly state version of the equilibrium equations in Section 2) in a monopoly state* $(e_1, 0)$ *with* $e_1 < M$*, given fixed buyer and seller continuation values if the buyer purchases from seller 1, if* $\lambda_1(e_1, 0) = 1$ *and* $\lambda_2(e_1, 0) = 0$.

Assumption 2 *There is a unique symmetric state-specific equilibrium (i.e., values of* p_1, p_2, V_1^S, V_2^S, $V_1^{S,INT}$, $V_2^{S,INT}$, V^B, $V^{B,INT}$ *satisfying the duopoly state equations in Section 2) in a symmetric duopoly state* (e_1, e_1) *with* $e_1 < M$*, given fixed buyer and seller continuation values if the buyer purchases from sellers 1 or 2, when* $\lambda_1(e_1, e_1) = \lambda_2(e_1, e_1) = 1$.

[40] One may notice that the algorithm does not solve for strategies in a state where seller 2 is the leader, e.g., (29,30). However, under the restriction that we are only solving for symmetric equilibria, then for the algorithm to be looping through e_2 states for $e_1 = 29$ it must be the case that $\lambda_2(30, 29) = 1$ on the path that is being tracked, so it follows that $\lambda_1(29, 30) = 1$.

[41] Of course, the strategies found in monopoly states where $e_1 < e_1'$ may not be consistent with equilibrium behavior, but they would have been consistent if, in search of an e_1 state meeting the SELPM-criteria, the algorithm had visited these states.

Assumption 3 *We are able to find all state-specific equilibria (i.e., values of p_1, p_2, λ_2, V_1^S, V_2^S, $V_1^{S,INT}$, $V_2^{S,INT}$, V^B, $V^{B,INT}$ satisfying the duopoly state version of the equations in Section 2) in an asymmetric duopoly state (e_1, e_2) with $e_1 > e_2$, given fixed buyer and seller continuation values if the buyer purchases from seller 2 or seller 1 (if $e_1 < M$), when $\lambda_1(e_1, e_2) = 1$.*

Assumption 1 concerns states $(e_1, 0)$ with $e_1 < M$. In state $(M, 0)$, a more general property must hold.

Property 1 *There is a unique state-specific equilibrium (i.e., values of p_1, V_1^S, $V_1^{S,INT}$, V^B, $V^{B,INT}$ satisfying the monopoly state version of the equilibrium equations in Section 2) in a monopoly state $(M, 0)$ if $\lambda_1(M, 0) = 1$ and $\lambda_2(M, 0) = 0$.*

Proof. If $\lambda_1(M, 0) = 1$ and $\lambda_2(M, 0) = 0$, then it is certain that the game will remain in state $(M, 0)$ whatever purchase decision the buyer makes. Therefore, from text equation (8), buyer demand will be identical to the demand of an atomistic buyer, whatever the value of b^p, and the monopolist's price choice can also not affect its future value. Therefore, the unique equilibrium will involve the seller setting the static monopoly price. ■

Assumption 2 concerns states (e_1, e_1) with $e_1 < M$. In state (M, M), a more general property must hold.

Property 2 *There is a unique state-specific equilibrium (i.e., values of p_1, V_1^S, $V_1^{S,INT}$, V^B, $V^{B,INT}$ satisfying the monopoly state version of the equilibrium equations in Section 2) in a monopoly state (M, M) if $\lambda_1(M, M) = \lambda_2(M, M) = 1$.*

Proof. If $\lambda_1(M, M) = \lambda_2(M, M) = 1$, then it is certain that the game will remain in state (M, M) whatever purchase decision the buyer makes. Therefore, from text equation (8), buyer demand will be identical to the demand of an atomistic buyer, whatever the value of b^p. The price choice of either seller will not affect their future values, so the number of equilibria consistent with Markov Perfect behavior will correspond to the number of equilibria in a one-shot game where sellers have the same marginal costs. The multinomial logit form of demand implies that the equilibrium will be unique (e.g., Mizuno (2003)). ■

As noted below, we have never found examples under which either of Assumptions 1 or 2 are violated. We detail below the procedures that we use to find equilibria in any state, and we provide evidence that explains why we believe all these assumptions hold, although, like BDK, we find that it can be challenging to find any equilibrium for low values of σ. For this reason, we do not report results for $\sigma < 0.5$.

Claim 2 *Under assumptions 1-3, if a SELPM equilibrium exists, then our algorithm will terminate in success.*

Proof. The assumptions and Properties 1 and 2 imply that the algorithm will follow, and evaluate, every possible SELPM-consistent equilibrium path before terminating in failure. Therefore if a SELPM-consistent state e_1^* exists, the algorithm will find it. ■

D.3 Methods for Solving for Equilibria in Specific States

We now describe how we solve for equilibria that are consistent with SELPM in specific states. We describe our routines assuming that $\sigma = 1$ to reduce notation. Our examples assume the illustrative parameters, with $\rho = 0.75$ and $\sigma = 1$, unless otherwise stated.

D.3.1 Solving for Equilibria in Monopoly States $(e_1, 0)$ assuming $\lambda_1(e_1, 0) = 1$ and $\lambda_2(e_1, 0) = 0$.

Consider a state $(e_1 < M, 0)$. Assuming $\lambda_1(e_1, 0) = 1$ and $\lambda_2(e_1, 0) = 0$, the following equations determine the equilibrium values of V^B, $V^{B,INT}$, V_1^S,$V_1^{S,INT}$ and p_1 where seller 1's marginal cost is c,

$$V^B = b^p \ln(\exp(V^{B,INT}) + \exp(v - p_1 + V^{B,INT}(e_1 + 1, 0)))+ \tag{D.1}$$
$$(1 - b^p)(D_1 V^{B,INT}(e_1 + 1, 0) + (1 - D_1)V^{B,INT})$$

$$V_1^S = (p_1 - c + V_1^{S,INT}(e_1 + 1, 0))D_1 + V_1^{S,INT}(1 - D_1) \tag{D.2}$$

$$D_1 + (p_1 - c + V_1^{S,INT}(e_1 + 1, 0) - V_1^{S,INT})\frac{\partial D_1}{\partial p_1} = 0 \tag{D.3}$$

$$V^{B,INT} = \beta V^B \tag{D.4}$$

$$V_1^{S,INT} = \beta V_1^S \tag{D.5}$$

where $D_1 = \frac{\exp\left(v_1 - p_1 + V^{B,INT}(e_1+1,0)\right)}{\exp(v_1 - p_1 + V^{B,INT}(e_1+1,0)) + \exp(V^{B,INT})}$ assuming, following BDK, that $v_0 = p_0$.

We could solve these sets of equations recursively for different monopoly states. However, we find it quicker to solve the equations for all of the monopoly states simultaneously in MATLAB using `fsolve`. We also reduce the number of variables by solving for V^B, V_1^S and p_1 and using these values to solve for $V^{B,INT}$ and $V_1^{S,INT}$ as needed.

Discussion of the Uniqueness Assumption. We have performed an analysis to check whether Assumption 1 is likely satisfied. Specifically, we can look at whether two equilibrium curves intersect more than once. The first curve solves the value of V^B as a function of p_1, reflecting equation (D.1). The second curve solves for the value of p_1 that maximizes the seller's value, given V^B, as determined by the first-order condition (D.3).

Figure D.2 presents examples of what these curves look like for state $(10, 0)$ using the illustrative parameters when $b^p = 0.25$, $0.5, 0.75$ and 1. The black curves denote the value of V^B given p_1, and the red curves reflect the value-maximizing choices of p_1 given values of V^B. The curves cross only once in every case, consistent with a single equilibrium. We have verified that there is only one intersection for a very large number of different values of ρ, σ, b^p, $V^S(e_1 + 1, 0)$ and $V^B(e_1 + 1, 0)$.[42]

D.3.2 Solving for Equilibrium in Absorbing Duopoly State (M, M).

(M, M) is an absorbing state in a SELPM equilibrium. This implies that there is a unique SELPM-consistent equilibrium where prices are the same as static Nash prices with non-strategic buyers (uniqueness of these prices follows from the multinomial logit form of demand (e.g., Mizuno (2003))).

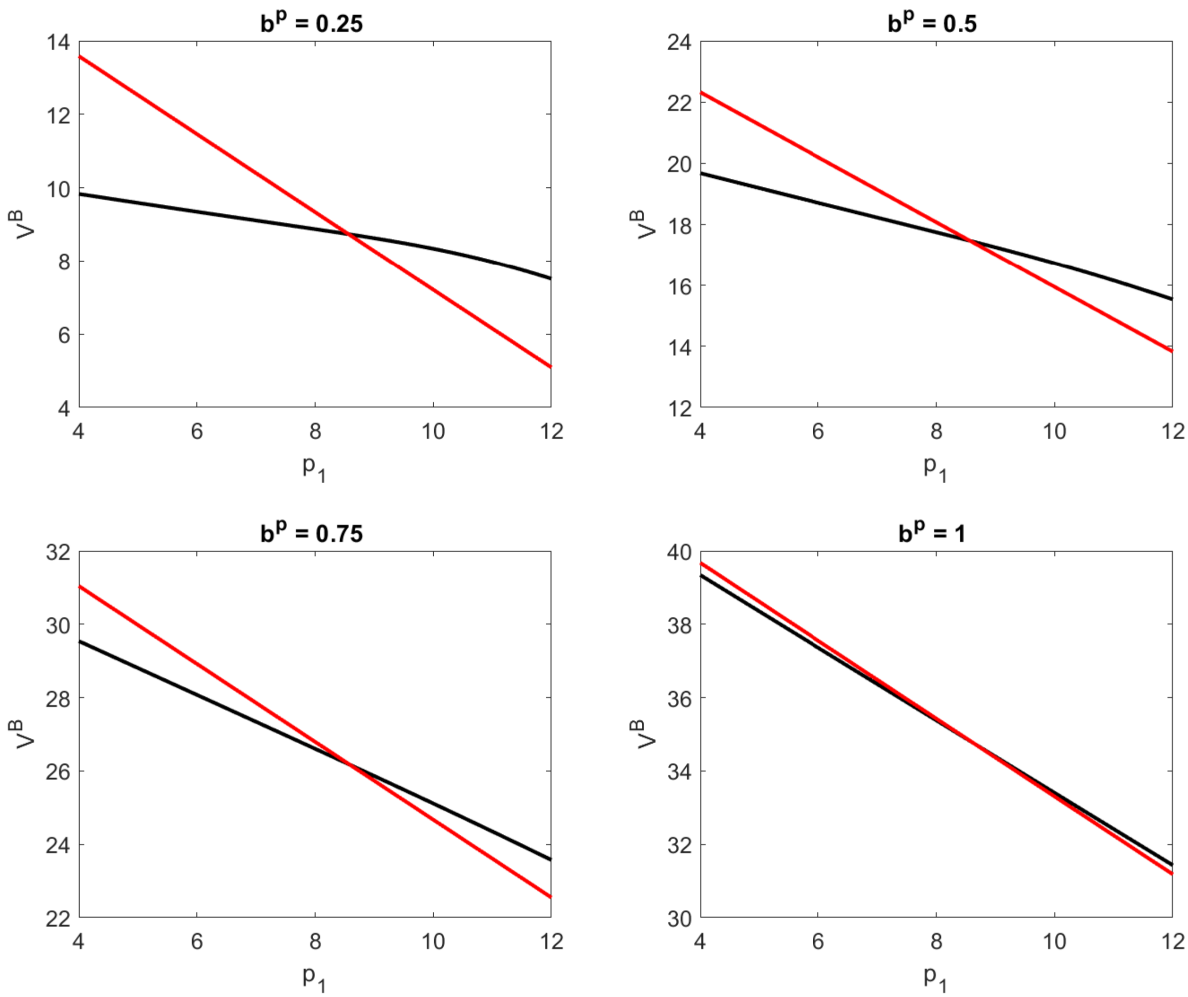
We find equilibrium prices by solving static pricing first-order conditions,

$$D_i + (p_i - c)\frac{\partial D_i}{\partial p_i} = 0,$$

and then calculating the implied buyer and seller values (V^S). We verify that βV^S is greater

[42]Specifically, we use b^p values on a grid [0.2,0.4,0.6,0.8,1], ρ values [0,0.1,0.2,..,0.9,1], σ values [0.5,0.6,..,1.1,1.2], $V^S(e_1 + 1, 0)$ values [60,65,..,95,100] and $V^B(e_1 + 1, 0)$ values b^p*[20,25,30,35,40]. This gives a total of 19,800 combinations that we check. We have also experimented with other values.

Figure D.2: Monopoly State Equations in State $(10, 0)$: black curve is the value of V^B as a function of p_1, red curve is the optimal p_1 given V^B. There is an equilibrium where the lines intersect.



than the maximum possible scrap value, so that exit is not optimal. If exit could be optimal, there is no SELPM equilibrium.

D.3.3 Solving for Equilibria in Other Duopoly States $(e_1, e_2), e_1 \geq e_2 > 0, e_2 < M$.

In a duopoly state we want to solve for all SELPM-consistent values of

- prices (p_1, p_2)
- values $(V_1^S, V_2^S, V_1^{S,INT}, V_2^{S,INT}, V^B, V^{B,INT})$
- continuation probability for seller 2 (λ_2), although SELPM implies that $\lambda_2 = 1$ if $e_1 = e_2$ for $e_1 \geq e_1^*$.

The continuation probability for seller 1 must be 1. The nine variables must satisfy the following nine equations

$$V_i^S - D_i(p_1, p_2, V^B)(p_i - c_i(e_i)) - \sum_{k=0,1,2} D_k(p_1, p_2, V^B) V_i^{S,INT}(\mathbf{e}'_k) = 0 \text{ for } i = 1, 2 \quad \text{(D.6)}$$

where $V_i^{S,INT}(\mathbf{e}'_0) = V_i^{S,INT}$,

$$V_1^{S,INT} = \beta \left(\lambda_2 V_1^S + (1 - \lambda_2) V_1^S(e_1, 0) \right) \text{ and } V_2^{S,INT} = \beta \lambda_2 V_2^S + (1 - \lambda_2) E(X|\lambda_2) \quad \text{(D.7)}$$

$$D_i(p_1, p_2, V^B) + \sum_{k=0,1,2} \frac{\partial D_k(p_1, p_2, V^B)}{\partial p_i} V_i^{S,INT}(\mathbf{e}'_k) + (p_i - c_i(e_i)) \frac{\partial D_i(p_1, p_2, V^B)}{\partial p_i} = 0 \text{ for } i = 1, 2 \quad \text{(D.8)}$$

$$\lambda_2 - F_{\text{scrap}}(\beta V_2^S) = 0 \quad \text{(D.9)}$$

$$V^B = b^p \log \left(\sum_{k=0,1,2} \exp \left(v_k - p_k + V^{B,INT}(\mathbf{e}'_k) \right) \right) - (1 - b^p) \sum_{k=0,1,2} D_k(p_1, p_2, V^B) V^{B,INT}(\mathbf{e}'_k), \quad \text{(D.10)}$$

where $V^{B,INT}(\mathbf{e}'_0) = V^{B,INT}$,

$$V^{B,INT} = \beta \left(\lambda_2 V^B + (1 - \lambda_2) V^B(e_1, 0) \right), \quad \text{(D.11)}$$

where $\mathbf{e}'_k$ is the state that the game transitions to when the buyer purchases from k, and

$E(X|\lambda_2)$ is the expected scrap value when seller 2 exits with probability $1-\lambda_2$. When this involves a change of state, we take the continuation values as given. For example, if $e_1 < M$[43],

$$V_1^{S,INT}(\mathbf{e}_1') = \beta\left(\lambda_2(e_1+1,e_2)V_1^S(e_1+1,e_2) + (1-\lambda_2(e_1+1,e_2))V_1^S(e_1+1,0)\right) \quad \text{(D.12)}$$

$$V_2^{S,INT}(\mathbf{e}_1') = \beta\left(\lambda_2(e_1+1,e_2)V_2^S(e_1+1,e_2) + (1-\lambda_2(e_1+1,e_2))E(X|\lambda_2(e_1+1,e_2))\right), \quad \text{(D.13)}$$

where $E(X|\lambda_2(e_1+1,e_2))$ is the expected scrap value if seller 2 exits with probability $1-\lambda_2(e_1+1,e_2)$.

$$V^{B,INT}(\mathbf{e}_2') = \beta\left(\lambda_2(e_1,e_2+1)V^B(e_1,e_2+1) + (1-\lambda_2(e_1,e_2+1))V^B(e_1,0)\right), \quad \text{(D.14)}$$

$$V^{B,INT}(\mathbf{e}_1') = \beta\left(\lambda_2(e_1+1,e_2)V^B(e_1+1,e_2) + (1-\lambda_2(e_1+1,e_2))V^B(e_1+1,0)\right) \quad \text{(D.15)}$$

If $e_1 = e_2 \geq e_1^*$ then a SELPM-consistent equilibrium must have $\lambda_2 = \lambda_1 = 1$. Therefore, for these states, we solve for equilibrium prices and values assuming that $\lambda_2 = 1$, and then we verify that the solution implies that βV_2^S is greater than the highest possible scrap value, implying that $\lambda_2 = 1$ is optimal. In practice, we solve for V_i^S, V^B and p_i for $i = 1, 2$, substituting in for $V_i^{S,INT}$ and $V^{B,INT}$.

If $e_1 > e_2 \geq e_1^*$ then a SELPM-consistent equilibrium may have $\lambda_2 < 1$, and we may find multiple equilibria. Our method for identifying the set of SELPM-consistent equilibria assumes that there is a unique equilibrium for a given value of λ_2.[44] We specify a grid of values of λ_2, with steps of 0.01, and for each of these values we solve the equations (D.6), (D.8) and (D.10) for p_i, V_i^S and V^B, substituting into equations (D.7) and (D.11) for the values of $V_i^{S,INT}$ and $V^{B,INT}$.[45] We then calculate the best response value of λ_2, $\lambda_2^{\text{BR}}(\lambda_2)$, given V_2^S using equation (D.9).

[43] Alternatively, if $e_1 = M$, $V_1^{S,INT}(\mathbf{e}_1') = \beta\left(\lambda_2 V_1^S + (1-\lambda_2)V_1^S(M,0)\right)$ and $V_2^{S,INT}(\mathbf{e}_1') = \beta\left(\lambda_2 V_2^S + (1-\lambda_2)E(X|\lambda_2)\right)$ and $V^{B,INT}(\mathbf{e}_1') = \beta\left(\lambda_2 V^B + (1-\lambda_2)V_2^B(e_1,0)\right)$, so they depend on the endogenous λ_2, V^B and V_1^S, because a sale by seller 1 does not change the state.

[44] Given that equilibrium prices directly affect V_2^S and λ_2 is a strictly increasing function of V_2^S for $\lambda_2 < 1$, we regard this assumption as weak for $\lambda_2 < 1$.

[45] Occasionally the equations do not solve using the starting values chosen, in which case we use a Pakes-McGuire type of routine to find alternative starting values.

Figure D.3: Best Response Continuation Probability Functions for Seller 2 Given Endogenous Pricing Choices by Both Sellers. Intersections with the 45^0-degree line are equilibria.

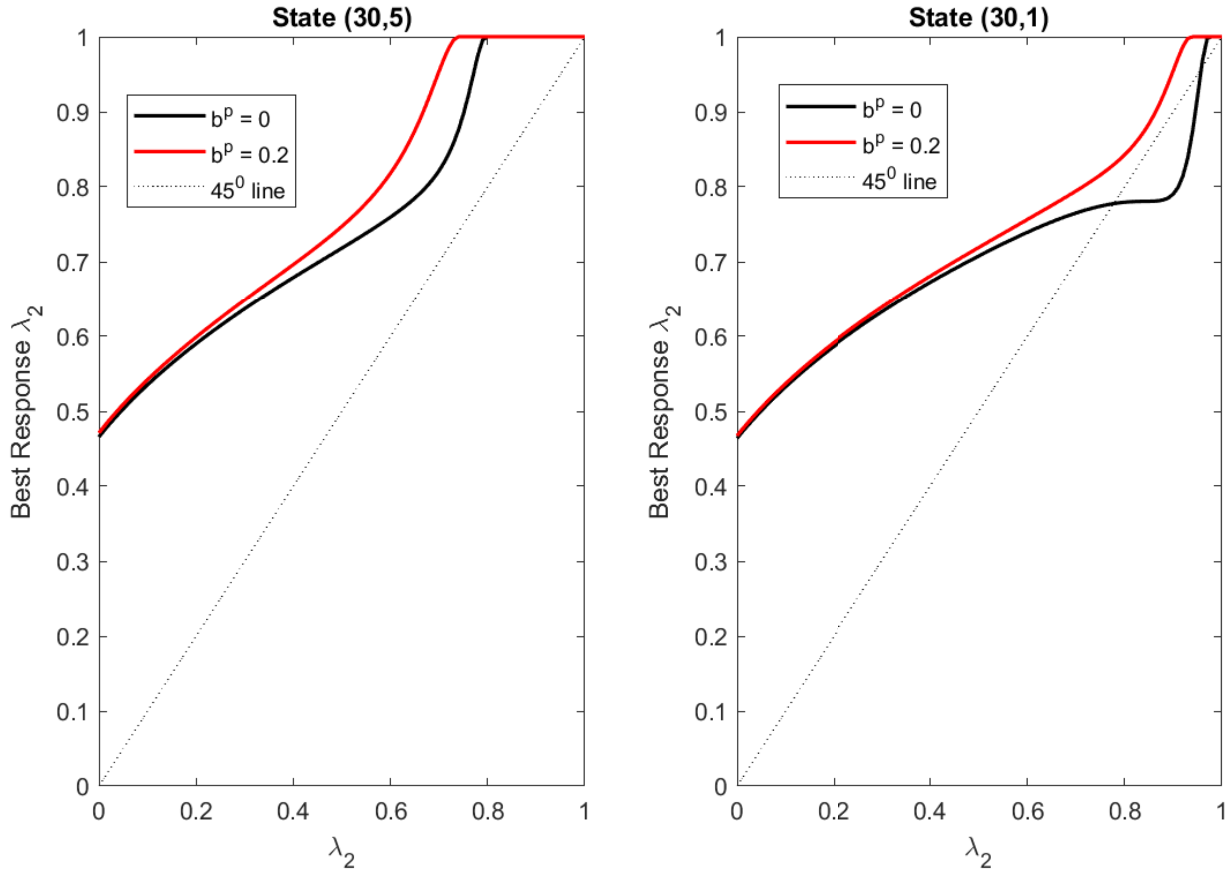


Figure D.3 shows examples of the function $\lambda_2^{\text{BR}}(\lambda_2)$ for the illustrative parameters, for states (30,1) and (30,5), with $b^p = 0$ and $b^p = 0.2$. There are equilibria at the points where the functions cross the 45-degree line. We find the precise intersection using locations between gridpoints either side of an intersection as starting points, before verifying that the solution is consistent with the leader continuing with probability 1, as required for SELPM.[46,47]

Discussion of the Uniqueness Assumption. As noted, our approach assumes that there is a unique pricing equilibrium given an assumed value of λ_2 when $\lambda_1 = 1$. There are

[46] We initially try to find the intersection by starting at the neighboring gridpoints, but if this fails, we use convex combinations of the gridpoints as starting values until the intersection is identified.

[47] As the figure suggests, it is possible that we would miss an intersection where the function is close to forming a tangent with the 45-degree line. We have found that gridpoints of 0.01 are adequate to identify whether SELPM equilibria exist, in the sense that our results do not change if we use a finer grid. This is partly because even if we do just miss an intersection in one particular state (e_1, e_2), there will often be a clearer intersection for state $(e_1, e_2 - 1)$ that we will capture, which may allow us to show that a SELPM equilibrium exists.

two types of evidence that support this presumption. First, we have never identified an instance of multiple equilibria for any of the parameters that we have considered, even when using multiple different starting points or alternative solution algorithms. Second, we have investigated whether there could be multiple equilibria by using a reaction function-type of analysis.

Specifically, for a given value of λ_2 and the continuation values, we solve the equations for V^B, V_1^S and the first-order condition for p_1 for a grid of alternative values of p_2. We then solve the equations for V^B, V_2^S and the first-order condition for p_2 for a grid of alternative values of p_1. We can then draw curves $p_1^*(p_2)$ and $p_2^*(p_1)$, which reflect optimal behavior of buyers and the other seller to the assumed price. The intersections correspond to equilibria, and we can test whether they intersect more than once. Figures D.4 presents some examples of these curves for the illustrative parameters, $b^p = 0$ or $b^p = 1$ and $e_1 = 30$ and $e_2 = 1$.

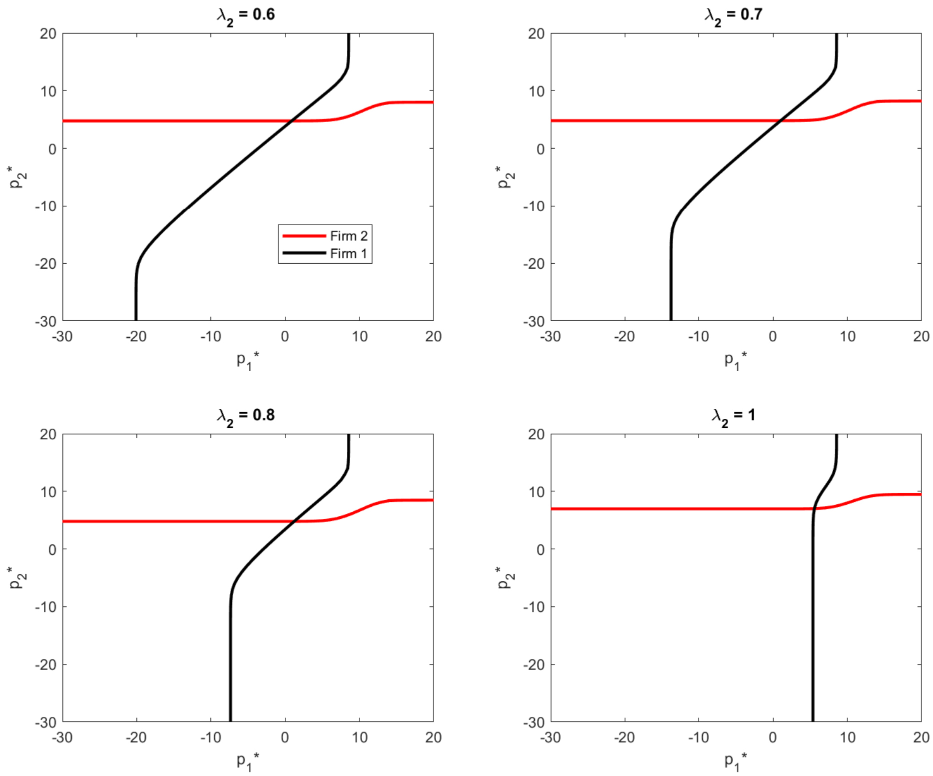
Recall that in the state (30,1), if the buyer purchases from seller 1, the state remains (30,1), whereas if seller 2 makes a sale, the state transitions to (30,2), where, for these parameters, there is always a unique equilibrium. If seller 2 is setting a much lower price in state (30,1) than in state (30,2), a strategic buyer will have an incentive to shift demand towards seller 1 in order to keep the state the same in future periods. As a result, seller 1's optimal price is less sensitive to seller 2's price in this state when $b^p = 1$, which accounts for the change in the slope of the reaction functions. However, in all cases, the reaction functions only intersect once, and there is a single equilibrium.[48]

In practice, it is prohibitive to perform this check for all values of λ_2 for all states for all parameters. However, our checking algorithm does perform this check in states where $e_1 = M$ for λ_2 =0.55, 0.65, 0.75, 0.85 and 0.95. We have never found parameters where there is ever more than one intersection. This is also the case when we have solved games for many different sets of arbitrary continuation values and parameters.

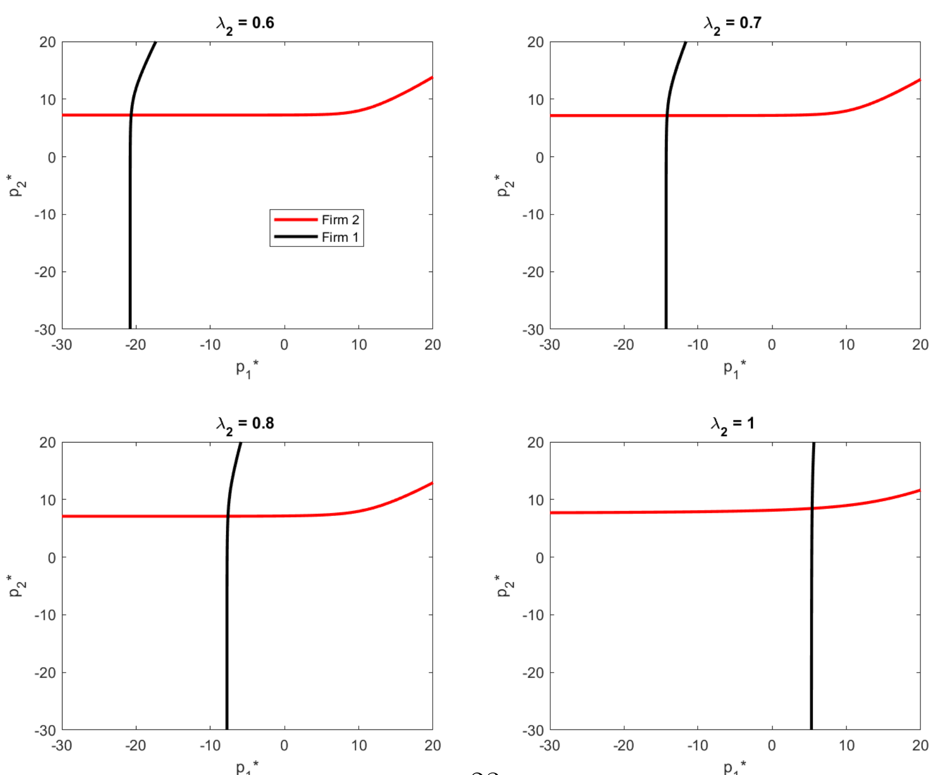
[48]Note that in a state (e_1, e_2) where $e_1 < M$, the buyer cannot keep the state the same by buying from seller 1. Therefore, for all values of b^p, reaction functions tend to look more like the case where $b^p = 0$.

Figure D.4: Pricing Best Response Functions in State (30,1) for Different Assumed Continuation Probabilities for Seller 2 (λ_2).

(a) $b^p = 0$



(b) $b^p = 1$



E Algorithm for Establishing Existence of an Accommodative Equilibrium

Definition *An equilibrium is **accommodative** if* $\lambda_1(e_1, e_2) = \lambda_2(e_1, e_2) = 1$ *for all states* (e_1, e_2) *where* $e_1 > 0$ *and* $e_2 > 0$.

In an accommodative equilibrium there is no exit by active sellers. If the industry starts off in state (1,1), it is guaranteed to arrive in state (M, M) in an accommodative equilibrium. This definition is the same as in BDK (2019), Appendix B.

E.1 Existence of an Accommodative Equilibrium

We establish whether an accommodative equilibrium exists by solving, using `fsolve` in MATLAB, for equilibrium prices and values assuming that there is no exit from any duopoly state, and then verifying that it is always optimal for each duopolist to continue in every duopoly state by checking that $\beta V^S(e_1, e_2)$ is greater than the highest possible scrap value.

E.2 Are Accommodative Equilibria Likely to Be Unique?

In an accommodative equilibrium the game is guaranteed to eventually end up in state (M, M), and remain there, and once a state has been left, because one of the sellers has made a sale and increased its know-how, it is guaranteed that the game will not return to it. This feature would guarantee a unique equilibrium if it is the case that there is a unique pricing equilibrium in any state given continuation values if the state changes. However, even though it can be shown that there is a unique price equilibrium in a one-shot Nash pricing game with a multinomial logit demand and an outside good that has a fixed price (e.g., Mizuno (2003)), this result is not sufficient in our model where the prices in the stage game affect sellers' continuation values (and a strategic buyer's continuation value if $b^p > 0$) if no sale is made.[49] The intuition for multiplicity would be that "at a low price equilibrium, each seller has a low opportunity cost of making a sale (when the other seller does not make a sale) as the state is unprofitable, whereas at a high price equilibrium, the opportunity

[49]The result is sufficient for state (M, M) as, whatever the buyer does, the state will be (M, M) in the next period. Therefore, the seller's pricing incentives in a Markov Perfect Equilibrium, will be the same as in a one-shot game.

cost of making a sale is higher". Note that this logic would tend to unravel with a strategic buyer, who would recognize that the possibility of them being the chosen buyer in the next period, which would make them keener to buy from one of the sellers when prices are high, lowering the probability that the state remains the same.

However, in practice, we have not found any examples of states with more than one accommodative pricing equilibrium despite extensive attempts to find an example for different values of b^p. One likely explanation for this is that the assumed value of $v_i = 10$ implies that the probability of the state remaining the same at prices that are close to equilibrium prices is small. For example, for all of the duopoly prices shown in text Table 1 the probability that the outside good is chosen is less than 0.02, and typically less than 0.01.

F Additional Results

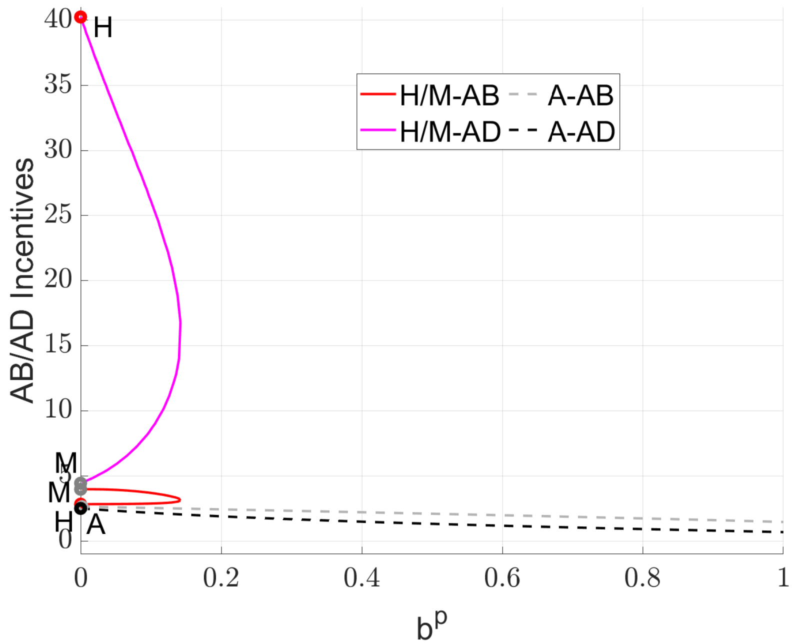
F.1 Equilibrium Buyer and Seller Incentives on b^p-Homotopy Paths for the Illustrative Parameters

Figure F.1(a) shows the equilibrium advantage-building and denying incentives for seller 1 in state (3,1). The decline in seller 1's demand and the falling probability that seller 2 will exit causes seller 1's advantage-denying incentive to fall sharply as we move from the High-HHI baseline equilibrium.

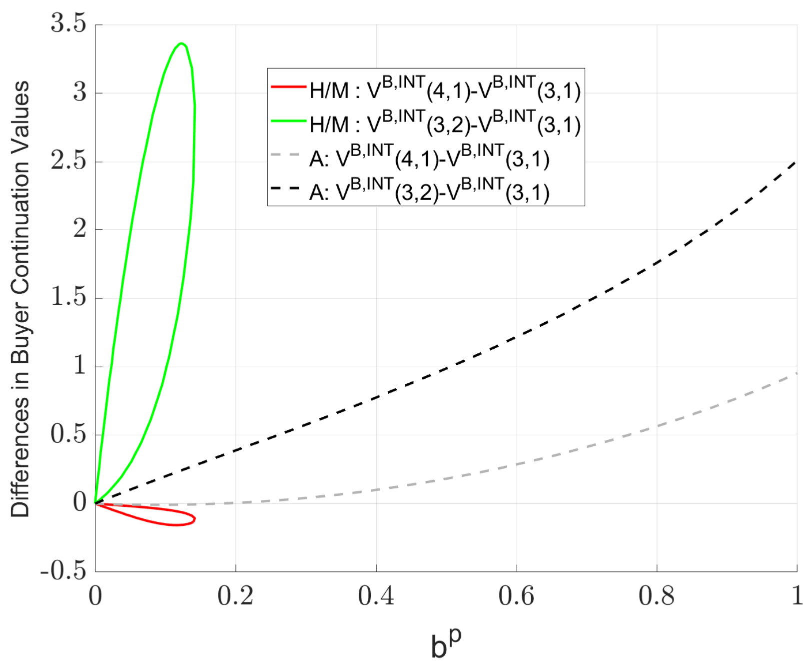
Figure F.1(b) shows the equilibrium dynamic incentives of a strategic buyer in state (3,1), measured by the change in the chosen buyer's continuation values when, compared to not buying, it buys from seller 1 ($V^{B,INT}(4,1) - V^{B,INT}(3,1)$) or seller 2 ($V^{B,INT}(3,2) - V^{B,INT}(3,1)$). These incentives are zero in all of the equilibria when $b^p = 0$. As b^p rises, the dynamic incentive to buy from seller 2 increases sharply in the non-accommodative equilibria, while there is an increasing dynamic disincentive to buy from seller 1. In an accommodative equilibrium there is a positive dynamic incentive to buy from the laggard as this lowers future prices, and, for $b^p > 0.2$ an incentive to buy from the leader which, relative to no purchase, lowers future costs.

Figure F.1: Equilibrium Dynamic Incentives Along b^p-Homotopy Paths for the Illustrative Parameters. H = High-HHI, M = Mid-HHI and A = Accommodative Baseline Equilibria, and AB=Advantage-Building and AD=Advantage-Denying Incentives.

(a) Seller 1 Equilibrium Incentives



(b) Buyer Equilibrium Incentives



F.2 Additional Welfare Results for the Illustrative Parameters

Text Figure 3 shows that, for $b^p = 0$, the present value of consumer surplus (PV CS) is highest in the Mid-HHI equilibrium and lowest in the High-HHI equilibrium, whereas the present value of total surplus (PV TS) is highest in the accommodative equilibrium and lowest in the High-HHI equilibrium. As b^p increases, both measures of surplus fall in the accommodative equilibrium as prices tend to increase.

The game will be in states (M, M) or $(M, 0)$ in the long-run, so that long-run expected consumer surplus will be higher in the accommodative equilibrium where (M, M) is the certain long-run outcome. PV CS is therefore higher in Mid-HHI equilibrium only because initial prices are lower, while the probability that the industry becomes a monopoly is not too large. To illustrate what happens to welfare in the first part of the game, Figure F.2(a) and (b) show the expected surplus measures for the first ten periods of a game beginning at (1,1). Note that the reported numbers are sums and there is no discounting.

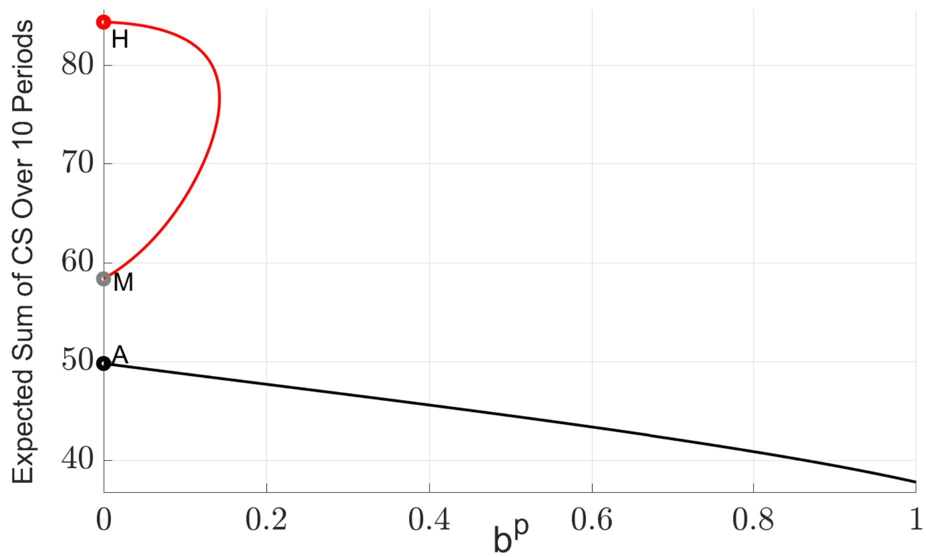
During the first ten periods, consumer surplus is highest in the High-HHI equilibrium due to the very low duopoly prices when one firm has not made a sale. This also tends to increase total surplus. Total surplus is also increased by the reduction in production costs which results from one seller tending to make most of the sales. This is illustrated in Figure F.2(c), which shows the sum of production costs over the first ten periods. The effect that strategic buyer behavior increases prices in the accommodative equilibrium causes both measures of surplus to fall in the accommodative equilibrium as b^p is increased.

The NPV of total surplus is affected by the number of sales that are made and the costs of production. Figure F.3(a) shows that the expected discounted production cost per sale is highest in the accommodative equilibrium, due to slower early learning, and it is lowest in the High-HHI equilibrium, where learning will tend to be quickest.[50] Figure F.3(b) shows the discounted total number of sales that are made. Even though low prices mean that more sales are made at the very beginning of the game in the High-HHI and Mid-HHI equilibria, the discounted number of sales is highest in the accommodative model as, despite higher average production costs, long-run margins are low.

[50]The reported number is the expected discounted total sum of production costs divided by the expected discounted total number of sales.

Figure F.2: Equilibrium Expected Consumer Surplus, Total Surplus and Production Costs Over the First 10 Periods for a Game Starting in State $(1, 1)$ Along b^p-Homotopy Paths for the Illustrative Parameters. The black line traces the homotopy path from the Accommodative (A) baseline equilibrium. The red line traces the overlapping paths from the High-HHI (H) and Mid-HHI (M) baseline equilibria.

(a) Consumer Surplus



(b) Total Surplus

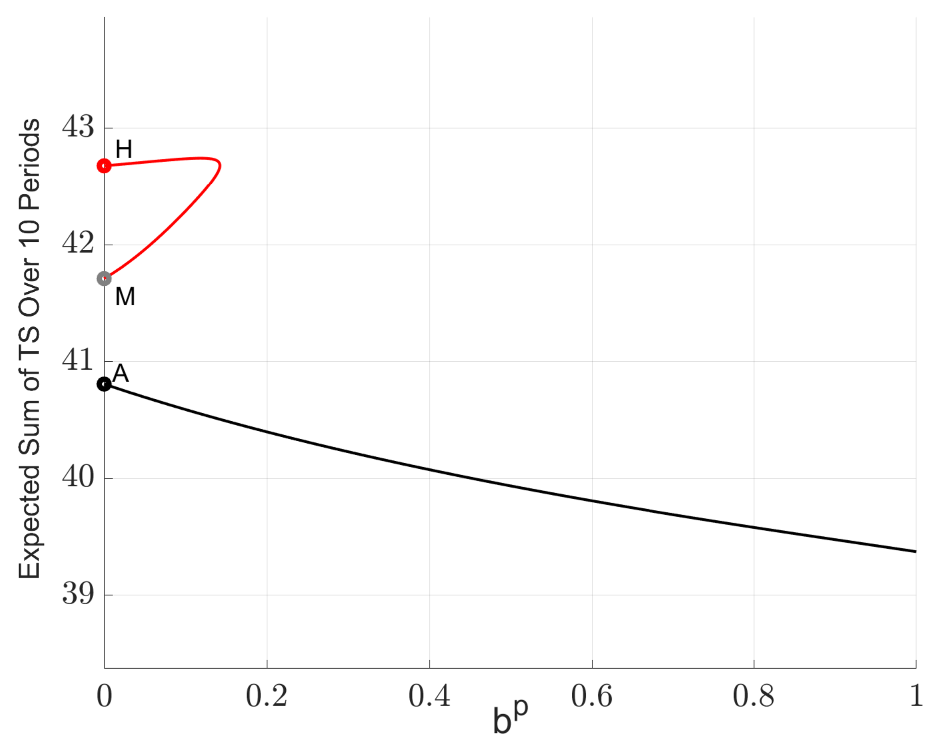


Figure F.2: cont.

(c) Expected Production Costs Over the First 10 Periods of the Game

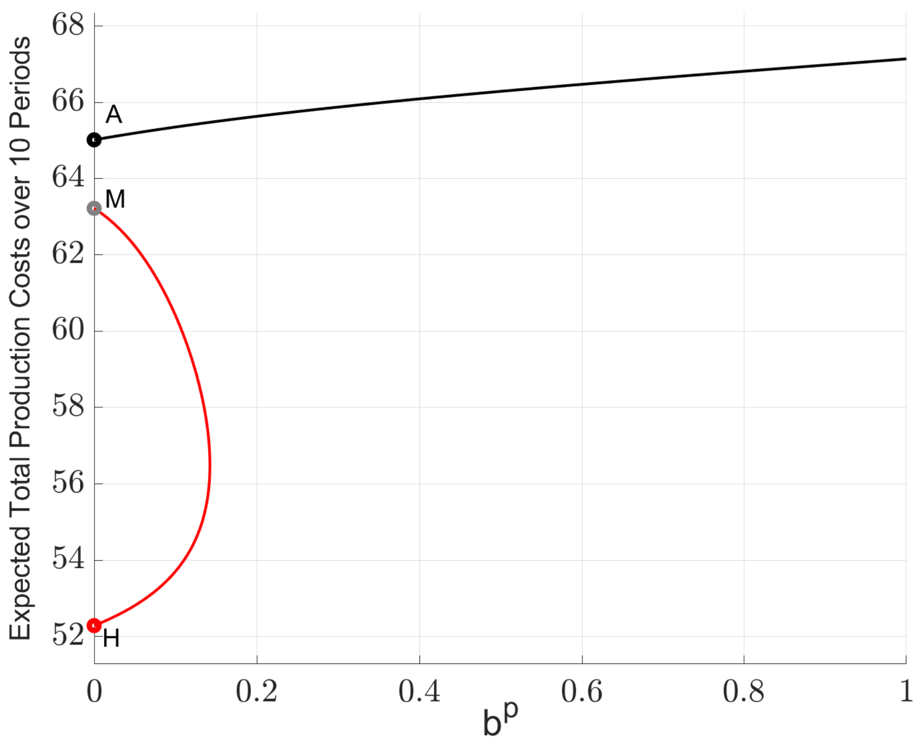
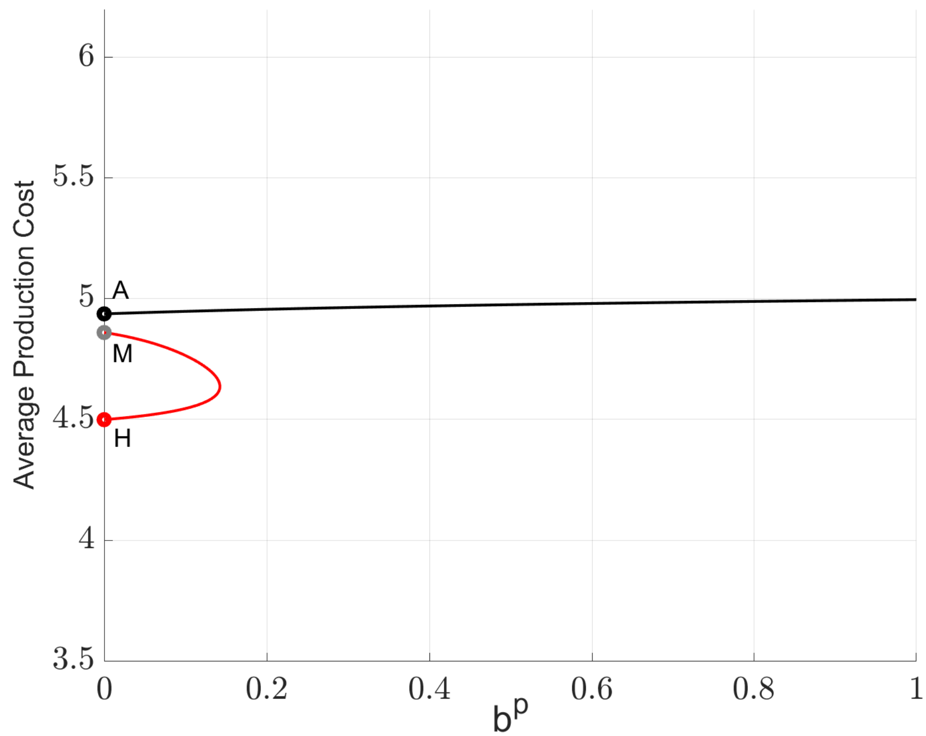
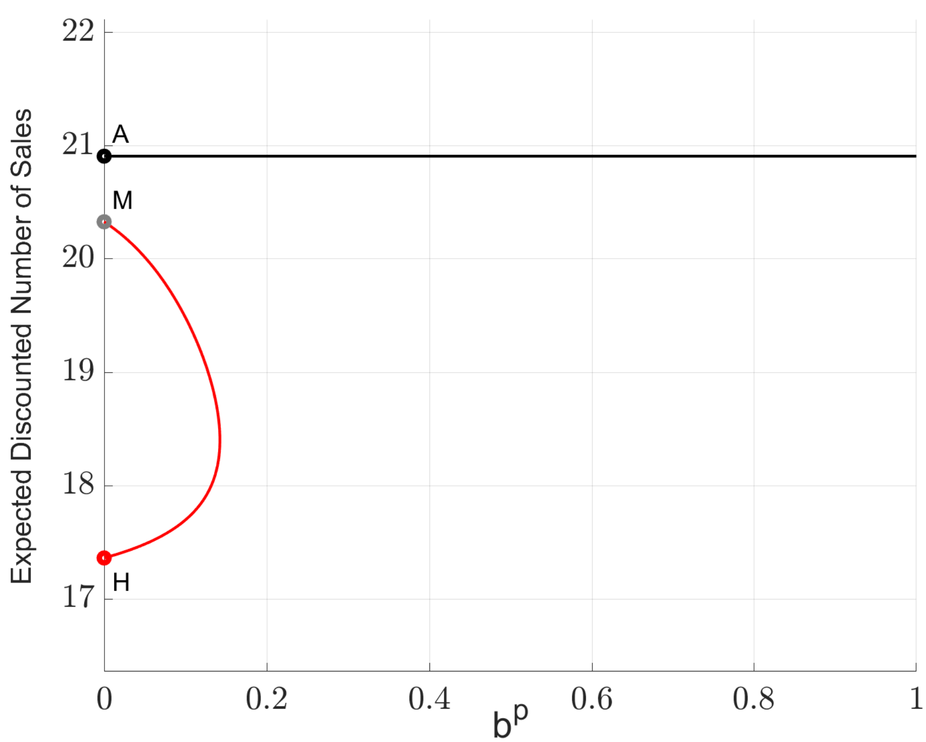


Figure F.3: Expected Present Value of Per-Sale Production Costs and the Expected Present Value (i.e., Discounted) Number of Sales for a Game Starting in State $(1,1)$ Along b^p-Homotopy Paths for the Illustrative Parameters.

(a) Expected Discounted Per-Sale Production Costs



(b) Expected Discounted Number of Sales

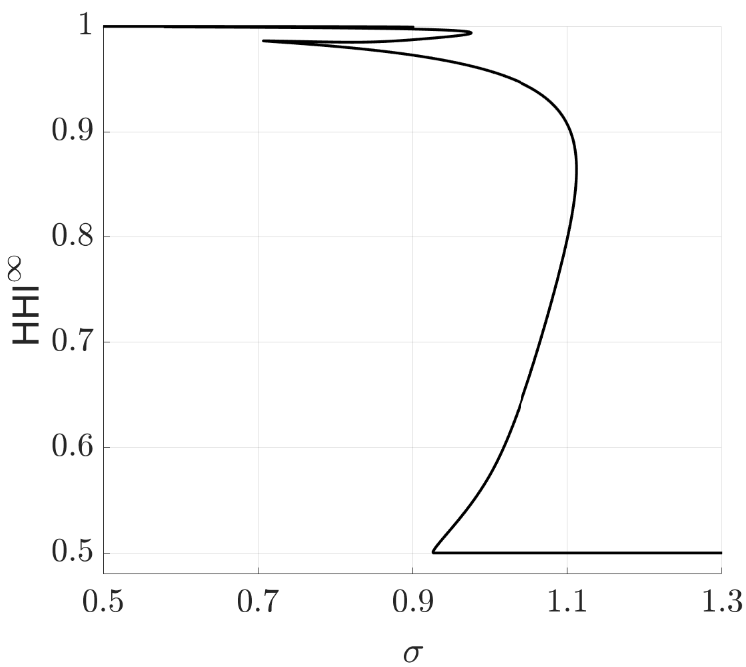


F.3 ρ and σ-Homotopy Paths for $b^p = 0$

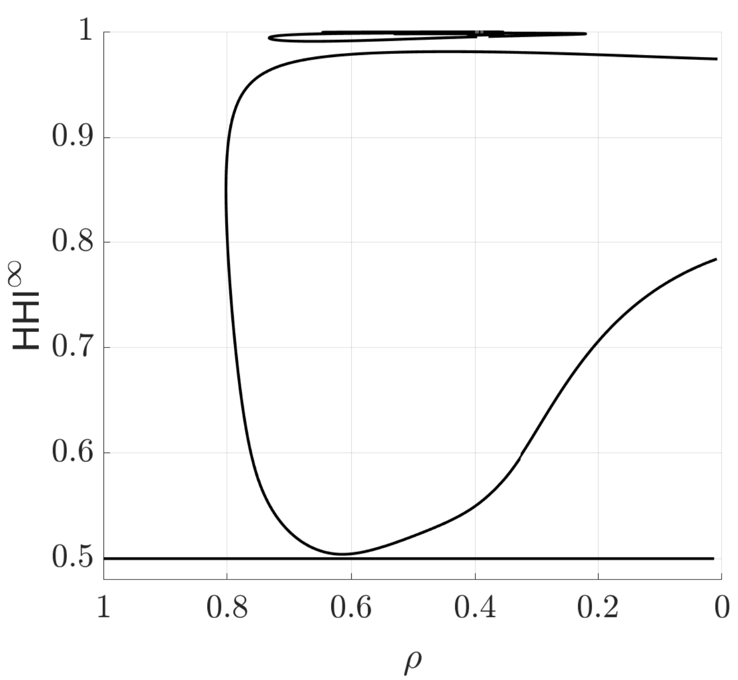
Text Figure 4(a)-(d) show ρ and σ homotopy paths for 11 different values of b^p. We reproduce the HHI^∞ plots for $b^p = 0$ in Figure F.4 for clarity, and so they can be compared with the figures in BDK1, Figure 2, panels A and B.

Figure F.4: Expected Long-Run HHI (HHI^{∞}) for Equilibria Identified by ρ- and σ-Homotopies when $b^p = 0$ and the Other Parameters are at their Illustrative Values.

(a) σ-Homtopies ($\rho = 0.75$)



(b) ρ-Homotopies ($\sigma = 1$)

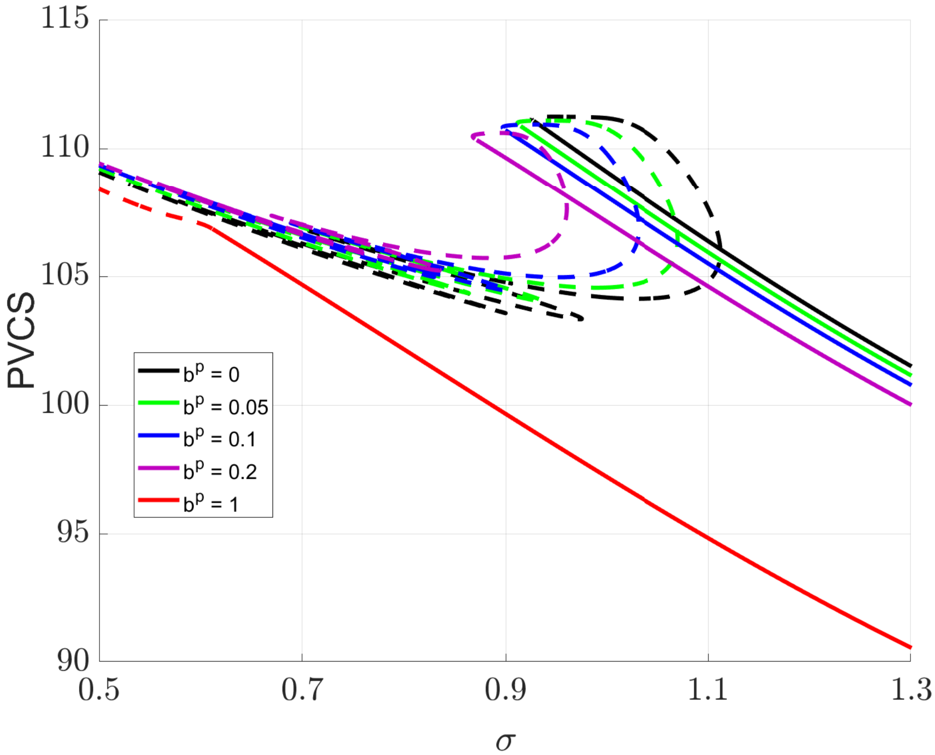


F.4 NPV of Consumer and Total Surplus on σ-Homotopy Paths for Multiple $b^p = 0$.

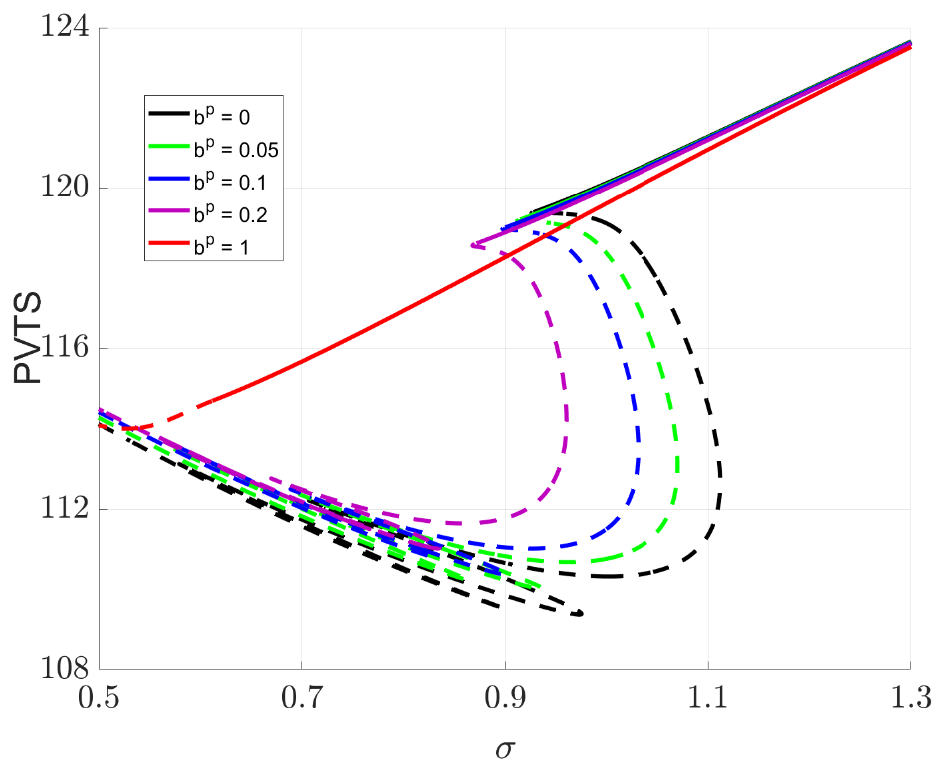
Text Figures 4(e) and (f) show the present value of consumer (PV CS) and total surplus (PV TS) for equilibria on ρ-homotopy paths for 11 different values of b^p. Here we provide similar plots for the σ-homotopies.

Figure F.5: Expected Present Value of Consumer and Total Surplus for Equilibria Along σ-Homotopy Paths for Multiple b^ps with Other Parameters are at their Illustrative Values.

(a) Consumer Surplus



(b) Total Surplus



G Extensions

Section 5 adapts our model in four ways to investigate how small changes to our very stylized assumptions affect our results. In this Appendix we detail these extensions and present some additional results.

G.1 Extension 1: Mixture of Strategic and Non-Strategic Buyers.

In this extension we assume that there are two types of buyers:

1. a mass of atomistic (A) buyers who, if they are chosen to the buyer, assume that they will never be in the market again (i.e., they act as if $b^p = 0$); and,
2. a group of 4 symmetric, strategic (NA, non-atomistic) buyers.

Each period nature picks a strategic buyer with probability γ, in which case each of the four strategic buyers is chosen with equal probability. Otherwise, an atomistic buyer is chosen. The sellers observe the chosen buyer's type before they set prices. If $\gamma = 0$, all buyers are atomistic and equilibrium play corresponds to play in the original BDK model. We run γ-homotopies, for the illustrative parameters, from the three $\gamma = 0$ equilibria.

G.1.1 Equilibrium Equations.

Values of the sellers and the strategic buyers are defined before nature has selected the chosen buyer's type (or the chosen buyer's identity). The values of atomistic buyers are equal to zero, so the only additional set of equations that we have to solve are the pricing first-order conditions of the sellers when selling to atomistic buyers.

Beginning of period value for seller 1 (V_1^S):

$$V_1^S(\mathbf{e}) - (1-\gamma)D_1^A(p^A(\mathbf{e}),\mathbf{e})(p_1^A(\mathbf{e}) - c_1(e_1)) - \gamma D_1^{NA}(p^{NA}(\mathbf{e}),\mathbf{e})(p_1^{NA}(\mathbf{e}) - c_1(e_1)) - \sum_{k=0,1,2} ((1-\gamma)D_k^A(p^A(\mathbf{e}),\mathbf{e}) + \gamma D_k^{NA}(p^{NA}(\mathbf{e}),\mathbf{e}))V_1^{S,INT}(\mathbf{e}'_k) = 0 \tag{G.1}$$

where

$$D_k^A(p^A, \mathbf{e}) = \frac{\exp(v_k - p_k^A)}{\sum_{j=0,1,2} \exp(v_j - p_j^A)}, D_k^{NA}(p^{NA}, \mathbf{e}) = \frac{\exp(v_k - p_k^{NA} + V^{INT,NA}(\mathbf{e}'_k))}{\sum_{k=0,1,2} \exp(v_j - p_j^{NA} + V^{INT,NA}(\mathbf{e}'_j))}, \tag{G.2}$$

$\mathbf{e}'_1 = (\min(e_1 + 1, M), e_2)$, $\mathbf{e}'_2 = (e_1, \min(e_2 + 1, M))$ and $\mathbf{e}'_0 = (e_1, e_2)$, i.e., the states that the game will transition to if there is a purchase from seller 1 or seller 2, or no purchase, respectively.

<u>Value for seller 1 before entry/exit stage ($V_1^{S,INT}$) :</u>

$$V_1^{S,INT}(\mathbf{e}) - \left(\begin{array}{c} \beta\lambda_1(\mathbf{e})\lambda_2(\mathbf{e})V_1^S(\mathbf{e}) + \beta\lambda_1(\mathbf{e})(1 - \lambda_2(\mathbf{e}))V_1^S(e_1, 0) + \\ (1 - \lambda_1(\mathbf{e}))E(X|\lambda_1(\mathbf{e})) \end{array} \right) = 0 \tag{G.3}$$

for $\mathbf{e} = (e_1, e_2)$ where $e_1, e_2 > 0$, with similar equations when one or both sellers is a potential entrant. $E(X|\lambda_1(\mathbf{e}))$ is the expected scrap value when seller 1 chooses to exit with probability $1 - \lambda_1(\mathbf{e})$.

<u>First-order condition for seller 1's price to non-strategic buyers (p_1^A) if $e_1 > 0$:</u>

$$D_1^A(p^A(\mathbf{e}), \mathbf{e}) + \sum_{k=0,1,2} \frac{\partial D_k^A(p^A(\mathbf{e}), \mathbf{e})}{\partial p_1^A} V_1^{S,INT}(\mathbf{e}'_k) + \left(p_1^A(\mathbf{e}) - c_1(e_1)\right) \frac{\partial D_1^A(p^A(\mathbf{e}), \mathbf{e})}{\partial p_1^A} = 0 \tag{G.4}$$

<u>First-order condition for seller 1's price to strategic buyers (p_1^{NA}) if $e_1 > 0$:</u>

$$D_1^{NA}(p^{NA}(\mathbf{e}), \mathbf{e}) + \sum_{k=0,1,2} \frac{\partial D_k^{NA}(p^{NA}(\mathbf{e}), \mathbf{e})}{\partial p_1^{NA}} V_1^{S,INT}(\mathbf{e}'_k) + \left(p_1^{NA}(\mathbf{e}) - c_1(e_1)\right) \frac{\partial D_1^{NA}(p^{NA}(\mathbf{e}), \mathbf{e})}{\partial p_1^{NA}} = 0 \tag{G.5}$$

<u>Seller 1's continuation probability in entry/exit stage (λ_1):</u>

$$\lambda_1(\mathbf{e}) - F_{\text{enter}}(\beta \left[\lambda_2(\mathbf{e})V_1^S(1, e_2) + (1 - \lambda_2(\mathbf{e}))V_1^S(1, 0)\right]) = 0 \text{ if } e_1 = 0 \tag{G.6}$$

$$\lambda_1(\mathbf{e}) - F_{\text{scrap}}(\beta \left[\lambda_2(\mathbf{e})V_1^S(e_1, \max(1, e_2)) + (1 - \lambda_2(\mathbf{e}))V_1^S(e_1, 0)\right]) = 0 \text{ if } e_1 > 0 \tag{G.7}$$

<u>Value for strategic buyer before entry/exit stage ($V^{INT,NA}$) :</u>

$$V^{INT,NA}(\mathbf{e}) - \beta \left(\sum_{\mathbf{e}'} \Pr(\mathbf{e}'|\mathbf{e}, \lambda_1(\mathbf{e}), \lambda_2(\mathbf{e})) V^{NA}(\mathbf{e}') \right) = 0. \tag{G.8}$$

where the sum is over the states that the game may transition to given entry/exit choices. Seller symmetry implies that, for buyers, $V^{INT,NA}(e_1, e_2) = V^{INT,NA}(e_2, e_1)$ and $V^{NA}(e_1, e_2) = V^{NA}(e_2, e_1)$.

<u>Beginning of period strategic buyer value (V^{NA}) :</u>

$$V^{NA}(\mathbf{e}) - \frac{1}{4}\gamma \log \left(\sum_{k=0,1,2} \exp\left(v_k - p_k^{NA} + V^{INT,NA}(\mathbf{e}'_k)\right) \right) - (1-\gamma) \sum_{k=0,1,2} D_k^A(p^A(\mathbf{e}), \mathbf{e}) V^{INT,NA}(\mathbf{e}'_k) - \tag{G.9}$$

$$\gamma(1 - \frac{1}{4}) \sum_{k=0,1,2} D_k^{NA}(p^{NA}(\mathbf{e}), \mathbf{e}) V^{INT,NA}(\mathbf{e}'_k) = 0$$

where $\frac{1}{4}$ is the probability that a given strategic buyer is chosen when one of them is selected.

G.2 Extension 2: Buyers with Persistent Preferences Over Sellers.

The Section 2 model also assumes that buyers always have identical preferences over sellers up to iid preference shocks. In reality, buyers may have systematic preferences for a particular seller (for example, because of geographic location or greater compatibility with existing equipment). We therefore extend the Section 2 model by assuming that there are equal numbers of two types of buyers. Type 1's indirect utility when it purchases from sellers 1 and 2 respectively are $v_1 + \frac{\theta}{2} - p_1 + \epsilon_1$ and $v_2 - \frac{\theta}{2} - p_2 + \epsilon_2$ respectively. For type 2 buyers, the signs on the $\frac{\theta}{2}$ terms are reversed. Sellers recognize the type of the buyer before setting prices. The model is equivalent to the Section 2 model when $\theta = 0$. Intuitively, it will become more attractive for a seller to remain in the market as θ increases, even when it has a marginal cost disadvantage, as it will have an increasing advantage when selling to half of the market.

G.2.1 Equilibrium Equations.

To the equations of the Section 2 model are added type-specific first-order conditions for prices, and equations for the values and intermediate values.[51] For example,

First-order condition for seller 1's price (p_1^{type1}) if $e_1 > 0$:

$$D_1^{type1}(p^{type1}(\mathbf{e}),\mathbf{e})+\sum_{k=0,1,2}\frac{\partial D_k^{type1}(p(\mathbf{e}),\mathbf{e})}{\partial p_1^{type1}}V_1^{S,INT}(\mathbf{e}'_k)+\left(p_1^{type1}(\mathbf{e})-c_1(e_1)\right)\frac{\partial D_1^{type1}(p^{type1}(\mathbf{e}),\mathbf{e})}{\partial p_1^{type1}}=0 \tag{G.10}$$

Value for type 1 buyer before entry/exit stage ($V^{type1,INT}$) :

$$V^{type1,INT}(\mathbf{e})-\beta\left(\sum_{\mathbf{e}'}\Pr(\mathbf{e}'|\mathbf{e},\lambda_1(\mathbf{e}),\lambda_2(\mathbf{e}))V^{type1}(\mathbf{e}')\right)=0. \tag{G.11}$$

Type 1 buyer value (V^{type1}) :

$$V^{type1}(\mathbf{e})-b^p\log\left(\sum_{k=0,1,2}\exp\left(v_k+[I(k=1)-I(k=2)]\frac{\theta}{2}-p_k^{type1}+V^{type1,INT}(\mathbf{e}'_k)\right)\right)- \tag{G.12}$$

$$(\frac{1}{2}-b^p)\sum_{k=0,1,2}D_k^{type1}(p^{type1}(\mathbf{e}),\mathbf{e})V^{type1,INT}(\mathbf{e}'_k)-\frac{1}{2}\sum_{k=0,1,2}D_k^{type2}(p^{type2}(\mathbf{e}),\mathbf{e})V^{type1,INT}(\mathbf{e}'_k)=0$$

with similar equations for type 2 buyers. Note that b^p is equal to the unconditional probability that the buyer will be the buyer in a future period, so the value of b^p with a single, rational buyer of each type would be $b^p = 0.5$.

Values for sellers then come from adding across the two types of buyers.

[51] The assumed functional forms imply that there will be symmetry across types, e.g., the price set by seller 1 to a type 1 buyer in state (4,1) will be the same as the price set by seller 2 to a type 2 buyer in state (1,4).

Beginning of period value for seller 1 (V_1^S):

$$V_1^S(\mathbf{e}) - \frac{1}{2}D_1^{type1}(p^{type1}(\mathbf{e}),\mathbf{e})(p^{type1}(\mathbf{e}) - c_1(e_1)) - \frac{1}{2}\sum_{k=0,1,2} D_k^{type1}(p^{type1}(\mathbf{e}),\mathbf{e})V_1^{S,INT}(\mathbf{e}_k') -$$
$$\frac{1}{2}D_1^{type2}(p^{type2}(\mathbf{e}),\mathbf{e})(p^{type2}(\mathbf{e}) - c_1(e_1)) - \frac{1}{2}\sum_{k=0,1,2} D_k^{type2}(p^{type2}(\mathbf{e}),\mathbf{e})V_1^{S,INT}(\mathbf{e}_k') = 0 \tag{G.13}$$

where

$$D_i^{type1}(p,\mathbf{e}) = \frac{\exp(v_i + [I(i=1) - I(i=2)]\frac{\theta}{2} - p_i^{type1} + V^{type1,INT}(\mathbf{e}_i'))}{\sum_{k=0,1,2}\exp(v_k + [I(k=1) - I(k=2)]\frac{\theta}{2} - p_k^{type1} + V^{type1,INT}(\mathbf{e}_k'))}. \tag{G.14}$$

$\mathbf{e}_1' = (\min(e_1+1, M), e_2)$, $\mathbf{e}_2' = (e_1, \min(e_2+1, M))$ and $\mathbf{e}_0' = (e_1, e_2)$, i.e., the states that the game will transition to if there is a purchase from seller 1 or seller 2, or no purchase, respectively.

G.3 Extension 3: Bargaining as a Constraint on Monopoly Power.

We consider a permutation of the model where we assume that, in the event that the industry becomes a monopoly, the buyer and seller engage in Nash bargaining rather than the seller simply setting a price. This formulation is somewhat ad-hoc because the Nash bargaining approach assumes that the buyer and seller have complete information about their values (i.e., the buyer's ϵs are publicly observed) whereas, to keep the model as similar to the BDK model as possible, we maintain the assumption that a buyer's ϵs are private information in duopoly states. However, the advantage of the Nash bargaining formulation is that it allows us to vary a single parameter, τ, that measures the buyer's share of the surplus from trade in monopoly states.

G.3.1 Details.

The equations for states with two active sellers are the same as for the Section 2 model. The following are the equations for a monopoly state $\mathbf{e} = (e_1 < M, 0)$.

Probability of trade when seller 1 is the monopolist

$$D_1 = \frac{\exp(v_1 + V^{S,INT}(\mathbf{e}_1') + V^{B,INT}(\mathbf{e}_1') - c(e_1))}{\begin{pmatrix} \exp(v_1 + V^{S,INT}(\mathbf{e}_1') + V^{B,INT}(\mathbf{e}_1') - c(e_1)) + \\ \exp(V^{S,INT}(\mathbf{e}) + V^{B,INT}(\mathbf{e})) \end{pmatrix}} \tag{G.15}$$

Beginning of period value for seller 1 (V_1^S):

$$V_1^S(\mathbf{e}) - D_1(\mathbf{e})(p(\mathbf{e}) - c_1(e_1)) - D_1(\mathbf{e})V_1^{S,INT}(\mathbf{e}_1') - (1 - D_1(\mathbf{e}))V_1^{S,INT}(\mathbf{e}) = 0 \tag{G.16}$$

Intermediate value for seller 1 ($V_1^{S,INT}$) :

$$V_1^{S,INT}(\mathbf{e}) - \begin{pmatrix} \beta\lambda_1(\mathbf{e})\lambda_2(\mathbf{e})V_1^S(e_1, 1) + \beta\lambda_1(\mathbf{e})(1 - \lambda_2(\mathbf{e}))V_1^S(\mathbf{e}) + \\ (1 - \lambda_1(\mathbf{e}))E(X|\lambda_1(\mathbf{e})) \end{pmatrix} = 0 \tag{G.17}$$

Value for buyer before entry/exit stage ($V^{B,INT}$) :

$$V^{B,INT}(\mathbf{e}) - \beta\left(\lambda_1(\mathbf{e})\lambda_2(\mathbf{e})V^B(e_1, 1) + \lambda_1(\mathbf{e})(1 - \lambda_2(\mathbf{e}))V^B(\mathbf{e})\right) = 0 \tag{G.18}$$

Beginning of period buyer value (V^B) :

$$\begin{aligned} &V^B(\mathbf{e}) - b^p\left(D_1(\mathbf{e})(v_1 - p(\mathbf{e}, \tau) - \log(D_1) + V^{B,INT}(\mathbf{e}_1')) + (1 - D_1(\mathbf{e}))(-\log(1 - D_1) + V^{B,INT}(\mathbf{e}))\right) \\ &-(1 - b^p)\left(D_1(\mathbf{e})V^{B,INT}(\mathbf{e}_1') + (1 - D_1(\mathbf{e}))V^{B,INT}(\mathbf{e})\right) = 0 \end{aligned} \tag{G.19}$$

Expected price in the event of trade is

$$p(\mathbf{e}) = \tau(c(e_1) + V^{S,INT}(\mathbf{e}) - V^{S,INT}(\mathbf{e}_1')) + (1-\tau)(\underbrace{v_1 - \log(D_1)}_{\text{exp. value of v+}\varepsilon_1\text{ given trade}} + V^{B,INT}(\mathbf{e}_1') - \underbrace{\frac{(1-D_1)\log(1-D_1)}{D_1}}_{\text{exp. value of }\varepsilon_0\text{when trade occurs}} - V^{B,INT}(\mathbf{e})) \text{ for } \mathbf{e} = (e_i, 0) \text{ and } \mathbf{e}_1' = (e_1+1, 0) \tag{G.20}$$

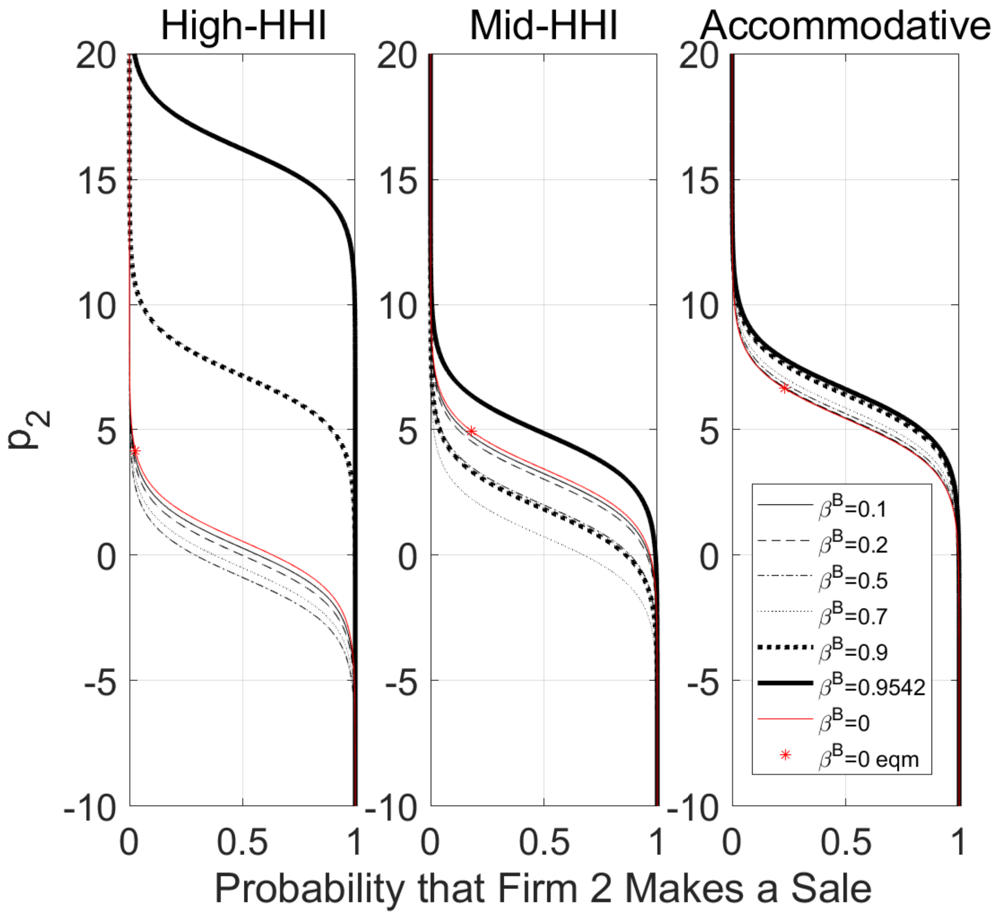
G.4 Extension 4: Buyer Discount Factors.

We investigate whether variation in b^p has a similar effect to variation in buyer patience using a model where $b^p = 1$ (i.e., monopsony) but the buyer's discount factor, $\beta^B \leq \beta = \frac{1}{1.05}$, the assumed discount factor of the sellers. The equations are the same as for the Section 2 model except that β in the $V^{B,INT}$ equation is replaced by β^B.

G.4.1 Effect of Variation on β^B on Seller 2 Demand Given Baseline Equilibrium Seller Strategies.

Figure G.1 shows the demand curve for seller 2 in state (3,1) when we assume that sellers use their baseline equilibrium seller strategies in all states, but we assume that there is a single strategic buyer ($b^p = 1$) with different discount factors. When sellers use accommodative equilibrium strategies, an increase in buyer patience tends to move demand towards seller 2 (the laggard), in the same way that an increase in b^p moved demand towards seller 2 in text Figure 2(a). However, in the Mid- and High-HHI equilibria, increases in β^B actually shift demand away from seller 2 until $\beta^B > 0.5$ in the High-HHI case, and until $\beta^B > 0.7$ in the Mid-HHI case, reflecting the fact that in these equilibria prices in state (4,1) are lower than in state (3,2) and that the loss that the buyer will experience from monopoly is likely to occur further into the future.

Figure G.1: Seller 2 Demand in State (3,1) as a Function of β^B.



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