Online Appendix

Constrained-Efficient Capital Reallocation

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A Stylized Model

This appendix provides additional details and results on the analysis of the stylized model of Section II.

A.1 Graphical Representation of Implementation of First Best

In Figure A1, we illustrate the stationary competitive equilibrium (solid lines) in our numerical example for the stylized model of Section II, and contrast it with the constrained-efficient allocation rule that supports the first-best outcome described in equation (31) (dashed lines). While total capital (bottom left) is weakly increasing in net worth in competitive equilibrium, inducing inefficient dispersion in marginal products, the constrained-efficient allocation equalizes the scale of production across all firms, increasing aggregate investment and reallocating old capital towards the most constrained firms, without incurring any equity issuance costs (bottom right).

A.2 Restrictions on Policy Instruments in the Stylized Model

In this section, we provide additional results on the policy experiments with restricted instruments in the stylized model from Section II.F.

A.2.1 No Taxes on Old Capital

We first consider the case in which the planner cannot tax old capital. In this case, and in the following one without subsidies on new capital, we assume that new and old capital are imperfect substitutes: $y = f(k)$ and $k = g(k^N, k^O)$, where g is a constant elasticity of substitution (CES) aggregator of new and old capital, as we assume in the quantitative model:

$$g(k^N, k^O) = \left[(\sigma^N)^{\frac{1}{\epsilon}}(k^N)^{\frac{\epsilon-1}{\epsilon}} + (1-\sigma^N)^{\frac{1}{\epsilon}}(k^O)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}.$$

Specifically, for our numerical example we set the elasticity of substitution $\epsilon = 50$, thus assuming high substitutability, similar to the baseline case for the stylized model, but retaining an interior solution for all firms. We further set $\sigma^N = 0.5$, thus treating new and

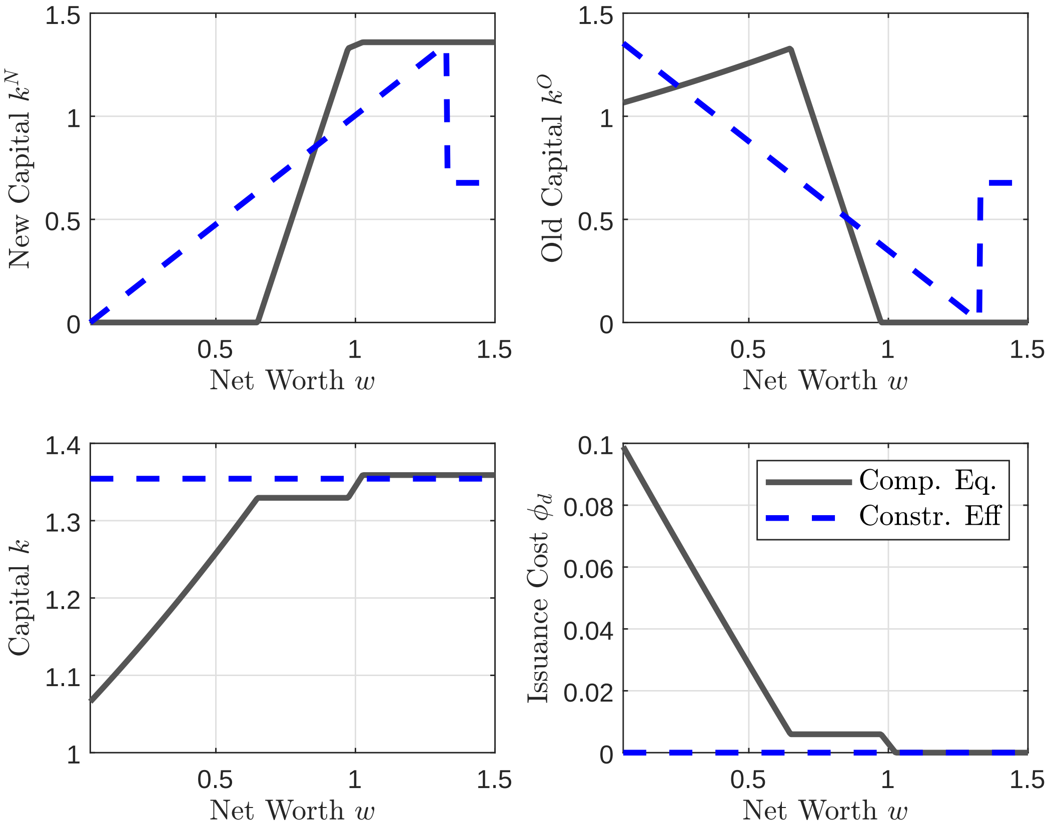

Figure A1: Stationary competitive equilibrium and constrained-efficient allocation – example. Top left: new capital k^N; top right: old capital k^O; bottom left: total capital k; bottom right: marginal cost of equity issuance ϕ_d. The x-axes report net worth w. Solid lines denote the competitive-equilibrium allocation, dashed lines the constrained-efficient allocation. See the caption of Figure 1 for the parameter values.

old capital symmetrically. All other functional forms and parameter values are as in the baseline numerical example (see caption of Figure 1).

The competitive-equilibrium optimality conditions for new and old capital are

$$1 + \phi_{d,t} = \beta \left(f_k(k_t) g_N(k_t^N, k_t^O) + q_{t+1}(1 + \theta\lambda_t) \right) \tag{A1}$$

$$q_t(1 + \phi_{d,t}) = \beta f_k(k_t) g_O(k_t^N, k_t^O), \tag{A2}$$

where we denote by g_N and g_O the partial derivatives of the bundle with respect to new and old capital, respectively.

To reflect the assumption that the planner cannot tax old capital, the planner faces the Euler equation for old capital (A2) as a constraint, for all w and t, with multiplier $\beta^t \psi_t^O$, where we omit the dependence of variables on w, but it is understood that both allocations and multiplier depend on firm net worth.

The planner's first-order conditions with respect to new capital, old capital, and debt are

$$1 + \phi_{d,t} = \beta\left(f_k(k_t) g_N(k_t^N, k_t^O) + q_{t+1}(1 + \theta\lambda_t)\right) + \beta\eta_{t+1} + \psi_t^O \frac{\partial O_t}{\partial k_t^N} \tag{A3}$$

$$q_t(1 + \phi_{d,t}) = \beta f_k(k_t) g_O(k_t^N, k_t^O) - \eta_t + \psi_t^O \frac{\partial O_t}{\partial k_t^O}$$

$$1 + \phi_{d,t} = 1 + \lambda_t - \psi_t^O \frac{\partial O_t}{\partial b_t}, \tag{A4}$$

with

$$\frac{\partial O_t}{\partial k_t^N} = q_t \phi_{dd,t} - \beta\left(f_{kk}(k_t) g_N(k_t^N, k_t^O) g_O(k_t^N, k_t^O) + f_k(k_t) g_{NO}(k_t^N, k_t^O)\right)$$

$$\frac{\partial O_t}{\partial k_t^O} = q_t^2 \phi_{dd,t} - \beta\left(f_{kk}(k_t)(g_O(k_t^N, k_t^O))^2 + f_k(k_t) g_{OO}(k_t^N, k_t^O)\right)$$

$$\frac{\partial O_t}{\partial b_t} = -q_t \phi_{dd,t}.$$

The first-order condition with respect to the price of old capital q_t is

$$\int k_t^O(1 + \phi_{d,t}) d\pi = \int k_{t-1}^N(1 + \theta\lambda_{t-1}) d\pi + \int \psi_t^O(1 + \phi_{d,t} + q_t \phi_{dd,t} k_t^O) d\pi.$$

The left-hand side reports the distributive externality on the buyers of old capital, whereas the first term on the right-hand side reports the distributive externality on the sellers as well as the collateral externality, as in our baseline case with taxes on new and old capital. Additionally, the second term of the right-hand side is a wedge due to the constraint (A2), which is positive and implies that the planner tolerates a distributive externality that is larger than the collateral externality in the constrained-efficient allocation. Because of this wedge, the optimal reduction in the price of old capital is smaller than when the planner can distort both new and old investment. Notice that the multipliers ψ_t^O are positive, because the planner would like to tax old capital.

The intuition for this result is that the planner would like to decrease the price of old capital substantially, as in the unrestricted case, but a low price of old capital reduces the left-hand side of constraint (A2) and incentivizes inefficiently large purchases of old capital, which the planner cannot offset, absent a tax on old capital.

We consider an implementation of the constrained-efficient allocation with a proportional tax on new capital. Moreover, the planner can also use a tax on debt to induce its desired value for the multiplier λ in competitive equilibrium, consistent with equation

(A4).[32] These taxes are then rebated lump-sum to each firm. To obtain the optimal tax rate on new capital, similar to the analysis of Section II.E, we first compute the allocation and then use the firm optimality condition

$$(1+\phi_{d,t})(1+\tau_t^N) = \beta\left(f_k(k_t)g_N(k_t^N, k_t^O) + q_{t+1}(1+\theta\lambda)\right)$$

to solve for τ_t^N. We find that the optimal tax is negative in stationary equilibrium, consistent with our main insight on the importance of the distributive externality. Increasing the supply of new capital reduces the price of old capital, improving the allocation. The incentive to subsidize capital is reflected in the last two terms on the right-hand side of equation (A3): The multiplier η is positive as in the baseline case with unrestricted instruments. Moreover, the term $\psi_t^O \frac{\partial O_t}{\partial k_t^N}$ is also positive because additional new investment contributes to relax the constraint due to the missing tax on old capital. In particular, for financially constrained firms additional new investment increases the marginal value of net worth, discouraging purchases of old capital, similar to a tax on old capital. Thus, as we explain in Section II.F, the optimal subsidy on new investment is larger for more financially constrained firms.

Figure A2 displays the optimal tax rate on new capital. Notice that there is a discontinuity around the level of net worth such that firms become unconstrained. This is because our numerical example assumes a quadratic cost of equity issuance, and thus the second derivative $\phi_{dd,t}$, which appears in the optimality conditions reported above, equals a positive constant for constrained firms, and zero for unconstrained firms.

A.2.2 No Subsidies on New Capital

We now consider the case in which the planner cannot subsidize new capital. In this case, the planner faces the Euler equation for new capital (A1) as constraint for all w and t, with multiplier $\beta^t\psi_t^N$. Notice that in formulating this constraint on the planning problem, we need to substitute a differentiable expression for the Lagrange multiplier on the collateral constraint $\beta^{t+1}\lambda_t$. We follow, for instance, Jeanne and Korinek (2019) and use the competitive-equilibrium optimality condition for debt to substitute $\lambda_t = \phi_{d,t}$ in the constraint. Henceforth in the derivations of this section, we use the notation λ_t^P to refer to the multiplier on the collateral constraint in the planning problem.

The planner's first-order conditions with respect to new capital, old capital, and debt

[32] The tax on debt is not relevant in the baseline case in which the planner can distort both new and old investment (Section II.E), because in that case the planner's first-order condition with respect to debt coincides with the competitive equilibrium one, (12).

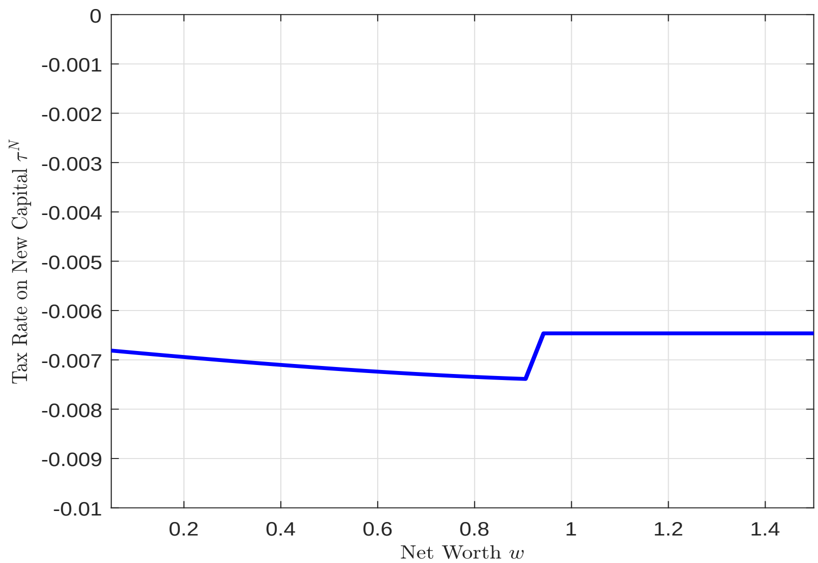

Figure A2: Tax Rate on New Capital without Tax on Old Capital. The x-axis reports net worth w. The y-axis reports the tax rate τ^N.

are

$$
\begin{aligned}
1+\phi_{d,t} &= \beta\left(f_k(k_t)g_N(k_t^N,k_t^O)+q_{t+1}(1+\theta\lambda_t^P)\right)+\beta\eta_{t+1}+\psi_t^N\frac{\partial N_t}{\partial k_t^N}\\
q_t(1+\phi_{d,t}) &= \beta f_k(k_t)g_O(k_t^N,k_t^O)-\eta_t+\psi_t^N\frac{\partial N_t}{\partial k_t^O}\\
1+\phi_{d,t} &= 1+\lambda_t^P-\psi_t^N\frac{\partial N_t}{\partial b_t},
\end{aligned}
$$

with

$$
\begin{aligned}
\frac{\partial N_t}{\partial k_t^N} &= \phi_{dd,t}(1-\beta\theta q_{t+1})-\beta\left(f_{kk}(k_t)(g_N(k_t^N,k_t^O))^2+f_k(k_t)g_{NN}(k_t^N,k_t^O)\right)\\
\frac{\partial N_t}{\partial k_t^O} &= q_t\phi_{dd,t}(1-\beta\theta q_{t+1})-\beta\left(f_{kk}(k_t)g_N(k_t^N,k_t^O)g_O(k_t^N,k_t^O)+f_k(k_t)g_{NO}(k_t^N,k_t^O)\right)\\
\frac{\partial N_t}{\partial b_t} &= -\phi_{dd,t}(1-\beta\theta q_{t+1}).
\end{aligned}
$$

The first-order condition with respect to the price of old capital q_t is

$$\int k_t^O(1+\phi_{d,t})d\pi=\int k_{t-1}^N(1+\theta\lambda_{t-1}^P)d\pi+\int\psi_t^N\phi_{dd,t}k_t^O(1-\beta\theta q_{t+1})d\pi-\int\psi_{t-1}^N(1+\theta\phi_{d,t-1})d\pi.$$

The left-hand side reports the distributive externality on the buyers of old capital, whereas the first term on the right-hand side reports the distributive externality on the sellers as well as the collateral externality. The remaining terms of the right-hand side represent again a wedge due to the constraint (A1). This wedge is positive in our numerical example, as in the previous section, and implies that the planner tolerates a distributive externality that is larger than the collateral externality in the constrained-efficient allocation. Hence, the optimal reduction in the price of old capital is smaller than when the planner can distort both new and old investment. Notice that the multipliers ψ_t^N are negative, because the planner would like to subsidize new investment.

We consider an implementation with proportional taxes on old capital and debt, rebated lump-sum to each firm. To obtain the optimal tax rate on old capital, we first compute the allocation and then use the following firm optimality condition

$$q_t(1+\phi_{d,t})(1+\tau_t^O) = \beta f_k(k_t) g_O(k_t^N, k_t^O)$$

to solve for τ_t^O. Figure A3 displays the stationary-equilibrium tax on old capital. We find that it is positive, consistent with our main insight on the importance of the distributive externality, which induces the planner to reduce the price of old capital. Moreover, this tax is smaller for more financially constrained firms. This is because the planner uses the tax on old capital to partially substitute for the missing subsidy on new capital and relax constraint (A1), which is more binding for unconstrained firms.

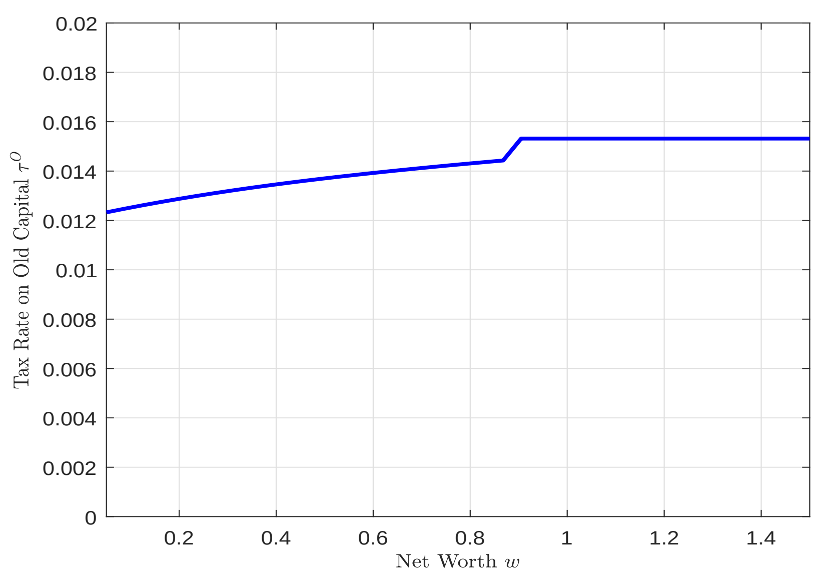

Figure A3: Tax Rate on Old Capital without Subsidy on New Capital. The x-axis reports net worth w. The y-axis reports the tax rate τ^O.

A.2.3 No Lump-Sum Transfers

We now consider the policy experiment with proportional taxes on new and old capital, but without lump-sum rebates. Figure A4 displays the overall tax liability of each firm as a function of net worth, confirming that firms with low net worth pay a positive tax and effectively subsidize firms with higher net worth under this policy.

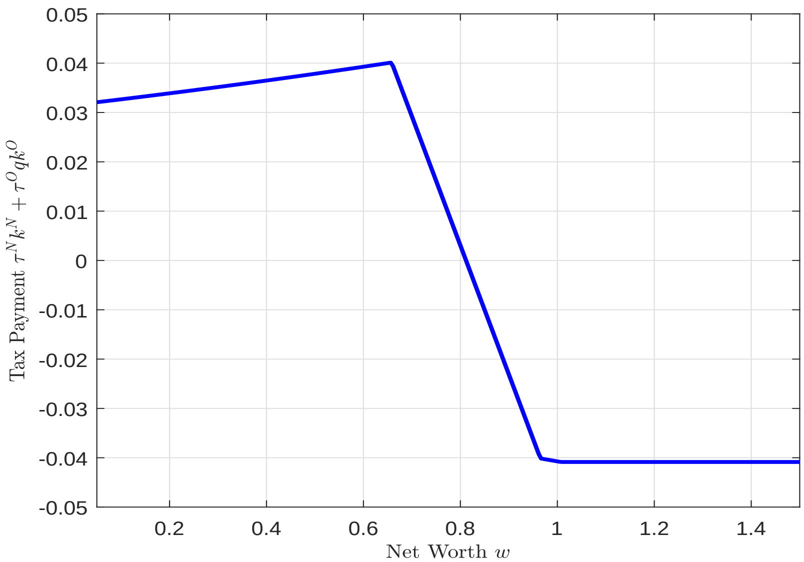

Figure A4: Tax Payment Without Lump-Sum Transfers in the Stylized Model. The x-axis reports net worth w. The y-axis reports the total tax payment $\tau^N k^N(w) + \tau^O q k^O(w)$ assuming that $\tau^N = -0.03$ and τ^O balances the government budget constraint.

B Extensions and Limitations of Efficiency Result

This appendix provides additional details and results on the analysis of Section III.

B.1 Risk-Averse Entrepreneurs

In this section, we analyze the case from Section III.A in which each firm is owned by a risk-averse entrepreneur whose consumption at each date equals the dividend paid by the firm.

Competitive Equilibrium with Financial Frictions. Given their initial net worth w and the price of old capital q_t, entrepreneurs maximize their utility by choosing consumption

c_{0t} and $c_{1,t+1}$, new and old capital k_t^N and k_t^O,and borrowing b_t, to solve[33]

$$\max_{\{c_{0t},c_{1,t+1},k_t^N,k_t^O,b_t\}\in\mathbb{R}_+^4\times\mathbb{R}} u\left(c_{0t}\right)+\beta u\left(c_{1,t+1}\right) \tag{B1}$$

where u is the utility function, with $u_c > 0$, $u_{cc} < 0$, and $\lim_{c\to 0} u_c(c) = +\infty$, subject to the budget constraints for the current and next period,

$$w_{0t}+b_t = c_{0t}+k_t^N+q_t k_t^O \tag{B2}$$

$$f(k_t^N+k_t^O)+q_{t+1}k_t^N = c_{1,t+1}+\beta^{-1}b_t, \tag{B3}$$

and the collateral constraint (9).

Denote the multipliers on the budget constraints by μ_{0t} and $\beta\mu_{1,t+1}$, on the collateral constraint by $\beta\lambda_t$, and on the non-negativity constraints for new and old capital by $\underline{\nu}_t^N$ and $\underline{\nu}_t^O$, respectively. The optimal demand for new capital, old capital, and borrowing, as functions of initial net worth w, satisfy the following first-order conditions

$$u_c(c_{0t}) = \beta u_c(c_{1,t+1})\left[f_k(k_t)+q_{t+1}\right]+\beta\theta\lambda_t q_{t+1}+\underline{\nu}_t^N \tag{B4}$$

$$q_t u_c(c_{0t}) = \beta u_c(c_{1,t+1})f_k(k_t)+\underline{\nu}_t^O \tag{B5}$$

$$u_c(c_{0t}) = u_c(c_{1,t+1})+\lambda_t, \tag{B6}$$

where $k_t = k_t^N + k_t^O$. Moreover, the firm's marginal value of net worth at date t is $\mu_{0,t} = u_c(c_{0t})$.

A *stationary competitive equilibrium* is a set of policy functions mapping initial net worth to an allocation $\{c_0(w), c_1(w), k^N(w), k^O(w), b(w)\}$, that is, consumption, investment, and debt choices, and a price of old capital q, such that entrepreneurs maximize their utility, $\forall w \in \mathcal{W}$, and the market for old capital clears, that is, $\int k^N(w)d\pi(w) = \int k^O(w)d\pi(w)$.

In a stationary equilibrium, the first-order conditions for new and old capital (B4) and (B5) can be written as investment Euler equations

$$1 \geq \beta\frac{u_c(c_1)}{u_c(c_0)}\frac{\left[f_k(k)+(1-\theta)q\right]}{\wp_N} \tag{B7}$$

$$1 \geq \beta\frac{u_c(c_1)}{u_c(c_0)}\frac{f_k(k)}{q}, \tag{B8}$$

[33]Because we now interpret dividends as consumption, we require that dividends are non-negative. We could alternatively allow for negative dividends, in which case this model becomes a generalization of our stylized model, which can be obtained as the special case $u(d) \equiv d - \phi(d)$.

with equality if $k^N > 0$ and $k^O > 0$, respectively, where we use the same definition of the down payments as in Section II. Using (B6), we can rewrite (B7) and (B8) as

$$u_N(w) \equiv u_N + \frac{\lambda}{u_c(c_1)}\wp_N = 1 - \beta q + \frac{\lambda}{u_c(c_1)}(1-\beta\theta q) \quad \geq \quad \beta f_k(k)$$
$$u_O(w) \equiv u_O + \frac{\lambda}{u_c(c_1)}\wp_O = q + \frac{\lambda}{u_c(c_1)}q \quad \geq \quad \beta f_k(k),$$

where we use the same definitions of the user cost as in Section II.

Combining (B7) and (B8) we moreover have

$$1 = \beta\frac{u_c(c_1)}{u_c(c_0)}\frac{(1-\theta)q}{\wp_N - \wp_O} + \frac{(\underline{\nu}^N - \underline{\nu}^O)/u_c(c_0)}{\wp_N - \wp_O}. \tag{B9}$$

Arguing as before, equation (B9) implies $\wp_N > \wp_O$, and thus, in equilibrium, $u_N \leq u_O$. Consider first entrepreneurs for which $\lambda = 0$. They invest $\overline{k}$ which solves $1 = \beta\frac{f_k(\overline{k})+(1-\theta)q}{\wp_N}$. Moreover $k^N = \overline{k}$ and $k^O = 0$ if $q > q^{FB}$ as we assume is the case in Proposition 3. Entrepreneurs with $\lambda = 0$ have net worth $w \geq \overline{w}$, where $\overline{w}$ solves $\overline{w} - \wp_N\overline{k} = f(\overline{k}) + (1-\theta)q\overline{k}$.

Entrepreneurs with sufficiently low w strictly prefer old capital, because as $w \to 0$, $f_k(k) \to +\infty$ and therefore $\frac{u_c(c_1)}{u_c(c_0)} \to 0$, and thus equation (B9) implies $\underline{\nu}^N > 0$. Hence, for sufficiently low w, $k^N = 0$ and $k^O > 0$. Moreover, k^O is strictly increasing in w. To see this, consider $w_+ > w$ and assume $k^O_+ \leq k^O$. Then, $c_{1,+} = f(k^O_+) \leq f(k^O) = c_1$ and $f_k(k^O_+) \geq f_k(k^O)$, whereas $\frac{u_c(c_{1,+})}{u_c(c_{0,+})} > \frac{u_c(c_{1,+})}{u_c(c_0)} \geq \frac{u_c(c_1)}{u_c(c_0)}$, which contradicts equation (B8).

For w sufficiently close to $\overline{w}$ and $w < \overline{w}$, $k^N > 0$ and $k^O = 0$. Hence (B7) holds with equality. Moreover, k^N is strictly increasing in w. To see this, consider $w_+ > w$ and assume $k^N_+ \leq k^N$. Then, $f_k(k^N_+) \geq f_k(k^N)$, whereas $c_{1,+} = f(k^N_+) + q(1-\theta)k^N_+ \leq f(k^N) + q(1-\theta)k^N = c_1$ and hence $\frac{u_c(c_{1,+})}{u_c(c_{0,+})} > \frac{u_c(c_{1,+})}{u_c(c_0)} \geq \frac{u_c(c_1)}{u_c(c_0)}$, which contradicts equation (B7).

Consider now entrepreneurs for which $\underline{\nu}^N = \underline{\nu}^O = 0$. Then, $1 = \beta\frac{u_c(c_1)}{u_c(c_0)}R_O$, where $R_O = \frac{(1-\theta)q}{\wp_N - \wp_O}$. With risk-averse entrepreneurs the value function is globally strictly concave, and the envelope condition implies that c_0 is strictly increasing in w, and thus so is c_1. Moreover $1 = R_O^{-1}\frac{[f_k(k)+(1-\theta)q]}{\wp_N} = R_O^{-1}\frac{f_k(k)}{q}$. Hence, $k = \underline{k} \leq \overline{k}$. Since $c_1 = f(\underline{k}) + (1-\theta)qk^N$, k^N is strictly increasing and $k^O = \underline{k} - k^N$ strictly decreasing in w.

Entrepreneurs who are indifferent between new and old capital have net worth $\underline{w}_N \leq w \leq \overline{w}_O \leq \overline{w}$ and these thresholds are implicitly characterized as follows: $c_0(\underline{w}_N) = \underline{w}_N - q\underline{k}$, $c_1(\underline{w}_N) = f(\underline{k})$, $1 = \beta R_O\frac{u_c(c_1(\underline{w}_N))}{u_c(c_0(\underline{w}_N))}$; $c_0(\overline{w}_O) = \overline{w}_O - \wp_N\underline{k}$, $c_1(\overline{w}_O) = f(\underline{k}) + (1-\theta)q\underline{k}$, $1 = \beta R_O\frac{u_c(c_1(\overline{w}_O))}{u_c(c_0(\overline{w}_O))}$.

Constrained Efficiency. Given an initial distribution of new and old capital, $k^N_{-1}(w)$ and

$k^O_{-1}(w)$, a utilitarian planner maximizes the total present discounted value of utility

$$\int \left[u\left(c_{10}(w)\right) + \sum_{t=0}^{\infty} \beta^t \left(u\left(c_{0t}(w)\right) + \beta u\left(c_{1,t+1}(w)\right)\right)\right] d\pi(w),$$

subject to the budget constraints (B2) and (B3) with multipliers $\beta^t \mu_{0,t}$ and $\beta^{t+1}\mu_{1,t+1}$, the collateral constraint (9) with multiplier $\beta^{t+1}\lambda_t$, the non-negativity constraints on new and old capital with multipliers $\beta^t \underline{\nu}^N_t$ and $\beta^t \underline{\nu}^O_t$, and the market clearing condition for old capital (3) with multiplier $\beta^t \eta_t$.

The first-order condition with respect to the price of old capital q_t for $t = 1, 2, \ldots$ is

$$\int k^O_t(w) u_c\left(c_{0t}(w)\right) d\pi(w) = \int k^N_{t-1}(w)\left[u_c\left(c_{1t}(w)\right) + \theta\lambda_{t-1}(w)\right] d\pi(w).$$

Thus, in the stationary constrained-efficient allocation, we have

$$\int k^O(w) u_c\left(c_0(w)\right) d\pi(w) = \int k^N(w)\left[u_c\left(c_1(w)\right) + \theta\lambda(w)\right] d\pi(w).$$

We now show that, in stationary competitive equilibrium, the distributive externality is larger than the collateral externality, that is,

$$\int k^O(w) u_c\left(c_0(w)\right) d\pi(w) > \int k^N(w)\left[u_c\left(c_1(w)\right) + \theta\lambda(w)\right] d\pi(w).$$

To do so, it is sufficient to prove that

$$\int k^O(w) u_c\left(c_0(w)\right) d\pi(w) > \int k^N(w) u_c\left(c_0(w)\right) d\pi(w), \tag{B10}$$

because $u_c\left(c_0(w)\right) = u_c\left(c_1(w)\right) + \lambda(w) \geq u_c\left(c_1(w)\right) + \theta\lambda(w)$.

We can bound the two sides of (B10) as follows. If there is positive mass between $\underline{w}_N$ and $\overline{w}_O$, we apply the following result: $\mathbb{E}[k^O u_c] = \mathbb{COV}(k^O, u_c) + \mathbb{E}[k^O]\mathbb{E}[u_c]$. Let $\overline{u}_c \equiv \int_{\underline{w}_N}^{\overline{w}_O} u_c\left(c_0(w)\right) d\pi(w) / \int_{\underline{w}_N}^{\overline{w}_O} d\pi(w)$. We have

$$\int k^O u_c(c_0) d\pi = \int^{\overline{w}_O} k^O u_c(c_0) d\pi \geq \overline{u}_c \int^{\overline{w}_O} k^O d\pi, \tag{B11}$$

because (i) $u_c(c_0) > \overline{u}_c$ for $w < \underline{w}_N$, and (ii) both k^O and $u_c(c_0)$ are strictly decreasing in w for $\underline{w}_N \leq w \leq \overline{w}_O$, thus their covariance is positive, implying $\int_{\underline{w}_N}^{\overline{w}_O} k^O u_c(c_0) d\pi \geq$

$\overline{u}_c \int_{\underline{w}_N}^{\overline{w}_O} k^O d\pi$.[34] Similarly, we have

$$\int k^N u_c(c_0) d\pi = \int_{\underline{w}_N} k^N u_c(c_0) d\pi \leq \overline{u}_c \int_{\underline{w}_N} k^N d\pi, \tag{B12}$$

because (i) $u_c(c_0) < \overline{u}_c$ for $w > \overline{w}_O$, and (ii) k^N is strictly increasing in w for $\underline{w}_N \leq w \leq \overline{w}_O$, implying its covariance with $u_c(c_0)$ is negative, and thus $\int_{\underline{w}_N}^{\overline{w}_O} k^N u_c(c_0) d\pi \leq \overline{u}_c \int_{\underline{w}_N}^{\overline{w}_O} k^N d\pi$. Furthermore, notice that at least one of the two inequalities (B11) and (B12) is strict, because the distribution of net worth $\pi(w)$ is non-degenerate. Thus, combining (B11), (B12), and the market-clearing condition $\int_{\underline{w}_N} k^N d\pi = \int^{\overline{w}_O} k^O d\pi$, we get (B10).

If there is no mass between $\underline{w}_N$ and $\overline{w}_O$, (B10) obtains more directly because all entrepreneurs investing in old capital have a marginal utility weakly greater than $u_c(c_0(\underline{w}_N))$, all entrepreneurs investing in new capital have a marginal utility weakly less than $u_c(c_0(\overline{w}_O))$, and $u_c(c_0(\underline{w}_N)) > u_c(c_0(\overline{w}_O))$. This proves Proposition 3.

We now discuss the case $q = q^{FB}$, which is not included in Proposition 3. In this case, $u_N = u_O = \wp_O = q = 1/(1+\beta) < 1 - \beta\theta q = \wp_N$ and $R_O = \beta^{-1}$. For entrepreneurs that are indifferent between new and old capital ($\underline{\nu}^N = \underline{\nu}^O = 0$), (B9) implies that $u_c(c_1) = u_c(c_0)$, so $c_1 = c_0$, and (B6) implies that they are unconstrained ($\lambda = 0$). The investment Euler equations moreover imply that $\underline{k} = \overline{k} = k_{FB}$. Further, $\underline{w}_N < \overline{w}_O = \overline{w}$; specifically, $c_0(\underline{w}_N) = \underline{w}_N - q\underline{k}$, $c_1(\underline{w}_N) = f(\underline{k})$, and $\underline{w}_N = f(\underline{k}) + q\underline{k}$, and $c_0(\overline{w}_O) = \overline{w}_O - \wp_N \underline{k}$, $c_1(\overline{w}_O) = f(\underline{k}) + (1-\theta) q\underline{k}$, and $\overline{w}_O = f(\underline{k}) + q\underline{k} + (1-\theta)\underline{k}$.

All entrepreneurs with $w \geq \underline{w}_N$ are indifferent between new and old capital at the margin, but entrepreneurs with $w \in (\underline{w}_N, \overline{w}_O)$ invest at least $k^O_{min}(w) = \frac{\overline{w}_O - w}{1-\theta}$ in old capital. For entrepreneurs with $w \geq \underline{w}_N$ we choose the following selection of their investment policy. Let $\kappa^O = (\int_{\underline{w}_N} k^O d\pi - \int_{\underline{w}_N}^{\overline{w}_O} k^O_{min} d\pi)/(\int k^N d\pi + \int_{\underline{w}_N} k^O d\pi - \int_{\underline{w}_N}^{\overline{w}_O} k^O_{min} d\pi)$ and select $k^O(w) = k^O_{min}(w) + \kappa^O(\underline{k} - k^O_{min}(w))$ for $w \in (\underline{w}_N, \overline{w}_O)$, and $k^O(w) = \kappa^O \overline{k}$ for $w \geq \overline{w}_O$.

Using this selection and defining $\overline{u}_c \equiv \int_{\underline{w}_N} u_c(c_0(w)) d\pi(w) / \int_{\underline{w}_N} d\pi(w)$, we have

$$\begin{aligned} \int k^O u_c(c_0) d\pi &= \int^{\underline{w}_N} k^O u_c(c_0) d\pi + \int_{\underline{w}_N} k^O u_c(c_0) d\pi \\ &\geq \int^{\underline{w}_N} k^O u_c(c_0) d\pi + \overline{u}_c \int_{\underline{w}_N} k^O d\pi \geq \overline{u}_c \int k^O d\pi, \end{aligned}$$

and $\int k^N u_c(c_0) d\pi \leq \overline{u}_c \int_{\underline{w}_N} k^N d\pi = \overline{u}_c \int k^N d\pi$. As long as some entrepreneurs are constrained, at least one of the inequalities is strict, and using $\int k^O d\pi = \int k^N d\pi$, (B10) obtains.

[34] See Schmidt (2003) for a proof of the sign of the covariance of monotone functions.

B.2 Heterogeneity in Productivity

In this section, we analyze the model with productivity heterogeneity from Section III.B in more detail.

Competitive Equilibrium with Financial Frictions. A firm that draws initial net worth w and productivity s maximizes (6) subject to the budget constraints (7) and

$$sf(k_t^N + k_t^O) + q_{t+1}k_t^N = d_{1,t+1} + \beta^{-1}b_t, \tag{B13}$$

and the collateral constraint (9). Let $v(w,s)$ denote the value function of the firm.

Denote the multipliers on the budget constraints by μ_{0t} and $\beta\mu_{1,t+1}$, on the collateral constraint by $\beta\lambda_t$, and on non-negativity constraint for new and old capital by $\underline{\nu}_t^N$ and $\underline{\nu}_t^O$, respectively. The optimality conditions are

$$\begin{aligned} 1 + \phi_{d,t} &= \beta\left[sf_k(k_t) + q_{t+1}\right] + \beta\theta\lambda_t q_{t+1} + \underline{\nu}_t^N && \text{(B14)} \\ q_t(1+\phi_{d,t}) &= \beta s f_k(k_t) + \underline{\nu}_t^O, && \text{(B15)} \end{aligned}$$

and (12), where $k_t = k_t^N + k_t^O$.

A *stationary competitive equilibrium* is a set of policy functions mapping initial net worth and productivity to an allocation, that is, dividends, investment, and borrowing choices, $\{d_0(w,s), d_1(w,s), k^N(w,s), k^O(w,s), b(w,s)\}$, and a price of old capital q, such that firms maximize the present discounted value of dividends net of equity issuance cost, $\forall(w,s) \in \mathcal{W}\times\mathcal{S}$, and the market for old capital clears, that is, $\int k^N(w,s)d\pi(w,s) = \int k^O(w,s)d\pi(w,s)$.

In a stationary equilibrium, we can rewrite equations (B14) and (B15) as

$$\begin{aligned} \wp_N(1+\phi_d) &= \beta\left[sf_k(k) + (1-\theta)q\right] + \underline{\nu}^N && \text{(B16)} \\ q(1+\phi_d) &= \beta s f_k(k) + \underline{\nu}^O && \text{(B17)} \end{aligned}$$

where $\wp_N = 1 - \beta\theta q$. Following the same arguments we develop in Section II.C, one can show that $q \geq q^{FB}$. Moreover, for each value of s, there are thresholds $\underline{w}_N(s) \leq \overline{w}_O(s) \leq \overline{w}(s)$ (with strict inequalities if $q > q^{FB}$) such that: firms with $w \leq \underline{w}_N(s)$ invest only in old capital; firms with $w \in (\underline{w}_N(s), \overline{w}_O(s))$ invest $\underline{k}(s)$ and invest in both new and old capital, and firms with $w \geq \overline{w}_O(s)$ invest only in new capital; firms with $w \geq \overline{w}(s)$ pay non-negative dividends and invest $\overline{k}(s) \geq k^{FB}(s) \geq \underline{k}(s)$.

We now show that the marginal equity issuance cost $\phi_d(w,s)$ (or equivalently the marginal value of net worth $v_w(w,s) = 1 + \phi_d(w,s)$) is weakly increasing in s, that is, higher productivity firms are more financially constrained, for a given level of net worth.

First, consider firms that pay positive dividends. For these firms, $\phi_d(w,s) = 0$.

Now consider firms with $\underline{\nu}^N > 0$ and $\underline{\nu}^O = 0$; for such firms rewrite equation (B16) as

$$1 + \phi_d(qk^O - w) = \beta \frac{sf_k(k^O)}{q}$$

where we use $d_0 = w - qk^O$; totally differentiating with respect to s, we obtain $\frac{dk^O}{ds} = \frac{\beta f_k(s) q^{-1}}{q\phi_{dd} - \beta s f_{kk}(k) q^{-1}} > 0$. Thus, d_0 is decreasing in s, which implies that $\phi_d(w,s)$ (and $v_w(w,s)$) is increasing in s.

Next, consider firms with $\underline{\nu}^N = 0$ and $\underline{\nu}^O = 0$. In this case, combining equations (B16) and (B17), we can write $1 + \phi_d = \beta R_O$, where $R_O = \frac{(1-\theta)q}{\wp^N - q}$. Thus, all firms that are indifferent between new and old capital issue the same level of equity ($\underline{d}_0$), and feature a constant marginal issuance cost $\overline{\phi}_d$, independent of productivity s. The total investment of such firms satisfies

$$\wp^N R_O = sf_k(\underline{k}(s)) + (1-\theta)q,$$

which implies that $\underline{k}(s)$ is increasing in s. Hence, also the indifference thresholds $\underline{w}_N(s) = \underline{d}_0 + q\underline{k}(s)$ and $\overline{w}_O(s) = \underline{d}_0 + \wp^N \underline{k}(s)$ are increasing in s.

Finally, for firms with $\underline{\nu}^N = 0$ and $\underline{\nu}^O > 0$, rewrite equation (B16) as

$$1 + \phi_d(\wp_N k^N - w) = \beta \frac{sf_k(k^N) + (1-\theta)q}{\wp_N}.$$

Totally differentiating with respect to s, we obtain $\frac{dk^N}{ds} = \frac{\beta f_k(s) \wp_N^{-1}}{\wp_N \phi_{dd} - \beta s f_{kk}(k) \wp_N^{-1}} > 0$. Thus, d_0 is decreasing in s, which implies that $\phi_d(w,s)$ is increasing in s. We conclude that $\phi_d(w,s)$ (and $v_w(w,s)$) is weakly increasing in productivity s for all firms.

Constrained Efficiency. The planner maximizes

$$\int \left[d_{10}(w,s) + \sum_{t=0}^{\infty} \beta^t \left(d_{0t}(w,s) - \phi(-d_{0t}(w,s)) + \beta d_{1,t+1}(w,s) \right) \right] d\pi(w,s),$$

subject to the budget constraints (7) and (B13), the collateral constraint (9), and the market-clearing condition for old capital. The first-order condition with respect to q_t is

$$\int k_t^O(w,s) \left(1 + \phi_{d,t}(w,s)\right) d\pi(w,s) = \int k_{t-1}^N(w,s) \left(1 + \theta \lambda_{t-1}(w,s)\right) d\pi(w,s),$$

which, in stationary equilibrium, can be rewritten as follows

$$\int k^O(w,s) \phi_d(w,s) d\pi(w,s) = \theta \int k^N(w,s) \phi_d(w,s) d\pi(w,s), \tag{B18}$$

where we used the market-clearing condition, as well as the fact that planner optimally sets the marginal equity issuance cost equal to the multiplier on the collateral constraint.

We now show that in stationary competitive equilibrium, the left-hand side of equation (B18) is larger than the right-hand side, that is, the distributive externality dominates the collateral externality. We can bound the two sides of equation (B18) as follows. First, notice that the marginal equity issuance cost $\overline{\phi}_d$ is the lower bound for the marginal equity issuance cost of any firms with productivity s purchasing old capital, and the upper bound for the marginal equity issuance cost of any firms with productivity s purchasing new capital. Thus, for any productivity level s, we get

$$\int_w k^O(w,s)\phi_d(w,s)d\pi(w,s) \geq \overline{\phi}_d \int_w k^O(w,s)d\pi(w,s),$$

and

$$\int_w k^N(w,s)\phi_d(w,s)d\pi(w,s) \leq \overline{\phi}_d \int_w k^N(w,s)d\pi(w,s).$$

Next, recall that $\overline{\phi}_d$ is independent of s. Hence, by summing both sides of these two inequalities over productivity levels, we obtain

$$\int k^O(w,s)\phi_d(w,s)d\pi(w,s) \geq \overline{\phi}_d \int k^O(w,s)d\pi(w,s),$$

and

$$\int k^N(w,s)\phi_d(w,s)d\pi(w,s) \leq \overline{\phi}_d \int k^N(w,s)d\pi(w,s).$$

The two bounds reported on the right-hand sides of these inequalities are equal to each other because of market clearing. Thus, $\theta < 1$ implies

$$\int k^O(w,s)\phi_d(w,s)d\pi(w,s) > \theta \int k^N(w,s)\phi_d(w,s)d\pi(w,s),$$

which proves Proposition 4.

B.3 Firm Life Cycle and Long-Lived Capital

In this section, we discuss the model with a stochastic firm life cycle and long-lived capital from Section III.C in more detail and we prove Proposition 5.

Competitive Equilibrium with Financial Frictions. The expected present discounted value of dividends, net of equity issuance costs, of a firm born at time t is

$$\sum_{a=0}^{\infty} \beta^a \gamma_a \left[d_{a,t+a} - \phi(-d_{a,t+a})\right] + \sum_{a=1}^{\infty} \beta^a \gamma_{a-1} \rho w_{a,t+a}$$

where d_{at} are dividends of continuing firms of age a at time t and w_{at} is net worth. We leave implicit the dependence of allocations on initial firm net worth w_0 to simplify notation.

The dividend of a continuing firm satisfies the following budget constraint:

$$d_{at} = w_{at} + b_{at} - k_{at}^N - q_t k_{at}^O$$

where k_{at}^N and k_{at}^O are investments in new and old capital, respectively, q_t is the price of old capital, and b_{at} is debt. Firm net worth evolves as follows. For $a > 0$, we have

$$w_{at} = f(k_{a-1,t-1}) + (1 - \delta^N(1 - q_t))k_{a-1,t-1}^N + q_t(1 - \delta^O)k_{a-1,t-1}^O - \beta^{-1} b_{a-1,t-1}$$

where $k_{a-1,t-1} = k_{a-1,t-1}^N + k_{a-1,t-1}^O$ and β^{-1} is the gross interest rate.

Firms face a collateral constraint, which states that debt cannot exceed a fraction θ of the resale value of new and old capital:

$$\theta \left[(1 - \delta^N(1 - q_{t+1}))k_{at}^N + q_{t+1}(1 - \delta^O)k_{at}^O\right] \geq \beta^{-1} b_{at}.$$

Denote the multiplier on the collateral constraint by $\beta^{t+1}\gamma_a \lambda_{at}$ and on the non-negativity constraints for new and old capital by $\beta^t \gamma_a \nu_{at}^N$ and $\beta^t \gamma_a \nu_{at}^O$, and the marginal equity issuance cost by $\phi_{d,at}$. The firm's optimality conditions for new capital, old capital, and debt, are

$$\begin{aligned}
1 + \phi_{d,at} &= \beta \left[f_k(k_{at}) + (1 - \delta^N(1 - q_{t+1}))\right](1 + (1 - \rho)\phi_{d,a+1,t+1}) \\
&\quad + \beta\theta\lambda_{at}(1 - \delta^N(1 - q_{t+1})) + \underline{\nu}_{at}^N \\
q_t(1 + \phi_{d,at}) &= \beta \left[f_k(k_{at}) + (1 - \delta^O)q_{t+1}\right](1 + (1 - \rho)\phi_{d,a+1,t+1}) \\
&\quad + \beta\theta\lambda_{at}(1 - \delta^O)q_{t+1} + \underline{\nu}_{at}^O \\
\phi_{d,at} &= (1 - \rho)\phi_{d,a+1,t+1} + \lambda_{at}.
\end{aligned}$$

The market-clearing condition for old capital is

$$\int \sum_{a=0}^{\infty} \gamma_a \left[\delta^N k_{a,t-1}^N + (1 - \delta^O)k_{a,t-1}^O\right] d\pi_0(w_0) = \int \sum_{a=0}^{\infty} \gamma_a k_{at}^O d\pi_0(w_0).$$

We define a stationary competitive equilibrium writing the firm problem and the market clearing condition recursively. A *stationary competitive equilibrium* is a set of policy functions mapping net worth to an allocation, that is, dividends, investment, and borrowing choices for continuing firms, $\{d(w), k^N(w), k^O(w), b(w)\}$, a stationary distribution of net worth $\pi(w)$, and a price of old capital q, such that firms maximize the present discounted value of dividends net of equity issuance cost, $\forall w$, the stationary distribution is consistent with firms' policy functions, and the market for old capital clears, that is, $\int \delta^N k^N(w) d\pi(w) = \int \delta^O k^O(w) d\pi(w)$. Notice that the stationary distribution of net worth π is an equilibrium object, whereas the distribution of net worth of new firms π_0 is taken as exogenous.

With long-lived capital, we define the down payment per unit of new and old capital as $\wp_N \equiv 1 - \beta\theta(1 - \delta^N(1 - q))$ and $\wp_O \equiv q\left(1 - \beta\theta(1 - \delta^O)\right)$, respectively, and the user cost of new and old capital to an unconstrained firm as $u_N \equiv 1 - \beta\left(1 - \delta^N(1 - q)\right)$ and $u_O \equiv q\left(1 - \beta(1 - \delta^O)\right)$, respectively. Analogously to (15) and (16) we define the user cost of new and old capital to a firm with net worth w as

$$u_N(w) \equiv u_N + \frac{\lambda}{1 + (1-\rho)\phi'_d}\wp_N = 1 - \beta(1 - \delta^N(1-q)) + \frac{\lambda}{1 + (1-\rho)\phi'_d}(1 - \beta\theta(1 - \delta^N(1-q)))$$

and

$$u_O(w) \equiv u_O + \frac{\lambda}{1 + (1-\rho)\phi'_d}\wp_O = q(1 - \beta(1 - \delta^O)) + \frac{\lambda}{1 + (1-\rho)\phi'_d}q(1 - \beta\theta(1 - \delta^O)),$$

respectively. The investment Euler equations for new and old capital are

$$1 = \beta\left(\frac{1 + (1-\rho)\phi'_d}{1 + \phi_d}\right)\frac{f_k(k) + \beta(1-\theta)(1 - \delta^N(1-q))}{\wp_N} + \frac{\underline{\nu}^N/(1 + \phi_d)}{\wp_N} \quad \text{(B19)}$$

$$1 = \beta\left(\frac{1 + (1-\rho)\phi'_d}{1 + \phi_d}\right)\frac{f_k(k) + \beta(1-\theta)q(1 - \delta^O)}{\wp_O} + \frac{\underline{\nu}^O/(1 + \phi_d)}{\wp_O}. \quad \text{(B20)}$$

Combining these two Euler equations we obtain

$$1 = \beta\left(\frac{1 + (1-\rho)\phi'_d}{1 + \phi_d}\right)\frac{(1-\theta)((1 - \delta^N(1-q)) - q(1 - \delta^O))}{\wp_N - \wp_O} + \frac{(\underline{\nu}^N - \underline{\nu}^O)/(1 + \phi_d)}{\wp_N - \wp_O}. \quad \text{(B21)}$$

To see that $q < 1$ in a stationary equilibrium, suppose instead that $q \geq 1$; then $u_O > u_N$ and $\wp_O > \wp_N$, implying that old capital would be dominated. To see that $\wp_N > \wp_O$ in a stationary equilibrium, suppose instead that $\wp_N \leq \wp_O$; then, since $1 - \delta^N(1-q) - q(1-\delta^O) = (1 - \delta^N)(1 - q) + q\delta^O > 0$, (B21) would imply that $\underline{\nu}^O > 0$, that is, no firms would invest in old capital, a contradiction. Note that $\wp_N > \wp_O$ is equivalent to

$$q < \frac{1 - \beta\theta(1 - \delta^N)}{1 + \beta\theta\delta^N - \beta\theta(1 - \delta^O)} < 1.$$

To see that $u_N \leq u_O$ in a stationary equilibrium, note that otherwise $u_N(w) > u_O(w)$ for all firms, so there would not be any new investment, which cannot be an equilibrium. Further, $u_N \leq u_O$ is equivalent to

$$q \geq q^{FB} \equiv \frac{1 - \beta(1 - \delta^N)}{1 + \beta\delta^N - \beta(1 - \delta^O)},$$

that is, the price of old capital in competitive equilibrium must be weakly higher than the price of old capital in a frictionless economy.

Consider first firms for which $\lambda = 0$. They invest $\overline{k}$ which solves $1 = \beta\frac{f_k(\overline{k})+(1-\theta)(1-\delta^N(1-q))}{\wp_N}$. Moreover $k^N = \overline{k}$ and $k^O = 0$ if $q > q^{FB}$. Firms with $\lambda = 0$ have net worth $w \geq \overline{w} \equiv \wp_N\overline{k}$.

Sufficiently constrained firms prefer old capital. Moreover, k^O is strictly increasing in w for such firms. To see this, first notice that the firm value function is concave, implying that the marginal value of net worth $1 + \phi_d$ is weakly decreasing in net worth. Consider $w_+ > w$ and assume $k^O_+ \leq k^O$. Then $f(k^O_+) \leq f(k^O)$, and thus $w'_+ \leq w'$ and $\phi'_{d,+} \geq \phi'_d$. Moreover, since $d_+ > d$ and $d < 0$, $\phi_d > \phi_{d,+}$, and thus $\frac{1+(1-\rho)\phi'_{d,+}}{1+\phi_{d,+}} > \frac{1+(1-\rho)\phi'_{d,+}}{1+\phi_d} \geq \frac{1+(1-\rho)\phi'_d}{1+\phi_d}$; but then equation (B20) implies that $k^O_+ > k^O$, a contradiction.

Consider now firms that are indifferent between investing in new and old capital. Let $R_O \equiv \frac{(1-\theta)((1-\delta^N(1-q))-q(1-\delta^O))}{\wp^N-\wp^O}$. Since $q \geq q^{FB}$, $R^O \geq \beta^{-1}$ (with equality iff $q = q^{FB}$). For firms in the indifference region, we can write (B21) as $1 = \beta\frac{1+(1-\rho)\phi'_d}{1+\phi_d}R_O$. For such firms, we can then write the investment Euler equation for new capital (B19) as

$$1 = R_O^{-1}\frac{f_k(k) + (1-\theta)(1-\delta^N(1-q))}{\wp_N},$$

implying that such firms all invest the same amount $\underline{k} \leq \overline{k}$. (If $q = q^{FB}$, then $\underline{k} = \overline{k} = k^{FB}$.) Moreover, in the indifference region, if $q > q^{FB}$, k^N (k^O) is strictly increasing (decreasing) in net worth. To see this, assume the opposite were true. Then, w' would be (weakly) decreasing in w, and thus ϕ'_d would be (weakly) increasing in w. Moreover, dividends $d = w - \wp^N k^N - \wp^O(\underline{k} - k^N)$ would be strictly increasing in w, implying that ϕ_d would be strictly decreasing in w, since in the indifference region $\frac{1+(1-\rho)\phi'_d}{1+\phi_d} < 1$ and hence $\phi_d > 0$ ($d < 0$). This contradicts the fact that $\frac{1+(1-\rho)\phi'_d}{1+\phi_d}$ is constant in the indifference region.

Firms that are indifferent between new and old capital have net worth $\underline{w}_N \leq w \leq \overline{w}_O \leq \overline{w}$ and these thresholds are implicitly characterized as follows: $d(\underline{w}_N) = \underline{w}_N - \wp^O\underline{k}$,

$w'(\underline{w}_N) = f(\underline{k}) + (1-\theta)(1-\delta^O)q\underline{k}$; $d(\overline{w}_O) = \overline{w}_O - \wp_N \underline{k}$, $w'(\overline{w}_O) = f(\underline{k}) + (1-\theta)(1-\delta^N(1-q))\underline{k}$; and $1 = \beta \frac{1+(1-\rho)\phi_d'}{1+\phi_d} R_O$.

If $q > q^{FB}$, for w sufficiently close to $\overline{w}$ and $w < \overline{w}$, $k^N > 0$ and $k^O = 0$. Hence (B19) holds with $\underline{\nu}^N = 0$. Moreover, k^N is strictly increasing in w, following a similar argument by contradiction to the one developed above for firms that only purchase old capital.

Constrained Efficiency. The planner maximizes the present discounted value of aggregate dividends net of equity issuance costs

$$\sum_{t=0}^{\infty} \beta^t \int \left[\sum_{a=0}^{\infty} \gamma_a \left[(d_{at} - \phi(-d_{at})) \right] + \sum_{a=1}^{\infty} \gamma_{a-1} \rho w_{at} \right] d\pi_0(w_0)$$

subject to the transition for net worth, the collateral constraint and the market-clearing condition for old capital, with multiplier $\beta^t \eta_t$.

The optimality condition for the price of old capital is

$$\int \sum_{a=0}^{\infty} \gamma_a k_{at}^O (1 + \phi_{d,at}) d\pi_0(w_0) = \int \sum_{a=0}^{\infty} \gamma_a \left[\delta^N k_{a,t-1}^N + (1-\delta^O) k_{a,t-1}^O \right] (1 + (1-\rho)\phi_{d,a+1,t} + \theta \lambda_{a,t-1}) d\pi_0(w_0). \quad \text{(B22)}$$

The summation on the left-hand side of equation (B22) represents the marginal cost of increasing the price q_t for firms that purchase old capital. The summation on the right-hand side represents the marginal benefit of increasing net worth for firms that own old capital, as well as the marginal effect of q_t on the borrowing capacity of constrained firms at $t-1$.

We now prove that in the stationary competitive equilibrium the distributive externality is larger than the collateral externality, that is,

$$\int \sum_{a=0}^{\infty} \gamma_a k_{at}^O (1 + \phi_{d,at}) d\pi_0(w_0) > \int \sum_{a=0}^{\infty} \gamma_a \left[\delta^N k_{a,t-1}^N + (1-\delta^O) k_{a,t-1}^O \right] (1 + (1-\rho)\phi_{d,a+1,t} + \theta \lambda_{a,t-1}) d\pi_0(w_0),$$

or, written recursively,

$$\int k^O(w)(1+\phi_d(w))d\pi(w) > \int \left[\delta^N k^N(w) + (1-\delta^O)k^O(w)\right](1+(1-\rho)\phi_d(w') + \theta\lambda(w))d\pi(w)$$

where w' denotes future net worth associated with current net worth w. Simplifying using the market-clearing condition, we have

$$\int k^O(w)\phi_d(w)d\pi(w) > \int \left[\delta^N k^N(w) + (1-\delta^O)k^O(w)\right]((1-\rho)\phi_d(w') + \theta\lambda(w))d\pi(w).$$

Using the first-order condition for debt to substitute out $\lambda(w)$, we obtain

$$\int k^O(w)\phi_d(w)d\pi(w) > \int \left[\delta^N k^N(w) + (1-\delta^O)k^O(w)\right](\theta\phi_d(w) + (1-\theta)(1-\rho)\phi_d(w'))d\pi(w). \tag{B23}$$

Notice that ϕ_d is weakly decreasing in net worth. Moreover, $\phi_d(w) \geq \theta\phi_d(w) + (1-\theta)(1-\rho)\phi_d(w')$. Hence, if inequality (B23) holds (weakly) for $\theta = 1$, it holds (strictly) for any $\theta < 1$. Accordingly, we now prove the following inequality:

$$\int k^O(w)\phi_d(w)d\pi(w) \geq \int \left[\delta^N k^N(w) + (1-\delta^O)k^O(w)\right]\phi_d(w)d\pi(w),$$

which, rearranging, we can equivalently express as follows

$$\delta^O \int k^O(w)\phi_d(w)d\pi(w) \geq \delta^N \int k^N(w)\phi_d(w)d\pi(w). \tag{B24}$$

As no firm invests in old capital above $\overline{w}_O$, market clearing implies:

$$\delta^O \int^{\overline{w}_O} k^O(w)d\pi(w) \geq \delta^N \int_{\underline{w}_N}^{\overline{w}} k^N(w)d\pi(w). \tag{B25}$$

Furthermore, we can bound the two sides of (B24) as follows. Since $\phi_d(w)$ is weakly decreasing in w,

$$\int^{\underline{w}_N} k^O(w)\phi_d(w)d\pi(w) \geq \phi_d(\underline{w}_N) \int^{\underline{w}_N} k^O(w)d\pi(w). \tag{B26}$$

In the region of indifference between new and old capital, i.e., between $\underline{w}_N$ and $\overline{w}_O$, if there is positive mass, we apply the following result: $\mathbb{E}[k^O\phi_d] = \mathbb{COV}(k^O, \phi_d) + \mathbb{E}[k^O]\mathbb{E}[\phi_d]$. Since k^O and ϕ_d are both decreasing in w, we have $\mathbb{COV}(k^O, \phi_d) \geq 0$. Thus,

$$\int_{\underline{w}_N}^{\overline{w}_O} k^O(w)\phi_d(w)d\pi(w) \geq \overline{\phi}_d \int_{\underline{w}_N}^{\overline{w}_O} k^O(w)d\pi(w) \tag{B27}$$

where $\overline{\phi}_d \equiv \int_{\underline{w}_N}^{\overline{w}_O} \phi_d(w)d\pi(w)/\int_{\underline{w}_N}^{\overline{w}_O} d\pi(w)$. Since $\phi_d(\underline{w}_N) \geq \overline{\phi}_d$, we can combine (B26) and (B27) and get

$$\int^{\overline{w}_O} k^O(w)\phi_d(w)d\pi(w) \geq \overline{\phi}_d \int^{\overline{w}_O} k^O(w)d\pi(w). \tag{B28}$$

Analogously, since k^N is increasing in w, we obtain

$$\int_{\underline{w}_N}^{\overline{w}} k^N(w)\phi_d(w)d\pi(w) \leq \overline{\phi}_d \int_{\underline{w}_N}^{\overline{w}} k^N(w)d\pi(w). \tag{B29}$$

Notice that our characterization of the stationary equilibrium implies $\int_{\overline{w}_O} k^O(w)\phi_d(w)d\pi(w) = \int_{\overline{w}} k^N(w)\phi_d(w)d\pi(w) = 0$. Hence, combining (B25), (B28), and (B29), we get

$$\delta^O \int k^O(w)\phi_d(w)d\pi(w) \geq \delta^N \int k^N(w)\phi_d(w)d\pi(w),$$

which, given $\theta < 1$, proves Proposition 5.

If there is no mass between $\underline{w}_N$ and $\overline{w}_O$, Proposition 5 obtains more directly because financially constrained firms investing in old capital have $\phi_d \geq \phi_d(\underline{w}_N)$, all firms investing in new capital have $\phi_d \leq \phi_d(\overline{w}_O)$, and $\phi_d(\underline{w}_N) > \phi_d(\overline{w}_O)$.

B.4 Current Price in the Collateral Constraint

To microfound the presence of the current price of old capital in the collateral constraint, we assume that firms can default on their debt at the beginning of the period, before production occurs. In the case of default, they can abscond a fraction $(1-\theta)$ of their assets and there is no exclusion from asset or financial markets.

Under these assumptions, the collateral constraint is

$$\theta(k_t^N + q_t k_t^O) \geq b_t,$$

with multiplier λ_t. As in Bianchi and Mendoza (2018), current asset prices determine the current borrowing capacity. However, different from their setup, it is the choice of investment at date t that firms pledge as collateral, instead of the capital owned at the beginning of date t. We make this small departure from their assumptions to preserve the property of our model that firms' net worth is a sufficient state variable. Bianchi and Mendoza (2018) argue that their mechanism is robust to a formulation like ours, where the collateral constraint depends on the choice of asset level at date t.

The first-order conditions of the firm problem with respect to new capital, old capital, and debt are

$$\begin{aligned} 1+\phi_{d,t} &= \beta\left[f_k(k_t)+q_{t+1}\right]+\theta\lambda_t+\underline{\nu}_t^N \\ q_t(1+\phi_{d,t}) &= \beta f_k(k_t)+\theta\lambda_t q_t+\underline{\nu}_t^O, \end{aligned}$$

and (12), respectively.

In stationary equilibrium, the expressions for user costs and down payments are as follows: $u_N = 1-\beta q$, $u_O = q$, $\wp_N = 1-\theta$, and $\wp_O = q(1-\theta)$. We can thus rewrite the investment Euler equations as follows:

$$\begin{aligned} u_N+\phi_d\wp_N &= 1-\beta q+\phi_d(1-\theta)\geq\beta f_k(k) \\ u_O+\phi_d\wp_O &= q+\phi_d q(1-\theta)\geq\beta f_k(k). \end{aligned}$$

Moreover, combining the optimality conditions for new and old capital, we have

$$1=\frac{(\theta+\beta)q-\theta}{(1+\phi_d)(\wp_N-\wp_O)}+\frac{\underline{\nu}^N-\underline{\nu}^O}{(1+\phi_d)(\wp_N-\wp_O)}. \tag{B30}$$

The price of old capital must be such that $q<1$ or, equivalently, $\wp_N>\wp_O$, otherwise all firms would invest in new capital. Hence, in order for some firms to invest in new capital, it must be that $u_N\leq u_O \Leftrightarrow q\geq\frac{1}{1+\beta}$.

Notice that the numerator of the first fraction on the right-hand side is positive, because $\frac{\theta}{\theta+\beta}<\frac{1}{1+\beta}$. Hence, equation (B30) implies that sufficiently constrained firms invest only in old capital. Firms that pay dividends weakly prefer new capital, and strictly so if $q>\frac{1}{1+\beta}$. Define $R_O\equiv\beta^{-1}\frac{(\theta+\beta)q-\theta}{\wp_N-\wp_O}$. Firms that are indifferent between new and old capital must have $\beta\frac{1}{1+\phi_d}=R_O^{-1}$ (from (B30)) and invest $\underline{k}$, which solves $1=R_O^{-1}\frac{f_k(\underline{k})+q-\beta^{-1}\theta}{\wp_N}$, where $\underline{k}\leq k^{FB}$ with equality iff $q=q^{FB}$. Firms are indifferent between new and old capital at the margin if $w\in(\underline{w}_N,\overline{w}_O)$, where $\underline{w}_N=\underline{d}_0+\wp_O\underline{k}$ and $\overline{w}_O=\underline{d}_0+\wp_N\underline{k}$, $\underline{d}_0=0$ if $q=q^{FB}$, and $\underline{d}_0$ solves $1+\phi_d=\beta R_O$ if $q>q^{FB}$.

Constrained Efficiency. The planner's first-order condition with respect to q_t for $t=1,2,\dots$ is

$$\int k_t^O(1+\phi_{d,t})d\pi=\int k_{t-1}^N d\pi+\theta\int k_t^O\phi_{d,t}d\pi.$$

Different from our baseline formulation with the future price in the collateral constraint, in this model a marginal change in q_t affects both budget constraints and collateral constraints at date t. Using the market-clearing condition for old capital and rearranging, we get

$$(1-\theta)\int k_t^O \phi_{d,t} d\pi = 0. \tag{B31}$$

In stationary competitive equilibrium, the left-hand side of equation (B31) is strictly positive as long as a positive mass of firms is financially constrained. Hence, also under these assumptions on the collateral constraint, the distributive externality dominates the collateral externality, showing that the main insight of our paper does not depend on specific timing assumptions related to limited enforcement.

B.5 Relation to Models with Representative Entrepreneur and Assets in Fixed Supply

In this section, we connect our results on constrained efficiency in capital reallocation with a related class of models, that feature a representative infinitely-lived entrepreneur subject to collateral constraints, possibly impatient relative to the equilibrium interest rate, and with an asset in fixed supply, which we will refer to as land. This class of models includes the seminal work of Kiyotaki and Moore (1997) (henceforth KM) and the small-open-economy models with collateral constraints analyzed by Bianchi and Mendoza (2018) and Jeanne and Korinek (2019). While there is some variation in the specification of collateral constraints across these papers, we maintain our formulation of collateral constraints that depend on future asset prices, as in KM and Rampini and Viswanathan (2010, 2013).

First, we show that the representative-entrepreneur assumption in this class of models implies that there is no reallocation of land in stationary equilibrium. This lack of reallocation precludes any distributive effects of asset prices: As Dávila and Korinek (2018) show, distributive externalities depend on (i) heterogeneity in the marginal valuation of resources, and (ii) non-zero equilibrium trading.[35] We also connect this insight to the effects of unexpected changes in collateral values described by KM.

Second, we show that the fact that land is in fixed supply in these models, different from capital, which is endogenously produced in our model, does not affect the main insights on distributive externalities vs. collateral externalities in reallocation at the core of our paper. To illustrate this point, we recover a version of our main result on inefficiency in a model of land reallocation with overlapping generations of entrepreneurs – and hence reallocation in stationary equilibrium. Finally, we use this model to briefly discuss the role of impatience

[35]This insight is also related to arguments developed in Stiglitz (1982) and Geanakoplos and Polemarchakis (1986), showing that incomplete-markets equilibria with no trading can be constrained efficient.

for the comparison of distributive and collateral externalities.

B.5.1 Representative Entrepreneur without Reallocation

We now describe an economy with a representative entrepreneur and a representative lender. The entrepreneur can be interpreted as the "farmer" in KM, or, alternatively, as the representative household residing in a small open economy. The lender can be interpreted as the "gatherer" in KM, or, alternatively, as a representative household in the rest of the world in small-open-economy models.

Model. A representative entrepreneur has preferences represented by the utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $\beta < 1$ is the discount factor, c_t is consumption, $u_c > 0$, and $u_{cc} < 0$. The entrepreneur has access to a technology $y_t = f(k_{t-1})$ with $f_k > 0$, $f_{kk} < 0$, and $\lim_{k \to 0} f_k(k) = +\infty$, where y_t is output and k_{t-1} is land.

A representative lender has preferences represented by the following utility function

$$\sum_{t=0}^{\infty} R^{-t} c_t^l, \tag{B32}$$

where $R \in (1, \beta^{-1}]$ is the inverse of the discount factor and c_t^l is consumption. The lender has access to a technology $y_t^l = f^l(k_{t-1}^l)$ with $f_k^l \geq 0$ and $f_{kk}^l \leq 0$, where y_t^l is output and k_{t-1}^l is land. The case of a small open economy corresponds to lenders having an exogenous endowment, and not operating land, that is, $f^l(k_{t-1}^l) = \bar{y}^l > 0$.

The resource constraint of the economy is $c_t + c_t^l = y_t + y_t^l$. Land is in exogenous fixed supply, $K = k_t + k_t^l$. Entrepreneur and lender can trade land at price q_t as well as a one-period bond b_t. Because of our assumptions on preferences, the gross interest rate is given by R. The budget constraints of entrepreneur and lender are as follows:

$$y_t + b_t = c_t + q_t(k_t - k_{t-1}) + Rb_{t-1} \tag{B33}$$

$$y_t^l + Rb_{t-1} = c_t^l + q_t(k_t^l - k_{t-1}^l) + b_t. \tag{B34}$$

Notice that our notation implicitly imposes equilibrium in the bond market, and we interpret a positive value of b_t as debt for the entrepreneur and assets for the lender.

The entrepreneur is also subject to the following collateral constraint

$$\theta q_{t+1} k_t \geq Rb_t, \tag{B35}$$

which limits borrowing to a fraction $\theta < 1$ of the future resale value of the entrepreneur's land.

We define the Lagrangian of the entrepreneur's problem as follows

$$\mathcal{L} \equiv \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \mu_t \left(f(k_{t-1}) + b_t - c_t - q_t(k_t - k_{t-1}) - Rb_{t-1} \right) + \beta \lambda_t \left(\theta q_{t+1} k_t - Rb_t \right) \right\},$$

where $\beta^t \mu_t$ and $\beta^{t+1}\lambda_t$ denote the multipliers on the budget constraint (B33) and the collateral constraint (B35), respectively.

The entrepreneur's optimality conditions with respect to land and debt are respectively

$$\begin{aligned} q_t u_c(c_t) &= \beta u_c(c_{t+1}) \left(f_k(k_t) + q_{t+1} \right) + \beta \lambda_t \theta q_{t+1} \\ u_c(c_t) &= \beta R u_c(c_{t+1}) + \beta R \lambda_t. \end{aligned}$$

The lender maximizes utility (B32) subject to the budget constraint (B34) and a non-negativity constraint on land holdings, with multiplier $R^{-t}\underline{\nu}_t^l$. The lender's optimality condition with respect to land is

$$q_t = R^{-1} \left(f_k^l(k_t^l) + q_{t+1} \right) + \underline{\nu}_t^l.$$

A *stationary competitive equilibrium* is defined as a time-invariant allocation $\left\{c, c^l, k, k^l, b\right\}$ and a price of land q that satisfy the entrepreneur's and lender's optimality conditions, as well as the market-clearing condition $K = k + k^l$.

Constrained Efficiency. To analyze the constrained-efficiency properties of the stationary competitive equilibrium, we now consider the marginal effect of a change in the price of land on welfare. For simplicity, we consider a planner who assigns zero weight on the lender's utility; our insights are unchanged if we allow for a positive weight on the lender. The derivative of the entrepreneur's Lagrangian with respect to q_t is

$$\frac{\partial \mathcal{L}}{\partial q_t} = -\beta^t \mu_t (k_t - k_{t-1}) + \beta^t \lambda_{t-1} \theta k_t. \tag{B36}$$

In stationary equilibrium, the first term is equal to zero, because the amount of land owned by the entrepreneur is constant, whereas the second term is weakly positive, and strictly so if the collateral constraint is binding. In this case, the only pecuniary externality is the collateral externality, and an increase in the price of land would lead to an increase in welfare by relaxing the collateral constraint. An equivalent way to formulate this insight is to observe that even though cheaper land at date t, taking as given the price at $t+1$, would seemingly benefit the entrepreneur by reducing the cost of investment at t, it would at the same time decrease the value of the entrepreneur's net worth by the same amount. Furthermore, it would make the collateral constraint tighter at date $t-1$, resulting in an overall negative effect.

This result arises because there is no net trading of land in stationary equilibrium, so no equilibrium capital reallocation. Clearly, even if the planner assigned positive weight to the lender's utility, there would be no distributive externality, as the lender's amount of land is also constant. This model may feature *misallocation* in stationary equilibrium, if, as in KM, the financial friction induces an allocation such that $f_k(k) > f^l_k(k^l)$. However, the model does not feature *reallocation*, in the sense that land is not traded in stationary equilibrium, resulting in no effect of the price of land on budget constraints, and thus no distributive externality.

Relation to the KM mechanism. To connect our efficiency analysis to the effects of asset-price changes in the KM model, we can rearrange equation (B35) as follows, after substituting out debt b_t from the budget constraint (B33):

$$(q_t - R^{-1}\theta q_{t+1})k_t - q_t k_{t-1} - f(k_{t-1}) + Rb_{t-1} + c_t \leq 0, \tag{B37}$$

or, equivalently,

$$k_t \leq \frac{1}{(q_t - R^{-1}\theta q_{t+1})}\left(w_t - c_t\right), \tag{B38}$$

where $w_t \equiv q_t k_{t-1} + f(k_{t-1}) - Rb_{t-1}$ denotes the entrepreneur's net worth, which, importantly, also depends on the price of land.

Constraints (B37) and (B38) correspond to equations (4) and (7) in KM (pages 219 and 220, respectively), which hold with equality in their model, determining the law of motion of the entrepreneur's land, whenever the collateral constraint is binding.[36] KM consider the following thought experiment: What would the effect of an *unexpected* increase in the *current and future* price of land on this constraint be? Notice that this marginal effect is different from the derivative (B36), which accounts for the effects of current prices on past debt, and moreover treats the price at each date as a distinct variable. Nevertheless, to consider the KM thought experiment, we assume there are two consecutive dates such that $q_t = q_{t+1} = q$ and differentiate both sides of inequality (B37) with respect to q. We denote this derivative by Δ^{KM}, and obtain

$$\Delta^{KM} = (1 - R^{-1}\theta)k_t - k_{t-1}.$$

As KM argue, as long as the constraint is binding and $f(k_{t-1}) - Rb_{t-1} - c_t < 0$, that is, as

[36] Under the technology assumption in KM, consumption c_t is a constant fraction of output, labeled as "perishable" output. Moreover, KM focus on the case in which $\theta = 1$.

long as there is sufficient leverage, we have $\Delta^{KM} < 0$, implying that an increase in the price of land would relax the collateral constraint, even after accounting for the effects of the price on the budget constraint. In stationary equilibrium, we have $\Delta^{KM} = -R^{-1}\theta k < 0$, because, as we argued, the effects of the price of land on the budget constraint other than through the collateral constraint cancel out, as the entrepreneur is keeping a constant amount of land.

We conclude that the collateral externality is the only pecuniary externality in the stationary equilibrium of this model, and this insight about efficiency is closely related to the fact that an unexpected change in current and future collateral values, as analyzed by KM, relaxes the collateral constraint.

B.5.2 Heterogeneous Entrepreneurs and Reallocation

We now consider a modification of this model and show how to recover our main insights on the importance of distributive externalities. Specifically, instead of assuming an infinitely-lived representative entrepreneur, we consider over-lapping generations of entrepreneurs, as in Section III.A (and as KM consider in the appendix of their paper), which introduces heterogeneity among entrepreneurs and reallocation in equilibrium.

Model. For simplicity, we assume that entrepreneurs live for two dates and that all entrepreneurs are endowed with a common initial level of net worth w. We maintain all other assumptions from the model in the previous subsection, namely an infinitely-lived lender with linear preferences and a productive asset in fixed supply.

The representative entrepreneur born at date t has utility

$$u(c_{0t}) + \beta u(c_{1,t+1})$$

with $\beta \leq R^{-1}$, $u_c > 0$, and $u_{cc} < 0$. The budget constraints are

$$\begin{aligned} w + b_t &= c_{0t} + q_t k_t \\ f(k_t) + q_{t+1} k_t &= c_{1,t+1} + R b_{t-1}, \end{aligned}$$

and the collateral constraint is given by equation (B35) as before.

The optimality conditions with respect to land and debt are respectively

$$q_t u_c(c_{0t}) = \beta u_c(c_{1,t+1}) \left(f_k(k_t) + q_{t+1} \right) + \beta \lambda_t \theta q_{t+1} \tag{B39}$$

$$u_c(c_{0t}) = \beta R u_c(c_{1,t+1}) + \beta R \lambda_t. \tag{B40}$$

where, again, $\beta\lambda_t$ denotes the multiplier on the collateral constraint. Notice that in this model there is always net trading of land, with young entrepreneurs being buyers and old entrepreneurs being sellers.

A *stationary competitive equilibrium* is defined as a time-invariant allocation $\left\{c_0, c_1, c^l, k, k^l, b\right\}$ and a price of land q that satisfy the entrepreneurs' and lender's optimality conditions, as well as the market-clearing condition $K = k + k^l$.

Constrained Efficiency. As in Section III.A and Online Appendix B.1, we consider a planner who maximizes the present discounted value of utilities of all generations of entrepreneurs, with discount factor β

$$u\left(c_{10}\right) + \sum_{t=0}^{\infty} \beta^t \left(u\left(c_{0t}\right) + \beta u\left(c_{1,t+1}\right)\right),$$

subject to all budget constraints, collateral constraints, and market-clearing conditions.

The marginal effect of the price of land q_t on aggregate welfare is given by

$$-\beta^t (u_c(c_{0t}) - u_c(c_{1t}))k_t + \beta^t \lambda_{t-1} \theta k_t.$$

The two marginal-utility terms represent the distributive externality, due to the fact that young entrepreneurs buy land, whereas old entrepreneurs sell land in equilibrium. Thus, as long as they have different marginal utility from consumption, the aggregate distributive effect of a price change is non-zero. The last term involving the multiplier λ_{t-1} denotes the collateral externality, because the price of land affects the collateral constraint in the previous period.

Using equation (B40) to substitute out λ_{t-1}, this expression can be rewritten as

$$-\beta^t \left[(u_c(c_{0t}) - u_c(c_{1t})) - \beta^{-1} R^{-1} \theta \left(u_c(c_{0,t-1}) - \beta R u_c(c_{1t})\right)\right] k_t. \tag{B41}$$

Case $\beta = R^{-1}$. We now show that under the baseline assumption on discounting in our paper, which is $\beta = R^{-1}$, this derivative is negative in stationary equilibrium, because buyers of land have a higher marginal utility than sellers, and moreover the distributive externality dominates the collateral externality. To see this, observe that $\beta = R^{-1}$ and equation (B40) imply $u_c(c_0) - u_c(c_1) \geq 0$. Furthermore, the expression in (B41) becomes

$$-\beta^t \left(u_c(c_0) - u_c(c_1)\right)(1-\theta)k \leq 0,$$

with strict inequality if the collateral constraint binds. Thus, when $\beta = R^{-1}$, introducing heterogeneity and equilibrium reallocation in the model with assets in fixed supply overturns the result on the sign of inefficiency obtained in representative-entrepreneur models.

The distributive externality dominates the collateral externality, consistent with the main insight in our model with endogenous investment.

Case $\beta < R^{-1}$*.* We now discuss the role of impatience for pecuniary externalities in the reallocation of land. In the case of relatively impatient entrepreneurs, that is, $\beta < R^{-1}$, we cannot sign the aggregate welfare effect of the price of land unambiguously in general. First, notice that under sufficient impatience, equation (B40) is consistent with young entrepreneurs having a *lower* marginal utility than old entrepreneurs. Second, notice that, as equation (B41) highlights, the distributive externality from the price q_t generates an aggregate welfare effect at date t, whereas the collateral externality relaxes a constraint at date $t-1$. When $\beta = R^{-1}$, this timing difference is exactly offset by the discounting of the value of collateral in the collateral constraint. In contrast, when $\beta < R^{-1}$, the difference in timing between the two externalities implies that the collateral externality is relatively more important, other things equal, as reflected by the factor $\beta^{-1}R^{-1} > 1$ that multiplies the corresponding terms in equation (B41).

B.6 Obtaining the Opposite Sign of Inefficiency

We have proved that the distributive externality dominates the collateral externality in stationary equilibrium in a large class of models. Both to highlight the role of several assumptions that lead to this result and to further relate to the large literature on pecuniary externalities, which in some cases obtains the opposite sign of inefficiency, we now show how one can modify our model to overturn our main efficiency result.

Specifically, we present three models. In the first model, the point of departure is the model with long-lived new and old capital, but we modify the assumptions on collateralizability of new and old capital. In the second and third model, the point of departure is the model with risk-averse entrepreneurs, but in one case we modify the assumptions on discount rates and the interest rate and in the other case we introduce saving constraints.

B.6.1 Role of Collateralizability

We consider the model of Section III.C with long-lived new and old capital. However, we generalize the model, by allowing for a different collateralizability parameter for new and old capital. Specifically, let θ^N be the collateralizability parameter for new capital and θ^O for old capital. Notice that our baseline assumption in Section III.C is $\theta^N = \theta^O$. We further assume $\delta^O < 1$ and, for simplicity, $\rho = 1$, that is, firms are only alive at two dates.

We show that if new capital serves as sufficiently better collateral than old capital, then it is possible that the collateral externality dominates the distributive externality. In particular, to obtain a stark characterization, we focus on the case $\theta^N = 1$ and $\theta^O = 0$, that is, new capital can by fully pledged, whereas old capital cannot serve as collateral.

In this case, the firm's optimality conditions for new capital, old capital, and debt, are

$$\begin{aligned}
1+\phi_{d,at} &= \beta\left[f_k(k_{at}) + (1-\delta^N(1-q_{t+1}))\right] + \beta\lambda_{at}(1-\delta^N(1-q_{t+1})) + \underline{\nu}^N_{at} \\
q_t(1+\phi_{d,at}) &= \beta\left[f_k(k_{at}) + (1-\delta^O)q_{t+1}\right] + \underline{\nu}^O_{at} \\
1+\phi_{d,at} &= 1+\lambda_{at}.
\end{aligned}$$

The definitions of user cost and down payment are as follows: $u_N = 1-\beta\left(1-\delta^N(1-q)\right)$, $u_O = q\left(1-\beta(1-\delta^O)\right)$, $\wp_N = 1-\beta\left(1-\delta^N(1-q)\right)$, and $\wp_O = q$. The investment Euler equations for new and old capital can be expressed as follows

$$u_N + \phi_d \wp_N \geq \beta f_k(k) \tag{B42}$$

$$u_O + \phi_d \wp_O \geq \beta f_k(k). \tag{B43}$$

Furthermore, combining the two investment Euler equations, we have

$$1 = \beta\frac{(1-\delta^O)q}{(1+\phi_d)(\wp_O-\wp_N)} + \frac{\underline{\nu}^O-\underline{\nu}^N}{(1+\phi_d)(\wp_O-\wp_N)}.$$

If $\wp_O \leq \wp_N$, then $\underline{\nu}^N > 0$, so no firm invests in new capital, which cannot be true in equilibrium. Therefore, $\wp_O > \wp_N$, or, equivalently,

$$q > \frac{1-\beta(1-\delta^N)}{1+\beta\delta^N}.$$

Equations (B42) and (B43) then imply $u_N \geq u_O$, as otherwise no firm would buy old capital, which cannot be true in equilibrium; equivalently, $q \leq q^{FB} = \frac{1-\beta(1-\delta^N)}{1+\beta\delta^N-\beta(1-\delta^O)}$.

Because new capital has a (weakly) higher user cost than old capital, but requires a lower down payment, the induced preference for new vs. old capital as a function of net worth is the opposite of that in our baseline model with $\theta^N = \theta^O = \theta$. In particular, more financially constrained firms invest in new capital and less financially constrained firms in old capital.

Consider first firms for which $\lambda = 0$. They invest $\overline{k}$ which solves $q\left(1-\beta(1-\delta^O)\right) = \beta f_k(\overline{k})$. Moreover, if $q < q^{FB}$, they set $k^O = \overline{k}$ and $k^N = 0$. Firms with $\lambda = 0$ have net worth $w \geq \overline{w} \equiv \wp_O\overline{k}$. Sufficiently constrained firms compare down payments and thus strictly prefer new capital. Consider now firms that are indifferent between investing in new and old capital. Let $R_N \equiv \frac{(1-\delta^O)q}{(1+\phi_d)(\wp_O-\wp_N)}$. These firms invest $\underline{k}$, which solves

$q = R_N^{-1}\left(f_k(\underline{k}) + (1-\delta^O)q\right)$. Because all indifferent firms have the same marginal value of net worth $1+\overline{\phi}_d$, and thus pay the same (negative) dividend, and old capital has a higher down payment than new capital, it follows that in the indifference region investment in new (old) capital is decreasing (increasing) in net worth.

Constrained Efficiency. The planner's optimality condition for the price of old capital is

$$\int k_t^O(1+\phi_{d,t})d\pi(w) = \int \left[\delta^N k_{t-1}^N(1+\lambda_{t-1}) + (1-\delta^O)k_{t-1}^O\right] d\pi(w).$$

Writing this condition with recursive notation in stationary equilibrium and using the market-clearing condition for old capital and substituting the multiplier on the collateral constraint out, we get

$$\int k^O(w)\phi_d(w)d\pi(w) = \int \delta^N k^N(w)\phi_d(w)d\pi(w).$$

We now show that for sufficiently large δ_N, in stationary equilibrium the collateral externality is larger than the distributive externality, that is,

$$\int k^O(w)\phi_d(w)d\pi(w) \leq \delta^N \int k^N(w)\phi_d(w)d\pi(w). \tag{B44}$$

Consider the case $\delta^N \leq \delta^O$. Then, market clearing for old capital implies $\int k^N(w)d\pi(w) \geq \int k^O(w)d\pi(w)$. Using the same arguments developed in Online Appendix B.3 and the characterization of equilibrium that we derived above, we get

$$\int k^O(w)\phi_d(w)d\pi(w) < \int k^N(w)\phi_d(w)d\pi(w),$$

because only sufficiently constrained firms ($\phi_d \geq \overline{\phi}_d$) purchase new capital, only sufficiently unconstrained firms ($\phi_d \leq \overline{\phi}_d$) purchase old capital, and aggregate new capital is at least as large as aggregate old capital.

By continuity, for sufficiently large δ^N, we obtain inequality (B44). Specifically, this result arises for $\delta^N = 1-\epsilon$, $\delta^O = 1-\frac{\epsilon}{\kappa}$, $\kappa \geq 1$, and $\epsilon > 0$ sufficiently small.

Finally, we highlight that we have focused on the case $\theta^N = 1$ and $\theta^O = 0$, but our results can be generalized to sufficiently high θ^N and sufficiently low θ^O. Overall, if new capital serves as sufficiently better collateral then old capital, it is possible that financially constrained firms prefer new capital and, as a result, the collateral externality may dominate the distributive externality.

B.6.2 Role of Discounting

We now consider the model with risk-averse entrepreneurs of Section III.A. However, we generalize the model to allow for different discount rates for planner and entrepreneurs, as well as a generic value for the interest rate, not necessarily tied to entrepreneurs' discount factor.

Specifically, let β be entrepreneurs' discount factor, $R \leq \beta^{-1}$ the gross interest rate entrepreneurs can borrow or lend at, and ξ the planner's discount factor for the utility of each generation. Notice that our baseline assumption in Section III.A is $\beta = R^{-1} = \xi$.

Given their initial net worth w and the price of old capital q_t, entrepreneurs maximize their utility (B1) by choosing consumption c_{0t} and $c_{1,t+1}$, new and old capital k_t^N and k_t^O, and borrowing b_t, to with the utility function u satisfying $u_c > 0$, $u_{cc} < 0$, and $\lim_{c \to 0} u_c(c) = +\infty$, subject to the budget constraints for the current and next period, (B2) and

$$f(k_t^N + k_t^O) + q_{t+1}k_t^N = c_{1,t+1} + Rb_t, \tag{B45}$$

and the collateral constraint

$$\theta q_{t+1}k_t^N \geq Rb_t. \tag{B46}$$

Denote the multipliers on the budget constraints by μ_{0t} and $\beta\mu_{1,t+1}$, on the collateral constraint by $\beta\lambda_t$, and on non-negativity constraint for new and old capital by $\underline{\nu}_t^N$ and $\underline{\nu}_t^O$, respectively. The optimal demand for new capital, old capital, and borrowing, as functions of initial net worth w, satisfy the first-order conditions (B4), (B5), and

$$u_c(c_{0t}) = \beta R u_c(c_{1,t+1}) + \beta R\lambda_t, \tag{B47}$$

where $k_t = k_t^N + k_t^O$.

In stationary equilibrium, the expressions for user costs and down payments are as follows: $u_N = 1 - R^{-1}q$, $u_O = q$, $\wp_N = 1 - R^{-1}\theta q$, and $\wp_O = q$. The first-order conditions for new and old capital can be rewritten as investment Euler equations (B7) and (B8), or, using both the definitions of user costs and down payments, as follows

$$\begin{aligned} u_N + \frac{\lambda}{u_c(c_1)}\wp_N &\geq R^{-1}f_k(k) \\ u_O + \frac{\lambda}{u_c(c_1)}\wp_O &\geq R^{-1}f_k(k). \end{aligned}$$

Combining equations (B7) and (B8), we get (B9). Hence, using the same arguments we develop in Section II.C, we obtain that $\frac{1}{1+R^{-1}} \leq q < \frac{1}{1+R^{-1}\theta}$. Moreover, the characterization of the choice between new and old capital is also analogous to the one we obtain when

$\beta R = 1$. Specifically, sufficiently constrained entrepreneurs only invest in old capital. Unconstrained entrepreneurs weakly prefer new capital, and strictly so when $q > \frac{1}{1+R^{-1}}$. In the indifference region, entrepreneurs substitute away from old capital and toward new capital as net worth increases.

Constrained Efficiency. Given an initial distribution of new capital, old capital, and debt, a utilitarian planner maximizes the total present discounted value of utility

$$\int \left[u\left(c_{10}(w)\right) + \sum_{t=0}^{\infty} \xi^t \left(u\left(c_{0t}(w)\right) + \beta u\left(c_{1,t+1}(w)\right)\right) \right] d\pi(w),$$

subject to the budget constraints (B2) and (B45) with multipliers $\xi^t \mu_{0,t}$ and $\xi^t \beta \mu_{1,t+1}$, the collateral constraint (B46) with multiplier $\xi^t \beta \lambda_t$, the non-negativity constraints on new and old capital with multipliers $\xi^t \underline{\nu}_t^N$ and $\xi^t \underline{\nu}_t^O$, and the market clearing condition for old capital (3) with multiplier $\xi^t \eta_t$.

The first-order condition with respect to the price of old capital q_t for $t = 1, 2, \ldots$ is

$$\int k_t^O(w) u_c\left(c_{0t}(w)\right) d\pi(w) = \xi^{-1} \beta \int k_{t-1}^N(w) \left[u_c\left(c_{1t}(w)\right) + \theta \lambda_{t-1}(w)\right] d\pi(w).$$

Thus, in the stationary constrained-efficient allocation, we have

$$\int k^O(w) u_c\left(c_0(w)\right) d\pi(w) = \xi^{-1} \beta \int k^N(w) \left[u_c\left(c_1(w)\right) + \theta \lambda(w)\right] d\pi(w).$$

We can further use equation (B47) to substitute out the multiplier on the collateral constraint and obtain

$$\int k^O(w) u_c\left(c_0(w)\right) d\pi(w) = \xi^{-1} \beta \int k^N(w) \left[\theta \beta^{-1} R^{-1} u_c\left(c_0(w)\right) + (1-\theta) u_c\left(c_1(w)\right)\right] d\pi(w). \tag{B48}$$

When we evaluate the left-hand side and the right-hand side of equation (B48) in the stationary competitive equilibrium, we can find two reasons why the collateral externality may dominate the distributive externality. First, the planner may be sufficiently impatient relative to entrepreneurs, that is, $\xi^{-1}\beta$ is sufficiently large. Second, entrepreneurs may be sufficiently impatient relative to the interest rate, that is, $\beta^{-1}R^{-1}$ is sufficiently large. Either of these factors would magnify the collateral externality relative to the distributive externality.

To see why this is the case, notice that a marginal increase in the price of old capital q_t affects budget constraints at date t—this is the distributive externality—and relaxes collateral constraints at date $t-1$. As a result, with a sufficient degree of impatience, this relaxation of collateral constraints may dominate the redistribution of financial resources. Notice that this argument does not apply under our baseline assumption $\beta R = 1$, because in this case the effect of impatience on the valuation of the collateral externality is exactly offset by the effect of a higher interest rate.

We believe this analysis of the role of discounting may be useful to connect our results to the literature on pecuniary externalities in small-open-economies, which focuses on the collateral externality and typically assumes that the interest rate is smaller than the inverse of the discount factor.

B.6.3 Role of Saving Constraints

We consider again the model with risk-averse entrepreneurs of Section III.A. To derive a sharp characterization, we assume that all entrepreneurs are born with a common initial endowment w_0. Nevertheless, the economy features heterogeneity between young and old entrepreneurs. Moreover, we assume that entrepreneurs cannot borrow or save using bonds; in this case, we replace the collateral constraint with the equality constraint $b_t = 0$. Because of these assumptions, the economy features distributive externalities, but no collateral externalities.

Alternatively, notice that this condition arises as an equilibrium condition if instead we assume that entrepreneurs can access a bond market, but the economy is closed and this market has to clear among entrepreneurs who are either homogeneous in all respects including their initial net worth or have heterogeneous initial net worth but $\theta = 0$.

The optimality conditions for new capital and old capital in this case are

$$\begin{aligned} u_c(c_{0t}) &= \beta u_c(c_{1,t+1}) \left[f_k(k_t) + q_{t+1} \right] + \underline{\nu}_t^N \\ q_t u_c(c_{0t}) &= \beta u_c(c_{1,t+1}) f_k(k_t) + \underline{\nu}_t^O \end{aligned}$$

Because all entrepreneurs face the same problem and have the same level of net worth, they choose the same level of capital and, in equilibrium, divide this level of capital equally between new and old capital.

Constrained Efficiency. The planner's first-order condition with respect to the price of old capital q_t for $t = 1, 2, \ldots$ is

$$k_t^O u_c(c_{0t}) = k_{t-1}^N u_c(c_{1t}).$$

Thus, in the stationary constrained-efficient allocation, we have

$$k^O u_c(c_0) = k^N u_c(c_1).$$

For sufficiently large initial endowment w_0, entrepreneurs would desire to save using bonds, if they were allowed to, and thus the constraint $b_t = 0$ is a binding saving constraint. As a result, in stationary equilibrium $u_c(c_1) > u_c(c_0)$, and, using the market-clearing condition $k^N = k^O$, we obtain that the distributive externality has the opposite sign relative to our baseline model with saving. Specifically, we have

$$k^O u_c(c_0) < k^N u_c(c_1),$$

and a higher price of old capital would increase welfare by redistributing resources from young entrepreneurs to old entrepreneurs, who have higher marginal utility, thus alleviating the effects of the saving constraint.

This analysis is useful in relating our results to the literature that focuses on fire-sale externalities and builds on Lorenzoni (2008). In that model, there is no collateral externality. However, the distributive externality has the opposite sign relative to our baseline results. Specifically, in some states of the world, financially constrained entrepreneurs are net sellers of assets. Hence, a higher price may induce higher welfare. To obtain this result, Lorenzoni (2008) assumes lack of commitment of both households and entrepreneurs, effectively preventing entrepreneurs from saving resources into those states. Our analysis confirms the importance of this assumption, by showing that saving constraints may induce a higher marginal utility for sellers of capital also in our framework.

Finally, we highlight that under the interpretation of the condition $b_t = 0$ as bond-market equilibrium among homogenous entrepreneurs, we have that the equilibrium interest rate is lower than the inverse of the discount factor, connecting this model with the previous subsection on the role of discounting.

C Quantitative Analysis

This appendix provides additional details and results on the analysis of the quantitative model of Sections IV, V, and VI.

C.1 Solution Method for Quantitative Model

In this section, we discuss the solution method for the quantitative model. We compute the stationary constrained-efficient allocation using the following iterative procedure:

1. Guess a value for the multiplier on the market clearing condition for old capital η.

   (a) Guess a value for the price of old capital q.

(b) Solve for the firm policy functions on a grid for net worth w and productivity s, using the optimality conditions (49), (50), and (47) evaluated in stationary equilibrium.

(c) Obtain the stationary distribution of net worth and productivity by simulating a continuum of firms.

(d) Check the market-clearing condition (37) and update the guess for the price q accordingly, until convergence.

2. Evaluate the optimality condition for the price of old capital (51) and update the guess for η accordingly, until convergence.

The stationary competitive equilibrium is a special case of steps (a)-(d) with $\eta = 0$.

C.2 Additional Quantitative Results and Sensitivity

This section provides additional results related to the quantitative analysis of Sections V and VI.

Figure C1 displays the optimal tax rates on new and old capital that implement the constrained-efficient allocation in our calibrated model.

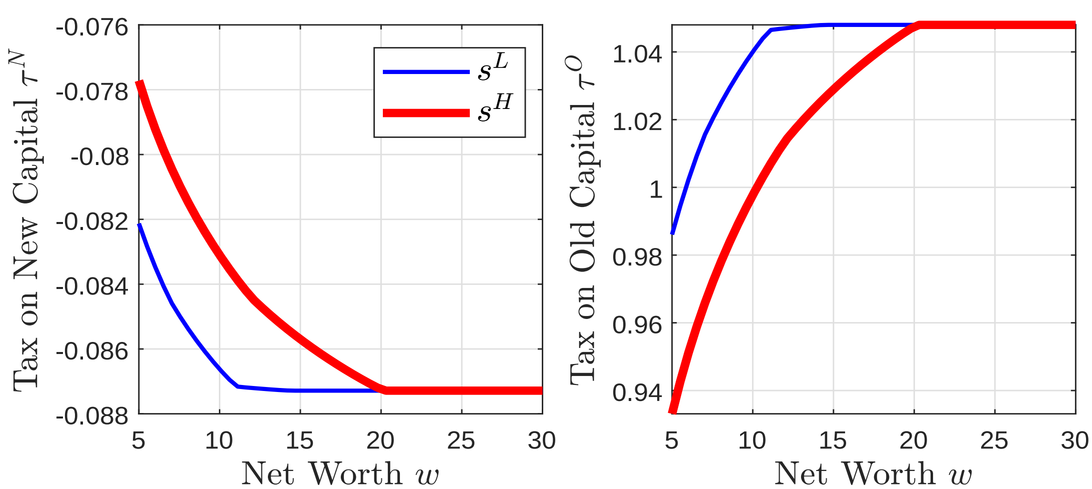

Figure C1: Optimal tax rates. Left panel: tax rate on new capital (a negative value denotes a subsidy); right panel: tax rate on old capital. The x-axes report net worth w. Thick red lines refer to the high productivity state; thin blue lines refer to the low state. See Table 1 for the parameter values.

Figure C2 displays the allocation implemented with uniform tax rates for all firms, equal to the average tax rates that implement the constrained-efficient allocation. The figure compares this allocation (solid lines) with the constrained-efficient allocation (dashed lines).

Figure C3 displays the effects of single tax instruments—only on new capital or only on old capital respectively—on the stationary-equilibrium price of old capital.

Figures C4 and C5 refer to the analysis of the balanced-budget policy without lump-sum transfers. Specifically, the figures display the tax payment for each firm, as a function of net worth and productivity, as well as the induced allocation, compared with the stationary equilibrium without policy, respectively.

Figure C6 plots the transition dynamics associated with the implementation of a tax rate on new capital (at $t = 0$), common for all firms and constant over time, starting from the competitive equilibrium without policy intervention (at $t = -1$). Net worth and productivity are sufficient firm state variables in the stationary equilibrium before the policy change and also along the perfect-foresight transition after the policy is announced. However, because the policy change is unanticipated, firms with equal net worth but different portfolios of new and old capital in the initial stationary equilibrium may be affected differently by the policy at $t = 0$. To maintain computational tractability and initialize the transition at the initial distribution of net worth and productivity, we assume that at $t = 0$, before the policy is announced, firms sell their initial holdings of old capital to an intermediary at the initial stationary equilibrium price.

Table C1 reports the results of the sensitivity analyses of Section VI.B.

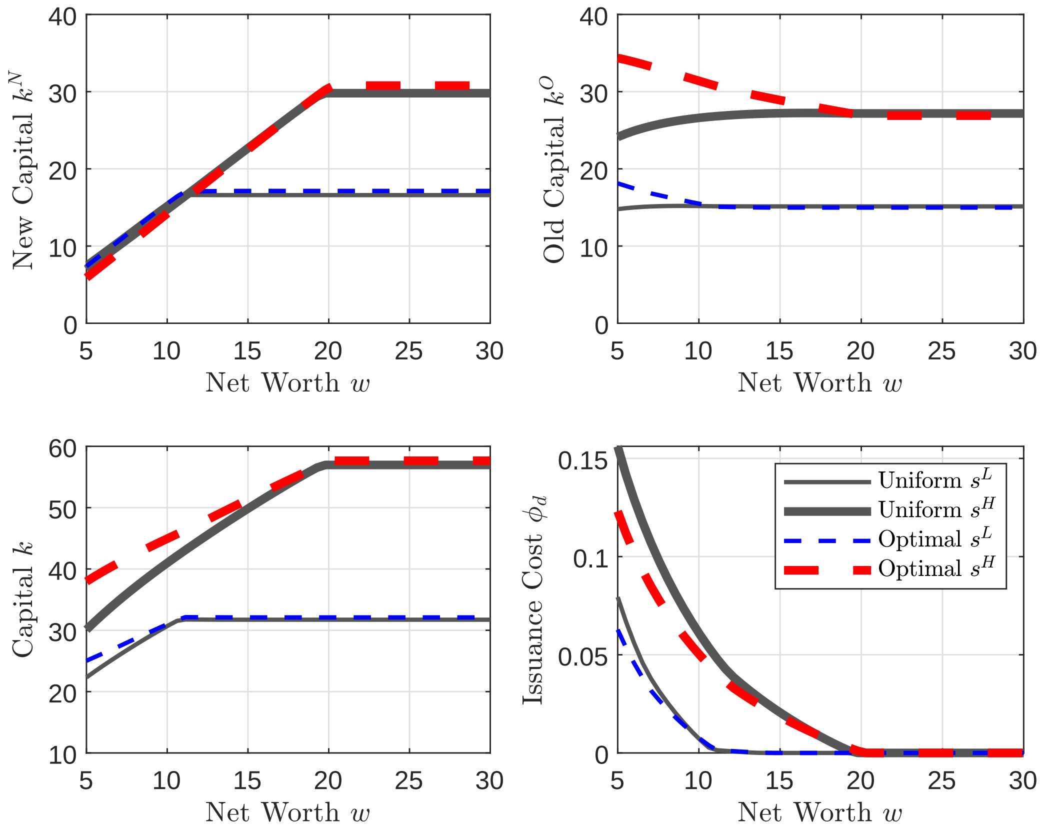

Figure C2: Firm-specific vs. Uniform Taxes. Top left: new capital k^N; top right: old capital k^O; bottom left: capital bundle k; bottom right: marginal cost of equity issuance ϕ_d. The x-axes report net worth w. Solid lines denote the allocation implemented with uniform tax rates $\tau^N = -0.086$ and $\tau^O = 1.037$; dashed lines denote the constrained-efficient allocation. Thick lines denote the high productivity state, thin lines the low state. See Table 1 for the parameter values.

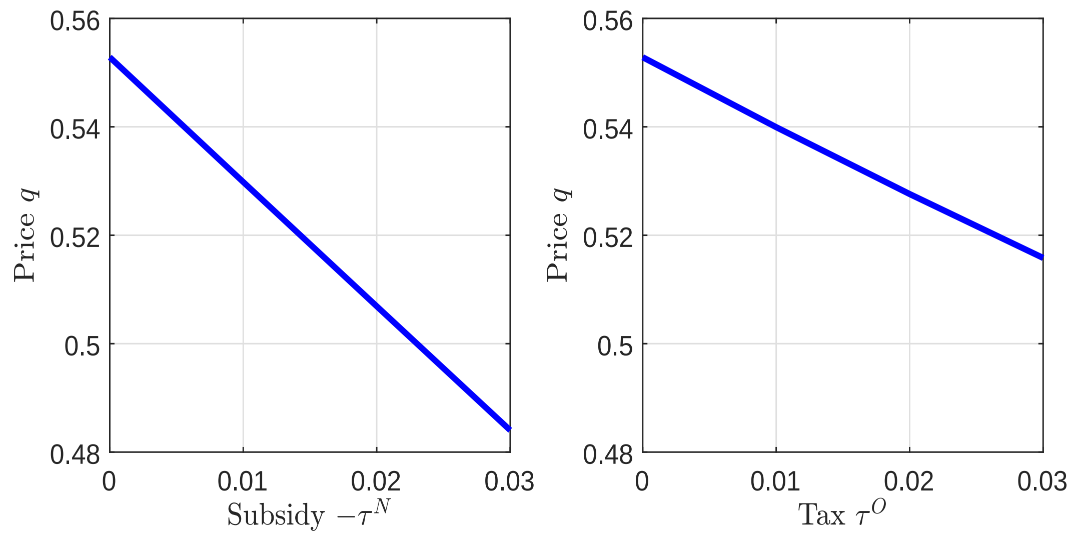

Figure C3: Effects of Single Tax Instruments on Price of Old Capital. The left panel refers to the case in which there are only subsidies on new capital, recovered from each firm in a lump-sum fashion. The x-axis reports the value of the subsidy rate on new capital ($-\tau^N$) and the y-axis reports the stationary-equilibrium price of old capital q. The right panel refers to the case in which there are only taxes on old capital, rebated to each firm in a lump-sum fashion. The x-axis reports the value of the tax rate on old capital (τ^O) and the y-axis reports the stationary-equilibrium price of old capital q.

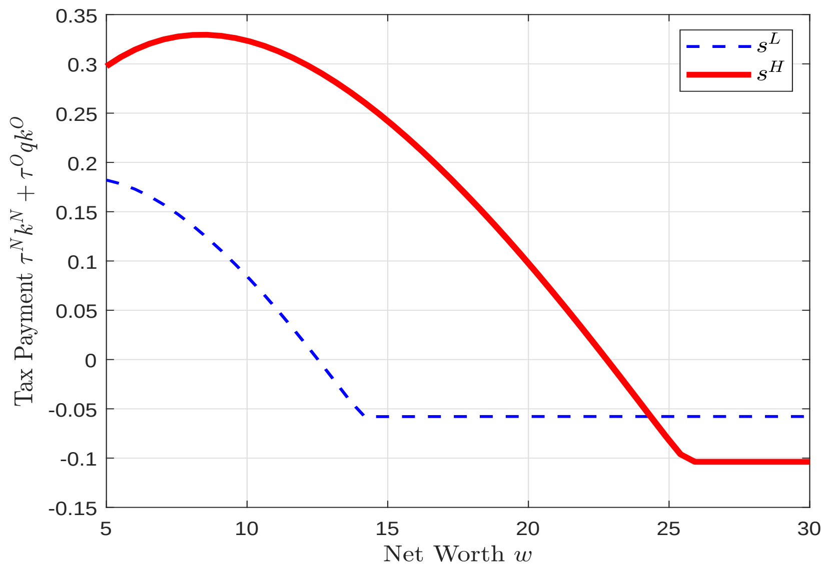

Figure C4: Tax Payment Without Lump-Sum Transfers in the Quantitative Model. The x-axis reports net worth w. The y-axis reports the total tax payment $\tau^N k^N(w) + \tau^O q k^O(w)$ assuming that $\tau^N = -0.03$ and τ^O balances the government budget constraint. Thick lines denote the high productivity state, thin lines the low state. See Table 1 for the parameter values.

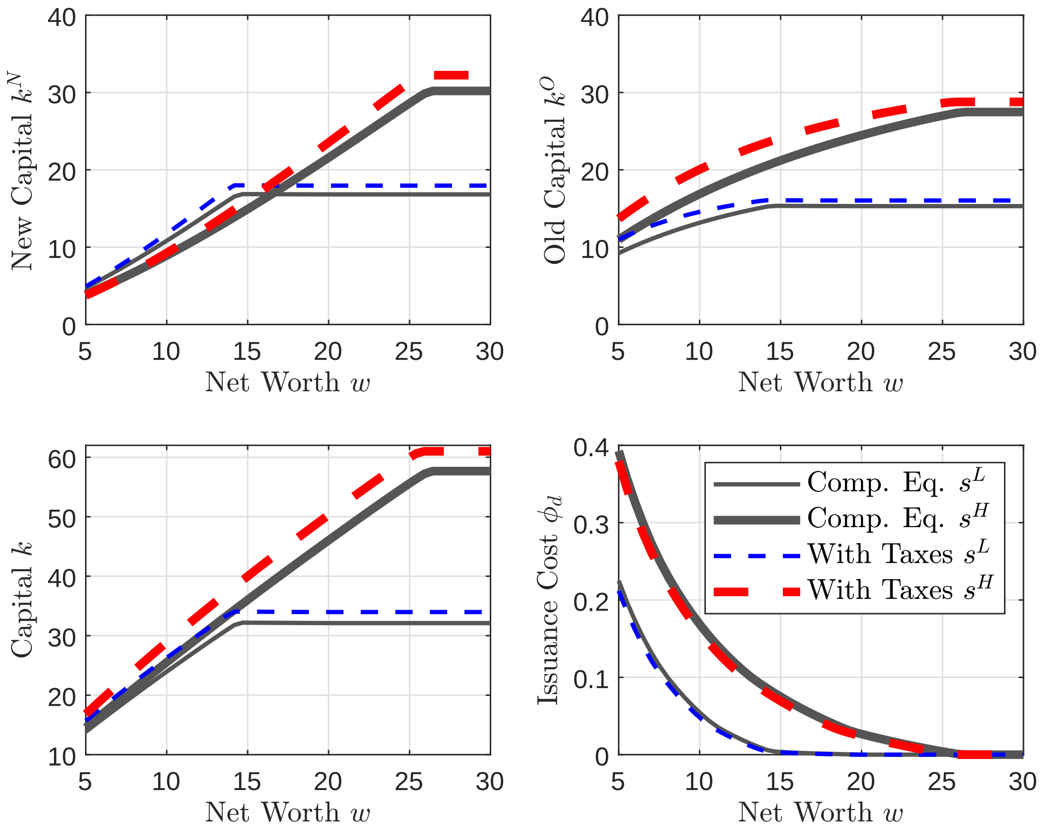

Figure C5: Uniform Taxes Without Lump-Sum Transfers. Top left: new capital k^N; top right: old capital k^O; bottom left: capital bundle k; bottom right: marginal cost of equity issuance ϕ_d. The x-axes report net worth w. Solid lines denote the stationary competitive equilibrium without policy interventions. Dashed lines denote the allocation with the following policy: $\tau^N = -0.03$ and τ^O balances to government budget constraint. Thick lines denote the high productivity state, thin lines the low state. See Table 1 for the parameter values.

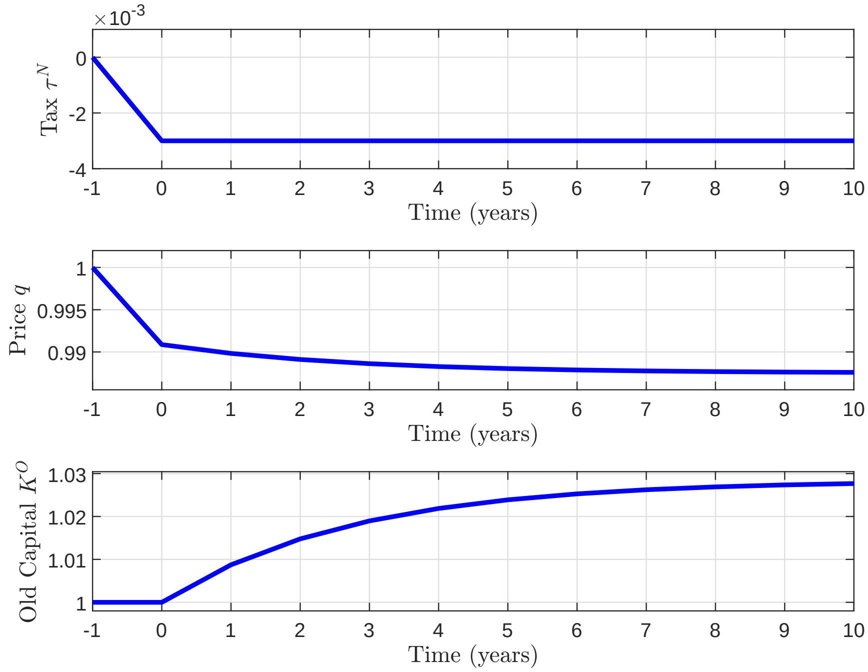

Figure C6: Equilibrium transition dynamics associated with the optimal constant tax rate τ^N, common for all firms. Top panel: tax rate τ^N; middle panel: price of old capital q_t; bottom panel: aggregate stock of old capital K_t^O. See Table 1 for the parameter values.

Table C1: Quantitative Results – Sensitivity Analysis

This table provides the sensitivity analysis of the quantitative results with respect the collateralizability θ (Panel A), elasticity of substitution ϵ (Panel B), and scrap value $\underline{q}$ (Panel C). Output, investment, consumption, and the price of used capital for the competitive equilibrium and constrained-efficient allocation are expressed as fractions of the corresponding first-best value, reported in parenthesis the first column of Panel A. See Table 1 for the baseline parameter values.

Panel A: Collateralizability θ

		$\theta = 0$		$\theta = 0.75$	
Variable	First Best	Comp. Eq.	Constr. Eff.	Comp. Eq.	Constr. Eff.
Output	(9.910)	0.808	0.949	0.949	0.985
Investment	(4.497)	0.736	0.929	0.925	0.978
Consumption	(5.413)	0.865	0.966	0.968	0.991
Price q	(0.547)	1.023	0.183	1.004	0.183
Average tax τ^N	0	0	-8.8%	0	-8.6%
Average tax τ^O	0	0	106.9%	0	102.9%

Panel B: Elasticity of Substitution ϵ

	$\epsilon = 1$		$\epsilon = 10$	
Variable	Comp. Eq.	Constr. Eff.	Comp. Eq.	Constr. Eff.
Output	0.894	0.944	0.905	0.985
Investment	0.850	0.919	0.864	0.978
Consumption	0.929	0.964	0.937	0.990
Price q	1.011	0.183	1.010	0.183
Average tax τ^N	0	-8.6%	0	-8.6%
Average tax τ^O	0	103.7%	0	103.3%

Panel C: Scrap Value $\underline{q}$

	$\underline{q} = 0.05$		$\underline{q} = 0.2$	
Variable	Comp. Eq.	Constr. Eff.	Comp. Eq.	Constr. Eff.
Output	0.899	0.979	0.899	0.959
Investment	0.857	0.969	0.857	0.942
Consumption	0.933	0.986	0.933	0.974
Price q	1.010	0.091	1.010	0.366
Average tax τ^N	0	-9.6%	0	-6.7%
Average tax τ^O	0	229.8%	0	40.5%