Online Appendix to *Smart Contracts and the Coase Conjecture*

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A An Equilibrium when $\mathcal{C} = \mathcal{D}$

This appendix is concerned with constructing an equilibrium when the seller's contract space is restricted to include only *simple and direct* contracts. In section A.1, we use a fixed-point argument to define a set of contracts $D^*(\mu)$ parameterised by $\mu \in [0,1]$, such that the seller deploys a contract in $D^*(\mu)$ when her belief about the buyer having high valuation is μ. In section A.2, we describe and analyse a class of auxiliary games whose sequential equilibria map to off-path parts of our equilibrium assessment. Finally in section A.3, we use the results from sections A.1 and A.2 to build a complete assessment and prove that it constitutes an equilibrium.

A.1 Deployed Contracts

Let $v_h > v_l > 0$ and $\delta \in (0,1)$. For $\mu \in [0,1]$, let $\bar{J}(\mu) = \max\{v_l, \mu v_h\}$ and

$$\underline{J}(\mu) = \begin{cases} v_l & \text{if } \mu < 1, \\ v_h & \text{if } \mu = 1. \end{cases}$$

A function $J : [0,1] \to \mathbb{R}$ is piece-wise linear if $[0,1]$ can be partitioned into countably many intervals such that J is affine on each of these intervals. We denote by $\mathcal{J}$ the set of non-decreasing, piece-wise linear and convex functions $J : [0,1] \to [v_l, v_h]$ such that, for all $\mu \in [0,1]$, $\bar{J}(\mu) \geq J(\mu) \geq \underline{J}(\mu)$.

Given $J \in \mathcal{J}$, we construct a function $\mathcal{A}J \in \mathcal{J}$. We set $\mathcal{A}J(0) = v_l$ and $\mathcal{A}J(1) = v_h$. For $\mu \in (0,1)$, $\mathcal{A}J(\mu)$ is the value to the maximisation problem described below.

Fix $\mu \in (0,1)$. A trading time s is a random time whose distribution $\langle s \rangle$ depends on the buyer's valuation $v \in \{v_l, v_h\}$. We identify $\langle s \rangle = (q^h, q^l) \in \mathcal{Q} \times \mathcal{Q}$, where $\mathcal{Q} = \{q \in [0,1]^{\mathbb{N}} : \sum_{t \geq 0} q_t \leq 1\}$. For $t \geq 0$ and $i \in \{h, l\}$, q_t^i is interpreted as the probability that trade occurs in period t if the buyer's valuation in v_i. Given $\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}$ and $t \geq 0$, if $\mathbb{P}(s \geq t) = 1 - \sum_{k=0}^{t-1} \left(\mu q_k^h + (1-\mu) q_k^l\right) > 0$, we define:

$$\mu_t = \mathbb{P}(v = v_h | s \geq t) = \frac{\mu\left(1 - \sum_{k=0}^{t-1} q_k^h\right)}{\mu\left(1 - \sum_{k=0}^{t-1} q_k^h\right) + (1-\mu)\left(1 - \sum_{k=0}^{t-1} q_k^l\right)}.$$

Define, for $\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}$, the objective function:

$$\Omega\big(\langle s \rangle \big| \mu\big) = \sum_{k=0}^{\infty} \delta^k \Big(q_k^h \mu v_h + q_k^l (v_l - \mu v_h)\Big),$$

and for $t \geq 1$, the constraint mapping:

$$G_t\big(\langle s \rangle \big| \mu, J\big) = \begin{cases} \mathbb{P}\big(s \geq t\big) J(\mu_t) - \sum_{k=t}^{\infty} \delta^{k-t} \Big(q_k^h \mu v_h + q_k^l (1-\mu) v_l\Big) & \text{if } \mathbb{P}\big(s \geq t\big) > 0, \\ 0 & \text{if } \mathbb{P}\big(s \geq t\big) = 0. \end{cases}$$

Note that $\Omega(\cdot|\mu)$ is linear and, for each $t \geq 1$, $G_t(\cdot|\mu, J)$ is convex on the convex set $\mathcal{Q} \times \mathcal{Q}$. Denoting $G\big(\langle s \rangle \big| \mu, J\big) = \Big(G_t\big(\langle s \rangle \big| \mu, J\big)\Big)_{t \geq 1}$, the maximisation problem is given by:

$$\begin{aligned} \mathcal{A}J(\mu) = \max_{\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}} \; & \Omega\big(\langle s \rangle \big| \mu\big) \\ \text{s.t. } & G\big(\langle s \rangle \big| \mu, J\big) \leq 0. \end{aligned} \tag{1}$$

If $\mu \leq \frac{v_l}{v_h}$, an obvious solution is given by $s = 0$, that is $q_0^h = q_0^l = 1$. Thus, in this case, $\mathcal{A}J(\mu) = v_l$. Therefore, we focus on the case $\mu > \frac{v_l}{v_h}$. We first prove preliminary results which help describe a candidate solution $\langle s^* \rangle(\mu, J)$ to (1). Then, we prove that $\langle s^* \rangle(\mu, J)$ indeed solves the maximisation problem, and we verify that $\mathcal{A}J$ indeed belongs to $\mathcal{J}$.

Lemma 1. *For $\tilde{\mu} \in (0,1)$, there exists $\tilde{\mu}' \in [0, \mu)$ such that:*

$$\delta J(\tilde{\mu}') = v_h - \frac{1 - \tilde{\mu}'}{1 - \tilde{\mu}} \big(v_h - J(\tilde{\mu})\big),$$

if and only if $\tilde{\mu} \geq \frac{(1-\delta) v_l}{v_h - \delta v_l} \equiv \bar{\mu} \in (0, \frac{v_l}{v_h})$. In this case, $\tilde{\mu}'$ is unique.

Proof. Uniqueness is guaranteed by the fact that J is convex, and $\delta J(\tilde{\mu}) < J(\tilde{\mu})$. If $\delta J(0) \geq v_h - \frac{v_h - J(\tilde{\mu})}{1-\tilde{\mu}}$, then a solution exists by the intermediate value theorem. Otherwise, no solution exists, by convexity. Therefore, a solution exists if and only if:

$$J(\tilde{\mu}) \leq \tilde{\mu} v_h + (1-\tilde{\mu})\delta v_l.$$

This inequality is satisfied if $\tilde{\mu} \geq \frac{v_l}{v_h}$ since $J(\tilde{\mu}) \leq \max\{\tilde{\mu} v_h, v_l\}$. For $\tilde{\mu} < \frac{v_l}{v_h}$, $J(\tilde{\mu}) = v_l$ and the inequality is equivalent to $\tilde{\mu} \geq \bar{\mu}$. □

In view of lemma 1, define on $[0,1)$ the function $\tilde{\mu}'$ such that $\tilde{\mu}'(\tilde{\mu}) = 0$ if $\tilde{\mu} \leq \bar{\mu}$, and:

$$\delta J(\tilde{\mu}') = v_h - \frac{1-\tilde{\mu}'}{1-\tilde{\mu}}\big(v_h - J(\tilde{\mu})\big),$$

otherwise. Now, given $\mu_1 \in (0,1)$, we construct the non-increasing sequence $(\mu_k)_{k\geq 1}$ such that, for $k \geq 1$, $\mu_{k+1} = \tilde{\mu}'(\mu_k)$.

Lemma 2. *There exists $T \geq 1$ such that $\mu_T < \bar{\mu}$.*

Proof. Otherwise, $(\mu_k)_{k\geq 1}$ has a limit $\mu_\infty \in [\bar{\mu}, 1)$. Since J is continuous on $[0,1)$, the limit must satisfy:

$$\delta J(\mu_\infty) = J(\mu_\infty),$$

which is impossible since $J \geq v_l > 0$ and $\delta < 1$. □

We define $T = \min\{t \geq 1 : \mu_t < \bar{\mu}\}$. Now, let:

$$\rho^J(\mu_1) = \frac{v_l}{v_h - v_l} \prod_{t=1}^{T} \Big(1 + \frac{1-\delta}{\delta} \frac{v_h}{v_h - J(\mu_t) - (1-\mu_t)J'_+(\mu_t)}\Big) \in (0, \infty],$$

where J'_+ is the right-derivative of J. By convexity and piece-wise linearity of J, ρ^J is a non-decreasing and right-continuous step function, which may take infinite values if $v_h = J(\mu_1) + (1-\mu_1)J'_+(\mu_1)$.

Lemma 3. *For any $\mu_1 \in (0,1)$, $\rho^J(\mu_1) > \frac{\mu_1}{1-\mu_1}$.*

Proof. If $T = 1$, then:

$$\frac{\mu_1}{1-\mu_1} < \frac{\bar{\mu}}{1-\bar{\mu}} = \frac{(1-\delta)v_l}{v_h - v_l} = \frac{\rho^J(\mu_1)}{\frac{1}{1-\delta} + \frac{1}{\delta}\frac{v_h}{v_h - v_l}} < \rho^J(\mu_1).$$

If $T > 1$, note that for $t < T$, since $\mu_{t+1} < \mu_t$ and J is convex:

$$J'_+(\mu_t) \geq \frac{J(\mu_t) - J(\mu_{t+1})}{\mu_t - \mu_{t+1}}.$$

As a result:

$$1+\frac{1-\delta}{\delta}\frac{v_h}{v_h-J(\mu_t)-(1-\mu_t)J'_+(\mu_t)} \geq \frac{(\mu_t-\mu_{t+1})v_h-\delta(1-\mu_{t+1})J(\mu_t)+\delta(1-\mu_t)J(\mu_{t+1})}{\delta\big[(\mu_t-\mu_{t+1})v_h-(1-\mu_{t+1})J(\mu_t)+(1-\mu_t)J(\mu_{t+1})\big]}.$$

Substitute in the numerator $\delta J(\mu_{t+1}) = v_h - \frac{1-\mu_{t+1}}{1-\mu_t}\big(v_h - J(\mu_t)\big)$, and in the denominator $J(\mu_t) = v_h - \frac{1-\mu_t}{1-\mu_{t+1}}\big(v_h - \delta J(\mu_{t+1})\big)$, to obtain:

$$1+\frac{1-\delta}{\delta}\frac{v_h}{v_h-J(\mu_t)-(1-\mu_t)J'_+(\mu_t)} \geq \frac{(1-\mu_{t+1})J(\mu_t)}{\delta(1-\mu_t)J(\mu_{t+1})}.$$

As a result:

$$\rho^J(\mu_1) \geq \frac{1-\mu_T}{\delta^T}\frac{v_h-\delta v_l}{(v_h-v_l)^2}\frac{J(\mu_1)}{1-\mu_1}.$$

Since $J(\mu_1) = v_h - \frac{1-\mu_1}{1-\mu_2}\big(v_h - \delta J(\mu_2)\big)$, we have:

$$\frac{1-\mu_T}{\delta^T}\frac{v_h-\delta v_l}{(v_h-v_l)^2}\frac{J(\mu_1)}{1-\mu_1} - \frac{\mu_1}{1-\mu_1} = \frac{1-\mu_T}{\delta^{T-1}}\frac{v_h-\delta v_l}{(v_h-v_l)^2}\frac{J(\mu_2)}{1-\mu_2} - \frac{\mu_2}{1-\mu_2} + \left(\frac{1-\mu_T}{\delta^T}\frac{v_h-\delta v_l}{(v_h-v_l)^2}v_h - 1\right)\left(\frac{1}{1-\mu_1}-\frac{1}{1-\mu_2}\right).$$

Now, since $\mu_T < \bar{\mu} = \frac{(1-\delta)v_l}{v_h-\delta v_l}$:

$$\frac{1-\mu_T}{\delta^T}\frac{v_h-\delta v_l}{(v_h-v_l)^2}v_h - 1 > \frac{1}{\delta^T}\frac{v_h}{v_h-v_l} - 1 > 0,$$

and since $\mu_1 > \mu_2$, we obtain:

$$\frac{1-\mu_T}{\delta^T}\frac{v_h-\delta v_l}{(v_h-v_l)^2}\frac{J(\mu_1)}{1-\mu_1} - \frac{\mu_1}{1-\mu_1} > \frac{1-\mu_T}{\delta^{T-1}}\frac{v_h-\delta v_l}{(v_h-v_l)^2}\frac{J(\mu_2)}{1-\mu_2} - \frac{\mu_2}{1-\mu_2}.$$

The same argument applies by induction to establish:

$$\frac{1-\mu_T}{\delta^T}\frac{v_h-\delta v_l}{(v_h-v_l)^2}\frac{J(\mu_1)}{1-\mu_1} - \frac{\mu_1}{1-\mu_1} > \frac{1}{\delta}\frac{v_h-\delta v_l}{(v_h-v_l)^2}v_l - \frac{\mu_T}{1-\mu_T} > \frac{v_l}{v_h-v_l}\frac{(1-\delta)v_h+\delta^2(v_h-v_l)}{\delta(v_h-v_l)} > 0.$$

To summarise:

$$\rho^J(\mu_1) > \frac{1-\mu_T}{\delta^T}\frac{v_h-\delta v_l}{(v_h-v_l)^2}\frac{J(\mu_1)}{1-\mu_1} > \frac{\mu_1}{1-\mu_1},$$

which proves the claim. □

Now, given the prior $\mu \in (\frac{v_l}{v_h}, 1)$, let:

$$\mu_1^* = \min\Big\{\mu_1 \in [0,1) : \rho^J(\mu_1) > \frac{\mu}{1-\mu}\Big\}.$$

As above, we iterate on $\tilde{\mu}'$ to construct the path $(\mu_1^*, \mu_2^*, ..., \mu_{T^*}^*)$, where $\mu_{T^*}^* \in [0, \bar{\mu})$. Defining $\mu_0^* = \mu$ and

$\mu^*_{T^*+1} = 0$, the candidate solution $\langle s^* \rangle(\mu, J)$ is characterised by:

$$\forall t \in \{0, ..., T^*\}, \quad q_t^{*h} = \frac{1-\mu}{\mu}\Big(\frac{1}{1-\mu_t^*} - \frac{1}{1-\mu_{t+1}^*}\Big),$$

$$q_{T^*}^{*l} = 1 - q_{T^*+1}^{*l} = \frac{(1-\delta)v_l - \mu_{T^*}^*(v_h - \delta v_l)}{(1-\delta)(1-\mu_{T^*}^*)v_l}.$$

Proposition 1. *$\langle s^* \rangle(\mu, J)$ solves problem (1).*

Proof. To simplify notations, we omit the dependence on μ and J of $\langle s^* \rangle$, Ω and G. We first define a sequence of non-negative Lagrange multipliers $(\lambda_t)_{t \geq 1}$ as follows.

If:

$$\frac{v_l}{v_h - v_l}\prod_{t=1}^{T^*}\Big(1 + \frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t^*) - (1-\mu_t^*)J'_-(\mu_t^*)}\Big) \leq \frac{\mu}{1-\mu} < \frac{v_l}{v_h - v_l}\prod_{t=1}^{T^*}\Big(1 + \frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t^*) - (1-\mu_t^*)J'_+(\mu_t^*)}\Big),$$

where $J'_-(0) = 0$, then there exists $\big(J'_*(\mu_t^*)\big)_{1 \leq t \leq T^*} \in \prod_{1 \leq t \leq T^*}\big[J'_-(\mu_t^*), J'_+(\mu_t^*)\big]$ such that:

$$\frac{\mu}{1-\mu} = \frac{v_l}{v_h - v_l}\prod_{t=1}^{T^*}\Big(1 + \frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t^*) - (1-\mu_t^*)J'_*(\mu_t^*)}\Big).$$

Otherwise, it must be that $\mu^*_{T^*} = 0$, and:

$$\frac{v_l}{v_h - v_l}\prod_{t=1}^{T^*-1}\Big(1 + \frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t^*) - (1-\mu_t^*)J'_-(\mu_t^*)}\Big) \leq \frac{\mu}{1-\mu} < \frac{v_l}{v_h - v_l}\prod_{t=1}^{T^*}\Big(1 + \frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t^*) - (1-\mu_t^*)J'_-(\mu_t^*)}\Big).$$

Then, we define $J'_*(\mu_t^*) = J'_-(\mu_t^*)$ for all $t \in \{1, ..., T^*\}$.

In both cases, for $t \in \{1, ..., T^*\}$, let:

$$\lambda_t = \frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t^*) - (1-\mu_t^*)J'_*(\mu_t^*)}\prod_{k=1}^{t-1}\Big(1 + \frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_k^*) - (1-\mu_k^*)J'_*(\mu_k^*)}\Big),$$

and for $t > T^*$, let $\lambda_t = 0$. We also introduce the notation, for $t \geq 0$:

$$\Lambda_t = \sum_{k=1}^{t}\lambda_k = -1 + \prod_{k=1}^{\min\{T^*, t\}}\Big(1 + \frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_k^*) - (1-\mu_k^*)J'_*(\mu_k^*)}\Big).$$

With these definitions, we establish below that $\langle s^* \rangle$ maximises on $\mathcal{Q} \times \mathcal{Q}$ the Lagrangian:

$$\Omega\big(\langle s \rangle\big) - \sum_{t=1}^{\infty}\delta^t\lambda_t G_t\big(\langle s \rangle\big).$$

It follows that $\langle s^* \rangle$ solves problem (1), since for any $\langle s \rangle$ feasible, $G\big(\langle s \rangle\big) \leq 0$, so:

$$\Omega\big(\langle s \rangle\big) \leq \Omega\big(\langle s \rangle\big) - \sum_{t=1}^{\infty} \delta^t \lambda_t G_t\big(\langle s \rangle\big) \leq \Omega\big(\langle s^* \rangle\big) - \sum_{t=1}^{\infty} \delta^t \lambda_t G_t\big(\langle s^* \rangle\big) = \Omega\big(\langle s^* \rangle\big).$$

For $\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}$, let:

$$\begin{aligned} f_{\langle s \rangle} &: [0,1] \to \mathbb{R} \\ &\qquad \alpha \mapsto \Omega\big(\alpha\langle s \rangle + (1-\alpha)\langle s^* \rangle\big) - \sum_{t=1}^{\infty} \delta^t \lambda_t G_t\big(\alpha\langle s \rangle + (1-\alpha)\langle s^* \rangle\big). \end{aligned}$$

The desired result is implied if, for any $\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}$, $f_{\langle s \rangle}$ is maximised at $\alpha = 0$. Since $f_{\langle s \rangle}$ is concave, it is sufficient to show that $f_{\langle s \rangle}$ is differentiable at $\alpha = 0$, with $f'_{\langle s \rangle}(0) \leq 0$.

Fix $\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}$ and denote $\langle s_\alpha \rangle = \alpha\langle s \rangle + (1-\alpha)\langle s^* \rangle$. Ω is linear and $\lambda_t = 0$ when $t > T^*$, thus it is sufficient to show differentiability of the term $\alpha \mapsto G_t\big(\langle s_\alpha \rangle\big)$, when $t \in \{1, ..., T^*\}$. In this case, if $\mathbb{P}(s \geq t) > 0$, then:

$$\begin{aligned} G_t\big(\langle s_\alpha \rangle\big) =& \mathbb{P}(s_\alpha \geq t) J\left(\alpha \frac{\mathbb{P}(s \geq t)}{\mathbb{P}(s_\alpha \geq t)} \mu_t + (1-\alpha) \frac{\mathbb{P}(s^* \geq t)}{\mathbb{P}(s_\alpha \geq t)} \mu_t^*\right) - \alpha\Big(\mathbb{P}(s \geq t) J(\mu_t) - G_t\big(\langle s \rangle\big)\Big) \\ & - (1-\alpha)\Big(\mathbb{P}(s^* \geq t) J(\mu_t^*) - G_t\big(\langle s^* \rangle\big)\Big), \end{aligned}$$

where $\mathbb{P}(s_\alpha \geq t) = \alpha\mathbb{P}(s \geq t) + (1-\alpha)\mathbb{P}(s^* \geq t)$. This expression has a right- and left-derivative at any α, which coincide if $\alpha \frac{\mathbb{P}(s \geq t)}{\mathbb{P}(s_\alpha \geq t)} \mu_t + (1-\alpha) \frac{\mathbb{P}(s^* \geq t)}{\mathbb{P}(s_\alpha \geq t)} \mu_t^*$ is not at a kink of J and write:

$$\begin{aligned} \frac{\partial G_t\big(\langle s_\alpha \rangle\big)}{\partial \alpha} =& \frac{\mathbb{P}(s \geq t)\mathbb{P}(s^* \geq t)}{\mathbb{P}(s_\alpha \geq t)} (\mu_t - \mu_t^*) J'\left(\alpha \frac{\mathbb{P}(s \geq t)}{\mathbb{P}(s_\alpha \geq t)} \mu_t + (1-\alpha) \frac{\mathbb{P}(s^* \geq t)}{\mathbb{P}(s_\alpha \geq t)} \mu_t^*\right) \\ & + \Big(\mathbb{P}(s \geq t) - \mathbb{P}(s^* \geq t)\Big) J\left(\alpha \frac{\mathbb{P}(s \geq t)}{\mathbb{P}(s_\alpha \geq t)} \mu_t + (1-\alpha) \frac{\mathbb{P}(s^* \geq t)}{\mathbb{P}(s_\alpha \geq t)} \mu_t^*\right) \\ & - \left(\Big(\mathbb{P}(s \geq t) J(\mu_t) - G_t\big(\langle s \rangle\big)\Big) - \Big(\mathbb{P}(s^* \geq t) J(\mu_t^*) - G_t\big(\langle s^* \rangle\big)\Big)\right), \end{aligned}$$

Now, using the fact that $G_t\big(\langle s \rangle\big) = \mathbb{P}\big(s \geq t\big) J(\mu_t) - \sum_{k=t}^{\infty} \delta^{k-t} \Big(q_k^h \mu v_h + q_k^l (1-\mu) v_l\Big)$ and $G_t\big(\langle s^* \rangle\big) = 0$, we obtain:

$$\left.\frac{\partial G_t\big(\langle s_\alpha \rangle\big)}{\partial \alpha}\right|_{\alpha=0} = \mathbb{P}(s \geq t)\Big(J(\mu_t^*) + (\mu_t - \mu_t^*) J'_{\to \mu_t}(\mu_t^*)\Big) - \sum_{k=t}^{\infty} \delta^{k-t} \Big(q_k^h \mu v_h + q_k^l (1-\mu) v_l\Big), \tag{2}$$

or equivalently:

$$\frac{\partial G_t\big(\langle s_\alpha\rangle\big)}{\partial \alpha}\bigg|_{\alpha=0} = \mu\Big(1-\sum_{k=0}^{t-1} q_k^h\Big)\Big(J(\mu_t^*)+(1-\mu_t^*)J'_{\to\mu_t}(\mu_t^*)\Big)+(1-\mu)\Big(1-\sum_{k=0}^{t-1} q_k^l\Big)\Big(J(\mu_t^*)-\mu_t^* J'_{\to\mu_t}(\mu_t^*)\Big)$$
$$-\sum_{k=t}^{\infty}\delta^{k-t}\Big(q_k^h\mu v_h+q_k^l(1-\mu)v_l\Big),$$

where:

$$J'_{\to\mu_t}(\mu_t^*)=\begin{cases} J'_-(\mu_t^*) & \text{if } \mu_t<\mu_t^*,\\ J'_+(\mu_t^*) & \text{if } \mu_t>\mu_t^*,\\ J'_*(\mu_t^*) & \text{if } \mu_t=\mu_t^*.\end{cases}$$

This expression is valid if $\mathbb{P}(s\geq t)=0$, in which case $G_t\big(\langle s_\alpha\rangle\big)=(1-\alpha)G_t\big(\langle s^*\rangle\big)=0$, if we extend the definition $J'_{\to\mu_t}(\mu_t^*)=J'_*(\mu_t^*)$ when $\mathbb{P}(s\geq t)=0$. Thus, the derivative at $\alpha=0$ of $f_{\langle s\rangle}$ writes:

$$f'_{\langle s\rangle}(0)=-\Omega\big(\langle s^*\rangle\big)+\sum_{k=0}^{\infty}\delta^k\Big(q_k^h\mu v_h+q_k^l(v_l-\mu v_h)\Big)-\sum_{t=1}^{T^*}\delta^t\lambda_t\Bigg(\mu\Big(1-\sum_{k=0}^{t-1}q_k^h\Big)\Big(J(\mu_t^*)+(1-\mu_t^*)J'_{\to\mu_t}(\mu_t^*)\Big)$$
$$+(1-\mu)\Big(1-\sum_{k=0}^{t-1}q_k^l\Big)\Big(J(\mu_t^*)-\mu_t^*J'_{\to\mu_t}(\mu_t^*)\Big)-\sum_{k=t}^{\infty}\delta^{k-t}\Big(q_k^h\mu v_h+q_k^l(1-\mu)v_l\Big)\Bigg).$$

Rearranging, we get:

$$\begin{aligned} f'_{\langle s\rangle}(0)=&-\Omega\big(\langle s^*\rangle\big)+\sum_{k=0}^{\infty}\delta^k\Big[q_k^h\mu v_h(1+\Lambda_k)+q_k^l\big(v_l-\mu v_h+(1-\mu)v_l\Lambda_k\big)\Big]\\ &+\sum_{k=0}^{T^*-1}\Bigg(\mu q_k^h\sum_{t=k+1}^{T^*}\delta^t\lambda_t\Big(J(\mu_t^*)+(1-\mu_t^*)J'_{\to\mu_t}(\mu_t^*)\Big)+(1-\mu)q_k^l\sum_{t=k+1}^{T^*}\delta^t\lambda_t\Big(J(\mu_t^*)-\mu_t^*J'_{\to\mu_t}(\mu_t^*)\Big)\Bigg)\\ &-\sum_{t=1}^{T^*}\delta^t\lambda_t\Big(J(\mu_t^*)+(\mu-\mu_t^*)J'_{\to\mu_t}(\mu_t^*)\Big). \end{aligned} \tag{3}$$

Using equation (2), note that: $f'_{\langle s\rangle}(0)\leq H\big(\langle s\rangle\big)$, where:

$$\begin{aligned} H\big(\langle s\rangle\big)=&-\Omega\big(\langle s^*\rangle\big)+\sum_{k=0}^{\infty}\delta^k\Big[q_k^h\mu v_h(1+\Lambda_k)+q_k^l\big(v_l-\mu v_h+(1-\mu)v_l\Lambda_k\big)\Big]\\ &+\sum_{k=0}^{T^*-1}\Bigg(\mu q_k^h\sum_{t=k+1}^{T^*}\delta^t\lambda_t\Big(J(\mu_t^*)+(1-\mu_t^*)J'_*(\mu_t^*)\Big)+(1-\mu)q_k^l\sum_{t=k+1}^{T^*}\delta^t\lambda_t\Big(J(\mu_t^*)-\mu_t^*J'_*(\mu_t^*)\Big)\Bigg)\\ &-\sum_{t=1}^{T^*}\delta^t\lambda_t\Big(J(\mu_t^*)+(\mu-\mu_t^*)J'_*(\mu_t^*)\Big). \end{aligned} \tag{4}$$

H is linear on $\mathcal{Q} \times \mathcal{Q}$. Denote, for all $t \geq 0$, γ_t^h and γ_t^l the terms multiplying q_t^h and q_t^l respectively. If $t \geq T^*$, $\gamma_t^h = \mu v_h \delta^t (1 + \Lambda_{T^*})$ is positive and decreasing in t. If $t \in \{1, ..., T^*\}$:

$$\begin{aligned}
\gamma_t^h &= \mu v_h \delta^t (1 + \Lambda_t) + \mu \sum_{k=t+1}^{T^*} \delta^k \lambda_k \Big(J(\mu_k^*) + (1 - \mu_k^*) J'_*(\mu_k^*) \Big) \\
&= \gamma_{t-1}^h - (1-\delta) \mu v_h \delta^{t-1} (1 + \Lambda_{t-1}) + \mu \delta^t \Big(v_h - J(\mu_t^*) - (1 - \mu_t^*) J'_*(\mu_t^*) \Big) \lambda_t \\
&= \gamma_{t-1}^h,
\end{aligned}$$

where we have used the fact that:

$$\lambda_t = 1 + \Lambda_t - (1 + \Lambda_{t-1}) = \frac{1-\delta}{\delta} \frac{v_h}{v_h - J(\mu_t^*) - (1 - \mu_t^*) J'(\mu_t^*)} (1 + \Lambda_{t-1}).$$

Now, for $t \in \{1, ..., T^*\}$:

$$\begin{aligned}
\gamma_t^l &= \delta^t \big(v_l - \mu v_h + (1-\mu) v_l \Lambda_t \big) + (1-\mu) \sum_{k=t+1}^{T^*} \delta^k \lambda_k \Big(J(\mu_k^*) - \mu_k^* J'_*(\mu_k^*) \Big) \\
&= \gamma_{t-1}^l + (1-\mu) \delta^t \lambda_t \Big(v_l - J(\mu_t^*) + \mu_t^* J'_*(\mu_t^*) \Big) + (1-\delta) \delta^{t-1} \Big(\mu (v_h - v_l) - (1-\mu) v_l (1 + \Lambda_{t-1}) \Big).
\end{aligned}$$

By convexity, $v_l = J(0) \geq J(\mu_t^*) - \mu_t^* J'_*(\mu_t^*)$. In addition:

$$1 + \Lambda_{t-1} \leq 1 + \Lambda_{T^*-1} \leq \frac{v_h - v_l}{v_l} \frac{\mu}{1-\mu}.$$

It follows that $\gamma_t^l \geq \gamma_{t-1}^l$. If $t \geq T^*$, $\gamma_t^l = \delta^t \big(v_l - \mu v_h + (1-\mu) v_l \Lambda_{T^*} \big)$. When $\mu_{T^*}^* > 0$, $1 + \Lambda_{T^*} = \frac{v_h - v_l}{v_l} \frac{\mu}{1-\mu}$, so $\gamma_t^l = 0$ for all $t \geq T^*$. In any case, $1 + \Lambda_{T^*} \geq \frac{v_h - v_l}{v_l} \frac{\mu}{1-\mu}$, and γ_t^l is non-negative and non-increasing in t for $t \geq T^*$.

It follows that H is maximised at $\langle s^* \rangle$, that is, for any $\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}$:

$$f'_{\langle s \rangle}(0) \leq H\big(\langle s \rangle\big) \leq H\big(\langle s^* \rangle\big) = f'_{\langle s^* \rangle}(0) = 0,$$

which proves the desired result. □

Remark 1. *For each μ and J, we have described a solution $\langle s^* \rangle(\mu, J)$. If there exists $\mu_1 \in [0,1)$ such that $\rho^J(\mu_1) = \frac{\mu}{1-\mu}$, we can describe an alternative solution $\langle \hat{s}^* \rangle(\mu, J)$ by setting:*

$$\hat{\mu}_1^* = \min \Big\{ \mu_1 \in [0,1) : \rho^J(\mu_1) = \frac{\mu}{1-\mu} \Big\},$$

and the continuing path of beliefs as before. All the arguments in the proof of proposition (1) directly apply. Since

ρ^J is a step function, $\langle \hat{s}^* \rangle(\mu, J)$ is only defined for a discrete set of priors μ.

Remark 2. *For each μ and J, there exists a direct and simple contract that implements $\langle s^* \rangle(\mu, J)$. The probabilities of trade for each type of the buyer are given by q^{*h} and q^{*l}. For every $t \geq 1$, the trading price if the buyer's report was v_i is $p_t^i = v_i$. In the initial period of deployment, the low-valuation buyer trades at price $p_0^l = v_l$, while the price for the high-valuation buyer is such that:*

$$q_0^{*h}(v_h - p_0^h) = \delta^{T^*}\big(q_{T^*}^{*l} + \delta(1 - q_{T^*}^{*l})\big)(v_h - v_l).$$

The same applies to $\langle \hat{s}^* \rangle(\mu, J)$. We denote $D(\mu, J)$ the set of those contracts (which contains either one or two elements).

Proposition 2. *$\mathcal{A}J$ belongs to $\mathcal{J}$.*

Proof. For $\mu \leq \frac{v_l}{v_h}$, $\mathcal{A}J(\mu) = v_l$. For $\mu \in \left(\frac{v_l}{v_h}, \frac{\rho^J(0)}{1+\rho^J(0)}\right]$, $\mathcal{A}J(\mu) = (1-\delta)\mu v_h + \delta v_l$. Note that $\mathcal{A}J$ is thus continuous at $\frac{v_l}{v_h}$. Similarly, if μ_1^* is a point of discontinuity of ρ^J, and $(\mu_1^*, ..., \mu_{T^*}^*)$ the corresponding path of beliefs obtained by iteration on $\tilde{\mu}'$, then for every $\mu \in \left[\frac{\rho_-^J(\mu_1^*)}{1+\rho_-^J(\mu_1^*)}, \frac{\rho^J(\mu_1^*)}{1+\rho^J(\mu_1^*)}\right]$, where $\rho_-^J(\mu_1^*)$ denotes the left limit of ρ^J at μ_1^*, we have:

$$\mathcal{A}J(\mu) = \left((1-\delta)\sum_{t=1}^{T^*} \frac{\delta^{t-1}}{1-\mu_t^*} + \delta^{T^*} \frac{\mu_{T^*}^*}{1-\mu_{T^*}^*} \frac{v_h - v_l}{v_l}\right)\mu v_h + \left(1 - (1-\delta)\sum_{t=1}^{T^*} \frac{\delta^{t-1}}{1-\mu_t^*} - \frac{\delta^{T^*}}{1-\mu_{T^*}^*} \frac{v_h - v_l}{v_h}\right) v_h.$$

Thus $\mathcal{A}J$ is piece-wise linear. In addition, by continuity at the boundaries, $\mathcal{A}J$ is non-decreasing. The term $\left((1-\delta)\sum_{t=1}^{T^*} \frac{\delta^{t-1}}{1-\mu_t^*} + \delta^{T^*} \frac{\mu_{T^*}^*}{1-\mu_{T^*}^*} \frac{v_h - v_l}{v_l}\right)$ is increasing in μ_1^*, therefore $\mathcal{A}J$ is convex. It is clear that $\mathcal{A}J \geq v_l$. Now, for $\mu > \frac{v_l}{v_h}$ and $\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}$:

$$\mu v_h - \Omega\big(\langle s \rangle \big| \mu\big) = \left(1 - \sum_{k=0}^{\infty} \delta^k q_k^h\right)\mu v_h + \sum_{k=0}^{\infty} \delta^k q_k^l (\mu v_h - v_l) \geq 0,$$

so $\mathcal{A}J \leq \bar{J}$. □

Lemma 4. *For all $J, \hat{J} \in \mathcal{J}$, if for all $\mu \in [0,1]$, $J(\mu) \geq \hat{J}(\mu)$, then for all $\mu \in [0,1]$, $\mathcal{A}J(\mu) \leq \mathcal{A}\hat{J}(\mu)$.*

Proof. Under the assumption of the lemma, for any $\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}$ and $\mu \in [0,1]$, $G\big(\langle s \rangle \big| \mu, J\big) \geq G\big(\langle s \rangle \big| \mu, \hat{J}\big)$. Thus $\langle s^* \rangle(\mu, J)$ is feasible in problem (1) for $\hat{J}$, from which the conclusion follows. □

Proposition 3. *The operator $\mathcal{A} : \mathcal{J} \to \mathcal{J}$ has a unique fixed point J^*.*

Proof. By lemma 4, and since $\bar{J}$ is an upper bound on any element of $\mathcal{J}$, any fixed point J^* of $\mathcal{A}$ must satisfy:

$$\mathcal{A}\bar{J} \leq J^* \leq \bar{J},$$

and by immediate induction:

$$\forall n \geq 0, \quad \mathcal{A}^{2n+1}\bar{J} \leq J^* \leq \mathcal{A}^{2n}\bar{J}.$$

Therefore, existence and uniqueness of a fixed point are guaranteed if $(\mathcal{A}^n\bar{J})_{n\geq 0}$ converges in $\mathcal{J}$. Note that for all $n, m \geq 0$, $\mathcal{A}^n\bar{J}$ and $\mathcal{A}^m\bar{J}$ coincide on $[0, \hat{\mu}_0)$, where $\hat{\mu}_0 = \frac{v_l}{v_h}$. Therefore, $\rho^{\mathcal{A}^n\bar{J}}$ and $\rho^{\mathcal{A}^m\bar{J}}$ also coincide on $[0, \hat{\mu}_0)$, thus $\mathcal{A}^{n+1}\bar{J}$ and $\mathcal{A}^{m+1}\bar{J}$ coincide on $[0, \hat{\mu}_1)$, where $\hat{\mu}_1 = \frac{\rho_-^{\bar{J}}(\hat{\mu}_0)}{1+\rho_-^{\bar{J}}(\hat{\mu}_0)} > \hat{\mu}_0$. By immediate induction, for every $k \geq 0$ and $n, m \geq k$, $\mathcal{A}^n\bar{J}$ and $\mathcal{A}^m\bar{J}$ coincide on $[0, \hat{\mu}_k)$, where the sequence $(\hat{\mu}_k)_{k\geq 0}$ is constructed such that $\hat{\mu}_{k+1} = \frac{\rho_-^{\mathcal{A}^k\bar{J}}(\hat{\mu}_k)}{1+\rho_-^{\mathcal{A}^k\bar{J}}(\hat{\mu}_k)}$. $(\hat{\mu}_k)_{k\geq 0}$ is an increasing sequence. By lemma 3, its limit must be $\hat{\mu}_\infty = 1$. It follows that for every $\mu \in [0, 1]$, $\mathcal{A}^n\bar{J}(\mu)$ remains constant for n sufficiently large (recall that $\mathcal{A}^n\bar{J}(1) = v_h$ for all $n \geq 0$). Denote $J^*(\mu)$ this constant.

By construction, $J^* \in \mathcal{J}$ and $\mathcal{A}J^* = J^*$. □

For $\mu \in [0, 1]$, we denote $D^*(\mu) = D(\mu, J^*)$. If a contract $d \in D^*(\mu)$ is deployed actively and truthfully in every period, the seller's payoff is $J^*(\mu)$ and the low-valuation buyer's payoff is always $U_l(\mu) = 0$. The high-valuation buyer's payoff depends on the contract, and we denote $U_h(\mu)$ the convex-hull of those payoffs. Finally, denote $d^*(\mu) \in D^*(\mu)$ the contract such that $\max U_h(\mu)$ is achieved.

A.2 Auxiliary Games

Given a contract $d = (x_\tau, p_\tau)_{\tau=0}^\infty \in \mathcal{D}$ and belief $\mu \in [0, 1]$, let $\Gamma(d, \mu)$ be the discontinuous dynamic psychological game defined as follows. The two players are the seller and the buyer. As in the main text, the buyer has two types v_l or v_h, and μ is the seller's common knowledge belief that the buyer has a high valuation. The game is played in discrete time. In the initial period, the contract d is deployed with $\tau = 0$. That is, the buyer chooses among $\{h, l, r\}$. If r is chosen, it is observed by the seller and the game proceeds to the next period. If instead $i \in \{h, l\}$ is selected, trade occurs with probability $x_0(v_i)$ at price $p_0(v_i)$, in which case the game ends. If trade does not occur, the game proceeds to the next period and the index in d is updated to $\tau = 1$. In any following period, the seller chooses among $\{C, S\}$. If S is selected, the game ends. If C is selected, the contract d is deployed again. That is, the buyer makes a report in $\{h, l, r\}$ if $\tau = 0$ or in $\{a, r\}$ if $\tau \geq 1$. As above, when r is selected, it is observed by the seller and the game proceeds to the next period. Otherwise, trade may occur or not as specified by the contract d.

The game ends either when trade occurs or when the seller chooses S. If trade occurs in period $t \geq 0$ at price p, the seller's payoff is $\delta^t p$ and the buyer's payoff is $\delta^t(v - p)$, where $v \in \{v_h, v_l\}$ is his valuation. If the game ends when the seller chooses S in period t, the payoffs are partially determined and depend on the seller's belief about the buyer's valuation at that information set $\hat{\mu}$. In particular, we use the approach of Simon and Zame (1990) and specify payoffs if the seller chooses S in period t with belief $\hat{\mu}$ as $\big(\delta^t J^*(\hat{\mu}), \delta^t U_l(\hat{\mu}), \delta^t U_h(\hat{\mu})\big)$ for the seller, low-valuation buyer and high-valuation buyer respectively. Since U_h is a correspondence, which element of $U_h(\hat{\mu})$ actually determines the high-valuation buyer's payoff is part of the solution concept.

As a result, we define a *payoff selection* u to be a mapping from terminal nodes following S to the real numbers. An *augmented assessment* is a triple (σ, α, u), where σ is a strategy profile, α is the seller's belief system and u is a payoff selection for the high-valuation buyer. u is said to be *consistent* with α if, at every terminal history $\mathfrak{h}$ following S, if the seller's belief that the buyer has a high valuation is $\mu_{\mathfrak{h}}$, then $u(\mathfrak{h}) \in U_h(\mu_{\mathfrak{h}})$. A *sequential equilibrium* of $\Gamma(d, \mu)$ is an augmented assessment (σ, α, u) such that (i) α is consistent with σ and the prior μ in the usual sense, (ii) u is consistent with α and (iii) σ is sequentially rational[1] given α and u.

Proposition 4. *Suppose that d specifies bounded transfers $(p_\tau)_{\tau \geq 0}$. Then $\Gamma(d, \mu)$ has a sequential equilibrium.*

Proof. We first consider the truncated version of the game $\Gamma_T(d, \mu)$ such that the game coincides with $\Gamma(d, \mu)$ until period T is reached, and for any period $t \geq T$, the seller's action space is restricted to $\{C\}$ and the buyer's action space is restricted to $\{r\}$.

For any $\varepsilon > 0$, there exists a continuous function $U_h^\varepsilon : [0, 1] \to [0, v_h - v_l]$ such that any point $\big(\tilde{\mu}, U_h^\varepsilon(\tilde{\mu})\big)$ in the graph of U_h^ε is at a distance less than ε to the graph of U_h. Let $\Gamma_T^\varepsilon(\mu, d)$ be the psychological game corresponding to $\Gamma_T(\mu, d)$ in which the high-valuation buyer's payoff after the seller chooses S with belief $\hat{\mu}$ is $U_h^\varepsilon(\hat{\mu})$. By Theorem 9 of Battigalli and Dufwenberg (2009), $\Gamma_T^\varepsilon(d, \mu)$ has a sequential equilibrium $(\sigma^\varepsilon, \alpha^\varepsilon)$ (where the first component refers to the strategy profile and the second to the belief system).

Given $(\sigma^\varepsilon, \alpha^\varepsilon)$, let $\vec{\mu}^\varepsilon$ the vector listing the seller's beliefs that the buyer has a high valuation at all her information sets in $\Gamma_T^\varepsilon(d, \mu)$, and $U_h^\varepsilon(\vec{\mu}^\varepsilon)$ the vector of high-valuation payoffs whenever S is chosen. Since $\big(\sigma^\varepsilon, \alpha^\varepsilon, U_h^\varepsilon(\vec{\mu}^\varepsilon)\big)_\varepsilon$ lives in a compact set, it possesses an accumulation point $(\sigma_T, \alpha_T, u_T)$ as $\varepsilon \to 0$.

We claim that $(\sigma_T, \alpha_T, u_T)$ is a sequential equilibrium of $\Gamma_T(d, \mu)$. Indeed, α must be consistent for σ since α^ε is consistent for σ^ε for any $\varepsilon > 0$. Moreover, at every terminal node following S, the seller's belief $\hat{\mu}^\varepsilon$ converges to $\hat{\mu}$, while the limit of $\big(\hat{\mu}^\varepsilon, U_h^\varepsilon(\hat{\mu}^\varepsilon)\big)$ must belong to the graph of U_h. Thus u_T must be a payoff selection for the high-valuation buyer. Finally, since the terminal payoffs of any player converge together with the assessment, for any strategy s_i of any player i, the evaluation of player i's expected payoff at any of her information sets under $(s_i, \sigma_{-i}^\varepsilon)$ also converges to that under (s_i, σ_{-i}). Since no profitable deviation exists under σ^ε, the same is true in the limit under σ.

σ_T specifies a full profile in $\Gamma(d, \mu)$. We can also extend α_T and u_T to construct a full assessment and payoff selection in $\Gamma(d, \mu)$. For the belief system α, take the limit of the belief system induced by a fully mixed buyer strategy after period T such that both types make a mistake with the same small probability at every decision

[1] The notion of sequential rationality extends naturally to psychological games. The reader is referred to Battigalli and Dufwenberg (2009) for details.

node. Given the completed infinite vector $\vec{\mu}$, whenever the seller chooses S after period T with belief $\hat{\mu}$, select $U_h^*(\hat{\mu}) = \max U_h(\mu)$. Denote $\left(\bar{\sigma}_T, \bar{\alpha}_T, \bar{u}_T\right)$ the completed assessment and payoff selection in the infinite-horizon game $\Gamma(d, \mu)$.

Since transfers are bounded, discounting guarantees that $\left(\bar{\sigma}_T, \bar{\alpha}_T, \bar{u}_T\right)$ is a $\bar{p}\delta^T$-equilibrium in $\Gamma(d, \mu)$. In addition, the set of augmented assessments is compact in the topology of Fudenberg and Levine (1983) (note that, given a belief system, a payoff selection is isomorphic to choosing a mixture between $\max U_h(\hat{\mu})$ and $\min U_h(\hat{\mu})$ at every information set of the seller, where $\hat{\mu}$ is her belief at that information set). Thus $\left(\bar{\sigma}_T, \bar{\alpha}_T, \bar{u}_T\right)$ has a converging subsequence, which must be a sequential equilibrium in $\Gamma(d, \mu)$. $\square$

A.3 Equilibrium Assessment

We construct an equilibrium in the version of the game presented in the main text in which the seller's contract space is restricted to include only simple and direct contracts, with the additional assumption that all transfers are bounded. Specifically, we assume that any transfer must be in the set $[\underline{p}, \bar{p}]$, where $\underline{p} < 0$ and $\bar{p} > v_h$.

The seller's prior is μ. Throughout we maintain that the seller's belief does not update following her own deviations. At the beginning of the game, the seller deploys the contract $d^*(\mu)$. Consider a history $\mathfrak{h}$ at which the seller's belief is $\mu_{\mathfrak{h}}$ and the seller offers a new contract d. If $d \in D^*(\mu_{\mathfrak{h}})$, the buyer accepts the contract and reports his valuation truthfully. In every following period, as long as trade does not occur, the seller's belief updates according to Bayes' rule, she deploys again d and the buyer accepts. If the buyer ever rejects d, the seller's belief does not update and she deploys d again. If $d \notin D^*(\mu_{\mathfrak{h}})$, the continuing assessment until the game ends or d is replaced is constructed as follows.

First note that, by the axiom of choice and proposition 4, we can select a sequential equilibrium of the auxiliary game $\Gamma(d, \mu_{\mathfrak{h}})$ for any $(d, \mu_{\mathfrak{h}})$. Let (σ, α, u) the chosen equilibrium. Note that, as long as d is deployed, all the decision nodes of the buyer are identical in $\Gamma(d, \mu_{\mathfrak{h}})$ and in the original game. Thus, the buyer's strategy at those nodes can be taken directly from the profile σ. Now, we translate the seller's strategy in σ to a strategy in the original game. We interpret the action C as deploying the contract d again. Note that, as long as d has been deployed, the seller's belief at every information set can be taken directly from α. Finally, when the seller chooses S in the auxiliary game, with belief $\hat{\mu}$, u specifies a payoff selection for the high-valuation buyer $\hat{u} \in U_h(\hat{\mu})$. We specify that the contract d is replaced by a contract $\hat{d} \in D^*(\hat{\mu})$, which results from the unique mixture across the contracts in $D^*(\hat{\mu})$ that delivers expected payoff $\hat{u}$ to the high-valuation buyer (recall that $D^*(\hat{\mu})$ contains at most two contracts), if $\hat{d}$ is to be deployed actively and truthfully forever after. Once d is replaced, the continuing assessment is described in the previous paragraph.

Next, we prove that the above assessment is indeed an equilibrium. By construction, the assessment satisfies updating consistency in the sense of Perea (2002). Therefore the one-shot deviation principle applies. It is clear that the buyer has no incentive to deviate. Next, we establish that the seller has no profitable one-shot deviation at any information set.

It is sufficient to show that, at any history $\mathfrak{h}$ with belief $\mu_{\mathfrak{h}}$, the seller's payoff from deploying a new contract cannot exceed $J^*(\mu_{\mathfrak{h}})$ given the buyer's strategy. Then, by construction, it is clear that the seller's behaviour is sequentially rational.

Suppose that the seller's belief is $\mu_{\mathfrak{h}}$ and a new contract is offered. This is a one-shot deviation, thus the continuation path is that specified in the above assessment. In particular, at every subsequent history $\hat{\mathfrak{h}}$, if the seller's belief is $\mu_{\hat{\mathfrak{h}}}$, her continuation payoff is at least $J^*(\mu_{\hat{\mathfrak{h}}})$. Let s the random time at which trade occurs induced by the continuation path. Denoting by V, θ_h and θ_l the continuation payoffs for the seller, the high-valuation buyer and the low-valuation buyer respectively, we can express the total continuation surplus as:

$$V + \mu_{\mathfrak{h}}\theta_h + (1-\mu_{\mathfrak{h}})\theta_l = \mathbb{E}_0[\delta^s|h]\mu_{\mathfrak{h}}v_h + \mathbb{E}_0[\delta^s|l](1-\mu_{\mathfrak{h}})v_l.$$

Since a new contract is deployed, the high-valuation buyer could mimic the low-valuation buyer's strategy, thus:

$$\theta_h \geq \theta_l + \mathbb{E}_0[\delta^s|l](v_h - v_l).$$

In addition, the buyer can guarantee a non-negative payoff by rejecting the contract in every period, thus:

$$\theta_l \geq 0.$$

Together, these inequalities imply that the seller's continuation payoff satisfies:

$$V \leq \mathbb{E}_0[\delta^s|h]\mu_{\mathfrak{h}}v_h + \mathbb{E}_0[\delta^s|l](v_l - \mu_{\mathfrak{h}}v_h).$$

Finally, since J^* is convex, in any period, the total continuation surplus must exceed $J^*(\mu_t)$, where μ_t is the average belief of the seller across all histories reaching period t. That is:

$$\mathbb{E}_t\big[\delta^s|h\big]\mu_t v_h + \mathbb{E}_t\big[\delta^s|l\big](1-\mu_t)v_l \geq J^*(\mu_t).$$

In other words, it must be the case that $V \leq \mathcal{A}J^*(\mu_{\mathfrak{h}}) = J^*(\mu_{\mathfrak{h}})$. Thus the seller has no profitable one-shot deviation, which concludes the proof.

B Continuous Types

In this appendix we will show how to modify the analysis in the main text to demonstrate that our result is robust in the case of continuous types. In particular, we first construct an abiding contract which achieves a payoff strictly bounded above the market clearing profit. This contract resembles the contract of Lemma 2 in the main text by treating the buyer as if his type was binary. In particular, the contract specifies a cut-off type $\hat{v} \in (\underline{v}, \overline{v})$ and treats all types $v < \hat{v}$ as the 'low type,' and all types $v \geq \hat{v}$ as the 'high type.' The allocation probabilities and prices across 'high' and 'low' types are structured similarly to the abiding contract in Lemma 2: only 'high' types trade in the initial period at a discount. Conditional on no trade in the initial period, all types trade at the same random time. 'High' types trade at the 'high' valuation, $\hat{v}$, while 'low' types trade at the 'low' valuation, $\underline{v}$.

The only significant difference is the abidance constraint. After no trade in the initial period the seller faces a residual market and we need to guarantee that the optimal monopoly price is the lowest valuation, $\underline{v}$. This is accomplished by choosing the cut-off type, $\hat{v}$, and the probability of trade in the initial period appropriately, so that the residual market is sufficiently deteriorated. Moreover, the payoff from continuing with the contract must exceed $\underline{v}$.

Proposition 5 shows that for each δ such an abiding contract can be found which delivers payoff greater than some $\underline{\pi} > \underline{v}$. Finally, in Proposition 6 we show that this $\underline{\pi}$ is a lower bound to the seller's best equilibrium profit. The argument proceeds similarly to the proof of Lemma 1 in the main text.

The Contract

Let F be the cdf of valuations on $[\underline{v}, \overline{v}]$, with $\underline{v} > 0$. We assume that $f(v) > 0$, for all $v \in [\underline{v}, \overline{v}]$ and that the revenue function $R(p) := p \cdot (1 - F(p))$ satisfies $R'(\underline{v}) > 0$. Note that this implies that market clearing is not the optimal monopoly price, $\max_p R(p) > \underline{v}$.

We consider the following stationary cut-off mechanism.

- The buyer submits a report $v \in [\underline{v}, \overline{v}]$.
- If $v \geq \hat{v}$, there is trade with probability α in the first period, at price p.
- In any future period trade occurs with probability β at price $\hat{v}$.
- If $v < \hat{v}$, there is no trade in the initial period.
- In any future period trade occurs with probability β at price $\underline{v}$.

We introduce the notation:

$$\psi = \frac{\beta\delta}{1-\delta+\beta\delta} \in [0,\delta]$$

for the expected present value of the discount factor at the time of trade.

Incentive Compatibility

First, we note that this contract is IC whenever:

$$\alpha \cdot (v-p) + (1-\alpha) \cdot \max\Big\{\psi \cdot (v-\hat{v}), 0\Big\} \geq \psi \cdot (v-\underline{v}) \quad \iff v \geq \hat{v}$$

where the $\max\{\cdot\}$ controls for the buyer's participation decision in future periods.

Then, the IC is written as:

$$\alpha \cdot (v-p) + (1-\alpha) \cdot \psi \cdot (v-\hat{v}) \geq \psi \cdot (v-\underline{v}) \quad \text{for all } v \geq \hat{v}$$

and

$$\psi \cdot (v-\underline{v}) \geq \alpha \cdot (v-p) \quad \text{for all } v \leq \hat{v}$$

The following lemma gives sufficient conditions for incentive compatibility.

Lemma 5. *The contract* $(\hat{v}, \alpha, \psi, p) \in (\underline{v}, \overline{v}) \times [0,1] \times [0,\delta] \times \mathbb{R}^+$ *is IC if:*

$$\alpha \cdot (\hat{v}-p) = \psi \cdot (\hat{v}-\underline{v}) \tag{IC+}$$

and

$$\alpha \geq \psi \tag{IC-}$$

Proof. Let $\Delta_+(v) = \alpha \cdot (v-p) + (1-\alpha) \cdot \psi \cdot (v-\hat{v}) - \psi \cdot (v-\underline{v})$. The IC for $v \geq \hat{v}$ is satisfied whenever $\Delta_+(v) \geq 0$ for $v \geq \hat{v}$. First, note that (IC+) implies that $\Delta_+(\hat{v}) = 0$. Hence, it is sufficient to verify that $\Delta_+(v)$ is increasing in v. We have:

$$\Delta_+'(v) = \alpha + (1-\alpha)\psi - \psi = \alpha \cdot (1-\psi) \geq 0$$

Therefore, the IC for $v \geq \hat{v}$ is satisfied.

Let $\Delta_-(v) = \psi \cdot (v-\underline{v}) - \alpha \cdot (v-p)$, so that the IC for $v \leq \hat{v}$ is satisfied if $\Delta_-(v) \geq 0$ for $v \leq \hat{v}$. Given (IC+) we have that the price p must be given by:

$$\alpha \cdot p = (\alpha - \psi) \cdot \hat{v} + \psi \cdot \underline{v}$$

and thus:

$$\Delta_-(v) = (\psi - \alpha) \cdot v - \psi \underline{v} + \alpha p = (\alpha - \psi) \cdot (\hat{v} - v) \geq 0$$

for $v \leq \hat{v}$, since $\alpha - \psi \geq 0$ by (IC-).

$\square$

Abidance

Abidance comprises of two features: (i) The monopoly price on the residual market is $\underline{v}$ (market clearing); and (ii) the seller's continuation payoff within the mechanism is greater than $\underline{v}$.

First, we compute the seller's continuation payoff. This is given by:

$$\psi \cdot \left[\mathbb{P}(v < \hat{v} | \neg \text{ trade}) \cdot \underline{v} + \mathbb{P}(v \geq \hat{v} | \neg \text{ trade}) \cdot \hat{v}\right]$$

Given the trading probabilities specified above, we have:

$$\mathbb{P}(v \geq \hat{v} | \neg \text{ trade}) = \frac{(1-\alpha) \cdot (1 - F(\hat{v}))}{F(\hat{v}) + (1-\alpha) \cdot (1 - F(\hat{v}))}$$

The abidance constraint is hence given by:[2]

$$\psi \cdot (\delta - \psi) \left[\underline{v} + \frac{(1-\alpha) \cdot (1 - F(\hat{v}))}{F(\hat{v}) + (1-\alpha) \cdot (1 - F(\hat{v}))} \cdot (\hat{v} - \underline{v})\right] \geq (\delta - \psi) \cdot \underline{v} \tag{AC1}$$

Residual Market

We now compute the residual market conditional on no trade in the initial period. In particular, we are interested in $\mathbb{P}(v \geq p | \neg \text{ trade})$, which is given by:

$$\mathbb{P}(v \geq p | \neg \text{ trade}) = \frac{\int_p^{\overline{v}} \left[1 - \alpha \cdot \mathbb{1}_{v \geq \hat{v}}\right] dF(v)}{\int_{\underline{v}}^{\overline{v}} \left[1 - \alpha \cdot \mathbb{1}_{v \geq \hat{v}}\right] dF(v)}$$

Hence, we get the residual market:

[2]Note that this is equivalent to the constraint in Lemma 2 of the main text. That is, continuing in the mechanism delivers more than the payoff from running the mechanism for a period more, and clearing immediately conditional on no trade.

$$D(p\,|\alpha,\hat{v}) = \mathbb{P}(v \geq p | \neg \text{ trade}) = \begin{cases} \frac{(1-\alpha)\cdot(1-F(p))}{F(\hat{v})+(1-\alpha)\cdot(1-F(\hat{v}))}, & \text{if } p \geq \hat{v} \\ \\ \frac{(1-\alpha)\cdot(1-F(\hat{v}))+F(\hat{v})-F(p)}{F(\hat{v})+(1-\alpha)\cdot(1-F(\hat{v}))} & \text{if } p < \hat{v} \end{cases}$$

Ex-ante payoffs

We now compute the seller's ex-ante payoff from a contract $(\hat{v}, \alpha, \psi, p) \in (\underline{v}, \overline{v}) \times [0,1] \times [0,\delta] \times \mathbb{R}^+$:

$$\Pi(\hat{v}, \alpha, \psi, p) = (1 - F(\hat{v})) \cdot (\alpha \cdot p + (1-\alpha)\psi\hat{v}) + F(\hat{v}) \cdot \psi\underline{v}$$

We can use (IC+) to eliminate the price, and re-write the problem as:

$$\Pi(\hat{v}, \alpha, \psi) = (1 - \psi)\alpha \cdot R(\hat{v}) + \psi\underline{v}$$

with the following constraints:

$$\alpha \geq \psi \tag{IC-}$$

$$\psi \cdot (\delta - \psi)\left[F(\hat{v}) \cdot \underline{v} + (1-\alpha) \cdot R(\hat{v})\right] \geq \Big[1 - \alpha \cdot \big(1 - F(\hat{v})\big)\Big](\delta - \psi) \cdot \underline{v} \tag{AC1}$$

$$\sup_p \; p \cdot D(p\,|\alpha, \hat{v}) = \underline{v} \tag{AC2a}$$

$$D(p\,|\alpha,\hat{v}) = \begin{cases} \frac{(1-\alpha)\cdot(1-F(p))}{F(\hat{v})+(1-\alpha)\cdot(1-F(\hat{v}))}, & \text{if } p \geq \hat{v} \\ \\ \frac{(1-\alpha)\cdot(1-F(\hat{v}))+F(\hat{v})-F(p)}{F(\hat{v})+(1-\alpha)\cdot(1-F(\hat{v}))} & \text{if } p < \hat{v} \end{cases} \tag{AC2b}$$

Parameters

We are looking for parameters $(\hat{v}, \alpha, \psi) \in (\underline{v}, \overline{v}) \times [0,1] \times [0,\delta]$ satisfying the following conditions:

$$\alpha \cdot R(\hat{v}) > \underline{v}$$

$$\psi < 1$$

$$\alpha \geq \psi$$

$$\psi \cdot [F(\hat{v}) \cdot \underline{v} + (1-\alpha) \cdot R(\hat{v})] \geq \Big[1 - \alpha \cdot \big(1 - F(\hat{v})\big)\Big]\underline{v}$$

$$\sup_p \; p \cdot D(p \,|\alpha, \hat{v}) = \underline{v}$$

The first two conditions guarantee the failure of the Coase Conjecture. The third ensures incentive compatibility for $v \leq \hat{v}$. The fourth is the abidance constraint for the seller. Note that the way this is written implicitly assumes that we can take $\psi < \delta$. This will turn out to be the case for sufficiently high δ—which is the relevant case. Finally, the last condition guarantees that market clearing is the monopoly price on the residual market.

Intuitively for all sufficiently low $\hat{v}$, there is a high enough α which deteriorates the residual market enough for market clearing to be the monopoly price.

We use this idea to specify for each $\hat{v}$, the following:

$$1 - \psi_{\hat{v}} = (\hat{v} - \underline{v})^2$$

$$1 - \alpha_{\hat{v}} = (\hat{v} - \underline{v})^2$$

with the intention of converging as $\hat{v} \downarrow \underline{v}$.

With this specification as $\hat{v}$ converges to $\underline{v}$, then $\hat{v} - \underline{v}$ converges to 0, slower than α and ψ are converging to 1. Note that $\alpha_{\hat{v}} = \psi_{\hat{v}}$ so (IC-) is taken care of;[3] while for $\hat{v} > \underline{v}$, $\psi_{\hat{v}} < 1$.

We now consider the remaining conditions.

Lemma 6. *There exists $\varepsilon_1 > 0$ such that for all $\hat{v} > \underline{v}$ with $\hat{v} - \underline{v} \leq \varepsilon_1$,*

$$\alpha_{\hat{v}} \cdot R(\hat{v}) > \underline{v}$$

Proof. Suppose that the inequality fails for all $\hat{v} > \underline{v}$. That is:

$$\begin{aligned}
\alpha_{\hat{v}} \cdot R(\hat{v}) &\leq \underline{v} \\
\alpha_{\hat{v}} \cdot \Big[R(\hat{v}) - R(\underline{v})\Big] &\leq (1 - \alpha_{\hat{v}})\underline{v} \\
\alpha_{\hat{v}} \cdot \Big[R(\hat{v}) - R(\underline{v})\Big] &\leq (\hat{v} - \underline{v})^2\underline{v} \\
\alpha_{\hat{v}} \cdot \frac{R(\hat{v}) - R(\underline{v})}{(\hat{v} - \underline{v})} &\leq (\hat{v} - \underline{v})\underline{v}
\end{aligned}$$

[3] In particular, this means that $p = \underline{v}$.

where $R(\underline{v}) = \underline{v}$. Taking limits as $\hat{v} \downarrow \underline{v}$ we arrive at:

$$R'(\underline{v}) \leq 0$$

which contradicts the hypothesis that $R'(\underline{v}) > 0$.

$\square$

Lemma 7. *There exists $\varepsilon_2 > 0$ such that for all $\hat{v} > \underline{v}$ such that $\hat{v} - \underline{v} \leq \varepsilon_2$,*

$$\psi_{\hat{v}} \cdot [F(\hat{v}) \cdot \underline{v} + (1 - \alpha_{\hat{v}}) \cdot R(\hat{v})] \geq \Big[1 - \alpha_{\hat{v}} \cdot \big(1 - F(\hat{v})\big)\Big]\underline{v}$$

Proof. Re-writing the inequality we have:

$$1 - \psi_{\hat{v}} \leq (1 - \alpha_{\hat{v}}) \frac{(\hat{v} - \underline{v}) \cdot (1 - F(\hat{v})}{[F(\hat{v}) \cdot \underline{v} + (1 - \alpha_{\hat{v}}) \cdot R(\hat{v})]}$$

Suppose the inequality fails for all $\hat{v} > \underline{v}$. We then have:

$$1 - \psi_{\hat{v}} > (1 - \alpha_{\hat{v}}) \frac{(\hat{v} - \underline{v}) \cdot (1 - F(\hat{v})}{[F(\hat{v}) \cdot \underline{v} + (1 - \alpha_{\hat{v}}) \cdot R(\hat{v})]} \Rightarrow 1 > \frac{(\hat{v} - \underline{v}) \cdot (1 - F(\hat{v}))}{[F(\hat{v}) \cdot \underline{v} + (\hat{v} - \underline{v})^2 \cdot R(\hat{v})]}$$

which can be re-arranged to:

$$\left[\frac{F(\hat{v})}{(\hat{v} - \underline{v})} \cdot \underline{v} + (\hat{v} - \underline{v}) \cdot R(\hat{v})\right] > (1 - F(\hat{v})) \Rightarrow \left[f(\underline{v}) \cdot \underline{v} + \frac{o((\hat{v} - \underline{v}))}{(\hat{v} - \underline{v})} + (\hat{v} - \underline{v}) \cdot R(\hat{v})\right] > (1 - F(\hat{v}))$$

Taking limits as $\hat{v} \downarrow \underline{v}$ we arrive at:

$$f(\underline{v}) \cdot \underline{v} \geq 1$$

This is a contradiction to $R'(\underline{v}) > 0$ since:

$$R'(x) = 1 - F(x) - f(x) \cdot x \Rightarrow R'(\underline{v}) = 1 - f(\underline{v}) \cdot \underline{v} > 0$$

$\square$

Lemma 8. *There exists* $\varepsilon_3 > 0$ *such that for all* $\hat{v} > \underline{v}$ *such that* $\hat{v} - \underline{v} \le \varepsilon_3$*,*

$$\sup_p \; p \cdot D(p \,|\alpha_{\hat{v}}, \hat{v}) = \underline{v}$$

Proof. First, we have that for any $p \ge \hat{v}$,

$$R(p|\hat{v}) := p \cdot D(p|\alpha_{\hat{v}}, \hat{v}) \le \overline{v} \cdot \frac{(\hat{v} - \underline{v})^2 \cdot (1 - F(\hat{v}))}{F(\hat{v}) + (\hat{v} - \underline{v})^2 \cdot (1 - F(\hat{v}))}$$

since the RHS is the profit from trading with all $v \ge \hat{v}$ at a price $p = \overline{v}$. Dividing both the numerator and the denominator by $(\hat{v} - \underline{v}) > 0$, we have:

$$\overline{v} \cdot \frac{(\hat{v} - \underline{v}) \cdot (1 - F(\hat{v}))}{F(\hat{v})/(\hat{v} - \underline{v}) + (\hat{v} - \underline{v}) \cdot (1 - F(\hat{v}))} \to 0$$

since $F(\hat{v})/(\hat{v} - \underline{v})$ converges to $f(\underline{v}) > 0$. Consequently, profits from prices $p \ge \hat{v}$ become arbitrarily small as $\hat{v} \downarrow \underline{v}$, so for sufficiently small $(\hat{v} - \underline{v}) > 0$ the optimal prices must be below $\hat{v}$.

We now consider prices $p < \hat{v}$. First note that:

$$\begin{aligned}
R'(\underline{v}|\hat{v}) \cdot (\hat{v} - \underline{v}) &= \frac{\hat{v} - \underline{v}}{F(\hat{v}) + (\hat{v} - \underline{v})^2 \cdot (1 - F(\hat{v}))} \cdot \Big[(\hat{v} - \underline{v})^2 \cdot (1 - F(\hat{v})) + F(\hat{v}) - F(\underline{v}) - \underline{v} \cdot f(\underline{v})\Big] \\
&= \frac{1}{\frac{F(\hat{v})}{\hat{v} - \underline{v}} + (\hat{v} - \underline{v}) \cdot (1 - F(\hat{v}))} \cdot \Big[(\hat{v} - \underline{v})^2 \cdot (1 - F(\hat{v})) + F(\hat{v}) - F(\underline{v}) - \underline{v} \cdot f(\underline{v})\Big] \\
&\to \frac{1}{f(\underline{v})} \cdot \Big[- \underline{v} \cdot f(\underline{v})\Big] = -\underline{v} \quad \text{as } \hat{v} \downarrow \underline{v}
\end{aligned}$$

Moreover, for $\underline{v} < p < \hat{v}$, we have:

$$\begin{aligned}
R(p|\hat{v}) - R(\underline{v}|\hat{v}) &= R'(\underline{v}|\hat{v}) \cdot (p - \underline{v}) + o(p - \underline{v}) \\
&< R'(\underline{v}|\hat{v}) \cdot (\hat{v} - \underline{v}) + o(p - \underline{v})
\end{aligned}$$

Since the RHS converges to a strictly negative limit as $\hat{v} \downarrow \underline{v}$, we can find $\varepsilon_3 > 0$, such that for all $\hat{v} > \underline{v}$ with $\hat{v} - \underline{v} \le \varepsilon_3$,

$$R(p|\hat{v}) - R(\underline{v}|\hat{v}) < 0, \quad \text{for all } \underline{v} < p < \hat{v}$$

The claim follows since $R(\underline{v}|\hat{v}) = \underline{v}$.

□

We now put lemmata 6-8 together to prove the failure of the Coase Conjecture:

Lemma 9. *There exists $\bar{\delta} < 1$ such that for all $\delta \geq \bar{\delta}$, we can pick a single abiding contract $(\hat{v}^*, \alpha^*, \psi^*, p^*) \in (\underline{v}, \overline{v}) \times [0,1] \times [0,\delta] \times \mathbb{R}^+$ with $\Pi(\hat{v}^*, \alpha^*, \psi^*, p^*) > \underline{v}$.*

Proof. We pick $\hat{v}^*$ such that $0 < (\hat{v}^* - \underline{v}) < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and set $\alpha^* = 1 - (\hat{v}^* - \underline{v})^2 = \psi^*$, and $p^* = \underline{v}$ given by (IC+). Lemma 3 and Lemma 4 imply that the contract is abiding. Lemma 2 delivers $\Pi(\hat{v}^*, \alpha^*, \psi^*, p^*) > \underline{v}$.

Finally, setting $\bar{\delta} = 1 - (\hat{v}^* - \underline{v})^2$ the feasibility constraint $\psi^* \leq \delta$, is satisfied for all $\delta \geq \bar{\delta}$.

□

Finally, we prove the analogue of Lemma 2 in the main text:

Proposition 5. *For each $\delta \in (0,1)$, there exists a δ-abiding contract $d_\delta \in \mathcal{D}$ such that $v(d_\delta, \delta) > \underline{\pi} > \underline{v}$.*

Proof. If $\delta \geq \bar{\delta}$ where $\bar{\delta}$ the cut-off in Lemma 9 we use the contract $(\hat{v}^*, \alpha^*, \psi^*, p^*)$. If $\delta < \bar{\delta}$, we consider the contract $(\hat{v}, \alpha, \psi, p) = (\hat{v}^*, 1, \delta, \tilde{p})$, with $\tilde{p} = \hat{v} - \delta \cdot (\hat{v} - \underline{v})$. This is abiding and delivers payoff:

$$(1-\delta)R(\hat{v}) + \delta \underline{v} > \underline{v}$$

The payoff from the contract $(\hat{v}^*, 1, \delta, \tilde{p})$ is strictly decreasing in δ so the payoff it generates for $\delta < \bar{\delta}$ is at least as large as $\Pi(\hat{v}^*, 1, \bar{\delta}, \tilde{p})$.

We set $\underline{\pi} = \min\{\Pi(\hat{v}^*, 1, \bar{\delta}, \tilde{p}), \Pi(\hat{v}^*, \alpha^*, \psi^*, p^*)\} > \underline{v}$. Consequently, for each δ we can construct an abiding contract d_δ such that $v(d_\delta, \delta) \geq \underline{\pi} > \underline{v}$.

□

Lower bound

We now prove that the best equilibrium payoff to the seller is bounded below by $\underline{\pi}$. To prove this we need to strengthen the equilibrium concept. In particular, an assessment is an equilibrium if it is a Weak PBE, and in addition, after *any* initial contract offer the continuation assessment forms a Weak PBE.[4] We will assume that for any prior distribution of types $G \in \Delta([\underline{v}, \overline{v}])$ such an equilibrium exists.

[4]Note that this is equivalent to a subgame-perfect Weak PBE assessment in a game-tree where Nature draws the type *after* every contract offer—at which point a well-specified subgame starts.

Proposition 6. *We have* $\pi(\mathcal{C}, \delta) \geq \underline{\pi}$.

Proof. Suppose in anticipation of a contradiction that $\pi(\mathcal{C}, \delta) < \underline{\pi}$. The argument follows exactly the same steps of Lemma 1 in the main text, apart from what happens along a history where only d_δ has been offered.

Note that d_δ is abiding so no matter what payoffs the seller gets from deviating to $c \neq d_\delta$, as long as they are consistent with sequential rationality, they cannot exceed market clearing which is the commitment payoff. So the best deviation payoff consistent with equilibrium in the continuation game is $\underline{v}$. Consequently, it is enough to show that *some* equilibrium assessment can be defined after any offer of contract $c \neq d_\delta$ when the seller's belief (residual demand) is given by $D(p\,|\alpha^*, \hat{v}^*)$.

The continuation game where $c \neq d_\delta$ is offered after τ periods of deployment of d_δ, is clearly isomorphic to the one arising from an initial offer of c, in a game where the prior is $D(p\,|\alpha^*, \hat{v}^*)$. By assumption, an assessment exists in such a game which induces a Weak PBE after the contract offer $c \neq d_\delta$. We can therefore specify the corresponding assessment to complete the proof.

□

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