# Online Appendix for "Bank Runs, Fragility, and Credit Easing"

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# A Proofs

## A.1 Proof of Lemma 1

The problem of a bank under default facing a sequence of prices $\{p_t\}_{t=0}^{\infty}$ is given by

$$V_t^D(k) = \max_{k',c} \log(c) + \beta V_{t+1}^D(k') \tag{A.1}$$
$$\text{subject to: } c = (p_t + z^D)k - p_t k'.$$

We conjecture that

$$V_t^D(k) = \mathbb{B}_t^D + \frac{1}{1-\beta}\log((z^D + p_t)k). \tag{A.2}$$

Replacing this conjecture into (A.1) and substituting out consumption from the budget constraint, we have that

$$V_t^D(k) = \max_{k'} \log(z^D k + p_t(k-k')) + \beta\left[\frac{1}{1-\beta}\log(k'(p_{t+1}+z^D)) + \mathbb{B}_{t+1}^D\right]. \tag{A.3}$$

The first-order condition with respect to $k'$ is given by

$$\frac{p_t}{z^D k + p_t(k-k')} = \left(\frac{\beta}{1-\beta}\right)\frac{1}{k'} \quad \Rightarrow \quad k' = \frac{\beta(z^D + p_t)}{p_t}k. \tag{A.4}$$

By the method of undetermined coefficients, we can now verify the conjecture and solve for $\mathbb{B}_t^D$. We substitute (A.4) into the right-hand side of (A.3) and replace the conjectured guess for $V_t^D(k)$ on the left-hand side of (A.3):

$$\mathbb{B}_t^D + \frac{1}{1-\beta}\log((z^D+p_t)k) = \log\big((1-\beta)(z^D+p_t)k\big) + \beta\left[\frac{1}{1-\beta}\log\big(\beta R_{t+1}^D(z^D+p_t)k\big) + \mathbb{B}_{t+1}^D\right].$$

where we have used the definition of $R_{t+1}^D$. Rearranging this equation, we can observe that the terms multiplying $\log(k)$ cancel out. After simplifying, we obtain that the conjectured value function is verified when $\mathbb{B}_t^D$ satisfies

$$\mathbb{B}_t^D = \log(1-\beta) + \frac{\beta}{1-\beta}\log(\beta) + \frac{\beta}{1-\beta}\log\big(R_{t+1}^D\big) + \beta\mathbb{B}_{t+1}^D. \tag{A.5}$$

Iterating forward on this equation and imposing $\lim_{\tau\to\infty}\beta^{\tau}\log\left(R^D_{\tau+1}\right)=0$, as in Condition 1, we have

$$\mathbb{B}_t^D=\frac{1}{1-\beta}\left[\frac{\beta}{1-\beta}\log(\beta)+\log(1-\beta)\right]+\frac{\beta}{1-\beta}\sum_{\tau\geq t}\beta^{\tau-t}\log\left(R^D_{\tau+1}\right). \tag{A.6}$$

Replacing (A.6) in (A.2), we obtain that the value under default is given by

$$V_t^D(k)=A+\frac{1}{1-\beta}\log((z^D+p_t)k)+\frac{\beta}{1-\beta}\sum_{\tau\geq t}\beta^{\tau-t}\log\left(R^D_{\tau+1}\right),$$

where $A=(\log(1-\beta)+\frac{\beta}{1-\beta}\log(\beta))/(1-\beta)$. We thus arrived at the value of $V^D$, as stated in the lemma. $\square$

## A.2 Proof of Lemma 2

We conjecture that the value function is

$$V_t^R(n)=\frac{1}{1-\beta}\log(n)+\mathbb{B}_t^R. \tag{A.7}$$

The borrowing constraint must be such that the bank does not default at $t+1$. That is,

$$\mathbb{B}_{t+1}^R+\frac{1}{1-\beta}\log(n')\geq\mathbb{B}_{t+1}^D+\frac{1}{1-\beta}\log((z^D+p_{t+1})k').$$

Replacing $n'$ for the law of motion and manipulating this expression, we arrive at

$$b'\leq\frac{\left[(z+p_{t+1})-(z^D+p_{t+1})e^{(1-\beta)(\mathbb{B}_{t+1}^D-\mathbb{B}_{t+1}^R)}\right]}{R}k'.$$

Therefore, the borrowing constraint takes a linear form, as conjectured. In particular,

$$b'\leq\gamma_t p_{t+1}k',$$

where $\gamma_t$ is the leverage parameter and is given by

$$\gamma_t=\frac{(z+p_{t+1})-(z^D+p_{t+1})e^{(1-\beta)(\mathbb{B}_{t+1}^D-\mathbb{B}_{t+1}^R)}}{Rp_{t+1}}. \tag{A.8}$$

We establish next that if $R^k_{t+1}>R$, the borrowing constraint binds at time $t$.

**Lemma A.1.** *If $R^k_{t+1}>R$, then the bank is against the borrowing constraint.*

*Proof.* The proof is by contradiction. Denote by $(c_t^*,k_{t+1}^*,b_{t+1}^*)$ the solution to the bank problem with $b_{t+1}^*<\gamma_t p_{t+1}k_{t+1}^*$. Consider the following alternative policy: $(c_t^*,\tilde{k}_{t+1}+\Delta,\tilde{b}_{t+1}+$

$\Delta p_t$), with $0 < \Delta < \frac{\gamma_t p_{t+1}\tilde{k}_{t+1} - \tilde{b}_{t+1}}{p_t - \gamma_t p_{t+1}}$. The alternative allocation is feasible and delivers higher net worth, since

$$\begin{aligned}\tilde{n}_{t+1} &= (\tilde{k}_{t+1} + \Delta)(z + p_{t+1}) - R\tilde{b}_{t+1} + \Delta p_t) \\ &= \tilde{k}_{t+1}(z + p_{t+1}) - R\tilde{b}_{t+1}) + \Delta(R^k_{t+1} - R) \\ &> \tilde{k}_{t+1}(z + p_{t+1}) - R\tilde{b}_{t+1} = n^*_{t+1},\end{aligned}$$

where $\tilde{n}_{t+1}$ and $n^*_{t+1}$ are respectively the net worth under the alternative and original allocations. Since the alternative allocation delivers the same consumption and higher net worth, this contradicts that the original allocation with a slack borrowing constraint is optimal. □

We now proceed to finish the proof of Lemma 2. Consider first the case with $R^k_{t+1} > R$. From Lemma A.1, we know that borrowing constraint binds, and hence we can use $b' = \gamma_t p_{t+1} k'$. Replacing this in the law of motion for net worth and consumption, we obtain

$$n' = k'(z + p_{t+1}) - \gamma_t p_{t+1} k' R$$

and $c = \ n - k'(p_t - \gamma_t p_{t+1})$. Replacing these two expressions and the conjectured value function (A.7) in the right-hand side of equation (2), we have

$$V^R_t(n) = \max_{k'} \log(n - k'(p_t - \gamma_t p_{t+1})) + \beta \left[\frac{1}{1-\beta} \log(k'(z + p_{t+1}(1 - \gamma_t R)) + \mathbb{B}^R_{t+1}\right], \quad \text{(A.9)}$$

The first-order condition with respect to $k'$ is

$$\frac{p_t - \gamma_t p_{t+1}}{n - k'(p_t - \gamma_t p_{t+1})} = \left(\frac{\beta}{1-\beta}\right)\frac{1}{k'}$$

and yields

$$k' = \frac{\beta n}{p_t - \gamma p_{t+1}}, \quad c = (1-\beta)n, \tag{A.10}$$

and

$$n' = \frac{\beta n}{p_t - \gamma_t p_{t+1}}(z + p_{t+1}(1 - \gamma_t R)).$$

Notice that by definition of $R^e_{t+1}$, we have that

$$R^e_{t+1} = \frac{z + p_{t+1}(1 - \gamma_t R)}{p_t - \gamma_t p_{t+1}}. \tag{A.11}$$

If we use (A.10) and (A.11) and replace (A.7), on the left-hand side of (A.9)

$$\mathbb{B}^R_t + \frac{1}{1-\beta}\log(n) = \log\left((1-\beta)n\right) + \beta\left[\frac{1}{1-\beta}\log(\beta R^e_{t+1} n) + \mathbb{B}^R_{t+1}\right].$$

Rearranging this equation, we can observe that the $\log(n)$ terms cancel out. We therefore obtain that the conjecture is verified when the $\mathbb{B}_t^R$ satisfies

$$\mathbb{B}_t^R = \frac{\beta}{1-\beta}\log(\beta) + \log(1-\beta) + \frac{\beta}{1-\beta}\log(R_{t+1}^e) + \beta\mathbb{B}_{t+1}^R. \tag{A.12}$$

Iterating forward and imposing $\lim_{t\to\infty}\beta^t\mathbb{B}_t^R = 0$, we have

$$\mathbb{B}_t^R = \frac{1}{1-\beta}\left[\frac{\beta}{1-\beta}\log(\beta) + \log(1-\beta)\right] + \frac{\beta}{1-\beta}\sum_{\tau\geq t}\beta^{\tau-t}\log\left(R_{\tau+1}^e\right), \tag{A.13}$$

so the value under repayment is given by

$$V_t^R(n) = \frac{1}{1-\beta}\log(n) + \mathbb{B}_t^R,$$

where $\mathbb{B}_t^R$ is given by (A.13). Equivalently, using the definitions of $R^e$ and $A$, we arrive at the expression for $V^R$ in the Lemma.

Notice also from (A.10) and (A.10) and the fact that $b' = \gamma_t p_{t+1} k'$ that we have also verified the policies in item (ii) of the lemma for the case of $R_{t+1}^k > R$.

Finally, it is straightforward to verify that in the case of $R_{t+1}^k = R$, the conjectured value function (A.7) solves the Bellman equation, and the bank is now indifferent across $b', k'$, while consumption remains given by (A.10). This completes the proofs of the three items in the lemma. □

## A.3 Proof of Proposition 1

Rearranging (A.8), we obtain

$$\frac{\beta}{1-\beta}\log\left(\frac{z + p_{t+1}(1-\gamma_t R)}{z^D + p_{t+1}}\right) = \beta(\mathbb{B}_{t+1}^D - \mathbb{B}_{t+1}^R). \tag{A.14}$$

To obtain an expression for the right-hand side of (A.14), we use (A.5) and (A.12), and obtain the result that the difference in the intercepts in the value functions is given by

$$\mathbb{B}_t^D - \mathbb{B}_t^R = \beta(\mathbb{B}_{t+1}^D - \mathbb{B}_{t+1}^R) + \frac{\beta}{1-\beta}\left[\log(R_{t+1}^D) - \log(R_{t+1}^e)\right]), \tag{A.15}$$

Using the definition of $R_{t+1}^D$ and $R_{t+1}^e$ and replacing (A.14), we get that

$$\mathbb{B}_t^D - \mathbb{B}_t^R = \beta(\mathbb{B}_{t+1}^D - \mathbb{B}_{t+1}^R) - \frac{\beta}{1-\beta}\left[\log\left(\frac{z + p_{t+1}(1-\gamma_t R)}{p_t - \gamma_t p_{t+1}}\right) - \log\left(\frac{z^D + p_{t+1}}{p_t}\right)\right].$$

Using that using that $\log(p_t - \gamma_t p_{t+1}) = \log\left(1 - \gamma_t\frac{p_{t+1}}{p_t}\right) + \log(p_t)$, simplifying, and replacing (A.14), we arrive at

$$\mathbb{B}_t^D - \mathbb{B}_t^R = \frac{\beta}{1-\beta}\left[\log\left(1 - \gamma_t\frac{p_{t+1}}{p_t}\right)\right]. \tag{A.16}$$

If we update (A.16) one period forward and replace in (A.14), we arrive at

$$\frac{z + p_{t+1}(1-\gamma_t R)}{z^D + p_{t+1}} = \left(1 - \gamma_{t+1}\frac{p_{t+2}}{p_{t+1}}\right)^\beta,$$

which is the expression in the proposition. □

## A.4 Proof of Lemma 4

The capital demand of a repaying bank with productivity $z_0$ can be written as

$$k_1^R(z_0) = \beta\frac{(z_0+p_0)\overline{K}-RB_0}{p_0-\gamma_0 p_1} = \beta\left(\frac{(z_0+\gamma_0 p_1)\overline{K}-RB_0}{p_0-\gamma_0 p_1}+\overline{K}\right).$$

We know from before that $k_1^R(z^F) \geq k_1^D$. We also have that $k_1^R(z^{Run}) \geq k_1^D$, as $z^{Run} \geq z^F$. So, independently of the default threshold, $\hat{z}$, we have

$$\int_{\hat{z}}^{z}(k_1^R(z_0)-k_1^D)dF(z_0) > 0,$$

where the inequality follows as the demand for capital is strictly increasing in $z_0$, and the threshold is interior. Market clearing at $t=0$ requires that

$$\int_{\hat{z}}^{z} k_1^R(z_0)dF(z_0) + k_1^D F(\hat{z}) = \overline{K}.$$

Subtracting the previous inequality, we have that

$$\int_{\hat{z}}^{z} k_1^R(z_0)dF(z_0) + k_1^D F(\hat{z}) - \int_{\hat{z}}^{z}(k_1^R(z_0)-k_1^D)dF(z_0) < \overline{K}.$$

And thus, $k_1^D < \overline{K}$. It follows then that $\int_{\hat{z}}^{z}(k_1^R(z_0)-\overline{K})dF(z_0) > 0$. The capital demand inequality implies

$$\int_{\hat{z}}^{z}\left(\beta\frac{(z_0+p_0)\overline{K}-RB_0}{p_0-\gamma_0 p_1}-\overline{K}\right)dF(z_0) > 0$$
$$\Rightarrow \beta\int_{\hat{z}}^{z}\left(\frac{(z_0+\gamma_0 p_1)\overline{K}-RB_0}{p_0-\gamma_0 p_1}\right)dF(z_0) > (1-\beta)\overline{K}(1-F(\hat{z})) > 0,$$

which delivers

$$\int_{\hat{z}}^{z}((z_0+\gamma_0 p_1)\overline{K}-RB_0)dF(z_0) > 0,$$

as $p_0 > \gamma_0 p_1$, an equilibrium requirement. We can then rewrite the capital demand of repaying banks as:

$$\int_{\hat{z}}^{z} k_1^R(z_0)dF(z_0) = \beta\left[\frac{\int_{\hat{z}}^{z}((z_0+\gamma_0 p_1)\overline{K}-RB_0)dF(z_0)}{p_0-\gamma_0 p_1}+\overline{K}(1-F(\hat{z}))\right].$$

Given what we have just shown, the numerator of the first term inside the square brackets is strictly positive, and thus it follows that an increase in $p_0$ strictly reduces demand from inframarginal repaying banks. $\square$