Online Appendix for "Learning from Manipulable Signals"

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Remark. Equations, claims and lemmas introduced in this Online Appendix are numbered as OA.#. Equations, claims, lemmas and theorems with regular numbering refer to those introduced in the printed manuscript.

I Omitted Proofs for Theorem 1

I.1 Proof of Lemma A1

Claim OA.1. *SP(a) is an open interval. That is, SP$(a)=(\underline{p},\bar{p})$.*

Proof. Clearly, SP(a) is a connected set because the sample path of X_t is almost surely continuous. Let $\underline{p}:=\inf \text{SP}(a)$, and $\bar{p}:=\sup \text{SP}(a)$. We need to show that $\bar{p},\underline{p}\notin \text{SP}(a)$. Suppose, toward a contradiction, that $\bar{p}\in \text{SP}(a)$. Then, consider a history that leads to the belief $\bar{p}$ and the continuation play starting from this history. Since the belief process is a martingale, we have $p_t=\bar{p}$ for all $t\leq \mathbb{T}$ and almost all sample paths. Agent optimality then implies $a(\bar{p})=0$, and thus the diffusion coefficient of the belief process at $\bar{p}$ is strictly positive. This contradicts $p_t=\bar{p}$ for all $t\leq \mathbb{T}$ and almost all sample paths. The same argument proves that $\underline{p}\notin \text{SP}(a)$. □

Claim OA.2. *The principal's equilibrium policy function b has a cutoff structure on SP(a). That is, there exists a unique $p^*\in[\underline{p},\bar{p}]$ such that $p\in(\underline{p},p^*)$ implies $b(p)=0$ and $p\in(p^*,\bar{p})$ implies $b(p)=1$.*

Proof. The proof is almost identical to that of Lemma A2, and is thus omitted. □

We continue with a technical claim that will be later used.

Claim OA.3. *Fix a positive integer T. For any $\varepsilon>0$ there exists $\eta>0$ satisfying the following property: Take any pair of adapted processes $dY_t^1=\mu_{1,t}dt+\sigma dB_t$ and $dY_t^2=\mu_{2,t}dt+\sigma dB_t$ such that $\mu_{j,t}\in[0,1]$ for $j=1,2$ and for every t. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be the probability distributions over $(C([0,T]),\mathbb{B}(C([0,T])))$[24] generated by such stochastic processes. If $A\in\mathbb{B}(C([0,T]))$ is such that $\mathbb{E}_{\mathbb{P}_1}[\mathbb{I}_A]<\eta$ then $\mathbb{E}_{\mathbb{P}_2}[\mathbb{I}_A]<\varepsilon$.*

Proof. The proof of this technical claim can be found in our working paper (Ekmekci et al., 2021). □

Claim OA.4. $\bar{p}=1$

Proof. Assume towards a contradiction that $\bar{p}<1$.
Case 1: $\bar{p}>p^*$.

The belief process p_t is a martingale, so for every $\varepsilon>0$ there exists an $\epsilon>0$ such that if $p_t>\bar{p}-\epsilon$, then $\mathbb{P}(\inf_{s>t}p_s>p^*+\varepsilon \mid \theta=NI)>1-\varepsilon$. This implies that $\mathbb{P}(\mathbb{T}=\mathbb{T}_\lambda \mid \theta=NI)>1-\varepsilon$ where $\mathbb{T}_\lambda$ is the arrival of the next Poisson-shock. Notice that for every $\eta>0$ we can take $\varepsilon_\eta>0$ such that the agent's payoff at p_t is no more than $(u+c)\left(\frac{r_1}{r_1+\lambda}\right)+\eta$. This implies that for every $\nu>0$ we can take η small enough (taking ε_η to satisfy the condition above) so that $\mathbb{E}\left(\int_t^{\min\{t+1,\mathbb{T}\}}(a(p_t))dt \mid \theta=NI\right)<\nu$. Hence, there exists $\varpi>0$ such that $\mathbb{E}\left(\int_t^{\min\{t+1,\mathbb{T}\}}(1-a(t))dt \mid \theta=NI\right)>\varpi$.

Consider the law of motion (A1) when $p_t\in[p^*,\bar{p}]$. Observe that the instantaneous variance of the belief process when the (noninvestible type) agent plays $a(\cdot)$ is bounded below by a positive constant

[24] $\mathbb{B}$ stands for the Borel sigma-field, and $C([0,T])$ is the set of continuous functions over $[0,T]$.

times $(1-a_t)^2\min\{p^*(1-p^*),\bar{p}(1-\bar{p})\}^2>0$. Because $\bar{p}<1$ and because $p_t-p_0=\int_0^t dp_t$, we obtain that $\mathbb{E}\left[\left|p_{\min\{t+1,\mathbb{T}\}}-p_t\right|^2\right]\geq\delta$ for some positive constant δ, hence $\mathbb{E}\left[\left|p_{\min\{t+1,\mathbb{T}\}}-p_t\right|\right]\geq\delta$. Because $\left(p_{\min\{t+1,\mathbb{T}\}}-p_t\right)$ has mean zero, we obtain that $\mathbb{E}\left[\left(p_{\min\{t+1,\mathbb{T}\}}-p_t\right)^+\right]\geq\frac{\delta}{2}$. Taking $\varepsilon<\frac{\delta}{4}$ we conclude that $\mathbb{P}\left(p_{\min\{t+1,\mathbb{T}\}}>\bar{p}+\frac{\varepsilon}{2}\right)>0$, which is a contradiction.

Case 2: $\bar{p}\leq p^*$.

Assume $\bar{p}\leq p^*$. Then, $b(p)=0$ for all $p\in\text{SP}(a)$. Claim OA.3 implies that, for every $T>0$, if the noninvestible agent plays $a_t=0$ for every $t\in[0,T]$, then the relationship terminates before T with probability zero, which implies that the agent's best response must satisfy $a_t=0$ for every $t>0$. This contradicts the assumption that $\bar{p}$ is never reached. □

Claim OA.5. $\underline{p}=0$.

Proof. The proof is analogous to that of Claim OA.4, and is thus omitted. □

Proof of Lemma A1. That $\text{SP}(a)=(0,1)$ follows directly from Claims OA.4 and OA.5. □

I.2 Translation Invariance

Lemma OA.1 (Translation Invariance of Agent's Problem)**.** *Fix an arbitrary strategy profile (α,β) and some $\epsilon\in\mathbb{R}$. Consider a new profile (α',β'), defined by $\alpha'_t:=\alpha_t\Big|\{Z_s-\epsilon\}_{s\leq t}$ and $\beta'_t:=\beta_t\Big|\{Z_s-\epsilon\}_{s\leq t}$. Then, the payoff of the noninvestible agent satisfies*

$$\mathbb{E}\{U_1(t,\alpha,\beta)|Z_0=z\}=\mathbb{E}\big\{U_1(t,\alpha',\beta')\big|Z_0=z+\epsilon\big\},$$

almost surely, for all $t\geq0$ and $z\in\mathbb{R}$.

Proof. The law of motion of the process $\{Z_t\}_{t\geq0}$ from the perspective of the noninvestible agent is:

$$dZ_t=\tfrac{1}{2}\psi^2(1-\alpha_t)^2dt+\psi(1-\alpha_t)dB_t.$$

This means that, if the principal perturbs her strategy using a constant displacement, the agent can maintain his distribution of payoffs intact by imitating the perturbation. □

Lemma OA.2 (Conditional Translation Invariance of Principal's Problem)**.** *Fix an arbitrary strategy profile (α,β) and some $\epsilon\in\mathbb{R}$. Consider a new profile (α',β'), defined by $\alpha'_t:=\alpha_t\Big|\{Z_s-\epsilon\}_{s\leq t}$ and $\beta'_t:=\beta_t\Big|\{Z_s-\epsilon\}_{s\leq t}$ for all $t\geq0$. Then, the conditional payoffs of the principal satisfy*

$$\mathbb{E}\{U_2(t,\alpha,\beta)|\theta=NI,Z_t=z\}=\mathbb{E}\big\{U_2(t,\alpha',\beta')\big|\theta=NI,Z_t=z+\epsilon\big\},$$

$$\mathbb{E}\{U_2(t,\alpha,\beta)|\theta=I,Z_t=z\}=\mathbb{E}\big\{U_2(t,\alpha',\beta')\big|\theta=I,Z_t=z+\epsilon\big\},$$

almost surely, for all $t\geq0$ and $z\in\mathbb{R}$.

Proof. In the case of conditioning on $\theta=NI$, the law of motion of $\{Z_t\}_{t\geq0}$ is as in the proof of Lemma OA.1. In the case of conditioning on $\theta=I$, the dynamics of $\{Z_t\}_{t\geq0}$ satisfies

$$dZ_t=-\tfrac{1}{2}\psi^2(1-\alpha_t)^2dt+\psi(1-\alpha_t)dB_t.$$

In both cases the dynamics are linear given α, so if the agent perturbs his strategy using a constant displacement in $z-$space, the principal can maintain her payoff distribution intact by imitating the perturbation. $\square$

I.3 Monotonicity and Curvature of Value Functions

Proof of Corollary A2. We first establish the claimed properties of $W(\cdot)$. From the law of motion of p_t given by (1), we know that the diffusion coefficient converges to 0 as $p\to 0$ or $p\to 1$. So it is easy to verify that $\lim_{p\to 0}W(p)=0$ and $\lim_{p\to 1}W(p)=\frac{\lambda}{r_2+\lambda}w_{NI}$. Also, since the principal can always ignore any information, $W(p)$ is bounded below by $\underline{W}(p)\equiv\frac{\lambda}{r_2+\lambda}\max\{0,R(p)\}$. The principal's HJB equation is

$$r_2W(p)=\max_{\tilde{b}\in[0,1]}\left\{\frac{1}{2}\psi^2[1-a(p)]^2\gamma(p)^2W''(p)+\lambda\tilde{b}[R(p)-W(p)]\right\}.$$

So we always have $W''(p)\geq 0$; that is, $W(\cdot)$ is convex on $(0,1)$. Since $\lim_{p\to 0}W(p)=0$ and $W(\cdot)\geq 0$, $W(\cdot)$ must be (weakly) increasing at 0, and because it is convex, $W(\cdot)$ is increasing on $(0,1)$.

Now we turn to $v(\cdot)$. Suppose first that $r_1\geq r^*$, so that in equilibrium $a(\cdot)\equiv 0$. From Claim A2 and conditions (A27) and (A29), $v(\cdot)$ is strictly decreasing and concave on $(-\infty,z^*)$, with $\lim_{z\to-\infty}v(z)=u+c$. From Claim A3 and conditions (A28) and (A30), $v(\cdot)$ is strictly decreasing and convex on (z^*,∞), with $\lim_{z\to\infty}v(z)=\frac{r_1}{r_1+\lambda}(u+c)$. Suppose now that $r_1<r^*$, so that in equilibrium $a(\cdot)$ is hump-shaped. In light of Claims A2 and A3 and conditions (A15), (A18), (A21) and (A24), it suffices to show that $v(\cdot)$ is strictly decreasing and concave on (z_L,z^*), and strictly decreasing and convex on (z^*,z_R). But these properties follow immediately from Claim A1 and the fact that $a(\cdot)$ is hump-shaped with $0<a(z)<1$ for $z\in(z_L,z_R)$. $\square$

II Omitted Proofs for Theorem 2

Proof of Lemma A7. Recall from the proof of Claim A8 that $r_1<r^*$ implies $v_R<v_L$. Then by Claim A10 and Corollary A4, we are done if we can find an $\lambda_2\geq\lambda_1$ and an $\underline{r}$ such that $\lambda>\lambda_2$ and $r_1<\underline{r}$ imply that $a^*_-(u;r_1)<a^*_+(u;r_1,\lambda)$. Using (A31) and (A32), we have

$$a^*_-(u;r_1)<a^*_+(u;r_1,\lambda)$$

$$\iff \frac{\frac{\sqrt{2}}{\psi}\phi(0)}{\frac{\sqrt{2r_1}}{\psi}\phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right)+\Phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right)-\Phi(0)}>\frac{\frac{1}{\sqrt{r_1}}\phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda\psi}{r_1c}u\right)}{\phi\left(\frac{v_R-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right)+\frac{\psi}{\sqrt{2(r_1+\lambda)}}\left[\Phi\left(\frac{v_R-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right)-\Phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda\psi}{r_1c}u\right)\right]} \tag{OA.1}$$

Since $\phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right)\leq\phi(0)$ and $\Phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right)\leq 1$, we can find a lower bound for the LHS of (OA.1) whenever $r_1<1$: $\frac{\frac{\sqrt{2}}{\psi}\phi(0)}{\frac{\sqrt{2}}{\psi}\phi(0)+1-\Phi(0)}$.

Now let us find an upper bound for the RHS of (OA.1). First, when $\lambda\geq 1$, we know

$$\frac{1}{\sqrt{r_1}}\phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda\psi}{r_1c}u\right)\leq\frac{1}{\sqrt{r_1}}\phi\left(\sqrt{\frac{2}{r_1+1}}\frac{\psi}{r_1c}u\right). \tag{OA.2}$$

Second, by direct calculation we have $\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}} = \frac{\sqrt{2}\psi}{4}\left(\frac{3}{\sqrt{\lambda+r_1}} + \sqrt{\frac{1}{r_1+\lambda}+\frac{8}{\psi^2}}\right)$, and when $\lambda \geq 1$, we have

$$\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}} \leq \frac{\sqrt{2}\psi}{4}\left(\frac{3}{\sqrt{1+0}} + \sqrt{\frac{1}{1+0}+\frac{8}{\psi^2}}\right) = \frac{\sqrt{2}}{4}\left(3\psi + \sqrt{\psi^2+8}\right).$$

Then,

$$\phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) \geq \phi\left(\frac{\sqrt{2}}{4}\left(3\psi + \sqrt{\psi^2+8}\right)\right). \tag{OA.3}$$

Third, $\frac{\psi}{\sqrt{2(r_1+\lambda)}}\left[\Phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) - \Phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda\psi}{r_1 c}u\right)\right] \geq -\frac{\psi}{\sqrt{2\lambda}}(1-\Phi(0))$. Apparently, there exists $\underline{\lambda} \geq \lambda_1$, such that for all $\lambda > \underline{\lambda}$, we have

$$\phi\left(\frac{\sqrt{2}}{4}\left(3\psi + \sqrt{\psi^2+8}\right)\right) - \frac{\psi}{\sqrt{2\lambda}}(1-\Phi(0)) \geq \frac{1}{2}\phi\left(\frac{\sqrt{2}}{4}\left(3\psi + \sqrt{\psi^2+8}\right)\right). \tag{OA.4}$$

Defining $\lambda_2 = \max\{1, \underline{\lambda}\}$ and applying conditions (OA.2), (OA.3) and (OA.4), we know that whenever $\lambda > \lambda_2$, we have the following upper bound for the RHS of (OA.1): $\frac{\frac{1}{\sqrt{r_1}}\phi\left(\sqrt{\frac{2}{r_1+1}}\frac{\psi}{r_1 c}u\right)}{\frac{1}{2}\phi\left(\frac{\sqrt{2}}{4}\left(3\psi+\sqrt{\psi^2+8}\right)\right)}$.

Now we compare the lower bound for the LHS of (OA.1) with the upper bound for the RHS of (OA.1). Taking limit $r_1 \to 0$, we have $\lim_{r_1 \to 0} \frac{\frac{1}{\sqrt{r_1}}\phi\left(\sqrt{\frac{2}{r_1+1}}\frac{\psi}{r_1 c}u\right)}{\frac{1}{2}\phi\left(\frac{\sqrt{2}}{4}\left(3\psi+\sqrt{\psi^2+8}\right)\right)} = 0 < \frac{\frac{\sqrt{2}}{\psi}\phi(0)}{\frac{\sqrt{2}}{\psi}\phi(0)+1-\Phi(0)}$. So there exists $\underline{r}' > 0$ such that this inequality holds for all $r_1 < \underline{r}'$.

Letting $\underline{r} = \min\{1, \underline{r}', r^*\}$, we have $a_-^\star(u; r_1) < a_+^\star(u; r_1, \lambda)$ whenever $\lambda > \lambda_2$ and $r_1 < \underline{r}$, as desired. □

Proof of Claim A11. Note that

$$a_+^*(u; r_1, \lambda) = 1 - \frac{1}{\phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) + \frac{\psi}{\sqrt{2(r_1+\lambda)}}\left[\Phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) - \Phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda\psi}{r_1 c}u\right)\right]}\frac{1}{\sqrt{2\pi}}e^{-\frac{\lambda^2}{\lambda+r_1}\frac{\psi^2}{r_1^2 c^2}u^2}. \tag{OA.5}$$

Note also that, for all $\lambda > \lambda_1$, $\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}} = \frac{\sqrt{2}\psi}{4}\left(\frac{3}{\sqrt{\lambda+r_1}} + \sqrt{\frac{1}{\lambda+r_1}+\frac{8}{\psi^2}}\right) \leq \frac{\sqrt{2}\psi}{4}\left(\frac{3}{\sqrt{\lambda_1}} + \sqrt{\frac{1}{\lambda_1}+\frac{8}{\psi^2}}\right)$. Thus for all $\lambda > \lambda_1$,

$$\begin{aligned} & \phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) + \frac{\psi}{\sqrt{2(r_1+\lambda)}}\left[\Phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) - \Phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda\psi}{r_1 c}u\right)\right] \\ \geq \quad & \phi\left(\frac{\sqrt{2}\psi}{4}\left(\frac{3}{\sqrt{\lambda_1}} + \sqrt{\frac{1}{\lambda_1}+\frac{8}{\psi^2}}\right)\right) - \frac{\psi}{\sqrt{2\lambda}}. \end{aligned} \tag{OA.6}$$

Let A' to be such that $\frac{1}{\sqrt{2\pi}A'} = \frac{1}{2}\phi\left(\frac{\sqrt{2}\psi}{4}\left(\frac{3}{\sqrt{\lambda_1}} + \sqrt{\frac{1}{\lambda_1}+\frac{8}{\psi^2}}\right)\right)$. Since $\lim_{\lambda\to\infty}\frac{\psi}{\sqrt{2\lambda}} = 0$, there must exist $\lambda_3' \geq \lambda_1$, such that for all $\lambda > \lambda_3'$,

$$\phi\left(\frac{\sqrt{2}\psi}{4}\left(\frac{3}{\sqrt{\lambda_1}} + \sqrt{\frac{1}{\lambda_1}+\frac{8}{\psi^2}}\right)\right) - \frac{\psi}{\sqrt{2\lambda}} > \frac{1}{\sqrt{2\pi}A'}. \tag{OA.7}$$

Finally, let $A = \max\{A', 2\}$. Conditions (OA.5), (OA.6) and (OA.7) then tell us that whenever $\lambda > \lambda_3'$, we have

$$a_+^*(u; r_1, \lambda) > 1 - A' e^{-\frac{\lambda^2}{\lambda + r_1} \frac{\psi^2}{r_1^2 c^2} u^2} \geq 1 - A e^{-\frac{\lambda^2}{\lambda + r_1} \frac{\psi^2}{r_1^2 c^2} u^2}. \qquad \square$$

III Omitted Proofs for Theorem 3

Proof of Claim A17. The proof is almost identical to Lemma OA.5's proof, and is thus omitted. □

IV Patient Limit: Toward a Proof of Theorem 4

For each $n \in \mathbb{N}$, take the unique Markov equilibrium (a_n, b_n) associated with the discount factor $r_{i,n}$ for $i = 1, 2$. Assume that $\lim_{n \to \infty} r_{i,n} = 0$ and $\lim_{n \to \infty} (r_{2,n}/r_{1,n}) := \chi \in (0, \infty)$. Let $V_n(\cdot)$ be the agent's value function in the equilibrium (a_n, b_n) and $W_n(\cdot)$ be the principal's value function. We will often use $z \equiv \log(p/1-p)$ as state variable when analyzing the agent's behavior. When doing so, we denote by $v_n(z) := V_n(p(z))$ the agent's value function in the $z-$space. Write z_n^* for the principal's equilibrium cutoff. Write $z_{L,n}$ for the infimum belief z at which the agent plays $a_n(z) > 0$ and write $z_{R,n}$ for the supremum. Write $\mathbb{T}$ for the equilibrium stopping time that stops the play of the game. *Without labeling explicitly, we note that the distribution of $\mathbb{T}$ depends on n and the current state z.* For $i = 1, 2$ and $\theta \in \{NI, I\}$, let $\mathbb{E}_n^\theta\{e^{-r_{i,n}\mathbb{T}}\}$ be the expected discount factor when the stopping action is taken in the equilibrium (a_n, b_n) discounted at rate $r_{i,n}$. When the game starts at state z, let

$$\mathbb{E}_n\left\{e^{-r_{i,n}\mathbb{T}}\right\} := p(z)\mathbb{E}_n^{NI}\left\{e^{-r_{i,n}\mathbb{T}}\right\} + (1 - p(z))\mathbb{E}_n^I\left\{e^{-r_{i,n}\mathbb{T}}\right\}.$$

Claim OA.6. *Take $(r_1, r_2) \in \mathbb{R}_{++}^2$ and let τ be any stopping time. Assume that $\mathbb{E}\{e^{-r_1\tau}\} = \xi \in (0, 1)$.*

i) If $r_2 \leq r_1$ then

$$\xi \leq \mathbb{E}\left\{e^{-r_2\tau}\right\} \leq \xi^{(r_2/r_1)}.$$

ii) If $r_2 > r_1$ then

$$\xi^{(r_2/r_1)} \leq \mathbb{E}\left\{e^{-r_2\tau}\right\} \leq \xi.$$

Moreover, for each $\xi \in (0, 1)$ and any inequality above, there exists a distribution over stopping times for which this inequality is tight.

Proof. The proof of this technical claim can be found in our working paper (Ekmekci et al., 2021). □

Claim OA.7. *For every $\varepsilon > 0$ there exists $z^\dagger \in \mathbb{R}$ and $\tilde{n}_1 \in \mathbb{N}$ such that, if $z \geq z^\dagger$ and $n \geq \tilde{n}_1$, then the continuation payoff of the agent at z is less than ε in the equilibrium (a_n, b_n).*

Proof. Otherwise we can find a sequence of equilibria (a_n, b_n) starting at $(z_n) \to +\infty$ in which the agent obtains a payoff weakly greater than ε. Since the agent's equilibrium payoff is bounded above by $(u + c)\left[1 - \mathbb{E}_n^{NI}\left(e^{-r_{1,n}\mathbb{T}}\right)\right]$, we have $\mathbb{E}_n^{NI}\left(e^{-r_{1,n}\mathbb{T}}\right) \leq \left(1 - \frac{\varepsilon}{u+c}\right)$, and hence the principal's payoff in equilibrium (a_n, b_n) at z_n is at most $\max\left\{\left(1 - \frac{\varepsilon}{u+c}\right), \left(1 - \frac{\varepsilon}{u+c}\right)^{(r_{2,n}/r_{1,n})}\right\} p(z_n) w_{NI}$, which is always strictly less than w_{NI}. Meanwhile, since $r_{2,n} \to 0$ and $z_n \to \infty$ as $n \to \infty$, the principal's payoff at z_n by terminating the relationship in the first opportunity satisfies $\lim_{n \to \infty}\left(\frac{\lambda}{r_{2,n} + \lambda}\right)[(1 - p(z_n))w_I + p(z_n)w_{NI}] = w_{NI}$. So the principal has a profitable deviation when n is sufficiently large. □

We assume that $n \geq \tilde{n}_1$ for the remainder of this proof.

Claim OA.8. *For every fixed* z_0*, we have* $\limsup_{n\to\infty} v_n(z_0) \leq u$.

Proof. Take any small $\varepsilon \in (0,u/2)$. For each $n \in \mathbb{N}$, let $z_\varepsilon^n := \inf\{z | a_n(z) = 1-\varepsilon\}$. There are two cases to consider. Let $z^\dagger$ be defined and delivered by Claim OA.7. Every sequence can be split into (at most) two subsequence, each one of them satisfying one of the cases below.

Case 1 $z_\varepsilon^n \leq z^\dagger$ for every $n \in \mathbb{N}$.

In this case, take $m \in \mathbb{N}$ such that $z^\dagger - m < z_0$ and let $z_0^n := z_\varepsilon^n - m$. Since $v_n(\cdot)$ is decreasing, it suffices to show that $\limsup_{n\to\infty} v_n(z_0^n) \leq u$.

Take any $\zeta > 0$. Suppose that the game starts at z_0^n and consider the stopping time $\hat{\mathbb{T}}_n$ that stops the play of the game at the first time $Z_n(t) = z_\varepsilon^n$ (setting $\hat{\mathbb{T}}_n = +\infty$ if this event does not happen in finite time). Note that $Z_n(t)$ is a submartingale under the strategy of the noninvestible type and that $a_n(z) \leq 1-\varepsilon$ with probability one before $\hat{\mathbb{T}}_n$. Using this observation and $Z_n(t)$'s law of motion (A2), it is straightforward to show that $\hat{\mathbb{T}}_n < +\infty$ with probability one under the strategy of the noninvestible type and that $\mathbb{E}_n^{NI}\left[e^{-r_{1,n}\hat{\mathbb{T}}_n}\right] \to 1$. Take $n^{**} \in \mathbb{N}$ for which $n > n^{**}$ implies $\mathbb{E}_n^{NI}\left[e^{-r_{1,n}\hat{\mathbb{T}}_n}\right] > 1-\varepsilon$. Next notice that, at the state z_ε^n, v_n is decreasing and concave (by Corollary A2), and hence

$$\begin{aligned} r_{1,n} v_n(z_\varepsilon^n) &= r_{1,n}[u + (1-a_n(z_\varepsilon^n))c] + \frac{1}{2}\psi^2[1-a_n(z_\varepsilon^n)]^2\left[v'(a_n(z_\varepsilon^n)) + v''(a_n(z_\varepsilon^n))\right] \\ &\leq r_{1,n}[u + (1-a_n(z_\varepsilon^n))c], \end{aligned}$$

which implies $v_n(z_n^\varepsilon) \leq u + \varepsilon c$, because $a_n(z_n^\varepsilon) = 1-\varepsilon$. It follows that the payoff of the noninvestible type converges to a number not greater than $(1-\varepsilon)(u+\varepsilon c) + \varepsilon(u+c)$, which proves the result as ε is arbitrary.

Case 2 $z_\varepsilon^n > z^\dagger$ for every $n \in \mathbb{N}$.

We may assume that $z_0 < z^\dagger$ for every n as otherwise the claim follows from Claim OA.7. Suppose that the game starts at z_0 and consider the stopping time $\hat{\mathbb{T}}_n$ that stops the play of the game at $z^\dagger$. As in Case 1, we have $\hat{\mathbb{T}}_n < +\infty$ with probability one under the noninvestible-type's strategy and $\mathbb{E}_n^{NI}\left\{e^{-r_{1,n}\hat{\mathbb{T}}_n}\right\} \to 1$. Since $\limsup_{n\to\infty} v_n(z^\dagger) \leq \varepsilon < u/2$, the rest of the proof follows the same argument as in Case 1. ☐

Claim OA.9. $\lim_{n\to\infty} z_{L,n} = -\infty$.

Proof. The proof follows verbatim from Claim A14's proof. ☐

Lemma OA.3. *For every* $z_0 < \liminf z_n^*$*, we have* $\lim_{n\to\infty} a_n(z_0) = 1$.

Proof. By Claim OA.9, $z_0 \in (z_{L,n}, z_n^*)$ for n sufficiently large. Then from condition (A11), we know that $a_n(\cdot)$ eventually satisfies the following differential equation

$$a_n'(z) = 1 - a_n(z) - 2\left(\tfrac{v_n(z)-u}{c}\right). \quad \text{(OA.8)}$$

Assume toward a contradiction that we can find a subsequence such that $\lim_{n\to\infty} a_n(z_0) = \bar{a} < 1$. Take $m \in \mathbb{N}$ such that $\frac{1-\bar{a}}{4} m > 2$. Claim OA.9 implies that $[z_0 - m, z_0] \subset (z_{L,n}, z_n^*)$ for n sufficiently large. Claim OA.8 and the monotonicity of $v_n(\cdot)$ imply that we can find $n^\dagger \in \mathbb{N}$ such that for every $n \geq n^\dagger$, for every $z \in [z_0 - m, z_0]$, we have $2\left(\frac{v_n(z)-u}{c}\right) < \frac{1-\bar{a}}{4}$. Given the contradiction assumption, we can find $n^\dagger \in \mathbb{N}$ such that for every $n \geq n^{\dagger}$ we have $a_n(z_0) < \frac{1+\bar{a}}{2}$. Since $a_n(\cdot)$ is strictly increasing on $[z_0 - m, z_0]$, this implies $a_n(z) < \frac{1+\bar{a}}{2}$ for all $z \in [z_0 - m, z_0]$. So (OA.8) implies $a_n'(z) > \frac{1-\bar{a}}{4}$ for all $z \in [z_0 - m, z_0]$ and hence

$$a_n(z_0 - m) < a_n(z_0) - \tfrac{1-\bar{a}}{4} m < a_n(z_0) - 2 < 0,$$

which leads to a contradiction as a_n is bounded below by 0. □

Lemma OA.4. *For every* $z_0 < \liminf_{n\to\infty} z_n^*$, *we have* $\lim_{n\to\infty} v_n(z_0) = u$.

Proof. By Claim OA.9, $z_0 \in (z_{L,n}, z_n^*)$ for n sufficiently large. Take $\vartheta > 0$ such that $z_0 + 2\vartheta < \liminf z_n^*$ and, taking a subsequence if necessary, assume that $z_0 + \vartheta < z_n^*$ for each one of its elements.

Assume toward a contradiction, taking a subsequence if necessary, that $\lim_{n\to\infty} v_n(z_0) < u - \varepsilon$, for some $\varepsilon > 0$. Because $v_n(\cdot)$ is strictly decreasing, we may take n^* such that $n \geq n^*$ implies $v_n(z) < u - \frac{\varepsilon}{2}$ for all $z \in \left[z_0, z_0 + \frac{\vartheta}{2}\right]$. In this case, we have $a_n'(z) = 1 - a_n(z) - 2\left(\frac{v_n(z) - u}{c}\right) \geq \frac{\varepsilon}{c}$ for every $z \in \left[z_0, z_0 + \frac{\vartheta}{2}\right]$. This implies that $\limsup_n a_n(z_0) \leq 1 - \left(\frac{\vartheta}{2}\right)\frac{\varepsilon}{c}$, contradicting Lemma OA.3. □

Lemma OA.5. *Fix a prior* $p_0 \in (0,1)$ *and some* $\bar{p} \in (p_0, 1)$. *For each* $r > 0$, *consider an adapted Markov function* $\alpha_r(\cdot)$ *and a belief process defined by substituting* $\alpha_r(\cdot)$ *into* (A1). *Take* $\varepsilon > 0$ *and let* $\bar{\mathbb{T}}$ *be the random time that stops the play in the first time that* $p \geq \bar{p}$. *Then we have:*

$$\limsup_{r\downarrow 0} \mathbb{E}^{NI}\left\{ r\int_0^{\bar{\mathbb{T}}} e^{-rt} \mathbb{I}_{\{\alpha_r(p_t) \leq 1-\varepsilon\}} dt \right\} = 0.$$

Proof. Take a small $\epsilon > 0$. Next, take $\zeta > 0$ and let $\mathbb{T}^\zeta$ be the stopping time that stops the play in the first time that the posterior reaches $(\zeta, \bar{p})^c$. Using the martingale property of beliefs whose law of motion is given by (A1), it is straightforward to show that we can take ζ small enough so that $\mathbb{P}^{NI}\{\mathbb{T}^\zeta < \infty, p(\mathbb{T}^\zeta) = \zeta\} < \frac{\epsilon}{2}$. Therefore, we have:

$$\mathbb{E}^{NI}\left\{ r\int_0^{\bar{\mathbb{T}}} e^{-rt} \mathbb{I}_{\{\alpha_r(p_t) \leq 1-\varepsilon\}} dt \right\} \leq \frac{\epsilon}{2} + \mathbb{E}^{NI}\left\{ r\int_0^{\mathbb{T}^\zeta} \mathbb{I}_{\{\alpha_r(p_t) \leq 1-\varepsilon\}} dt \right\}.$$

We must then show that $\limsup_{r\downarrow 0} \mathbb{E}^{NI}\left\{ r\int_0^{\mathbb{T}^\zeta} \mathbb{I}_{\{\alpha_r(p_t)\leq 1-\varepsilon\}} dt \right\} < \frac{\epsilon}{2}$. Let $\xi_r(t)$ be a function that is 1 whenever $a_r(p_t) \leq 1 - \varepsilon$ and 0 otherwise. It suffices to show that $\limsup_{r\downarrow 0} r\mathbb{E}^{NI}\left\{ \int_0^{\mathbb{T}^\zeta} \xi_r(t) dt \right\} < \frac{\epsilon}{2}$.

For that we will consider a different stopping time $\mathbb{T}^*$ and a new process $\xi_r^*(t)$ which are built from $\mathbb{T}^\zeta$ and $\xi_r(t)$ in the following way. Whenever $\mathbb{T}^\zeta < \infty$ and $\int_0^{\mathbb{T}^\zeta} \xi_r(t) dt \in (m-1, m)$ for some $m \in \mathbb{N}$, we will set

$$\xi_r^*(t) := \begin{cases} \xi_r(t) & t \leq \mathbb{T}^\zeta, \\ 1 & t > \mathbb{T}^\zeta. \end{cases}$$

We will also set $\mathbb{T}^* := \mathbb{T}^\zeta + \tilde{t}$, where $\tilde{t}$ is defined by $\int_0^{\mathbb{T}^\zeta} \xi_r(t) dt + \tilde{t} = m$. Whenever $\mathbb{T}^\zeta < +\infty$ and $\int_0^{\mathbb{T}^\zeta} \xi_r(t) dt = m - 1$ for some $m \in \mathbb{N}$, we set $\xi_r^*(t) := \xi_r(t)$ and $\mathbb{T}^* := \mathbb{T}^\zeta$. Clearly it suffices to show that $\limsup_{r\downarrow 0} \mathbb{E}^{NI}\left\{ r\int_0^{\mathbb{T}^*} \xi_r^*(t) dt \right\} < \frac{\epsilon}{2}$. Next, we build a family of stochastic processes $\left\{\xi_{r,m}^*(t)\right\}_{m\in\mathbb{N}}$ from

$\xi_r^*(t)$ by setting

$$\xi_{r,m}^*(t) := \begin{cases} \xi_r^*(t) & \int_0^t \xi^*(t)dt \in (m-1,m], \\ 0 & \text{otherwise.} \end{cases}$$

This immediately implies that $r\mathbb{E}^{NI}\left\{\int_0^{\mathbb{T}^*} \xi_r^*(t)dt\right\} = r\sum_{m=1}^{\infty}\mathbb{E}^{NI}\left\{\int_0^{\mathbb{T}^*} \xi_{r,m}^*(t)dt\right\}$.

Next, observe that, conditional on $\theta = NI$, p_t is a bounded submartingale. Thus, for any adapted function $\tilde{\xi}(t) \in \{0,1\}$ and any stopping time $\tilde{\mathbb{T}}$, we have

$$1 \geq p_{\mathbb{T}} - p_0 = \mathbb{E}^{NI}\left[\int_0^{\mathbb{T}} dp_t\right] = \mathbb{E}^{NI}\left[\int_0^{\mathbb{T}} \tilde{\xi}(t)dp_t\right] + \mathbb{E}^{NI}\left[\int_0^{\mathbb{T}} \left(1-\tilde{\xi}(t)\right)dp_t\right] \geq \mathbb{E}^{NI}\left[\int_0^{\mathbb{T}} \tilde{\xi}(t)dp_t\right]$$

because $\left(1-\tilde{\xi}(t)\right)$ being an adapted process and p_t being a submartingale jointly imply that $\mathbb{E}^{NI}\left[\int_0^{\mathbb{T}} \left(1-\tilde{\xi}(t)\right)dp_t\right] \geq 0$. As a result, we have

$$1 \geq \mathbb{E}^{NI}\left[\int_0^{\mathbb{T}} \xi_r^*(t)dp_t\right] = \sum_{m=1}^{\infty}\mathbb{E}^{NI}\left[\int_0^{\mathbb{T}} \xi_{r,m}^*(t)dp_t\right]. \quad \text{(OA.9)}$$

Next, since $0 < \zeta < \bar{p} < 1$, from condition (A1) it is straightforward to show that there exists a positive constant $\vartheta > 0$ such that, for any $m \in \mathbb{N}$, we have

$$\mathbb{E}^{NI}\left\{\int_0^{\mathbb{T}^*} \xi_{r,m}^*(t)dp_t\right\} \geq \vartheta\mathbb{E}^{NI}\left\{\int_0^{\mathbb{T}^*} \xi_{r,m}^*(t)dt\right\}. \quad \text{(OA.10)}$$

Therefore, combining (OA.9) and (OA.10) we have

$$\sum_{m=1}^{\infty}\mathbb{E}^{NI}\left\{\int_0^{\mathbb{T}^*} \xi_{r,m}^*(t)dt\right\} \leq \frac{1}{\vartheta}\sum_{m=1}^{\infty}\mathbb{E}^{NI}\left\{\int_0^{\mathbb{T}^*} \xi_{r,m}^*(t)dp_t\right\} \leq \frac{1}{\vartheta},$$

implying that $\sum_{m=1}^{\infty} r\mathbb{E}^{NI}\left\{\int_0^{\mathbb{T}^*} \xi_{r,m}^*(t)dt\right\} \leq \frac{r}{\vartheta}$, which is smaller than $\frac{\epsilon}{2}$ when r is sufficiently small. □

Lemma OA.6. $\lim_{n\to\infty} z_n^* = z^{**}$.

Proof. Suppose toward a contradiction that we can find a subsequence for which $\lim_{n\to\infty} z_n^* := \bar{z} > z^{**}$. Let z^m be the midpoint between $\bar{z}$ and z^{**}. Take $\varepsilon > 0$. Consider the game starting at z^m. Notice that Lemma OA.4 implies that $\lim_{n\to\infty} v_n(z^m) = u$, while Lemma OA.5 implies that, for each $\nu > 0$, we have

$$\lim_{n\to\infty}\mathbb{E}_n^{NI}\left\{r_{1,n}\int_0^{\mathbb{T}} e^{-r_{1,n}t}\mathbb{I}_{\{a_n(z_t)\leq 1-\nu\}}dt\right\} = 0.$$

These two observations imply that $\lim_{n\to\infty}\mathbb{E}_n^{NI}\left(e^{-r_{1,n}\mathbb{T}}\right) = 0$, which, by the same argument as Claim OA.6's proof, implies that $\lim_{n\to\infty}\mathbb{E}_n^{NI}\left(e^{-r_{2,n}\mathbb{T}}\right) = 0$; that is, conditional on $\theta = NI$, the principal derives zero discounted payoff from the game. It follows that the principal obtains a limit payoff bounded above by zero at z^m. But then, for n sufficiently large, if the stopping opportunity arrives at $z = z^m$, the principal can profitably deviate by stopping the game to obtain $p(z^m)w_{NI} + (1-p(z^m))w_I > 0$, a contradiction. □

Lemma OA.7. *For every $z_0 > z^{**}$ and $i = 1,2$, we have* $\lim_{n\to\infty}\mathbb{E}_n\left\{e^{-r_{i,n}\mathbb{T}}\right\} = 1$.

Proof. Fix $z_0 > z^{**}$. By Claim OA.6, it suffices to show that $\lim_{n\to\infty}\mathbb{E}_n\left\{e^{-r_{2,n}\mathbb{T}}\right\} = 1$. Taking a subsequence if necessary, assume toward a contradiction that $\lim_{n\to\infty}\mathbb{E}_n\left\{e^{-r_{2,n}\mathbb{T}}\right\} < 1$.

Let $\tilde{\tau}$ be the stopping time that stops the play in the first time that either the state reaches $[0, p(z_n^*)]$ or when $\mathbb{T}$ happens. Let $x = e^{-r_{2,n}t}$. Let Q_n be the distribution of $p_{\tilde{\tau}}$ and $H_n(\cdot|p_{\tilde{\tau}})$ be the conditional distribution of x given $p_{\tilde{\tau}}$.

Step 1. We show that the contradiction assumption implies that, the discounted amount of time that the relationship continues with beliefs close to $p(z_n^*)$ is nonnegligible (i.e., condition (OA.13) holds).

Note that

$$W_n(p(z_0)) = \int_{p(z_n^*)}^1 \int_0^1 x\left[\mathbb{I}_{\{p_{\tilde{\tau}} > p(z_n^*)\}} R(p_{\tilde{\tau}}) + \mathbb{I}_{\{p_{\tilde{\tau}} \le p(z_n^*)\}} W_n(p(z_n^*))\right] H_n(dx|p_{\tilde{\tau}}) dQ_n(dp_{\tilde{\tau}}).$$

Because $\lim_{n\to\infty} W_n(p(z_n^*)) = \lim_{n\to\infty} R(p(z_n^*)) = 0$, we have

$$\limsup_{n\to\infty} W_n(p(z_0)) = \limsup_{n\to\infty} \int_{p(z_n^*)}^1 \int_0^1 x R(p_{\tilde{\tau}}) H_n(dx|p_{\tilde{\tau}}) Q_n(dp_{\tilde{\tau}}). \tag{OA.11}$$

Moreover, since $R(p^{**}) = 0$ and $p(z_n^*) \to p^{**}$ (by Lemma OA.6), for every $\varepsilon > 0$ there exists $\zeta > 0$ such that when n is sufficiently large, $R(p) > \zeta$ for every $p > p(z_n^*) + \varepsilon$. Combining this observation with condition (OA.11), it is easy to show that, for every $\varepsilon > 0$, if

$$\limsup_{n\to\infty} \int_{p(z_n^*)+\varepsilon}^1 \int_0^1 (1-x) H_n(dx|p_{\tilde{\tau}}) Q_n(dp_{\tilde{\tau}}) > 0,$$

then we would have $\limsup_{n\to\infty} W_n(p(z_0)) < R(p(z_0))$, which contradicts $b_n(z_0) = 1$ (principal optimality) when n is sufficiently large. Hence, for every $\varepsilon > 0$, we have

$$\limsup_{n\to\infty} \int_{p(z_n^*)+\varepsilon}^1 \int_0^1 (1-x) H_n(dx|p_{\tilde{\tau}}) Q_n(dp_{\tilde{\tau}}) = 0. \tag{OA.12}$$

Therefore, the assumption that $\lim_{n\to\infty}\mathbb{E}_n\left(e^{-r_{2,n}\mathbb{T}}\right) < 1$ implies that, for every $\varepsilon > 0$, we have

$$\limsup_{n\to\infty} \int_{p(z_n^*)}^{p(z_n^*)+\varepsilon} \int_0^1 (1-x) H_n(dx|p_{\tilde{\tau}}) Q_n(dp_{\tilde{\tau}}) > 0. \tag{OA.13}$$

For the remainder of this proof, we take $\varepsilon > 0$ such that $p(z_n^*) + \varepsilon < \left(\frac{z_0 + z^{**}}{2}\right)$.

Step 2. We show that condition (OA.13) implies that, the noninvestible type has a profitable deviation by fully mimicking the investible type.

Lemma OA.5 implies that if we let $\bar{\mathbb{T}}_m$ be the random time that stops the play in the first time that the posterior leaves $\left(m^{-1}, 1-m^{-1}\right)$ or that $\mathbb{T}$ happens, then for each $\upsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{E}_n^{NI}\left\{ r_{1,n} \int_0^{\bar{\mathbb{T}}_m} e^{-r_{1,n}t} \mathbb{I}_{\{a_n(p_t) \le 1-\upsilon\}} dt \right\} = 0.$$

By the martingale property of beliefs we can take $m \in \mathbb{N}$ large enough to make $\limsup_{n\to\infty} \mathbb{P}^{NI}\left\{\inf_{t\le\mathbb{T}} p_t \le m^{-1}\right\}$ as small as we want. Analogously, we can take m large enough to guarantee that whenever the posterior starts at $(1-m^{-1},1)$ then $\limsup_{n\to\infty}\mathbb{P}^{NI}\{\inf_{t\le\mathbb{T}}p_t\le p(z_n^*)+\varepsilon\}$ is as small as we want. These two observations then imply that

$$\limsup_{n\to\infty}\mathbb{E}_n^{NI}\left\{r_{1,n}\int_0^{\mathbb{T}}e^{-r_{1,n}t}(1-a_n(p_t))dt\right\}=0. \tag{OA.14}$$

Next, let $y=e^{-r_{1,n}t}$. For $\theta\in\{NI,I\}$, let Q_n^θ stand for the distribution of $p_{\mathbb{T}}$ (not $p_{\tilde{\tau}}$ as above) given the strategy of type θ and let $H_n^\theta(\cdot\,|\,p_{\mathbb{T}})$ stand for the conditional distribution of y given $p_{\mathbb{T}}$ and the strategy of type θ. On the one hand, using (OA.12) and (OA.14), it is straightforward to see that, taking a subsequence if necessary, the limit payoff of the noninvestible type from following his equilibrium strategy is given by:

$$\lim_{n\to\infty}u\int_0^{p(z_n^*)+\varepsilon}\int_0^1(1-y)H_n^{NI}(dy\,|\,p_{\mathbb{T}})Q_n^{NI}(dp_{\mathbb{T}})>0, \tag{OA.15}$$

where the positive sign follows from (OA.13). On the other hand, the limit payoff of the noninvestible type from following the strategy of the investible type (i.e., always boosting performance with probability 1) is given by:

$$\lim_{n\to\infty}u\int_0^{p(z_n^*)+\varepsilon}\int_0^1(1-y)H_n^{I}(dy\,|\,p_{\mathbb{T}})Q_n^{I}(dp_{\mathbb{T}})>0. \tag{OA.16}$$

Next, a straightforward application of Bayes rule implies that $H_n^{NI}(\cdot\,|\,p_{\mathbb{T}})=H_n^I(\cdot\,|\,p_{\mathbb{T}})$ for every $p_{\mathbb{T}}\in(0,1)$. Moreover, using $p(z_n^*)+\varepsilon<\left(\frac{z_0+z^{**}}{2}\right)$ and Bayes rule, one can find $\xi>1$ such that $Q_n^I(A)\ge\xi Q_n^{NI}(A)$ for every (Borel-measurable) $A\subset[0,p(z_n^*)+\varepsilon]$. Hence, subtracting (OA.15) from (OA.16) we obtain an expression at least as large as

$$\lim_{n\to\infty}(\xi-1)u\int_0^{p(z_n^*)+\varepsilon}\int_0^1(1-y)H_n^{NI}(dy\,|\,p_{\mathbb{T}})Q_n^{NI}(dp_{\mathbb{T}})>0.$$

This implies that the noninvestible type can profitably deviate by fully mimicking, which leads to a contradiction and concludes the proof. □

Proof of Theorem 4. First, for the agent, Lemmas OA.4 and OA.6 tell us that $\lim_{n\to\infty}v_n(z)=u$ for all $z<z^{**}$, and Lemma OA.7 implies that $\lim_{n\to\infty}v_n(z)=0$ for all $z>z^{**}$.

Next, for the principal, we first argue that $W_n(\cdot)$ converges pointwise to $\max\{0,R(\cdot)\}$. In light of Corollary A2, we continuously extend $W_n(\cdot)$ from $(0,1)$ to $[0,1]$ by setting $W_n(0)=0$ and $W_n(1)=\frac{\lambda}{r_{2,n}+\lambda}w_{NI}$. Lemma OA.7 implies that $\lim_{n\to\infty}W_n(p(z))=R(p(z))$ for all $z>z^{**}$. We now show that $\lim_{n\to\infty}W_n(p(z))=0$ for all $z\le z^{**}$. To see this, fix any $z\le z^{**}$ and take any $\varepsilon>0$. Since $R(p(z^{**}))=0$, there exists $\delta>0$ such that $R(p(z^{**})+\delta)<\frac{\varepsilon}{2}$. But then, we can find n^* such that for every $n>n^*$, $W_n(p(z^{**})+\delta)<\varepsilon$. Since $W_n(\cdot)$ is increasing, it follows that $W_n(p(z))<\varepsilon$ for every $n>n^*$. So we must have $\lim_{n\to\infty}W_n(p(z))=0$, because ε is arbitrary and $W_n(\cdot)$ is bounded below by 0.

To show uniform convergence, note that for any fixed n, $W_n(\cdot)$ is bounded below by $\frac{\lambda}{r_{2,n}+\lambda}\max\{0,R(\cdot)\}$ such that $W_n(1)=\frac{\lambda}{r_{2,n}+\lambda}R(1)$. Because $W_n(\cdot)$ is convex and increasing, $|W_n'(\cdot)|$ is bounded above by $(w_{NI}-w_I)$, and hence $\{W_n\}_n$ is uniformly equicontinuous. Since W_n converges pointwise to $\max\{0,R\}$, invoking Arzelà–Ascoli theorem we conclude that the convergence is uniform. □