Online Appendix
Optimal Procurement With Quality Concerns

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A Proofs for Section III

A.1 Proof of Theorem 1

For any pair p_L and p_H such that $c_L \leq p_L \leq p_H \leq c_H$, consider the LoLA with threshold prices p_L and p_H. Sincere bidding in the LoLA induces the following outcome:

$$q_i^{\mathrm{L}}(c_i, c_{-i}; p_L, p_H) \equiv \begin{cases} 1 & \text{if } \; p_L \leq c_i < c_{-i}^{(1)} \\ 1 & \text{if } c_i \leq p_L < c_{-i}^{(1)} \\ \frac{1}{\kappa+1} & \text{if } \max\left\{c_{-i}^{(\kappa)}, c_i\right\} \leq p_L < c_{-i}^{(\kappa+1)} \\ 0 & \text{else} \end{cases} \tag{21}$$

and

$$m_i^{\mathrm{L}}(c_i, c_{-i}; p_L, p_H) \equiv \begin{cases} c_{-i}^{(1)} & \text{if } \; p_L \leq c_i < c_{-i}^{(1)} \\ c_{-i}^{(1)} & \text{if } c_i \leq p_L < c_{-i}^{(1)} \\ \frac{1}{\kappa+1} \cdot p_L & \text{if } \max\left\{c_{-i}^{(\kappa)}, c_i\right\} \leq p_L < c_{-i}^{(\kappa+1)} \\ 0 & \text{else,} \end{cases} \tag{22}$$

where $c_{-i}^{(\kappa)}$ denotes the κ-th lowest cost among all supplier i's opponents. For expositional simplicity, events where two or more bidders have the same cost are ignored in (21, 22) because they happen with probability zero.

The functions $(q^{\mathrm{L}}, m^{\mathrm{L}})$ may also be interpreted as a direct revelation mechanism. We now show that, in this direct revelation mechanism, truthful reporting is a (weakly) dominant strategy.

Lemma 1. *$(q^L,\ m^L)$ satisfies, $\forall i = 1, ..., N$,*

$$\forall c_i, c_i', c_{-i}, \qquad m_i(c_i, c_{-i}) - c_i \cdot q_i(c_i, c_{-i}) \geq m_i(c_i', c_{-i}) - c_i \cdot q_i(c_i', c_{-i}) \tag{23}$$

and

$$\forall c_i, c_{-i}, \qquad m_i(c_i, c_{-i}) - c_i \cdot q_i(c_i, c_{-i}) \geq 0. \tag{24}$$

Proof. It is well known in mechanism design that conditions (23-24) hold if and only if the following conditions hold jointly: $\forall c_{-i} \in [c_L, c_H]^{N-1}$

$$m_i^{\mathrm{L}}(c_H, c_{-i}; p_L, p_H) \geq c_H \cdot q_i^{\mathrm{L}}(c_H, c_{-i}; p_L, p_H) \tag{25}$$

$$q_i^{\mathrm{L}}(\cdot, c_{-i}; p_L, p_H) \text{ is nonincreasing,} \tag{26}$$

and

$$\forall\, c_i \in [c_L, c_H] \qquad m_i^{\mathrm{L}}(c_i, c_{-i}; p_L, p_H) = c_i \cdot q_i^{\mathrm{L}}(c_i, c_{-i}; p_L, p_H) + \int_{c_i}^{c_H} q_i^{\mathrm{L}}(t, c_{-i}; p_L, p_H)\, dt. \tag{27}$$

Therefore, it suffices to show that (25-27) hold. To this end, observe that the inequalities in (25) and the monotonicity in (26) are immediate. The envelope condition in (27) holds because both m^{L} and q^{L} are constant in c_i on $[c_L, p_L)$ and on $(p_H, c_H]$, and

$$p_L \cdot \left[\lim_{x\uparrow p_L} q_i^{\mathrm{L}}(x, c_{-i}; p_L, p_H) - \lim_{x\downarrow p_L} q_i^{\mathrm{L}}(x, c_{-i}; p_L, p_H)\right] = \lim_{x\uparrow p_L} m_i^{\mathrm{L}}(x, c_{-i}; p_L, p_H) - \lim_{x\downarrow p_L} m_i^{\mathrm{L}}(x, c_{-i}; p_L, p_H)$$

■

Our strategy of proof will involve restricting attention to candidate mechanisms that are symmetric, and this will be without loss of generality. Next, we introduce a formal definition of symmetric mechanism.

Definition 2. *A mechanism $(q_i, m_i)_{i=1,...,N}$ is symmetric if, for all i,*

$$q_i(c_{\pi(1)}, c_{\pi(2)}, ..., c_{\pi(N)}) = q_{\pi(i)}(c_1, c_2, ..., c_N),$$

and

$$m_i(c_{\pi(1)}, c_{\pi(2)}, ..., c_{\pi(N)}) = m_{\pi(i)}(c_1, c_2, ..., c_N),$$

for every permutation π of $\{1, 2, ..., N\}$. A symmetric mechanism is given by two functions

$$q \equiv q_1 : [c_L, c_H]^N \to [0,1] \quad \text{and} \quad m \equiv m_1 : [c_L, c_H]^N \to [0,1]$$

which are invariant to permutations of the last $N-1$ variables, i.e., letting $\mathcal{N}$ be the set of numbers $\{1, ..., N\}$, $\forall i \in \mathcal{N}$, $\forall$ permutation π of $\mathcal{N}$ we have:

$$q_i(c_1, c_2, ..., c_n) = q(c_i, c_2, ..., c_{i-1}, c_1, c_{i+1}, ..., c_N),$$

and

$$m_i(c_1, c_2, ..., c_n) = m(c_i, c_2, ..., c_{i-1}, c_1, c_{i+1}, ..., c_N).$$

If we restrict attention to symmetric mechanisms, the original weighted welfare maximization problem (6- 10) can be written more simply. We write down the simplified problem next and then, in Lemma 2, we show that the two maximization problems are equivalent. Define:

$$\begin{aligned} Q(c_1) &\equiv \int_{[c_L,c_H]^{N-1}} q(c_1,\, c_{-1}) \cdot \prod_{j>1} dF(c_j) \\ M(c_1) &\equiv \int_{[c_L,c_H]^{N-1}} m(c_1,\, c_{-1}) \cdot \prod_{j>1} dF(c_j). \end{aligned} \tag{28}$$

First reformulation of the weighted welfare maximization problem

$$\max_{Q,M} \; N \int_{[c_L,c_H]} \left[\left[v(c_i, \xi) - (1-\beta) \cdot c_i \right] \cdot Q(c_i) - \beta \cdot M(c_i) \right] f(c_i)\, dc_i \tag{29}$$

subject to, for all $c_i, c_i' \in [c_L, c_H]$:

$$M(c_i) - c_i \cdot Q(c_i) \geq M(c_i') - c_i \cdot Q(c_i'), \tag{30}$$

$$M(c_i) - c_i \cdot Q(c_i) \geq 0, \tag{31}$$

$$Q(c_i) \geq 0, \tag{32}$$

and

$$N \int_{c_L}^{c_i} Q(y)\, f(y)\, dy \;\leq\; 1 - \left[1 - F(c_i)\right]^N. \tag{33}$$

Lemma 2. *Restrict attention to symmetric mechanism. The value of the weighted welfare maximization problem (6- 10) is the same as the value of problem (29- 33).*

Proof. Because in solving problem (6- 10) we are restricting attention to mechanisms $(q_i, m_i)_{i=1,\dots,N}$ that are symmetric, the objective function (6) can be re-written as (29). Similarly, the constraints (9) and (10) can be re-written as: (30) and (31). Furthermore, Border (1991) proves that, if the function q is symmetric in the sense of Definition 2, the demand constraints (7) and nonnegativity constraints (8) hold if and only if (32) and (33) are satisfied.

■

Problem (29- 33) can be further simplified, as follows.

Second reformulation of the weighted welfare maximization problem

$$\max_{Q} \; N \int_{c_L}^{c_H} w(c;\xi,\beta) \cdot Q(c) \cdot f(c)\, dc \tag{34}$$

where $w(c;\xi,\beta)$ is defined in (4), subject to:

$$Q \text{ is nonincreasing,} \tag{35}$$

and, for all $c \in [c_L, c_H]$:

$$Q(c) \geq 0, \tag{36}$$

and

$$N \int_{c_L}^{c} Q(y)\, f(y)\, dy \;\leq\; 1 - [1 - F(c)]^N . \tag{37}$$

Lemma 3. *The weighted welfare maximization problem (29- 33) can be reformulated as (34- 37).*

Proof. The incentive constraints (30) and (31) can be replaced without loss of generality by (35) and the envelope condition:

$$\forall\, c \in [c_L, c_H] \qquad M(c) = c \cdot Q(c) + \int_{c}^{c_H} Q(t) dt. \tag{38}$$

(This result is standard: see, e.g., Proposition 5.2 at p. 66 of Krishna 2010). Next, we use (38) to eliminate M from the problem. Substituting it into (29) and simplifying yields (34). Finally, (36) and (37) are identical to (32) and (33). ■

Next is the final reformulation of the problem.

Final (relaxed) formulation of the weighted welfare maximization problem

$$\max_{Q} \ N \int_{c_L}^{c_H} w(c;\xi,\beta) \cdot Q(c) \cdot f(c)\, dc \tag{39}$$

where $w(c;\xi,\beta)$ is defined in (4), subject to:

$$N \int_{c_L}^{c_H} w(c;\xi,\beta) \cdot Q(c) \cdot f(c)\, dc \ \leq \ N \int_{c_L}^{c_H} w(c;\xi,\beta) \cdot Q^L(c, p_L^*, p_H^*) \cdot f(c)\, dc, \tag{40}$$

where $Q^L(c, p_L^*, p_H^*)$ is given by expression (28) with q being replaced by $q_i^{\mathrm{L}}(c_i, c_{-i}; p_L^*, p_H^*)$ from expression (21).

Problem (39-40) below is actually a relaxation of (34- 37). Aggregating constraints (35-37) into the single inequality (40) is the most innovative part of the proof. This aggregation is proved in the next lemma.

Lemma 4. *Any allocation function Q that satisfies (35-37) also satisfies (40).*

Proof. The proof consists in multiplying both sides of each inequality (35-37) by a non-negative multiplier (which does not change the constraint), and then integrating over c on both sides of each constraint, and finally summing the three resulting inequalities. The resulting inequality identifies a superset of the original feasible set, and happens to equal (40).

The multipliers equal zero except:

$$\begin{cases} \forall c \in (p_H^*, c_H] : & \eta(c) \equiv -w(c;\xi,\beta) \cdot f(c) \\ \\ \forall c \in (p_L^*, p_H^*) : & \delta(c) \equiv -w_c(c;\xi,\beta) \\ \\ \forall c \in [c_L, p_L^*) : & \mu(c) \equiv \dfrac{F(c)}{F(p_L^*)} \displaystyle\int_{c_L}^{p_L^*} w(t;\xi,\beta)\, dF(t) - \int_{c_L}^{c} w(t;\xi,\beta)\, dF(t) \end{cases} \tag{41}$$

To save on notation, in the rest of this proof we omit the dependence of w on (ξ, β).

Let us first show that the multipliers are nonnegative. We have $\eta(c) \geq 0 \ \ \forall c \in (p_H^*, c_H]$, because w is negative on the interval $(p_H^*, c_H]$. We have $\delta(c) \geq 0 \ \ \forall c \in (p_L^*, p_H^*)$, because w is decreasing on the interval $[p_L^*, p_H^*]$. Finally, consider μ on $[c_L, p_L^*)$. First note that

$$\mu(c_L) = \mu(p_L^*) = 0 \tag{42}$$

If $c_L < p_L^*$, then the definition of p_L^* in (12) implies $w(c_L) < w(p_L^*)$. Since w is quasiconcave, there exists a point p_0 such that $w(p_0) = w(p_L^*)$ and

$$\begin{aligned} &\forall c \in [c_L, p_0) \quad w(p_L^*) - w(c) \geq 0, \text{ and} \\ &\forall c \in (p_0, p_L^*] \quad w(p_L^*) - w(c) \leq 0, \end{aligned}$$

Thus the derivative

$$\mu'(c) = f(c)\,[w(p_L^*) - w(c)]$$

is positive for $c < p_0$, and negative for $c > p_0$, that is μ is single-peaked on $[c_L, p_L^*]$. This, together with (42), implies that μ is nonnegative on $[c_L, p_L^*]$. Thus nonnegativity is established.

Now, we multiply both sides of: (35) by $\mu(c)$, (36) by $\eta(c)$, (37) by $\delta(c)$. We then integrate over c. Finally, we sum the three resulting inequalities. We arrive at:

$$\int_{c_L}^{p_L^*} \mu(y)\, dQ(y) \; + \int_{p_L^*}^{p_H^*} \delta(t) \int_{c_L}^{t} Q(y) f(y)\, dy\, dt - \int_{p_H^*}^{c_H} \eta(y)\, Q(y)\, dy \;\; \leq \int_{p_L^*}^{p_H^*} \delta(c)\, B(c)\, dc. \tag{43}$$

where

$$B(c) \equiv \frac{1}{N} \cdot \Big(1 - [1 - F(c)]^N\Big) \quad c \in [c_L, c_H]. \tag{44}$$

To see that (43) is equivalent to (40), let's focus first on the LHS of (43). The first integral can be rewritten as:

$$\begin{aligned}
&\overbrace{\mu(p_L^*)}^{=0}\cdot Q(p_L^*) - \overbrace{\mu(c_L)}^{=0}\cdot Q(c_L) - \int_{c_L}^{p_L^*} Q(y)\,\mu'(y)\,dy \\
&= -\int_{c_L}^{p_L^*} \mu'(y)\,Q(y)\,dy. \qquad (45)
\end{aligned}$$

The second integral on the LHS of (43) can be rewritten as:

$$\begin{aligned}
&\int_{c_L}^{p_L^*}\left(\int_{p_L^*}^{p_H^*}\delta(t)dt\right)Q(y)f(y)dy + \int_{p_L^*}^{p_H^*}\left(\int_{y}^{p_H^*}\delta(t)dt\right)Q(y)f(y)dy \\
=&\int_{c_L}^{p_L^*}\left(\int_{p_L^*}^{p_H^*}\delta(t)dt\right)Q(y)f(y)dy + \int_{p_L^*}^{p_H^*} w(y)Q(y)f(y)dy, \qquad (46)
\end{aligned}$$

where equality holds because:

$$\int_{y}^{p_H^*}\delta(t)dt = w(p_H^*) - \int_{y}^{p_H^*} w'(c)dc = w(y).$$

Adding (45) and (46) yields:

$$\begin{aligned}
&\int_{c_L}^{p_L^*}\left(\int_{p_L^*}^{p_H^*}\delta(t)dt - \frac{\mu'(y)}{f(y)}\right)Q(y)\,f(y)\,dy \\
=&\int_{c_L}^{p_L^*} w(y)\,Q(y)\,f(y)\,dy, \qquad (47)
\end{aligned}$$

where equality holds because:

$$\begin{aligned}
\int_{p_L^*}^{p_H^*} \delta(t)\,dt - \frac{\mu'(y)}{f(y)} &= \overbrace{w(p_H^*) - \int_{p_L^*}^{p_H^*} w'(t)dt}^{=w(p_L^*)} - \frac{\mu'(y)}{f(y)} \\
&= w(p_L^*) - \left(\frac{1}{F(p_L^*)} \cdot \int_{c_L}^{p_L^*} w(t)f(t)dt - w(y)\right) \\
&= \overbrace{w(p_L^*) - \frac{1}{F(p_L^*)} \cdot \int_{c_L}^{p_L^*} w(t)f(t)dt}^{=0} + w(y) \\
&= w(y)
\end{aligned}$$

The third integral on the LHS of (43) can be rewritten as:

$$-\int_{p_H^*}^{c_H} \eta(y)Q(y)dy = \int_{p_H^*}^{c_H} w(y)f(y)Q(y)dy. \tag{48}$$

Combining (47) and (48) we can rewrite the LHS of (43) as

$$\int_{c_L}^{c_H} w(y)Q(y)f(y)dy.$$

Let's now focus on the RHS of (43). Plugging in the expression for δ and simplifying yields:

$$\begin{aligned}
&-\int_{p_L^*}^{p_H^*} w'(c)\,B(c)\,dc \\
&= w(p_L^*)\,B(p_L^*) + \int_{p_L^*}^{p_H^*} w(c)\,B'(c)\,dc \\
&= \int_{p_L^*}^{p_H^*} w(c)\,(1-F(c))^{N-1}\,f(c)\,dc + \int_{c_L}^{p_L^*} \frac{1-[1-F(p_L)]^N}{N \cdot F(p_L)} w(c)\,f(c)\,dc,
\end{aligned} \tag{49}$$

where the second equality follows from the definition of B in (44). Now observe that:

$$
\begin{aligned}
Q^L(c_1; p_L^*, p_H^*) &\equiv \int_{[c_L, c_H]^{N-1}} q(c_1,\ c_{-1}; p_L^*, p_H^*) \cdot \prod_{j>1} dF(c_j) \\
&= \begin{cases} 0, & c_1 \in (p_H^*, c_H]\,; \\ [1 - F(c_1)]^{N-1}\,, & c_1 \in (p_L^*, p_H^*]\,; \\ \frac{1-\left[1-F(p_L^*)\right]^N}{N\cdot F(p_L^*)}, & c_1 \in [c_L, p_L^*]\,. \end{cases}
\end{aligned}
\tag{50}
$$

Hence (49) boils down to:

$$
\begin{aligned}
&\int_{p_L^*}^{p_H^*} w(c)\, Q^L(c, p_L^*, p_H^*)\, f(c)\, dc + \int_{c_L}^{p_L^*} w(c)\, Q^L(c, p_L^*, p_H^*)\, f(c)\, dc \\
=& \int_{c_L}^{p_H^*} w(c)\, Q^L(c, p_L^*, p_H^*)\, f(c)\, dc \\
=& \int_{c_L}^{c_H} w(c)\, Q^L(c, p_L^*, p_H^*)\, f(c)\, dc.
\end{aligned}
$$

This completes the proof.

■

We are now ready to prove Theorem 1.

Proof of Theorem 1

Proof. Lemma 1 shows that the direct mechanism $(q^{\mathrm{L}}, m^{\mathrm{L}})$ satisfies both IC and IR *ex post*. Therefore, sincere bidding in the LoLA is a (weakly) dominant strategy equilibrium.

Moreover, $(q^{\mathrm{L}}, m^{\mathrm{L}})$ is a feasible mechanism, i.e., it satisfies constraints (7 - 10). Indeed, unit demand (7) and nonnegativity (8) can be checked directly from the definition (21), and the fact that $(q^{\mathrm{L}}, m^{\mathrm{L}})$ satisfy the ex-post incentive constraints, as proved in Lemma 1, immediately implies that it also satisfies their interim counterparts (9) and (10).

It remains to show that the mechanism $(q^{\mathrm{L}}, m^{\mathrm{L}})$ defined in (21) and (22) solves the weighted welfare problem. We proceed in two steps.

Maskin and Riley (1986, footnote 11) show that, in our setting, given any optimal mechanism for the weighted welfare problem, there is a symmetric mechanism that attains the same (maximal) value. Therefore, we can restrict the search for an optimal mechanism to the set of symmetric mechanisms (of which $(q^{\mathrm{L}}, m^{\mathrm{L}})$ is one) without loss of generality.

After restricting to symmetric mechanisms, Lemmas 2 -4 yield a relaxed problem with a set of feasible mechanisms (40) that contains the original feasible set. If a LoLA solves this relaxed problem, then a fortiori the LoLA solves the original problem. The LoLA defined by (50) solves this relaxed problem because $Q^L(c, p_L^*, p_H^*)$ satisfies (40) with equality.

■

A.2 Proof of Proposition 2

Proof. The weighted welfare generated by the optimal LoLA is:

$$N \cdot \int_{c_L}^{p_H^*} w(c_i) \cdot Q_*^{\mathbb{L}}(c_i) \cdot dF(c_i)$$

$$= N \cdot \overbrace{\frac{1-[1-F(p_L^*)]^N}{N \cdot F(p_L^*)}}^{\bar{Q}} \cdot \int_{c_L}^{p_L^*} w(c_i) \cdot dF(c_i) + N \cdot \overbrace{\int_{p_L^*}^{p_H^*} w(c_i) \cdot [1-F(c_i)]^{N-1} \cdot dF(c_i)}^{Q_*^{\mathbb{L}}(c_i) \text{ above } p_L}$$

$$= \frac{1-[1-F(p_L^*)]^N}{F(p_L^*)} \cdot \int_{c_L}^{p_L^*} w(c_i) \cdot dF(c_i) + N \cdot \int_{p_L^*}^{c_H} \max\{w(c_i), 0\} \cdot [1-F(c_i)]^{N-1} \cdot dF(c_i)$$

$$= \left(1-[1-F(p_L^*)]^N\right) \cdot \overbrace{\int_{c_L}^{p_L^*} w(c_i) \cdot \frac{f(c_i)}{F(p_L^*)}\, dc_i}^{w(p_L^*)} + \int_{p_L^*}^{c_H} \max\{w(c_i), 0\}\, dG(c_i)$$

$$= \left(1 - \overbrace{[1-F(p_L^*)]^N}^{\equiv P(N)}\right) \cdot w(p_L^*) + \overbrace{[1-F(p_L^*)]^N}^{\equiv P(N)} \cdot \overbrace{\mathbb{E}\left[\max\left\{w(c^{(1)}), 0\right\} \mid p_L^* < c^{(1)}\right]}^{\equiv \mathcal{E}(N)}$$

$$= w(p_L^*) - P(N) \cdot \left(w(p_L^*) - \mathcal{E}(N)\right), \tag{51}$$

where $G(c_i) \equiv 1-(1-F(c_i))^N$ is the c.d.f. of the lowest cost, and $g(c_i) = G'(c_i) = N\,(1-F(c_i))^{N-1} f(c_i)$ is its density.

The first equality follows from replacing $Q_*^{\mathbb{L}}$ with (56). The second equality follows from canceling N in the first term, and extending the second integral to c_H. The third equality follows from pulling $F(p_L^*)$ inside the integral, and using the definition of G. The fourth equality makes use of the fact that

$$w(p_L^*) = \frac{1}{F(p_L^*)} \int_{c_L}^{p^*} w(c)\, dF(c)\, dc, \tag{52}$$

which follows from integrating by parts the integral in(12).

Now note that, by definition (12), if p_L^* is greater than c_L it must lie in the region where the function w is decreasing. This region extends all the way to c_H by Assumption 1. Because $\max\{w(\cdot), 0\}$ is positive at p_L^* (see (52)) and nonincreasing on $[p_L, c_H]$, stochastic dominance implies that the conditional expectation $\mathcal{E}(N)$ in (51) is strictly increasing in N. This implies that the term in parenthesis in (51) is positive. Because $P(N)$ in (51) is decreasing in N, expression (51) is increasing in N. ∎

B Proofs for Section IV

B.1 Proofs for Section IVA

In this extension, each supplier $i = 1, ..., N$ draws its type $\theta_i = (c_i, x_i)$ independently from a distribution with density ϕ and support $\Theta_1 \equiv [c_L, c_H] \times [x_L, x_H]$. Let $\Theta \equiv \Theta_1^N$.

A direct mechanism $\mathcal{M}$ consists of $2N$ functions

$$\mathcal{M} \equiv \{q_i(c, x), m_i(c, x) \mid (c, x) \in \Theta\}_{i=1}^N$$

The restriction to symmetric mechanisms is wlog in the case as well.

Define supplier i's profit function as

$$\Pi(c_i, x_i) \equiv \sup \{M_i(c_i', x_i') - c_i \cdot Q_i(c_i', x_i') \mid (c_i', x_i') \in \Theta_1\}$$

Lemma 5. *If a mechanism is incentive compatible, then its reduced form Q must be independent of x_i, except possibly at for a zero measure set, i.e., for all i, all c_i, x_i and x_i'*

$$Q_i(c_i, x_i) = Q_i(c_i, x_i')$$

Proof. Standard mechanism design arguments imply that Π is convex and absolutely continuous.
The envelope theorem implies

$$\frac{\partial \Pi(c_i, x_i)}{\partial x_i} = 0 \quad \text{a.e.}$$

and

$$\frac{\partial \Pi(c_i, x_i)}{\partial c_i} = -Q(c_i, x_i) \quad \text{a.e.}$$

For any types (c_i, x_i) and (c_i', x_i'), the profit difference $\Pi(c_i', x_i') - \Pi(c_i, x_i)$ is equal to the line integral of the gradient of Π along any path. Therefore we have

$$\int_{c_i}^{c_i'} \overbrace{\frac{\partial \Pi(t, x_i)}{\partial c_i}}^{=-Q(t,x_i)} dt + \int_{x_i}^{x_i'} \overbrace{\frac{\partial \Pi(c_i', t)}{\partial x_i}}^{=0} dt \;=\; \int_{x_i}^{x_i'} \overbrace{\frac{\partial \Pi(c_i, t)}{\partial x_i}}^{=0} dt + \int_{c_i}^{c_i'} \overbrace{\frac{\partial \Pi(t, x_i')}{\partial c_i}}^{=-Q(t,x_i')} dt$$

If $(c_i, x_i) < (c_i', x_i')$, the path in the LHS is "first east and then north"; and the path in the RHS is "first north and then east".

Thus we have

$$\int_{c_i}^{c_i'} Q(t, x_i)dt \ = \ \int_{c_i}^{c_i'} Q(t, x_i')dt$$

Because the last equality must hold for any c_i and c_i', Q must be independent of x_i, except possibly for a zero measure set. ■

Lemma 5 implies that Π is also independent of x_i. and thus M must satisfy the envelope condition

$$M(c_i) = \Pi(c_L) + \int_{c_L}^{c_i} Q_i(t)\, dt,$$

and thus must be independent of x_i.

B.2 Proof for Section IVD

Lemma 6. *Fix a LoLA with floor price p_L and reserve price p_H, and denote by $\bar{Q}$ the probability of winning for any type with cost below p_L. The FPLoLA with the same reserve price and minimum bid given by*

$$b_L = \frac{\bar{Q} - [1 - F(p_L)]^{N-1}}{\bar{Q}} \cdot p_L + \frac{[1 - F(p_L)]^{N-1}}{\bar{Q}} \cdot \beta(p_L; p_H) \tag{53}$$

is equivalent in the sense that it generates the same interim expected payments and profits for each supplier, and the same buyer's expected surplus.

The following strategy is a symmetric equilibrium in the equivalent first-price LoLA:

$$\beta^{f\mathbb{L}}(c_i; p_L, p_H) \equiv \begin{cases} b_L & \text{if } c_i \le p_L \\ \beta(c_i; p_H) & \text{if } p_L < c_i \le p_H \\ \text{no bid} & \text{if } c_i > p_H, \end{cases} \tag{54}$$

where

$$\beta(c_i; p_H) = \mathbb{E}\left[\min\left\{c_{-i}^{(1)}, p_H\right\} \mid c_i < c_{-i}^{(1)}\right]$$

is the equilibrium bidding strategy in the standard (reverse) first-price auction with reserve price p_H and no minimum bid, and $c_{-i}^{(1)}$ denotes the lowest cost among i's opponents.

Proof. The proof proceeds as follows. First, we show that when bidder i computes its best response in the FPLoLA, there is no loss of generality in ignoring the interval of "unused bids" $(b_L, \beta(p_L))$. Since all the remaining bids are used by some type, we can restate the best response problem as reporting a type in the direct revelation mechanism induced by $\beta^{f\mathbb{L}}(c_i; p_L, p_H)$.

Next, we compute the interim probability of winning and expected payment for each type c_i in the direct revelation mechanism induced by $\beta^{f\mathbb{L}}(c_i; p_L, p_H)$, and show that these functions coincide with their counterparts in the sincere equilibrium in the LoLA. This implies the equivalence of the two auction formats in terms of buyer expected surplus, interim expected payments, and expected profits.

Finally, because sincere bidding is an equilibrium in the LoLA, truthful reporting must also be an equilibrium in the direct revelation mechanism induced in the FPLoLA by $\beta^{f\mathbb{L}}(c_i; p_L, p_H)$. Equivalently, $\beta^{f\mathbb{L}}(c_i; p_L, p_H)$ is an equilibrium in the FPLoLA.

No loss of generality in ignoring "unused bids"

Suppose that all bidders in the FPLoLA except i follow the strategy $\beta^{f\mathbb{L}}$ given in (54). Then bidder i's expected payoff function is:

$$\Pi_i(b, c_i) = \begin{cases} 0 & \text{if no bid} \\ (b - c_i)\ [1 - F(\beta^{-1}(b))]^{N-1} & \text{if } \beta(b_L; p_H) \le b \le p_H \\ (b - c_i)\ [1 - F(b_L)]^{N-1} & \text{if } b_L < b < \beta(b_L; p_H) \\ (b_L - c_i)\ \bar{Q} & \text{if } b = b_L \end{cases}$$

For any $c_i \in [c_L, c_H]$, the payoff function $\Pi_i(\cdot, c_i)$ is linear and strictly increasing on the middle interval $(b_L, \beta(b_L; p_H)]$. Therefore all bids in this interval cannot be optimal for any type. Once all bids in $(b_L, \beta(b_L; p_H)]$ are removed from consideration, all remaining bids are in the range of $\beta^{f\mathbb{L}}$. Since all the remaining bids are used by some type, we can interpret choosing the best response in the FPLoLA as choosing a type report in the direct revelation mechanism induced by $\beta^{f\mathbb{L}}$.

The direct revelation mechanism induced by the LoLA

In the LoLA, for any type profile (c_i, c_{-i}) supplier i sells with probability

$$q_i^{\mathbb{L}}(c_i, c_{-i}) = \begin{cases} 1 & \text{if } c_i < \min\{c_{-i}^{(1)}, p_H\} \text{ and } p_L < c_{-i}^{(1)} \\ \frac{1}{k+1} & \text{if } c_i \le p_L \text{ and } c_{-i}^{(k)} \le p_L < c_{-i}^{(k+1)} \\ 0 & \text{else} \end{cases} \tag{55}$$

The resulting *interim* probability of selling is

$$
\begin{aligned}
Q^{\mathbb{L}}(c_i) &\equiv \int_{[c_L, c_H]^{N-1}} q_i^{\mathbb{L}}(c_i, c_{-i}) \prod_{j \neq i}^{N} f(c_j)\, dc_j \\
&= \begin{cases} \bar{Q} & \text{if } c_i \in [c_L, p_L] \\ [1 - F(c_i)]^{N-1} & \text{if } c_i \in (p_L, p_H] \\ 0 & \text{if } c_i \in (p_H, c_H] \end{cases}
\end{aligned} \tag{56}
$$

where

$$
\begin{aligned}
\bar{Q} &= \sum_{j=0}^{N-1} \overbrace{\binom{N-1}{j} \cdot F(p_L)^j \cdot [1 - F(p_L)]^{N-1-j}}^{\Pr[j \text{ opponents have cost below } p_L]} \cdot \overbrace{\frac{1}{j+1}}^{\text{uniform tie-breaking}} \\
&= \sum_{j=0}^{N-1} \frac{(N-1)!}{j!\,(N-1-j)!} F(p_L)^j (1 - F(p_L))^{N-1-j} \frac{1}{j+1} \\
&= \frac{1}{N\,F(p_L)} \sum_{j=0}^{N-1} \frac{N!}{(j+1)!\,(N-1-j)!} F(p_L)^{j+1} (1 - F(p_L))^{N-1-j} \\
&= \frac{1}{N\,F(p_L)} \sum_{t=1}^{N} \binom{N}{t} F(p_L)^t (1 - F(p_L))^{N-t} \\
&= \frac{1}{N\,F(p_L)} \left[1 - (1 - F(p_L))^N\right]
\end{aligned}
$$

In a LoLA, the *ex post* expected payment function is

$$
m_i^{\mathbb{L}}(c_i, c_{-i}) = \begin{cases} \min\{c_{-i}^{(1)},\, p_H\} & \text{if } c_i < \min\{c_{-i}^{(1)},\, p_H\} \text{ and } p_L < c_{-i}^{(1)} \\ \frac{1}{k+1}\, p_L & \text{if } c_i \le p_L \text{ and } c_{-i}^{(k)} \le p_L < c_{-i}^{(k+1)} \\ 0 & \text{else} \end{cases} \tag{57}
$$

and the resulting *interim* expected payment function is

$$M^{\mathbb{L}}(c_i) \equiv \int_{[c_L,c_H]^{N-1}} m_i^{\mathbb{L}}(c_i, c_{-i}) \prod_{j\neq i}^{N} f(c_j)\, dc_j$$

$$= \begin{cases} \bar{M} & \text{if } c_i \in [c_L, p_L] \\ \beta(c_i; p_H) \cdot [1 - F(c_i)]^{N-1} & \text{if } c_i \in (p_L, p_H] \\ 0 & \text{if } c_i \in (p_H, c_H] \end{cases} \tag{58}$$

We have

$$\bar{M} = \bar{Q} \cdot\ p_L + [1 - F(p_L)]^{N-1} \cdot [\beta(p_L; p_H) - p_L]$$

The first term in the RHS captures the fact that any bidder with type below p_L wins with probability $\bar{Q}$ and is paid at least p_L. The second term captures the event in which the costs of all the bidder's opponents exceed p_L; in this case, which happens with probability $[1 - F(p_L)]^{N-1}$, the bidder is paid more.

The second line in (58) holds because any type $c_i \in [p_L, p_H]$ sells at price $\min\left\{c_{-i}^{(1)}, p_H\right\}$ when $c_i < c_{-i}^{(1)}$. Therefore its expected payment is

$$\int_{c_i}^{c_H} \min\{y, p_H\}\, dF_{-i}^{(1)}(y) = \beta(c_i; p_H) \cdot [1 - F(c_i)]^{N-1}.$$

The direct revelation mechanism induced by $\beta^{f\mathbb{L}}$ in the FPLoLA coincides with its LoLA counterpart.

Type c_i's interim probability of winning in the direct revelation mechanism induced by $\beta^{f\mathbb{L}}$ is the same as the probability of winning in the FPLoLA assuming that all other bidders follow the strategy $\beta^{f\mathbb{L}}$ given in (54) and i bids according to $\beta^{f\mathbb{L}}(c_i; p_L, p_H)$. Because the strategy $\beta^{f\mathbb{L}}$ is strictly increasing in the region above p_L and flat below p_L, the regions of the type space in which the lowest type wins with probability 1 are the same as in the sincere equilibrium of the equivalent LoLA. Similarly, the regions in which multiple suppliers win with positive probability are the same as in the two auctions. Therefore, both the ex post and interim probability of winning in the FPLoLA are the same as in the sincere equilibrium of the equivalent LoLA.

To see that the interim expected payment function in the first-price LoLA is equal to

$M^{\mathbb{L}}$, note first that all types above p_H do not bid and thus are paid zero. Next, note that any type between p_L and p_H sells at price $\beta(c_i; p_H)$ when all other suppliers bid above. Therefore the expected payment of all types between p_L and p_H is $\beta(c_i; p_H) \cdot [1 - F(c_i)]^{N-1}$, as in (58). Finally, for all types below p_L, the *interim* expected payment

$$b_L \cdot \bar{Q} = \left[\bar{Q} - [1 - F(p_L)]^{N-1}\right] \cdot p_L + [1 - F(p_L)]^{N-1} \cdot \beta(p_L; p_H)$$

is equal to $\bar{M}$ in (58).

Equivalence between FPLoLA and LoLA

Because the direct revelation mechanism induced by $\beta^{f\mathbb{L}}$ in the FPLoLA coincides with its LoLA counterpart, the two auction formats are equivalent in terms of buyer expected surplus, interim expected payments, and expected profits.

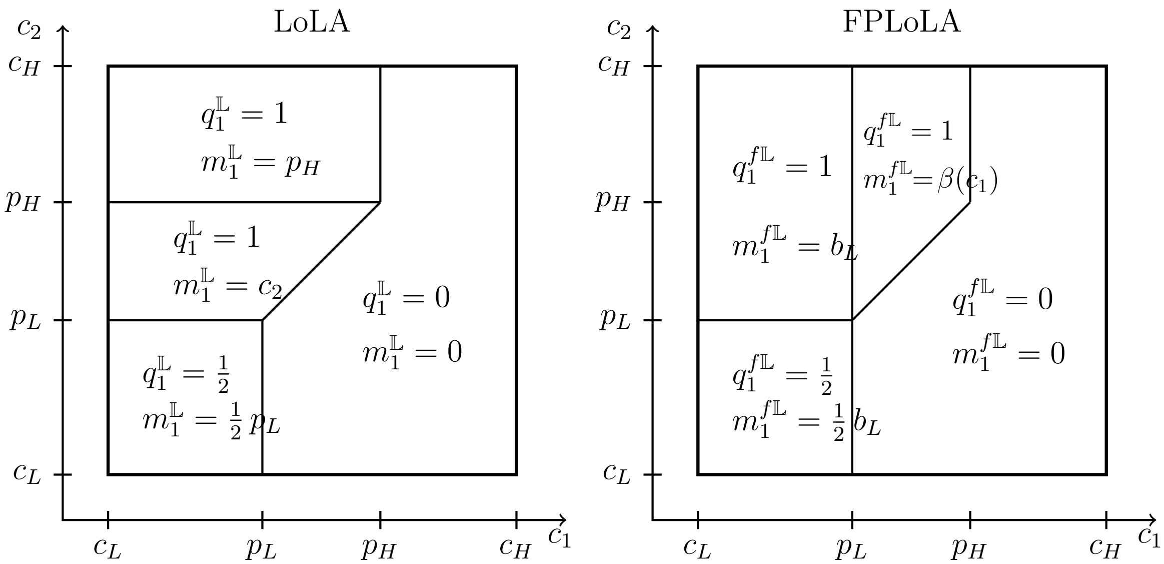
$\beta^{f\mathbb{L}}(c_i; p_L, p_H)$ is an equilibrium in the FPLoLA

Because sincere bidding is an equilibrium in the LoLA, truthful reporting must also be an equilibrium in the direct revelation mechanism induced in the FPLoLA by $\beta^{f\mathbb{L}}(c_i; p_L, p_H)$. Equivalently, $\beta^{f\mathbb{L}}(c_i; p_L, p_H)$ is an equilibrium in the FPLoLA.

■

The next figure compares the equilibrium outcomes in a LoLA and its equivalent

FPLoLA.



LoLA
c_2
c_H
p_H
p_L
c_L
$q_1^{\mathbb{L}} = 1$
$m_1^{\mathbb{L}} = p_H$
$q_1^{\mathbb{L}} = 1$
$m_1^{\mathbb{L}} = c_2$
$q_1^{\mathbb{L}} = 0$
$m_1^{\mathbb{L}} = 0$
$q_1^{\mathbb{L}} = \frac{1}{2}$
$m_1^{\mathbb{L}} = \frac{1}{2}\, p_L$
c_L
p_L
p_H
c_H
c_1
FPLoLA
c_2
c_H
p_H
p_L
c_L
$q_1^{f\mathbb{L}} = 1$
$m_1^{f\mathbb{L}} = b_L$
$q_1^{f\mathbb{L}} = 1$
$m_1^{f\mathbb{L}} = \beta(c_1)$
$q_1^{f\mathbb{L}} = 0$
$m_1^{f\mathbb{L}} = 0$
$q_1^{f\mathbb{L}} = \frac{1}{2}$
$m_1^{f\mathbb{L}} = \frac{1}{2}\, b_L$
c_L
p_L
p_H
c_H
c_1

C Material for Section V

C.1 Semi-parametric identification of $\hat{D}$ and $\hat{O}$

We seek to recover the unobserved distribution of supplier quality conditional on cost c, that gives rise to the empirical distributions g_D and g_O in Figure 5. We take a guess-and-verify approach. In the next definition we guess a semi-parametric form of the distribution of supplier quality conditional on c; then we verify that the guess gives rise to the empirical distributions g_D and g_O, as it should.

Definition 3. ***(guess: distribution of supplier quality conditional on supplier cost)*** *For any $\xi \in [0,1]$ define:*

$$\hat{D}(c,\xi) = \begin{cases} \delta(c) & w.p.\ \xi \\ D & w.p.\ 1-\xi \end{cases}$$

$$\hat{O}(c,\xi) = \begin{cases} \omega(c) & w.p.\ \xi \\ O & w.p.\ 1-\xi, \end{cases}$$

where $\delta(c) = G_D^{-1}\left(\left[1-\hat{F}(c)\right]^N\right)$ and $\omega(c) = G_O^{-1}\left(\left[1-\hat{F}(c)\right]^N\right)$, and D and O are the random variables with distributions depicted in Figure 5.

Intuitively, $\hat{D}(c,\xi)$ is a random variable that represents the delay associated with a generic supplier with cost c. With probability ξ this delay is identically equal to the number $\delta(c)$; with complementary probability this delay is a random draw from the random variable D whose distribution is depicted in Figure 5, panel B. The same intuition holds for $\hat{O}(c,\xi)$. The functions $\delta(c)$ and $\omega(c)$ are specifically constructed so that the random variables D and O give rise to the "empirically correct marginals," in the following sense.

Lemma 7. ***(verify: $\hat{D}$ and $\hat{O}$ have the correct marginals)*** *Denote: $C_{(1)} = \min\{C_1,...,C_N\}$. Then for any $\xi \in [0,1]$ we have: $\hat{D}\left(C_{(1)},\xi\right) \sim D$ and $\hat{O}\left(C_{(1)},\xi\right) \sim O$.*

Proof. We show the proof for the random variable D.

$$\begin{aligned}
\Pr\left(\delta\left(C_{(1)}\right) \leq d\right) &= \Pr\left[G_D^{-1}\left(\left[1-\hat{F}\left(C_{(1)}\right)\right]^N\right) \leq d\right] \\
&= \Pr\left[\left[1-\hat{F}\left(C_{(1)}\right)\right]^N \leq G_D(d)\right] \\
&= \Pr\left[1-\left[G_D(d)\right]^{1/N} \leq \hat{F}\left(C_{(1)}\right)\right] \\
&= \Pr\left[\hat{F}^{-1}\left(1-\left[G_D(d)\right]^{1/N}\right) \leq C_{(1)}\right]
\end{aligned}$$

Since

$$\Pr\left(x \leq C_{(1)}\right) = \left[1-\hat{F}(x)\right]^N,$$

then:

$$\begin{aligned}
\Pr\left(\delta\left(C_{(1)}\right) \leq d\right) &= \left\{1-\hat{F}\left(\hat{F}^{-1}\left(1-\left[G_D(d)\right]^{1/N}\right)\right)\right\}^N \\
&= \left\{1-\left(1-\left[G_D(d)\right]^{1/N}\right)\right\}^N \\
&= \left\{G_D(d)^{1/N}\right\}^N \\
&= G_D(d).
\end{aligned}$$

The proof for the random variable O is virtually identical. ■

This lemma proves that, if C is distributed according to $\hat{f}$, the delays and overruns of a bidder with cost c are drawn from $\hat{D}(c,\xi)$ and $\hat{O}(c,\xi)$, and there are N bidders, then the lowest bidder's marginal distributions of delays and overruns equals the observed marginal distributions of D and O from Figure 5. This property holds *for any value of the parameter* ξ. The parameter ξ encodes the correlation between cost and quality.

The calibrated buyer surplus function reads:

$$\begin{aligned}
\hat{v}(c,\xi) &= \mathit{const} - K\mathbb{E}\left[\hat{D}(c,\xi)+\hat{O}(c,\xi)\right] \\
&= \mathit{const} - (1-\xi)K\mathbb{E}\left[D+O\right] - \xi K\left[\delta(c)+\omega(c)\right] \\
&= \mathit{const}(\xi) - \xi K\left[\delta(c)+\omega(c)\right]. \qquad (59)
\end{aligned}$$

C.2 Calibration of $\hat{v}$

From expression (19), the calibrated buyer's payoff reads:

$$\hat{v}(c,\xi) = const(\xi) - \xi K \left[G_D^{-1}\left(\left[1-\hat{F}\left(c\right)\right]^N\right) + G_O^{-1}\left(\left[1-\hat{F}\left(c\right)\right]^N\right)\right]. \qquad (60)$$

Our goal is to fully calibrate this function of (c, ξ). The constant $const(\xi)$ reads, from (59):

$$const(\xi) = const - (1-\xi)K\mathbb{E}\left[D+O\right]. \qquad (61)$$

We set *const* large enough that the virtual valuation $\hat{w}$ is everywhere positive,[30] and K large enough that, as ξ varies between 0 and 1, the slope of the social welfare function (dashed red line in Figure 6) changes from positive to negative, while keeping at a magnitude that is reasonable. Specifically, we set $const = 1.0775 \times 10^6$ and $K = 2 \times 10^3$. With these values $\hat{w}$ is always positive (albeit barely so when c is small and ξ is large). Furthermore, the variation of the social surplus caused by a variation in supplier cost is reasonable. Indeed, given that the standard deviation of the distribution $\hat{f}$ (Figure 5, left-hand panel) equals 4.76×10^4, increasing the supplier's cost by one standard deviation around the mean (about one tick on the c-axis in Figure 6) yields variations in social surplus (dashed red line in Figure 6) that are plausible in magnitude, that is, not too large relative to average cost. With this choice of *const* and K, the social welfare evaluated at mean cost is of the same magnitude as the average cost for any ξ, which we view as a reassuring sanity check.

The three quantities $\hat{F}$, G_D, and G_O are given in Figure 5.

The number of bidders N is set equal to 7, the average number of bidders in the (first price) auctions studied by Decarolis (2014, 2016).

[30] This guarantees that the optimal LoLA does not involve a reserve price.

Log-concavity of $\hat{F}$

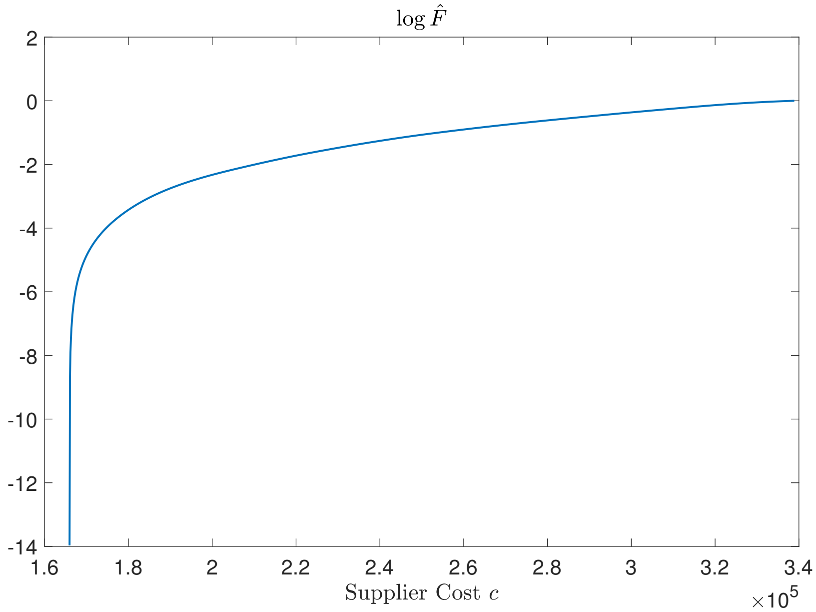


Figure 8: $\log(\hat{F})$ is concave.

D Software applications

This appendix describes two software applications that we have created and made publicly available.[31] These applications compute the buyer-optimal procurement mechanisms in the presence of quality concerns. The purpose of disseminating these applications is twofold. First, we wish to allow business practitioners to assess whether they can benefit from a buyer-optimal LoLA and, if so, with what floor and reserve prices. Second, for pedagogical purposes, we want to facilitate the teaching of this paper in an engaging way.

D.1 Software 1

This software is a visually handy procedure realized in Matlab that does not require IBM ILOG CPLEX. An Excel-based visual interface asks the user to input a probability distribution of costs (corresponding to $f(c)$ in our theoretical model), a function $v(c)$ (corresponding to $v(c, \xi)$ for some fixed ξ), and the number of bidders N. The application assumes that bidder costs are drawn independently from the cost distribution, and requires that $v(c_L) > c_L$. The application's output displays the buyer and social surplus functions as a function of the LoLA floor price p_L, and displays the optimal floor and reserve prices (analogous to the right-hand panel of Figure 6). The program also displays the ratio between the social (or buyer) surplus under a LoLA with reserve price p_L, over a first price auction.

The user specifies three inputs in an excel spreadsheet called "Input.xlsx", as shown in Figure 9 (where input cells are colored in orange). There are four inputs: (i.) the minimum cost c_L (cell D21) and the maximum cost c_H (cell M21) used by the spreadsheet to automatically generate a linear cost grid with 10 nodes, (ii.) the 10 relative weights used to infer the cost distribution $f(c)$ (cells D20:M20), (iii.) the 10 values that represent the willingness to pay $v(c)$ (cells R20:AA20), and (iv.) the number of bidders N (cell R25).

[31] Downloadable from https://www.alessandrotenzinvilla.com/research.html.

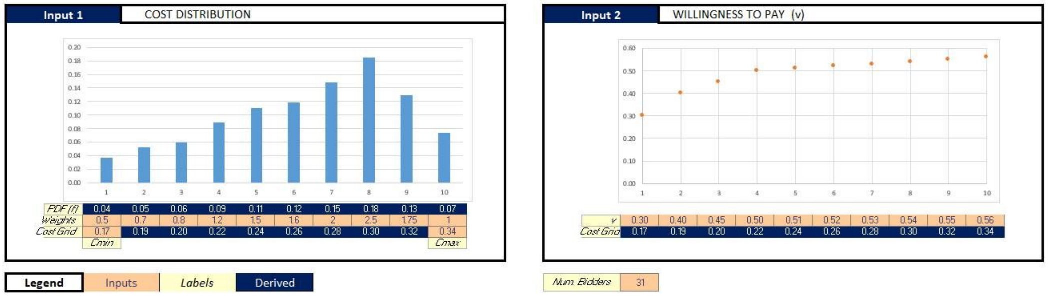


Figure 9: The figure shows the inputs of the visual program that solves for the optimal LoLA among all LoLAs.

The Matlab script "FindOptimalLola.m" (which needs to be located in the same folder of the input file "Input.xlsx") reads the 4 aforementioned inputs and calculates the virtual valuation function w. The script also re-samples all inputs on a grid with $T = 100$ nodes to increase the precision of the calculation. Given a grid $\{c_i\}_{i=1}^{T}$, the virtual valuation w is calculated as

$$\begin{cases} w_i = v_i - c_i, & i = 1 \\ w_i = v_i - c_i - (c_i - c_{i-1}) \cdot \frac{F_i}{f_i}, & \forall i > 1 \end{cases} \tag{62}$$

The result for w is showed to the user as in figure 11. The user is asked to check whether w is single-peaked in accordance to assumption 1.

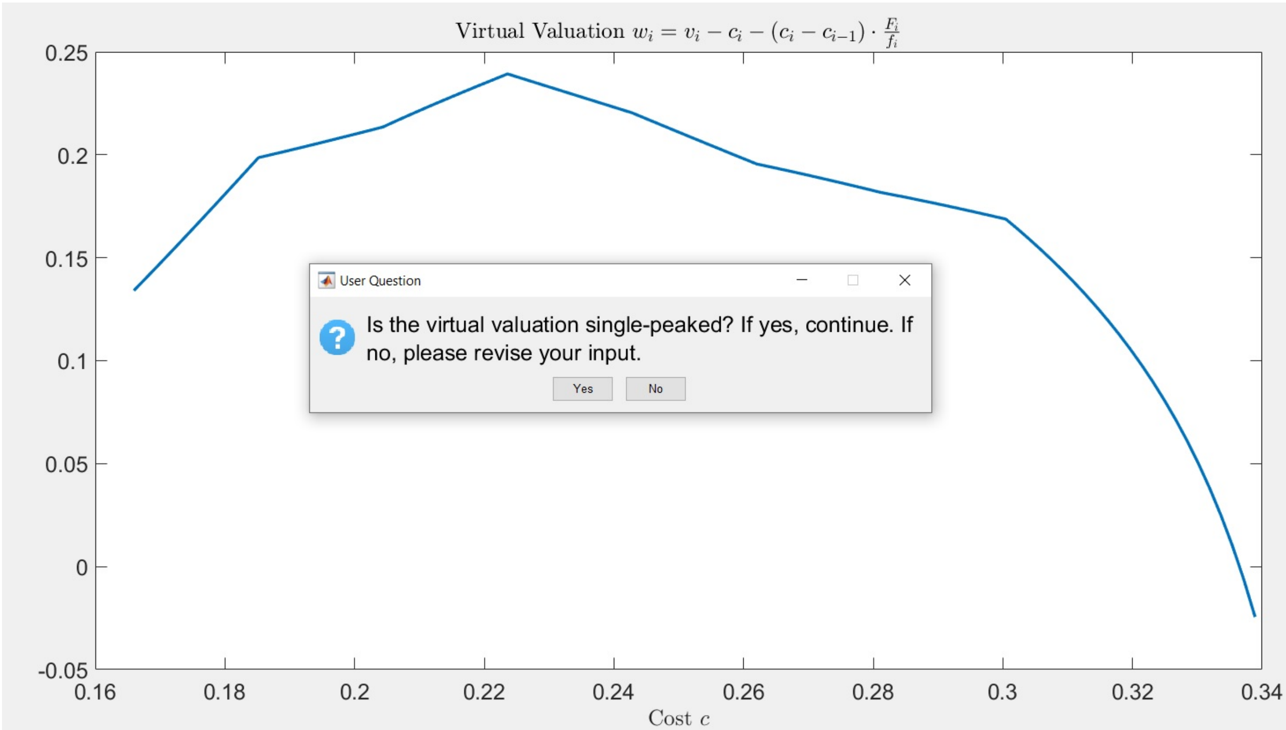


Figure 10: The figure shows w and it involves the user's participation by asking whether or not w is single-peaked.

If the user clicks "yes" the procedure continues, otherwise it stops as assumption 1 is violated. If "yes" is clicked, the procedure checks whether w has a root. If it does have a root, the software shows it in a new pop-up window as shown by figure 12. Hence, the software asks for the user's confirmation to set the root of w as a reservation price p_H.

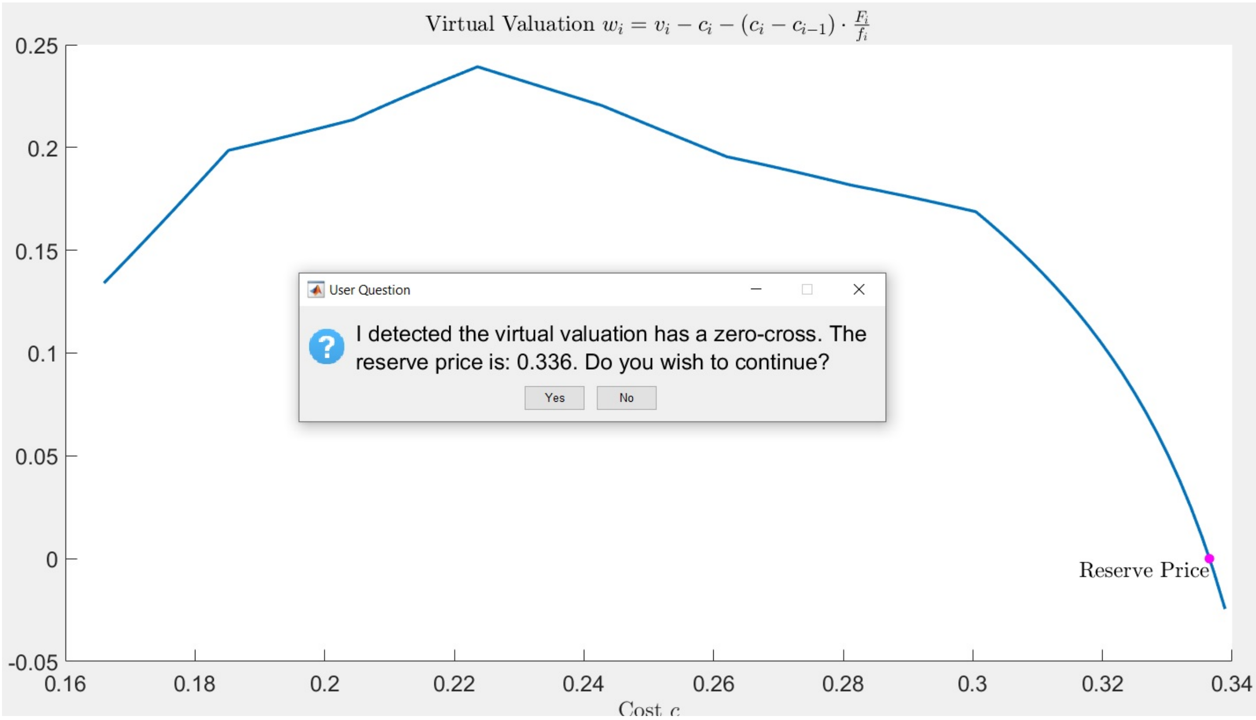


Figure 11: The figure shows the root of w calculated with a solver and using piece-wise linear interpolation on w. It involves the user's participation by acknowledging the root will be used as the reservation price.

Hence, the procedure iterates on all possible floor prices $\{p_{L,j}\}_{j=1}^{T}$ between c_L and c_H. For each floor price $p_{L,j}$, it calculates the associated buyer surplus $\sum_{i=1}^{T} w_i \cdot f_i \cdot Q_{i,j}$ and social surplus $\sum_{i=1}^{T}(v_i - c_i) \cdot f_i \cdot Q_{i,j}$. Note that $Q_{i,j} = Q(c_i, p_{L,j}, p_H)$ is calculated piece-wise as in equation 28 and it is function of the number of bidders N. The script terminates by showing the two resulting surpluses, optimal prices and benchmarks against the associated First Price Auction (FPA). The program shows results as reported in figure 12 and 13.

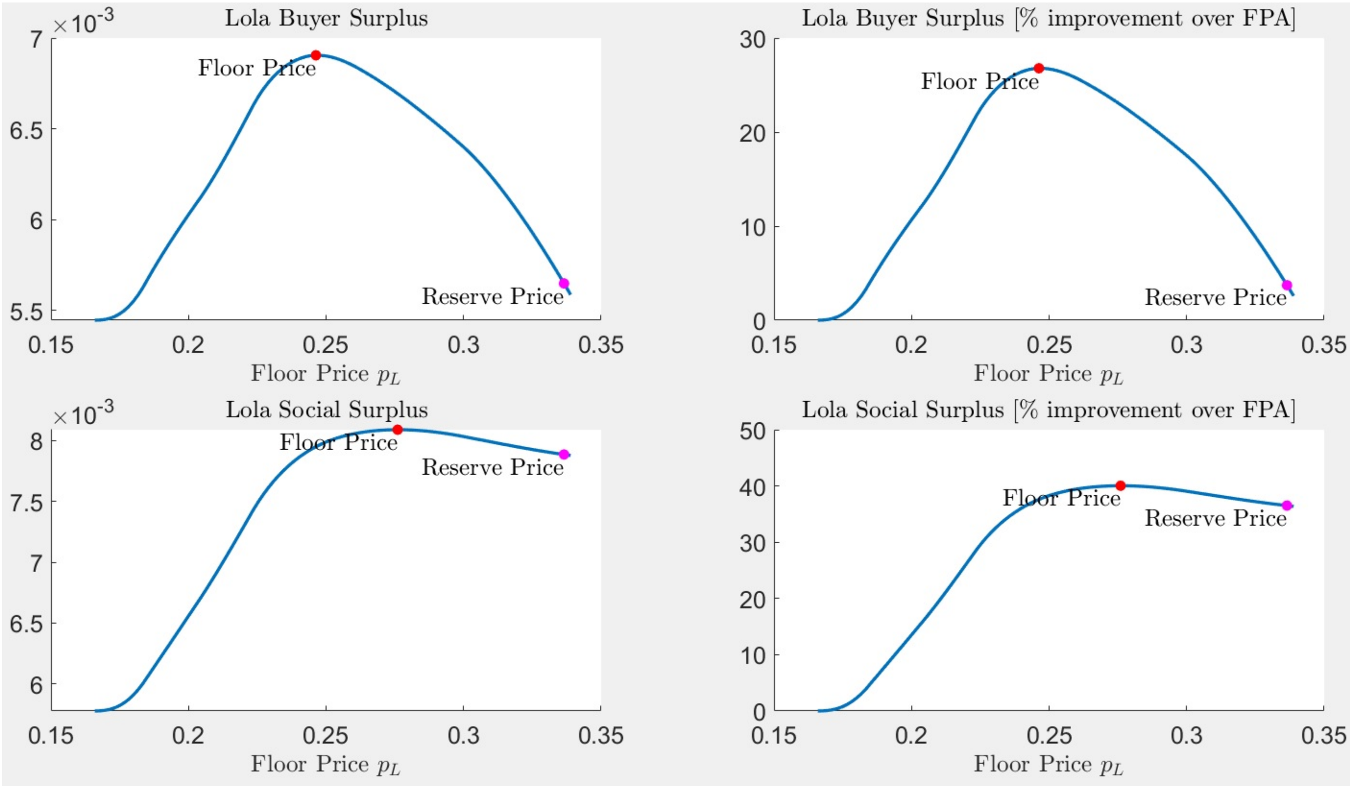


Figure 12: The figure shows the buyer surplus and social surplus in function of the floor price $p_{L,j}$. The points at which these functions are maximized correspond to the respective optimal LoLAs. In addition, the reservation price is also reported.

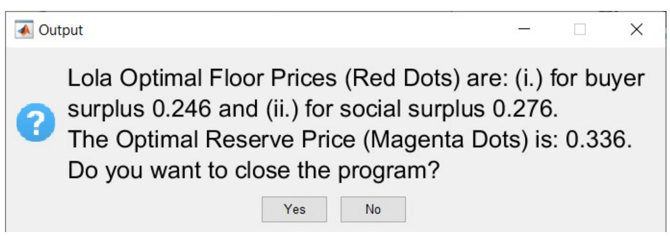


Figure 13: The figure shows the final report with the optimal floor and reservation prices.

D.2 Software 2

This software is realized in Matlab and IBM ILOG CPLEX. The application requires the same inputs as Software 1, and it computes the optimal mechanism *even when that mechanism is not a LoLA*. Therefore, Software 2 dispenses with Assumption 1 and with the requirement that $v(c_L) > c_L$. The application yields the buyer-optimal direct revelation mechanism, expressed through the interim probability $Q(c)$ that a generic bidder with cost

c wins the auction. This application is helpful to deal with settings where assumptions made in the paper are violated, and so Theorem 1 does not apply.

The entry point is "main.m". There are 5 inputs: (i.) the number of nodes T of the cost grid, (ii.) the minimum cost c_L, (iii.) the maximum cost c_H, (iv.) a vector of the willingness to pay $[v_1, \cdots, v_T]$, (v.) a vector of the cost distribution $[f_1, \cdots, f_T]$.

Given a distribution f, the virtual valuation is calculated as in (62). Then, the software passes all inputs to the script "CallCPLEX.m" in order to solve the linear program. This script generates two files: (i.) AMPL and (ii.) DAT.

The AMPL's file tells CPLEX how to generate the objective function and all constraints. In particular, it embeds the logic to generate: (i.) the demand constraints, (ii.) the non-negativity constraints, and (iii.) the monotonicity constraints.[32] The DAT's file specifies all numerical inputs.

Then, the program calls CPLEX to perform the high-scale optimization.

[32] CPLEX is preferable to Matlab because the optimization problem is large.