ONLINE APPENDIX TO "CROWDING IN SCHOOL CHOICE"

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Appendix A. Existence of RCE and Maximal RCE

We proceed in four subsections. In Section A.1 we study the set of *fair* allocations, which contains the set of RCEs, and which can be trivially shown to be non-empty on our domain. In Section A.2 we show that RCEs, if they exist, induce an upper lattice in welfare space, which then leads to Theorem 3. Proposition 2 is necessary for this argument, so we give its proof here as well. In Section A.3, we uncover some dominance relations in the set of *fair* allocations. These imply that RCEs will lie in the welfare-upper-envelope of the *fair* allocations, and so will exist if the *fair* allocations induce a closed set in welfare space, which we show in Section A.4.

A.1. **Existence of *Fair* Allocations.**

Given profile $\boldsymbol{R} \in \mathcal{R}^N$, schools s_1 and s_{k+1} are **connected by indifference*** (CBI*) if they satisfy condition (1) of connection by indifference for $\rho_{s_1} = \max\{b_{s_1}, 1/|N|\}$ and $\rho_{s_{k+1}} = \max\{b_{s_{k+1}}, 1/|N|\}$, and the same condition (2). A preference domain is thus NCBI* if it contains no profiles at which a pair of schools is CBI*. The maximal NCBI* domain is a proper superset of the NBCI domain.

Proposition 4. *A* fair *allocation $(\boldsymbol{\rho}, \sigma)$, with each $\rho_s \geq \max\{b_s, 1/|N|\}$, exists on any NCBI* domain.*

Proof. Consider $\boldsymbol{\rho} \in [0,1]^S$ such that each $\rho_s = \max\{b_s, 1/|N|\}$. Then we have a standard school choice problem where the capacity of school $s \in S$ is $\min\{b_s^{-1}, |N|\}$ (Section II.A). NCBI* implies that student preferences on $\{(\rho_s, s) : s \in S\}$ are strict, so the set of matchings that are *non-wasteful* and satisfy *no justified envy* is non-empty [Roth and Sotomayor, 1990]. If σ is such a matching for this problem, then $(\boldsymbol{\rho}, \sigma)$ is clearly a *fair* allocation in our model. □

Note that by *exhaustiveness* and *inferior empty schools*, if $(\boldsymbol{\rho}, \sigma)$ is an RCE, then each $\rho_s > \frac{1}{|N|+1}$. This fact, combined with the foregoing Proposition and our proof strategy, make it without loss of generality to assume henceforth that any *fair* allocation $(\boldsymbol{\rho}, \sigma)$ has each $\rho_s > \frac{1}{|N|+1}$.

A.2. **The Upper-Lattice Property.**

Fix a preference profile $\boldsymbol{R} \in \mathcal{R}^N$. Given two allocations, $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$, construct the labeled, directed **transfer graph** T on vertices S so that labeled arc $s \xrightarrow{i} t \in T$ if $\sigma(i) = s$, $\tau(i) = t$, and $s \neq t$.[26] We may omit the label when the identity of the

[26] A labeled, directed graph T on vertices S is a set of ordered triples $(s,t,i) \in S \times S \times N$ where (s,t) represents the arc and i is the label.

student generating the arc is not important. Define the sets

$$
\begin{aligned}
S^+ &= \{s \in S : \gamma_s > \rho_s\} & N^+ &= \{i \in N : (\boldsymbol{\gamma}, \tau(i))\ P_i\ (\boldsymbol{\rho}, \sigma(i))\}\\
S^- &= \{s \in S : \rho_s > \gamma_s\} & N^- &= \{i \in N : (\boldsymbol{\rho}, \sigma(i))\ P_i\ (\boldsymbol{\gamma}, \tau(i))\}\\
S^= &= \{s \in S : \gamma_s = \rho_s > b_s\} & N^= &= \{i \in N : (\boldsymbol{\gamma}, \tau(i))\ I_i\ (\boldsymbol{\rho}, \sigma(i))\}\\
S^* &= \{s \in S : \gamma_s = \rho_s = b_s\}.
\end{aligned}
$$

We denote by $s \rightsquigarrow t \subseteq T$ a path in the transfer graph with no repeated arcs.[27] Note that since our graph may contain several arcs with the same orientation between a given pair of vertices, there may be many distinct paths from s to t, even on the same ordered list of vertices. We distinguish between different paths by decoration, so that $s \rightsquigarrow' t \neq s \rightsquigarrow t$. A path is positive (negative) if it contains an N^+-labeled arc (N^--labeled arc). A positive (negative) path is totally-positive (totally-negative) if it contains only labels from $N^+ \cup N^=$ ($N^- \cup N^=$). A cycle is a path in which the first and last vertices are identical and there are otherwise no repeated vertices. The in-degree of a set of vertices $V \subseteq S$ is the number of arcs $s \to t \in T$ with $s \notin V$ and $t \in V$. Symmetrically, the out-degree is the number of such arcs where $s \in V$ and $t \notin V$.

Let $\bar{T} \supseteq T$ be the **extended transfer graph** that includes self cycles. That is, we drop the requirement that $s \neq t$; so, if $\sigma(i) = \tau(i) = s$, then $s \xrightarrow{i} s \in \bar{T}$. Abusing terminology, a **cycle decomposition** $\mathcal{F}$ of extended transfer graph $\bar{T}$ is a collection of cycles of $\bar{T}$ where each student $i \in N$ appears in exactly one cycle (possibly a self cycle).[28] Let c denote a generic element of $\mathcal{F}$. Let

$$
\begin{aligned}
\mathcal{F}^= &= \{c \in \mathcal{F} :\ (s,t,i) \in c \Rightarrow s \neq t \ \&\ (\boldsymbol{\rho}, \sigma)\ I_i\ (\boldsymbol{\gamma}, \tau)\}\\
\mathcal{F}^+ &= \{c \in \mathcal{F} \setminus \mathcal{F}^= :\ (s,t,i) \in c \Rightarrow s \neq t \ \&\ (\boldsymbol{\gamma}, \tau)\ R_i\ (\boldsymbol{\rho}, \sigma)\}\\
\mathcal{F}^- &= \{c \in \mathcal{F} \setminus \mathcal{F}^= :\ (s,t,i) \in c \Rightarrow s \neq t \ \&\ (\boldsymbol{\rho}, \sigma)\ R_i\ (\boldsymbol{\gamma}, \tau)\}\\
\mathcal{F}^0 &= \{c \in \mathcal{F} :\ c = \{(s,s,i)\} \text{ for some } s \text{ and } i\}.
\end{aligned}
$$

A cycle in $\mathcal{F}^=$ is one where each involved student is indifferent between their old and new school. Similarly, $\mathcal{F}^+$ is the set of totally-positive cycles; $\mathcal{F}^-$, totally-negative cycles; $\mathcal{F}^0$, self cycles.

Recall that Proposition 2 posits the existence of a cycle decomposition $\mathcal{F}$ with certain properties. We present a key tool in its proof, namely the existence of a subfamily of cycles which will eventually constitute $\mathcal{F}^+ \subseteq \mathcal{F}$.

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[27] A path in T is a sequence of distinct arcs $\{(s_k, t_k, i_k) : k = 1, \ldots, K\}$ with $t_{k-1} = s_k$ for each $k \in \{2, \ldots, K\}$.

[28] In the language defined in Section IV, each student appears in exactly one trading cycle or self cycle.

Proposition 5. *Let $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$ be two RCEs for a profile $\boldsymbol{R}$ satisfying NCBI*. Let T be the transfer graph from $(\boldsymbol{\rho}, \sigma)$ to $(\boldsymbol{\gamma}, \tau)$. Every path in T beginning in a positive arc is totally-positive, and every positive arc is part of a totally-positive cycle. Every totally-positive cycle is confined to $\overline{S} = S^+ \cup S^= \cup S^*$. Finally, the totally-positive cycles in T can be (non-uniquely) decomposed into a disjoint family of cycles.*

Proof. We first establish several claims.

Claim 1. *For each $s \in \overline{S}$ with $\sigma[s] \neq \emptyset$, the out-degree of s in T is at least as large as its in-degree.*

Proof of claim. For $s \in \overline{S}$, with $\sigma[s] \neq \emptyset$, exhaustiveness yields $\lfloor \rho_s^{-1} \rfloor = |\sigma[s]|$. Since $|\tau[s]| \in \mathbb{N}$, distribution feasibility gives $|\tau[s]| \leq \lfloor \gamma_s^{-1} \rfloor$. Combining these, since $\gamma_s \geq \rho_s$ ($s \in \overline{S}$), we have

$$|\tau[s]| \leq \lfloor \gamma_s^{-1} \rfloor \leq \lfloor \rho_s^{-1} \rfloor = |\sigma[s]|,$$

Thus, for each arc entering s, there must be at least one exiting. □

Claim 2. *If $s \xrightarrow{i} t \in T$ and $i \in N^+ \cup N^=$, then $t \in \overline{S}$.*

Proof of claim. Let $s \xrightarrow{i} t \in T$. Assume $t \notin \overline{S}$, and so $\gamma_t < \rho_t$. This also implies that $\rho_t > b_t$. Then since $(\boldsymbol{\rho}, \sigma)$ is an RCE and $\sigma(i) = s$, by *fairness*, $(\boldsymbol{\rho}, s)\ R_i\ (\boldsymbol{\rho}, t)$. Since $\gamma_t < \rho_t$, $(\boldsymbol{\rho}, t)\ P_i\ (\boldsymbol{\gamma}, t)$. Combining these, we get

$$(\boldsymbol{\rho}, \sigma(i)) = (\boldsymbol{\rho}, s)\ P_i\ (\boldsymbol{\gamma}, t) = (\boldsymbol{\gamma}, \tau(i)) .$$

Thus, we have shown that $t \notin \overline{S}$ implies $i \notin N^+ \cup N^=$, the contrapositive of the desired conclusion. □

Claim 3. *$\sigma[S^+] \subseteq N^+$ and $\tau(N^+) \subseteq S^+ \cup S^*$.*

Proof of claim. For $s \in S^+$, $\gamma_s > \rho_s \geq b_s$. Therefore, for each $i \in N$, *fairness* yields $(\boldsymbol{\gamma}, \tau(i))\ R_i\ (\boldsymbol{\gamma}, s)$. In particular, for $i \in \sigma[s]$, preference monotonicity gives

$$(\boldsymbol{\gamma}, \tau(i))\ R_i\ (\boldsymbol{\gamma}, s)\ P_i\ (\boldsymbol{\rho}, s) = (\boldsymbol{\rho}, \sigma(i)) .$$

Thus, $i \in N^+$.

Let $i \in N^+$. If $\tau(i) \notin S^+$, then $\rho_{\tau(i)} \geq \gamma_{\tau(i)}$ and by preference monotonicity

$$(\boldsymbol{\rho}, \tau(i))\ R_i\ (\boldsymbol{\gamma}, \tau(i))\ P_i\ (\boldsymbol{\rho}, \sigma(i)) ,$$

where the strict relation holds because $i \in N^+$. Then since $(\boldsymbol{\rho}, \sigma)$ is an RCE and therefore *fair*, $b_{\tau(i)} = \rho_{\tau(i)} \geq \gamma_{\tau(i)} \geq b_{\tau(i)}$. This yields $\tau(i) \in S^*$. □

Claim 4. *Consider a path in T of the following form:*

$$t \xrightarrow{i} u \xrightarrow{j} v$$

with $i \in N^+$ *and* $j \notin N^+$*. Then* $j \in N^=$*,* $u \in S^*$*,* $v \in S^+ \cup S^=$*, and* $\sigma[v] \neq \emptyset$*.*

Proof of claim. By Claim 3 via contraposition, $j \notin N^+$ implies $\sigma(j) = u \notin S^+$, and so $\gamma_u \leq \rho_u$. By preference monotonicity,

$$(\boldsymbol{\rho}, u) \; R_i \; (\boldsymbol{\gamma}, u) = (\boldsymbol{\gamma}, \tau(i)) \; P_i \; (\boldsymbol{\rho}, \sigma(i)) \,,$$

where the strict relation holds because $i \in N^+$. It follows that $\sigma(j) = u \in S^*$ (so $\gamma_u = \rho_u$) and $j \succ_u i$. Thus if $j \in N^-$, we have

$$(\gamma_u, u) = (\rho_u, u) = (\boldsymbol{\rho}, \sigma(j)) \; P_j \; (\boldsymbol{\gamma}, \tau(j))$$

implying, since $\tau(i) = u$, that $i \succ_u j$. In sum, we have $j \succ_u i$ and $i \succ_u j$, a contradiction. Thus, $j \in N^+ \cup N^=$. By assumption we have that $j \notin N^+$, so $j \in N^=$. Since $u \in S^*$, if $\gamma_v = b_v$, then j is indifferent between (b_u, u) and (b_v, v), contradicting NCBI*. Moreover, if $\rho_v > \gamma_v$, then $\rho_v > b_v$ and

$$(\boldsymbol{\rho}, v) \; P_j \; (\boldsymbol{\gamma}, v) = (\boldsymbol{\gamma}, \tau(j)) \; I_j \; (\boldsymbol{\rho}, \sigma(j)) \,,$$

where the indifference relation holds because $j \in N^=$. Since $(\boldsymbol{\rho}, \sigma)$ is an RCE, *fairness* implies that $\rho_v = b_v$, contradicting $\rho_v > \gamma_v$. Thus, $\gamma_v \geq \rho_v$ and $\gamma_v > b_v$, and so $v \in S^+ \cup S^=$. Finally, if $\sigma[v] = \emptyset$, then

$$(\boldsymbol{\rho}, u) = (\boldsymbol{\rho}, \sigma(j)) \; P_j \; (\boldsymbol{\rho}, v) = (1, v) \; R_j \; (\boldsymbol{\gamma}, v) = (\boldsymbol{\gamma}, \tau(j)) \,,$$

where the strict relation is by *inferior empty schools*, contradicting that $j \in N^=$. Thus, $\sigma[v] \neq \emptyset$. □

Let $s \xrightarrow{1} u \in T$ have $1 \in N^+$. We shall extend this to a path $s \rightsquigarrow t$. By Claim 2, $u \in \overline{S}$. If it were the case that $\sigma[u] = \emptyset$, then since $(\boldsymbol{\rho}, \sigma)$ is an RCE, *inferior empty schools* implies

$$(\boldsymbol{\rho}, s) \; P_1 \; (\boldsymbol{\rho}, u) = (1, u) \; R_1 \; (\boldsymbol{\gamma}, u) \,,$$

contradicting that $1 \in N^+$. Thus, $\sigma[u] \neq \emptyset$. By Claim 1, there is $u \xrightarrow{2} v \in T$. If $2 \in N^+$, we could then start with this arc instead. Continuing inductively, let j be the first student on the path who is not in N^+ (if such an student does not exist, the argument yields a totally-positive cycle, as desired). We have, thus far, a path of the form

$$s \xrightarrow{1} u \rightsquigarrow v \xrightarrow{j-1} \sigma(j) \xrightarrow{j} w,$$

where $u \rightsquigarrow v$, if it exists, is labeled by N^+ students and $j-1 \in N^+$. By Claim 4, $j \in N^=$, $\sigma(j) \in S^*$, $w \in S^+ \cup S^=$, and $\sigma[w] \neq \emptyset$. If $w \in S^+$, then the path is extended by an arc $w \xrightarrow{k} w'$ with $k \in N^+$ (Claims 1 and 3). In this case, our argument has returned to its starting point. Thus, we proceed constructively, and when we arrive at an arc with an N^+ label, call this the *escape condition* of our proof.

Assume, therefore, that $w \in S^=$. Claim 1 implies there is $w \xrightarrow{k} w' \in T$. Note that $k \in N^+ \cup N^=$, as otherwise,

$$(\gamma_w, w) = (\rho_w, w) = (\boldsymbol{\rho}, \sigma(k))\ P_k\ (\boldsymbol{\gamma}, \tau(k))\,,$$

violating that $(\boldsymbol{\gamma}, \tau)$ is an RCE (recall the definition of $S^=$). If $k \in N^+$, we have encountered the escape condition again, so assume $k \in N^=$. By Claim 2, $w' \in \overline{S}$. If $w' \in S^+$, then $\rho_{w'} < 1$ so $\sigma[w'] \neq \emptyset$ (*inferior empty schools*), and there must be an outgoing N^+ arc from w' (Claims 1 and 3); again we have the escape condition. Thus, to continue the argument, assume $w' \in S^* \cup S^=$. Now if $w' \in S^*$, we have

$$\sigma(j) \xrightarrow{j} w \xrightarrow{k} w'$$

with $\sigma(j) \in S^*$, $w \in S^=$, and $j, k \in N^=$. This is an indifference path, given $\boldsymbol{\rho}$, connecting two S^* schools, contradicting NCBI*. Conclude that $w' \in S^=$, and recalling that $k \in N^=$,

$$(\boldsymbol{\rho}, w)\ I_k\ (\boldsymbol{\gamma}, w') = (\boldsymbol{\rho}, w')\,.$$

Since $(\boldsymbol{\rho}, \sigma)$ is an RCE, by *inferior empty schools*, $\sigma[w'] \neq \emptyset$. We have returned to the same situation as encountered at the beginning of this paragraph, a non-empty school in $S^=$ with an incoming $N^=$ arc in T. We can then therefore repeat the foregoing arguments and continue the path. In particular, w' must have an outgoing arc $w' \xrightarrow{k'} w''$, and $k' \in N^+ \cup N^=$. If $k' \in N^+$, we get the escape condition, and if $k' \in N^=$, we again conclude that $w'' \in S^=$ and $\sigma[w''] \neq \emptyset$.

Thus, we construct path $s \rightsquigarrow t$ which can be decomposed as follows:

$$s \xrightarrow{1} u \rightsquigarrow v \rightarrow \sigma(j) \xrightarrow{j} w \rightsquigarrow x \rightarrow t$$

where

(1) $u \rightsquigarrow v$, is labeled by N^+ students and touches only $\overline{S}$ schools,
(2) $j \in N^=$ and $\sigma(j) \in S^*$,
(3) $w \rightsquigarrow x$, is within $S^=$ and labeled by $N^=$ students,
(4) $t \in S^= \cup S^+$.

These segments need not all exist, but we have shown, segment by segment, that any N^+ labeled arc $s \rightarrow u$ induces a path that is always labeled by $N^+ \cup N^=$ students, is always within $\overline{S}$ (except possibly for the very first vertex, s), and can always be extended. It follows that we can find a cyclic sub-path, not necessarily including s. However, by deleting the cycle from T (recording its existence in $\mathcal{F}^+$ if appropriate), we preserve the vertex degree inequality of Claim 1, and none of the other claims are affected. Thus, we may repeat the argument. Eventually, we must find a cycle involving s, implying $s \in \overline{S}$. By further repeating the argument we construct a family of disjoint cycles.

Note that in our construction each time we extend the path there may be multiple candidate arcs; thus, choosing different arcs subsequently results in different families of cycles.

□

Now we apply Proposition 5 to the NCBI domain and complete the proofs of Proposition 2 and Theorem 3.

Proof of Proposition 2. First apply Proposition 5 to get a collection $\mathcal{F}^+$ of disjoint, totally-positive cycles. Then apply it to the change from $(\boldsymbol{\gamma}, \tau)$ to $(\boldsymbol{\rho}, \sigma)$ to get a collection $\mathcal{F}^-$ of disjoint, totally-negative cycles. Collection $\mathcal{F}^0$ is constructed in the obvious way. To complete the proof of statement (1) of the Proposition, we decompose $T' = \overline{T} \setminus (\mathcal{F}^+ \cup \mathcal{F}^- \cup \mathcal{F}^0)$ into disjoint cycles.

By construction, T' contains only indifference arcs; equivalently, each arc in T' is $N^=$-labeled. Recall that Claim 3 in the proof of Proposition 5 yields $\sigma[S^+] \subseteq N^+$. It follows that, for each $u \xrightarrow{i} v \in T'$, $u \notin S^+$. Equivalently, if $u \in S^+$, then u has no outgoing arcs in T'. Let $s \rightsquigarrow t \subseteq T'$. We have then that each vertex on this path, except possibly t, belongs to $S \setminus S^+$. To arrive at a contradiction, assume $t \in S^+$. Then t has no outgoing arcs in T', and so its in-degree is greater than its out-degree in T'. Since $\mathcal{F}^+$, $\mathcal{F}^-$, and $\mathcal{F}^0$ are all families of cycles, the in-degree of t in $\overline{T}$ is also greater than its out-degree in $\overline{T}$. Thus, $|\tau[t]| > |\sigma[t]|$. Then by *exhaustiveness* we have

$$\lfloor \rho_t^{-1} \rfloor = |\sigma[t]| < |\tau[t]| = \lfloor \gamma_t^{-1} \rfloor,$$

yielding $\gamma_t < \rho_t$, contradicting $t \in S^+$. In sum, every vertex touching $s \rightsquigarrow t$ belongs to $S \setminus S^+$. Applying the same argument to the extended transfer graph from $(\boldsymbol{\gamma}, \tau)$ to $(\boldsymbol{\rho}, \sigma)$, we conclude that every vertex belongs to $S \setminus S^-$, and so, in fact, every vertex belongs to $S^* \cup S^=$.

The presence of the path $s \rightsquigarrow t$ implies that only s or t could be empty in one of the two RCEs. We show that neither s nor t is empty at either of the two RCEs. Since there is $u \xrightarrow{i} t \subseteq s \rightsquigarrow t$, $\tau[t] \neq \emptyset$, so suppose $\sigma[t] = \emptyset$. Then by *inferior empty schools*, $(\boldsymbol{\rho}, u)\ P_i\ (1, v) = (\boldsymbol{\gamma}, v)$, contradicting that $i \in N^=$. If $\tau[s] = \emptyset$ then again consider the change from $(\boldsymbol{\gamma}, \tau)$ to $(\boldsymbol{\rho}, \sigma)$. Thus, no school on the path is empty at either RCE.

Since $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$ are both exhaustive, for each vertex u touched by the path, $\lfloor \rho_u^{-1} \rfloor = |\sigma[u]|$ and $\lfloor \gamma_u^{-1} \rfloor = |\tau[u]|$. Since $u \in S^* \cup S^=$, $\rho_u = \gamma_u$ and so $|\sigma[u]| = |\tau[u]|$. Thus, in $\overline{T}$, the in-degree and out-degree of u are equal. Removing cycles preserves this fact for T'. For $u = s$, since s has an outgoing arc in T', it has an incoming arc in T'. For $u = t$, since t has an incoming arc in T', it has an outgoing arc in T'. Thus, $s \rightsquigarrow t$ can be extended to $s' \to s \rightsquigarrow t \to t'$, with $s' \rightsquigarrow t' \subseteq T'$. We can repeat the argument, and since there are finitely many vertices, we will eventually find a cycle that can itself be removed and the entire argument restarted. Thus, we decompose T' into a family of cycles, and arrive at $\mathcal{F}^=$. In conclusion, we have $\mathcal{F} = \mathcal{F}^+ \cup \mathcal{F}^- \cup \mathcal{F}^0 \cup \mathcal{F}^=$ as our desired cycle decomposition.

To show statement (2), let $\mathcal{F}$ be a cycle decomposition. Let $s \to t \in c \in \mathcal{F}$ with $s \neq t$. By Proposition 5, if $s \to t$ is positive, then $c \in \mathcal{F}^+$. Symmetrically, if $s \to t$ is negative, it is in $\mathcal{F}^-$. □

Proof of Theorem 3. Let $\bar{T}$ be the extended transfer graph from $(\boldsymbol{\rho}, \sigma)$ to $(\boldsymbol{\gamma}, \tau)$. Given a cycle decomposition $\mathcal{F}$ of $\bar{T}$, let μ be the matching that results from executing all cycles in $\mathcal{F}^+$ on σ. That is, if i labels an arc on a cycle in $\mathcal{F}^+$, then $\mu(i) = \tau(i)$, and otherwise $\mu(i) = \sigma(i)$. Let $\boldsymbol{\zeta} = \boldsymbol{\rho} \vee \boldsymbol{\gamma}$. We show that $(\boldsymbol{\zeta}, \mu)$ is an RCE.

By Proposition 2, the number of students at each school is unchanged from σ to μ, so $(\boldsymbol{\zeta}, \mu)$ satisfies *exhaustiveness*. To check that $(\boldsymbol{\zeta}, \mu)$ is an allocation, it is sufficient to check the schools whose distribution has increased from $(\boldsymbol{\rho}, \sigma)$ to $(\boldsymbol{\zeta}, \mu)$. Pick $s \in S^+$, where S^+ is defined as above. Since $|\tau[s]| = |\sigma[s]|$, and $\zeta_s = \gamma_s$, distribution feasibility at school s then follows from the distribution feasibility of $(\boldsymbol{\gamma}, \tau)$, and since each $\zeta_s \geq b_s$ it *respects capacities*.

So $(\boldsymbol{\zeta}, \mu)$ is an exhaustive allocation and the remainder of the argument verifies that it satisfies *fairness* and *inferior empty schools*. Let N' be the set of students in (1) a cycle in $\mathcal{F}^+$, or (2) a self cycle $(s, s, \cdot)$ with $s \in S^+$. If $i \in N'$ satisfies (1), then $i \in N^+ \cup N^=$, and so $(\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\gamma}, \tau(i))\ R_i\ (\boldsymbol{\rho}, \sigma(i))$. If $i \in N'$ satisfies (2), then $\sigma(i) = \tau(i) = \mu(i) \in S^+$, and so $i \in N^+$. Combining, we have

$$\forall i \in N', (\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\gamma}, \tau(i))\ R_i\ (\boldsymbol{\rho}, \sigma(i))\,. \tag{2}$$

Let $i \in N^+$. If $\tau(i) \neq \sigma(i)$, then there is positive arc $\sigma(i) \xrightarrow{i} \tau(i) \in T$, and by Proposition 5, there is a cycle $c \in \mathcal{F}^+$ with $\sigma(i) \xrightarrow{i} \tau(i) \in c$. If $\tau(i) = \sigma(i)$, then since $i \in N^+$ we must have $\gamma_{\sigma(i)} > \rho_{\sigma(i)}$. In sum, $N^+ \subseteq N'$. Now let $i \notin N'$, so $i \notin N^+$. Let $s = \sigma(i)$, and therefore $\mu(i) = s$, recalling that only agents labeling $\mathcal{F}^+$ cycles have $\mu(\cdot) \neq \sigma(\cdot)$. Recall also that by Claim 3 in the proof of Proposition 5, $\sigma[S^+] \subseteq N^+$. Thus, as $i \notin N^+$, $\rho_s \geq \gamma_s$ and so $\zeta_s = \rho_s$. Moreover, $(\boldsymbol{\rho}, \sigma(i))\ R_i\ (\boldsymbol{\gamma}, \tau(i))$. We conclude that

$$\forall i \in N \setminus N', (\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\rho}, \sigma(i))\ R_i\ (\boldsymbol{\gamma}, \tau(i))\,. \tag{3}$$

Lines (2) and (3) yield

$$\forall i \in N, (\boldsymbol{\zeta}, \mu(i))\ R_i\ \max_{R_i}\{(\boldsymbol{\rho}, \sigma(i))\,, (\boldsymbol{\gamma}, \tau(i))\}. \tag{4}$$

By Theorem 2, the set of empty schools remains the same in $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$, and so also in $(\boldsymbol{\zeta}, \mu)$. Thus, by line (4), and the fact that both $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$ satisfy *inferior empty schools*, $(\boldsymbol{\zeta}, \mu)$ does as well.

Finally, suppose $(\boldsymbol{\zeta}, s)\ P_i\ (\boldsymbol{\zeta}, \mu(i))$, which by line (4) implies

$$(\boldsymbol{\zeta}, s)\ P_i \max_{R_i}\{(\boldsymbol{\rho}, \sigma(i)), (\boldsymbol{\gamma}, \tau(i))\}.$$

By *inferior empty schools* there is $j \in \mu[s]$, so since $(\boldsymbol{\zeta}, \mu(j)) \in \{(\boldsymbol{\rho}, \sigma(j)), (\boldsymbol{\gamma}, \tau(j))\}$, plugging the appropriate case into the previous line yields $\zeta_s = b_s$ and $j \succ_s i$, since $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$ are RCEs. □

A.3. **Domination Lemmas.** Let $\Gamma = \{s \xrightarrow{i} t :\ \sigma(i) = s \neq t \text{ and } (\boldsymbol{\rho}, t)\ R_i\ (\boldsymbol{\rho}, s)\}$ be the *weak-envy graph* of allocation $(\boldsymbol{\rho}, \sigma)$.

Recall that a *source set* in a directed graph is a set of vertices that no arc enters. Formally, it is a set $S' \subseteq S$ such that if $s \to t \in \Gamma$ and $s \notin S'$, then $t \notin S'$. The set of *vertices upstream of* s is

$$U_s = \{t \in S : t \rightsquigarrow s \subseteq \Gamma\}.$$

Similar to the usage above, let $S^* = \{s \in S : \rho_s = b_s \text{ and } \sigma[s] \neq \emptyset\}$. Say a school $s \in S$ is *totally exhausted* at $(\boldsymbol{\rho}, \sigma)$ if $|\sigma[s]|\rho_s = 1$.

Lemma 1. *Let $(\boldsymbol{\rho}, \sigma)$ be a* fair *allocation for $\boldsymbol{R} \in \mathcal{R}^N$ with weak-envy graph Γ. Suppose $S' \subseteq S \setminus S^*$, with $\sigma[S']$ not empty, is a source set in Γ and that no school in S' is totally exhausted. Then there is a fair allocation $(\boldsymbol{\gamma}, \tau)$, Pareto-dominating $(\boldsymbol{\rho}, \sigma)$ and with $\gamma \gneq \rho$.*

Proof. Let $N' = \sigma[S']$. For each $s \in S'$, let $n_s = |\sigma[s]|$. We shall construct a non-transferable utility assignment game locally isomorphic to the problem we currently face when restricted to N' and S'. Let $\mathfrak{S}$ be a set of $\sum_{s \in S'} n_s$ elements. Let $f : \mathfrak{S} \to S'$ have $|f[s]| = n_s$.[29] We view $\mathfrak{S}$ as the set of copies of the elements of S'. Note that schools empty under $(\boldsymbol{\rho}, \sigma)$ have an empty pre-image under f, and so are effectively excluded from the assignment game. Thus, we now assume that for each $s \in S'$, $\sigma[s] \neq \emptyset$, and so $\rho_s > b_s$.

Each $\mathfrak{s} \in \mathfrak{S}$ consumes a bundle $(l, i) \in \mathbb{R} \times N'$ and has simple preferences represented by utility function $W_{\mathfrak{s}}(l, i) = l$; copies of schools care only about resources. Each copy has an outside option denoted $\underline{w}_{\mathfrak{s}} \in \mathbb{R}$, so that $\mathfrak{s}$ will withdraw from the matching (now an option) before accepting a bundle giving utility less than $\underline{w}_{\mathfrak{s}}$. With slight abuse, we retain the same notation for the students. Each $i \in N'$ consumes a bundle $(r, \mathfrak{s}) \in \mathbb{R} \times \mathfrak{S}$ and has preferences so that

$$(r, \mathfrak{s})\ R_i\ (r', \mathfrak{s}') \iff (r, f(\mathfrak{s}))\ R_i\ (r', f(\mathfrak{s}')).$$

[29]Recall, we use $g^{-1}(x)$ to denote the *unique* inverse of a bijection and $g[x]$ to denote the set-valued pre-image.

Let U_i be a continuous utility function representation of R_i, extended to $\mathbb{R} \times S$, and with the property that for each $(r, s) \in \mathbb{R} \times S$ and each $t \in S$, there is $q \in \mathbb{R}$ with $U_i(q, t) = U_i(r, s)$. Assume that the outside option utility for student i is less than $\min_{(r,s)\in[0,1]\times S} U_i(r, s)$. When a student and a school match, one unit of divisible resource is produced, independent of their identities. A feasible outcome in this assignment game is a vector $\hat{\boldsymbol{\rho}} \in [0, 1]^{\mathfrak{S}}$ of resources together with a bijection $\hat{\sigma} : N' \to \mathfrak{S}$. Each $i \in N'$ consumes $(\hat{\boldsymbol{\rho}}, \hat{\sigma}(i))$, where we reuse our notation from the model of school choice with crowding, and each $\mathfrak{s} \in \mathfrak{S}$ consumes $(1 - \hat{\rho}_{\mathfrak{s}}, \hat{\sigma}^{-1}(\mathfrak{s}))$. Agents i and $\mathfrak{s}$ *block* $(\hat{\boldsymbol{\rho}}, \hat{\sigma})$ if there is $\hat{\rho}'_{\mathfrak{s}} \in [0, 1]$ such that $U_i(\hat{\rho}'_{\mathfrak{s}}, \mathfrak{s}) \geq U_i(\hat{\boldsymbol{\rho}}, \hat{\sigma}(i))$ and $\hat{\rho}'_{\mathfrak{s}} < \hat{\rho}_{\mathfrak{s}}$, where this latter of course implies the block gives $\mathfrak{s}$ higher utility. An allocation is stable if there are no blocks and each agent finds it at least as good as their outside option.

Demange and Gale [1985] (henceforth D&G) show that, in this assignment game, there is a unique student-optimal utility profile $(\boldsymbol{u}, \boldsymbol{w}) \in \mathbb{R}^{N'} \times \mathbb{R}^{\mathfrak{S}}$ in the space of utility profiles induced by stable outcomes. There may be several stable outcomes that induce it. Moreover, because of student optimality, there is at least one $\mathfrak{s} \in \mathfrak{S}$ with $w_{\mathfrak{s}} = \underline{w}_{\mathfrak{s}}$.

Let $(\hat{\boldsymbol{\rho}}, \hat{\sigma})$ be the assignment game outcome induced by $(\boldsymbol{\rho}, \sigma)$ in the statement of the Lemma, but restricted to N' and S'. We show in this paragraph that, so long as $\underline{w}_{\mathfrak{s}} \leq 1 - \rho_{f(\mathfrak{s})}$, this outcome is stable. Clearly, each agent finds $(\hat{\boldsymbol{\rho}}, \hat{\sigma})$ at least as good as their outside option. Since $(\boldsymbol{\rho}, \sigma)$ is *fair* in the original model with crowding, for $f(\mathfrak{s}) = \sigma(i)$ and $f(\mathfrak{s}') \in S'$, $(\rho_{f(\mathfrak{s})}, f(\mathfrak{s}))\ R_i\ (\rho_{f(\mathfrak{s}')}, f(\mathfrak{s}'))$. Thus, for i and $\mathfrak{s}'$ to block in the assignment game, given the construction of the preferences for students, i must get $\hat{\rho}'_{\mathfrak{s}'} \geq \rho_{f(\mathfrak{s}')}$. Since $\hat{\rho}_{\mathfrak{s}'} = \rho_{f(\mathfrak{s}')}$, we then have $\hat{\rho}'_{\mathfrak{s}'} \geq \hat{\rho}_{\mathfrak{s}'}$. Recalling that a block requires $\hat{\rho}'_{\mathfrak{s}'} < \hat{\rho}_{\mathfrak{s}'}$ to raise the utility of the copied schools, we have what was desired.

Fix $\varepsilon > 0$ and set each $\underline{w}_{\mathfrak{s}} = 1 - \hat{\rho}_{\mathfrak{s}} - \varepsilon$. Let $(\hat{\boldsymbol{\gamma}}, \hat{\mu})$ be a student-optimal outcome for this problem. Since $(\hat{\boldsymbol{\rho}}, \hat{\sigma})$ is stable, Property 3 of D&G yields that, for each $\mathfrak{s} \in \mathfrak{S}$, $\hat{\gamma}_{\mathfrak{s}} \geq \hat{\rho}_{\mathfrak{s}}$; student and school interests are opposed in the stable set.

Suppose there are $\mathfrak{s}, \mathfrak{s}' \in f[s]$ with $\hat{\gamma}_{\mathfrak{s}} > \hat{\gamma}_{\mathfrak{s}'}$. Student $i = \hat{\mu}^{-1}(\mathfrak{s}')$ gets $\hat{\gamma}_{\mathfrak{s}'}$ resource at $\mathfrak{s}'$. Since i cannot distinguish $\mathfrak{s}$ and $\mathfrak{s}'$, $U_i(\hat{\gamma}_{\mathfrak{s}'}, \mathfrak{s}) = U_i(\hat{\gamma}_{\mathfrak{s}'}, \mathfrak{s}')$. Clearly then i and $\mathfrak{s}$ block with resource level $\hat{\gamma}_{\mathfrak{s}'}$. Conclude that copies of the same school all get the same level of utility, and so we may define partial distribution $\boldsymbol{\gamma} \in [0, 1]^{S'}$ by setting each γ_s equal to the common $\hat{\gamma}_{\mathfrak{s}}$ of its copies. Complete the distribution by setting, $\gamma_s = \rho_s$ for each $s \notin S'$. We therefore have that $\boldsymbol{\gamma} \geq \boldsymbol{\rho}$. Recalling that, at the student-optimal stable outcome, some school $\mathfrak{s}$ gets its utility lower bound $\underline{w}_{\mathfrak{s}} = 1 - \hat{\rho}_{\mathfrak{s}} - \varepsilon$ (D&G Lemma 3), it follows that, for $s = f(\mathfrak{s})$, $\gamma_s = \rho_s + \varepsilon$. Thus, $\boldsymbol{\gamma} \gneq \boldsymbol{\rho}$, and

Lemma 1 in D&G also implies that for $i = \hat{\mu}^{-1}(\mathfrak{s})$, $(\hat{\boldsymbol{\gamma}}, \hat{\mu}(i))\ P_i\ (\hat{\boldsymbol{\rho}}, \hat{\sigma}(i))$, yielding that $(\hat{\boldsymbol{\gamma}}, \hat{\mu})$ Pareto dominates $(\hat{\boldsymbol{\rho}}, \hat{\sigma})$ for the N' students. Finally, define μ so that, for each $s \in S'$, $\mu[s] = \{i \in N : \hat{\mu}(i) \in f[s]\}$, and for $s \notin S'$, $\mu[s] = \sigma[s]$.

To check that $(\boldsymbol{\gamma}, \mu)$ is an allocation, we need only check distribution feasibility for the S' schools. Each $s \in S'$ has $n_s \rho_s < 1$, being not totally exhausted. Then for ε small enough and any $\mathfrak{s} \in f[s]$, since $1 - \hat{\gamma}_{\mathfrak{s}} \geq \underline{w}_{\mathfrak{s}}$,

$$n_s \gamma_s \leq n_s(1 - \underline{w}_{\mathfrak{s}}) = n_s(\rho_s + \varepsilon) < 1.$$

Next, we show that $(\boldsymbol{\gamma}, \mu)$ is *fair*. Recall that for each $s \in S'$, $\rho_s > b_s$, so there is a violation of *fairness* in $(\boldsymbol{\gamma}, \mu)$ at $s \in S'$ if there is $i \in N$ with $(\boldsymbol{\gamma}, s)\ P_i\ (\boldsymbol{\gamma}, \mu(i))$. Since S' is a source set in Γ, for ε small enough, it remains a source set in the weak-envy graph for $(\boldsymbol{\gamma}, \mu)$. Thus, there is no violation with students outside N'. Since $(\hat{\boldsymbol{\gamma}}, \hat{\mu})$ is stable, for each $i \in N'$ and each $s \in S'$, $(\boldsymbol{\gamma}, \mu(i))\ R_i\ (\boldsymbol{\gamma}, s)$. Thus the only possible *fairness* violations involve $i \in N'$ and $s \notin S'$. For such a potential pair, we have

$$(\boldsymbol{\rho}, s) = (\boldsymbol{\gamma}, s)\ P_i\ (\boldsymbol{\gamma}, \mu(i))\ R_i\ (\boldsymbol{\rho}, \sigma(i))\,,$$

where the last relation is because the stable match in the one-to-one problem is the optimal stable match for the N' students. Since $(\boldsymbol{\rho}, \sigma)$ is fair, $\rho_s = \gamma_s = b_s$ and each $j \in \sigma[s] = \mu[s]$ has $j \succ_s i$.

Conclude by observing that, as students in $N \setminus N'$ are indifferent between $(\boldsymbol{\gamma}, \mu)$ and $(\boldsymbol{\rho}, \sigma)$, while the former Pareto dominates the latter when restriced to N', $(\boldsymbol{\gamma}, \mu)$ Pareto dominates $(\boldsymbol{\rho}, \sigma)$. □

Let Γ be the weak-envy graph of allocation $(\boldsymbol{\rho}, \sigma)$ at preferences $\boldsymbol{R}$, and let $t{\leadsto}s \subseteq \Gamma$. Construct τ so that for each $i \in N$ with $u \xrightarrow{i} v \in t{\leadsto}s$, $\tau(i) = v$, and otherwise $\tau(i) = \sigma(i)$. We allow for $s = t$, so that the path may be a cycle. We say that τ is the matching that results from *executing* the path on matching σ. Clearly, all agents find τ at least as good as σ.

Lemma 2. *Given profile $\boldsymbol{R}$ from the NCBI domain, let $(\boldsymbol{\rho}, \sigma)$ be a* fair *allocation that is not an RCE. Then there is another* fair *allocation $(\boldsymbol{\gamma}, \tau)$ for $\boldsymbol{R}$ such that either 1) $\boldsymbol{\gamma} \gneq \boldsymbol{\rho}$ and $\tau = \sigma$, or 2) $(\boldsymbol{\gamma}, \tau)$ Pareto dominates $(\boldsymbol{\rho}, \sigma)$.*

Proof. We begin by showing the following two claims.

Claim 5. *Let $(\boldsymbol{\rho}, \sigma)$ be a* fair *allocation, and $s \in S$ be such that $|\sigma[s]| \leq \lfloor \rho_s^{-1} \rfloor - 1$. If there is $i \in N$ such that $(\rho_s, s)\ P_i\ (\boldsymbol{\rho}, \sigma(i))$, then (2) in the conclusion of Lemma 2 holds.*

Proof of claim. Let $N' = \{j \in N : (\rho_s, s)\ P_j\ (\boldsymbol{\rho}, \sigma(j))\}$. Since $i \in N'$, N' is non-empty. Let $j = \max_{\succ_s} N'$. Since $|\sigma[s] \leq \lfloor \rho_s^{-1} \rfloor - 1$, we can move at least one student to s. Define matching τ so that $\tau(j) = s$ and $\tau(j') = \sigma(j')$ for each $j' \neq j$. Finally, let $\boldsymbol{\gamma} = \boldsymbol{\rho}$. It is easy to see that we introduce no violation of *fairness*, and $(\boldsymbol{\gamma}, \tau)$ is *fair* and Pareto-dominates $(\boldsymbol{\rho}, \sigma)$ as desired. □

Claim 6. *Let $(\boldsymbol{\rho}, \sigma)$ be a* fair *allocation. Assume $s \in S$ is not totally exhausted and has $\rho_s < 1$ and $U_s = \emptyset$, where U_s are the upstream vertices of s in weak-envy graph Γ of $(\boldsymbol{\rho}, \sigma)$. Then (1) in the conclusion of Lemma 2 holds.*

Proof of claim. Since $U_s = \emptyset$, it holds that for each $i \in N \setminus \sigma[s]$, $(\boldsymbol{\rho}, \sigma(i))\ P_i\ (\rho_s, s)$. Since s is not totally exhausted and $\rho_s < 1$, there is $\epsilon > 0$ such that $|\sigma[s]|\, \rho_s < |\sigma[s]|\, (\rho_s + \epsilon) \leq 1$ and for each $i \in N \setminus \sigma[s]$, $(\boldsymbol{\rho}, \sigma(i))\ P_i\ (\rho_s + \epsilon, s)$. Let γ be such that $\gamma_s = \rho_s + \epsilon$ and $\gamma_{s'} = \rho_{s'}$ for $s' \neq s$. Keeping the same matching so $\tau = \sigma$, it is easy to see that we introduce no violation of *fairness*, and we have that $(\boldsymbol{\gamma}, \tau)$ is *fair* and the desired result. □

Let $s \in S$ be such that

$$|\sigma[s]| \leq \lfloor \rho_s^{-1} \rfloor - 1. \tag{5}$$

In the following (Case 1 and 2 below), we study the relation between a school satisfying line (5) and conclusions of Lemma 2 (or other properties). We then show that a *fair* allocation violating either *exhaustiveness* or *inferior empty schools*, so not an RCE, will lead to the conclusion of Lemma 2.

Recall $S^* = \{s' \in S :\ \sigma[s'] \neq \emptyset\ \&\ \rho_{s'} = b_{s'}\}$, and let $S^{**} = S^* \cup \{s' \in S :\ \sigma[s'] = \emptyset\ \&\ \rho_{s'} = b_{s'}\}$. Let U_s be the upstream vertices of s in the weak-envy graph Γ of $(\boldsymbol{\rho}, \sigma)$.

Case 1: $s \in S^{**}$.

Case 1.1: $U_s = \emptyset$. If $\sigma[s] \neq \emptyset$, then combined with line (5), we have that $\rho_s < 1$ and s is not totally exhausted. We therefore invoke Claim 6. Assume, then, that $\sigma[s] = \emptyset$. If $\rho_s < 1$, then we invoke Claim 6 again. If $\rho_s = 1$, then since $U_s = \emptyset$, it holds that for each $i \in N$, $(\boldsymbol{\rho}, \sigma(i))\ P_i\ (\rho_s, s)$. So s satisfies *inferior empty schools* (and *exhaustiveness* vacuously).

Case 1.2: $U_s \neq \emptyset$. Let $N' = \{j \in N :\ (\rho_s, s)\ P_j\ (\boldsymbol{\rho}, \sigma(j))\}$. In the case of $N' \neq \emptyset$, we invoke Claim 5 to arrive at conclusion (2) of the lemma. It follows that $N' = \emptyset$ and so for each $j \in N \setminus \sigma[s]$, we have $(\boldsymbol{\rho}, \sigma(j))\ R_j\ (\rho_s, s)$. Since $U_s \neq \emptyset$ and $N' = \emptyset$, each arc $\sigma(j) \xrightarrow{j} s \in \Gamma$ represents indifference.

If there is $t' \in U_s$ with $\rho_{t'}^{-1} \in \mathbb{N}$, then by taking sub-paths, assume $t' \rightsquigarrow s$ is a minimal path starting from such a t'. Then, for every $s' \in S$ touched by the path except s and t', we have $\rho_{s'}^{-1} \notin \mathbb{N}$, and so $\rho_{s'} > b_{s'}$ (otherwise, since $b_{s'}^{-1} \in \mathbb{N}$, there is a shorter sub-path from s' to s, contradicting minimality). If $t \rightsquigarrow s = t' \to s$, then by the previous paragraph, $t' \to s$ represents indifference and we have a violation of NCBI. Otherwise, decompose $t' \rightsquigarrow s$ as $t' \to u \rightsquigarrow v \to s$. Then $u \rightsquigarrow v$ touches no S^{**} vertices and so, since $(\boldsymbol{\rho}, \sigma)$

is a *fair* allocation, the arc $t' \to u$ and all arcs in $u \rightsquigarrow v$ represent indifference. Again, $v \to s$ represents indifference. Since $s \in S^{**}$ and so $\rho_s^{-1} \in \mathbb{N}$, recalling that $\rho_{t'}^{-1} \in \mathbb{N}$, this path is a contradiction to NCBI. Conclude that U_s contains no vertex t' with $\rho_{t'}^{-1} \in \mathbb{N}$, which implies it contains neither a totally exhausted vertex, nor a S^{**} vertex. So, $s \notin U_s$. By definition, U_s is a source set in Γ. Since $U_s \neq \emptyset$ (so $\sigma[U_s] \neq \emptyset$), and $U_s \cap S^{**} = \emptyset$ implies $U_s \cap S^* = \emptyset$, we invoke Lemma 1 to arrive at conclusion (2) of the Lemma.

Case 2: $s \notin S^{**}$.
Case 2.1: $U_s = \emptyset$. The reasoning as in Case 1.1 holds.
Case 2.2: $U_s \neq \emptyset$ and $U_s \cap S^{**} \neq \emptyset$. Find a minimal path $t \rightsquigarrow s \subseteq \Gamma$ with $t \in S^{**}$. That is, by taking sub-paths, $t \rightsquigarrow s$ touches S^{**} only at t. Execute the path to arrive at a new allocation $(\boldsymbol{\rho}, \mu)$ with weak-envy graph Γ_μ. Observe that $(\boldsymbol{\rho}, \mu)$ is a *fair* allocation, as no student has entered a S^{**}-school and so no violations of *fairness* can be introduced. Since t is the only school on the path in S^{**} and $(\boldsymbol{\rho}, \sigma)$ is *fair*, all arcs represent indifferences and so $(\boldsymbol{\rho}, \mu)$ is welfare equivalent to $(\boldsymbol{\rho}, \sigma)$. In particular, the student j who leaves t has $\mu(j) \xrightarrow{j} t \in \Gamma_\mu$. Thus, the set of upstream vertices of t in Γ_μ, denoted U_t^μ, is non-empty. Observe now that

$$|\mu[t]| = |\sigma[t]| - 1 \leq \lfloor \rho_t^{-1} \rfloor - 1.$$

Thus, since $(\boldsymbol{\rho}, \mu)$ is *fair*, $t \in S^{**}$ satisfies inequality (5), and $U_t^\mu \neq \emptyset$, we have arrived at Case 1.2 for allocation $(\boldsymbol{\rho}, \mu)$ and with school t in place of school s.
Case 2.3: $U_s \neq \emptyset$ and $U_s \cap S^{**} = \emptyset$. Since $U_s \neq \emptyset$, $\sigma[U_s] \neq \emptyset$. By $U_s \cap S^{**} = \emptyset$ and $S^* \subseteq S^{**}$, we have that $U_s \cap S^* = \emptyset$ or $U_s \subseteq S \setminus S^*$. By definition, U_s is a source set in Γ. If there are no totally exhausted schools in U_s, then we may invoke Lemma 1 to arrive at conclusion (2) of the lemma. Assume, then, that there is $t \in U_s$ that is totally exhausted. Since $\{s\} \cup U_s \subseteq S \setminus S^{**}$, and since $(\boldsymbol{\rho}, \sigma)$ is *fair*, all the arcs between these vertices in Γ represent indifferences. Suppose there is $t' \in U_s$, $t' \neq t$, that is also totally exhausted. Then there are two paths of indifference, $t \rightsquigarrow s$ and $t' \rightsquigarrow' s$, in Γ, confined to U_s vertices. The concatenation of these, $t \rightsquigarrow s \leftsquigarrow' t'$, represents a sequence of indifferences connecting t and t'. This violates NCBI as both of these vertices are totally exhausted and so $\rho_t^{-1}, \rho_{t'}^{-1} \in \mathbb{N}$. Therefore, t is the *only* member of U_s that is totally exhausted.

Execute $t \rightsquigarrow s$ on $(\boldsymbol{\rho}, \sigma)$ to arrive at allocation $(\boldsymbol{\rho}, \tau)$ with associated weak-envy graph Γ_τ. Since the arcs in $t \rightsquigarrow s$ all represent indifferences, $(\boldsymbol{\rho}, \tau)$ is welfare equivalent to $(\boldsymbol{\rho}, \sigma)$ and no violations of *fairness* are introduced with $S \setminus S^{**}$ schools. Moreover, as $U_s \cap S^{**} = \emptyset$, there is no change in enrollment at S^{**} schools, and therefore no violation of *fairness* is introduced in these either. In sum, $(\boldsymbol{\rho}, \tau)$ is *fair*. We show that $\{s\} \cup U_s$ is a source set in Γ_τ.

Let $u \xrightarrow{i} v \in \Gamma_\tau$ have $u \notin \{s\} \cup U_s$. We show that $v \notin \{s\} \cup U_s$. First, we argue that $u = \tau(i) = \sigma(i)$. Suppose by contradiction that $\tau(i) \neq \sigma(i)$, then i labels some arc on the path $t \rightsquigarrow s \subseteq \Gamma$. Stated formally, there is $u' \xrightarrow{i} u \in t \rightsquigarrow s$, with $\sigma(i) = u'$, and $\tau(i) = u$. However, $t \rightsquigarrow s$ touches only $\{s\} \cup U_s$ schools, contradicting that $u \notin \{s\} \cup U_s$. Since $u = \tau(i) = \sigma(i)$, clearly $u \xrightarrow{i} v \in \Gamma$, and since U_s is a source set in Γ, $v \notin U_s$. Moreover, if $v = s$ then $u \in U_s$, contradicting that $u \notin U_s$. Thus, $v \notin \{s\} \cup U_s$.

Recall that our original path $t \rightsquigarrow s \subseteq \Gamma$ represented only indifferences. Since t is totally exhausted at $(\boldsymbol{\rho}, \sigma)$, $\rho_t^{-1} \in \mathbb{N}$. By NCBI, it follows that ρ_s^{-1} is *not* an integer, implying that $\rho_s^{-1} > \lfloor \rho_s^{-1} \rfloor$. Thus,

$$|\tau[s]| = |\sigma[s]| + 1 \leq \lfloor \rho_s^{-1} \rfloor < \rho_s^{-1},$$

and so s is not totally exhausted at $(\boldsymbol{\rho}, \tau)$. The schools in the middle of the path have not changed the number of students they admit from σ to τ, so they remain not totally exhausted. Clearly, t is not totally exhausted at $(\boldsymbol{\rho}, \tau)$. Since t was the *only* totally exhausted school in U_s under $(\boldsymbol{\rho}, \sigma)$, we now have that $\{s\} \cup U_s$ is a source set in Γ_τ with $\tau[\{s\} \cup U_s] \neq \emptyset$ and no totally exhausted schools. Further, $(U_s \cup \{s\}) \cap S^{**} = \emptyset$ implies $(U_s \cup \{s\}) \cap S^{*} = \emptyset$. We therefore invoke Lemma 1 to arrive at conclusion (2) of the lemma.

We now complete the proof of the lemma. Since $(\boldsymbol{\rho}, \sigma)$ is *fair* but not an RCE, it fails either *inferior empty schools* or *exhaustiveness*.

Suppose that $(\boldsymbol{\rho}, \sigma)$ does not satisfy *inferior empty schools*, i.e., there is $s \in S$ with $\sigma[s] = \emptyset$ such that either (i) $\rho_s < 1$, or (ii) for some $i \in N$, $(\rho_s, s)\ R_i\ (\boldsymbol{\rho}, \sigma(i))$. Since $\sigma[s] = \emptyset$ and $\rho_s \leq 1$, line (5) holds for s. If only (i) holds but (ii) does not, i.e., for each $i \in N$, $(\boldsymbol{\rho}, \sigma(i))\ P_i\ (\rho_s, s)$, then $U_s = \emptyset$, and we invoke Claim 6 to arrive at conclusion (1) of the lemma. Now suppose that (ii) holds, so $U_s \neq \emptyset$. If there is $i \in N$ such that $(\rho_s, s)\ P_i\ (\boldsymbol{\rho}, \sigma(i))$, then since $\sigma[s] = \emptyset$ and $\rho_s \leq 1$, it holds that $|\sigma[s]| \leq \lfloor \rho_s^{-1} \rfloor - 1$. Then we invoke Claim 5 to arrive at conclusion (2) of the lemma. If there is no such i, then for each $i \in N$ with $(\rho_s, s)\ R_i\ (\boldsymbol{\rho}, \sigma(i))$, we have $(\rho_s, s)\ I_i\ (\boldsymbol{\rho}, \sigma(i))$. If $\rho_s = b_s$, then we are in Case 1.2. In the case of $\rho_s > b_s$, we are in Case 2.2 or 2.3. Thus, if *inferior empty schools* is violated, the lemma holds.

Suppose that $(\boldsymbol{\rho}, \sigma)$ does not satisfy *exhaustiveness*. Recall that *exhaustiveness* requires that for each $s \in S$ with $\sigma[s] \neq \emptyset$, $|\sigma[s]| = \lfloor \rho_s^{-1} \rfloor$; so a violation of the property leads to line (5). Then Cases 1 and 2 cover all possibilities, and all lead to either conclusion (1) or (2) of the lemma. □

A.4. **Topological Argument to Complete the Proof.**

Theorem 6. *Given $\boldsymbol{R} \in \mathcal{R}^N$ satisfying NCBI, let $\mathcal{E}$ be the set of RCEs for $\boldsymbol{R}$. Then*

(1) *$\mathcal{E}$ is not empty,*
(2) *$\mathcal{E}$ induces a closed upper-lattice in welfare space, and*
(3) *the set of distributions supporting the elements of $\mathcal{E}$ has a $\leq$-greatest element, $\boldsymbol{\rho}^*(\boldsymbol{R})$, which itself supports the welfare-greatest elements of $\mathcal{E}$.*

Proof. For each $i \in N$, let u_i be a continuous utility function representation of R_i. Fixing a matching σ, the function $\boldsymbol{\rho} \in [0,1]^S \overset{U^\sigma}{\mapsto} (u_i(\boldsymbol{\rho}, \sigma(i)))_{i \in N}$ is continuous. Closed subsets of $[0,1]^S$ are compact and so map to compact sets under this function. We will show that the set of distributions $\boldsymbol{\rho}$ such that $(\boldsymbol{\rho}, \sigma)$ is a *fair* allocation is closed: If $(\boldsymbol{\rho}, s)\ P_i\ (\boldsymbol{\rho}, \sigma(i))$ and $\boldsymbol{\rho}^n \to \boldsymbol{\rho}$, then for n sufficiently large, $(\boldsymbol{\rho}^{\boldsymbol{n}}, s)\ P_i\ (\boldsymbol{\rho}^{\boldsymbol{n}}, \sigma(i))$ by the continuity of preferences. Thus, if $(\boldsymbol{\rho}^{\boldsymbol{n}}, \sigma)$ is a sequence of *fair* allocations then $\rho_s^n = b_s$ for n sufficiently large—yielding $\rho_s = b_s$—and each $j \in \sigma[s]$ has $j \succ_s i$, so $(\boldsymbol{\rho}, \sigma)$ is *fair*. Since there are at most finitely many matchings, the set $\mathcal{A} = \{(\boldsymbol{u}, \boldsymbol{\rho}) \in \mathbb{R}^N \times \mathbb{R}^S : (\boldsymbol{\rho}, \sigma) \text{ is } \textit{fair}, u_i = u_i(\boldsymbol{\rho}, \sigma(i))\}$ is non-empty (Proposition 4) and compact, being the finite union of compact sets. In particular, $\mathcal{A}$ is the union of the graphs of the U^σ restricted to the compact domains of distributions that induce *fair* allocations for the respective matching σ.

Let $(\boldsymbol{u}, \boldsymbol{\rho}) \in \mathcal{A}$ be $\leq$-maximal.[30] By definition, there is a *fair* allocation that induces $(\boldsymbol{u}, \boldsymbol{\rho})$. If there are no RCEs that induce $(\boldsymbol{u}, \boldsymbol{\rho})$, then by Lemma 2, there is a *fair* allocation $(\boldsymbol{\gamma}, \tau)$ inducing $(\boldsymbol{v}, \boldsymbol{\gamma}) \gneq (\boldsymbol{u}, \boldsymbol{\rho})$, contradicting maximality of $(\boldsymbol{u}, \boldsymbol{\rho})$. Thus, the $\leq$-upper envelope of $\mathcal{A}$ is the image of a non-empty set $E \subseteq \mathcal{E}$. By Theorem 3 then, the $\leq$-upper envelope of $\mathcal{A}$ is an upper-lattice. Therefore, $\mathcal{A}$ has a $\leq$-greatest element. □

Proof of Theorem 1. It follows directly from statement (1) of Theorem 6. □

Proof of Proposition 3. By Lemma 2 and Theorem 6, it follows that the correspondence of welfare-greatest RCE, i.e., the maximal RCE, on the NCBI domain is non-empty, single-valued in welfare, and satisfies *student-optimal fairness*. □

Appendix B. Proof of Theorem 4: *Strategy-proofness*

To begin, we establish two lemmas regarding the weak-envy graph at a maximal RCE. The first is an immediate consequence of lemmas in Section A.3, but highlights an important structural feature useful for proving

[30] Here, $\leq$ on $\mathbb{R}^k$ is the standard extension of $\leq$ on $\mathbb{R}$.

strategy-proofness. The second shows that at a maximal RCE no student is indifferent between what they get and a school that is at capacity.

Lemma 3 (Connectedness)**.** *Let* Γ *be the weak-envy graph of* $\varphi(\boldsymbol{R})$*, a maximal RCE mechanism, for* $\boldsymbol{R} \in \mathcal{D}$*. Let* $s \in S$ *be non-empty and not totally exhausted. Then there is* $t \rightsquigarrow s \subseteq \Gamma$*, consisting entirely of indifference arcs, such that* t *is totally exhausted and the only such school touched by the path.*

Proof. Let $S^* = \{s' \in S : \sigma[s'] \neq \emptyset$ and $\boldsymbol{\rho}^*_{s'}(\boldsymbol{R}) = b_{s'}\}$. By *exhaustiveness*, each $s' \in S^*$ is totally exhausted. Let U_s be the set of upstream vertices of s in Γ, which is a source set in Γ by definition. Moreover, if $t \to s \in \Gamma$, then $t \in U_s$, and so $U_s \cup \{s\}$ is also a source set in Γ. If U_s contains no totally exhausted schools, then $(U_s \cup \{s\}) \cap S^* = \emptyset$. Moreover, $\sigma[U_s \cup \{s\}] \supseteq \sigma[s]$ is non-empty, and so by Lemma 1, there is a *fair* allocation Pareto-dominating $\varphi(\boldsymbol{R})$, contradicting Proposition 3. It follows that U_s contains a totally exhausted school t and thus there is a path $t \rightsquigarrow s \subseteq \Gamma$. If there is a totally exhausted $u \neq t$ on the path, then we can take the sub-path $u \rightsquigarrow s \subsetneq t \rightsquigarrow s$ such that u is the only totally exhausted school on the sub-path. Thus, we may assume each $u \neq t$ on the path $t \rightsquigarrow s$ is not totally exhausted. We show that for each $u \neq t$, it must be that $\rho_u > b_u$. Suppose by contradiction that there is $u \neq t$ with $\rho_u = b_u$. If $u = s$, then by assumption $\sigma[s] \neq \emptyset$; if $u \neq s$, then u on the path $u \rightsquigarrow s$ means $\sigma[u] \neq \emptyset$. Together with $\rho_u = b_u$, we have $u \in S^*$. Thus, u is totally exhausted—contradicting the assumption above. To complete the proof of the lemma, observe that since $\varphi(\boldsymbol{R})$ is *fair*, each arc in $t \rightsquigarrow s$ represents indifference. ☐

Lemma 4. *Let* $\boldsymbol{R} \in \mathcal{D}$*,* $(\boldsymbol{\rho}, \sigma) = \varphi(\boldsymbol{R})$*, and* $S^* = \{s \in S : \sigma[s] \neq \emptyset$ *and* $\rho_s = b_s\}$.

1. *For each* $i \in N$ *and each* $s \in S^* \setminus \{\sigma(i)\}$ *such that* $(\rho_s, s)\ R_i\ (\boldsymbol{\rho}, \sigma(i))$*, we have* $(\rho_s, s)\ P_i\ (\boldsymbol{\rho}, \sigma(i))$.
2. *For each* $i \in N$ *and* $s \in S \setminus \{\sigma(i)\}$ *such that* $(\rho_s, s)\ I_i\ (\rho, \sigma(i))$*, we have* $s \in S \setminus S^*$.

Proof. (1). Let $i \in N$, $s \in S^*$, and $(\rho_s, s)\ R_i\ (\boldsymbol{\rho}, \sigma(i))$. If $\sigma(i)$ is totally exhausted, then $(\rho_s, s)\ I_i\ (\boldsymbol{\rho}, \sigma(i))$ is a violation of NCBI, and so the conclusion holds. Let $\sigma(i)$ be not totally exhausted. By Lemma 3, there is a path $t \rightsquigarrow \sigma(i) \subseteq \Gamma$ consisting entirely of indifferences with t being totally exhausted. If $(\rho_s, s)\ I_i\ (\boldsymbol{\rho}, \sigma(i))$, then $t \rightsquigarrow \sigma(i) \xrightarrow{i} s \subseteq \Gamma$ and t and s are connected by indifference in violation of NCBI. Thus, $(\rho_s, s)\ P_i\ (\boldsymbol{\rho}, \sigma(i))$.

(2). Let $i \in N$ and $(\rho_s, s)\ I_i\ (\boldsymbol{\rho}, \sigma(i))$, with $s \neq \sigma(i)$. Then $(\rho_s, s)\ R_i\ (\boldsymbol{\rho}, \sigma(i))$, and so $s \in S^*$ would contradict (1). ☐

Given preference relation $R_i \in \mathcal{R}$ and bundle (r, s), let $UC(R_i, (r, s)) = \{(r', s') \in [0, 1] \times S : (r', s')\ R_i\ (r, s)\}$ denote the upper contour set of R_i

at (r, s). Say R_i' is a Maskin monotonic transformation of R_i at (r, s), which we denote $R_i' \in \mathcal{T}(R_i, (r, s))$, if $UC(R_i', (r, s)) \subseteq UC(R_i, (r, s))$. Say $R_i' \in \mathcal{T}^*(R_i, (r, s))$ if $UC(R_i', (r, s)) = UC(R_i, (r, s))$. Note that, by the monotonicity of preferences, each $R_i' \in \mathcal{T}^*(R_i, (r, s))$ has the same indifference set as R_i through point (r, s). Given a profile $\boldsymbol{R} \in \mathcal{R}^N$ and an allocation $(\boldsymbol{\rho}, \sigma)$, $\mathcal{T}(\boldsymbol{R}, (\boldsymbol{\rho}, \sigma))$ is the set of profiles $\boldsymbol{R}'$ such that $R_i' \in \mathcal{T}(R_i, (\boldsymbol{\rho}, \sigma(i)))$ for each $i \in N$. The set $\mathcal{T}^*(\boldsymbol{R}, (\boldsymbol{\rho}, \sigma))$ is defined analagously. It is straightforward to verify that if $(\boldsymbol{\rho}, \sigma)$ is an RCE for $\boldsymbol{R}$ and $\boldsymbol{R}' \in \mathcal{T}(\boldsymbol{R}, (\rho, \sigma))$, then $(\boldsymbol{\rho}, \sigma)$ is an RCE for $\boldsymbol{R}'$. We now prove a stronger property on the smaller set of transformations $\mathcal{T}^*$.

Theorem 7 (Locality)**.** *Assuming $\boldsymbol{R} \in \mathcal{D}$, for each $\boldsymbol{R}' \in \mathcal{T}^*(\boldsymbol{R}, \varphi(\boldsymbol{R})) \cap \mathcal{D}$, $\boldsymbol{\rho}^*(\boldsymbol{R}') = \boldsymbol{\rho}^*(\boldsymbol{R})$.*

Proof. Let $(\boldsymbol{\rho}, \sigma) = \varphi(\boldsymbol{R})$ and $(\boldsymbol{\gamma}, \tau) = \varphi(\boldsymbol{R}')$. Since $\boldsymbol{R}'$ is a Maskin monotonic transform of $\boldsymbol{R}$ at $(\boldsymbol{\rho}, \sigma)$, $(\boldsymbol{\rho}, \sigma)$ is an RCE for $\boldsymbol{R}'$. By Proposition 3 applied to $\boldsymbol{R}'$, $\boldsymbol{\gamma} \geq \boldsymbol{\rho}$. To show $\boldsymbol{\gamma} = \boldsymbol{\rho}$, suppose by contradiction that $\boldsymbol{\gamma} \gneq \boldsymbol{\rho}$. Recall the definitions of S^+, $S^=$, S^*, N^+, and $N^=$ from Section A.2. Since $\boldsymbol{\gamma} \gneq \boldsymbol{\rho}$, $S^+ \cup S^* \cup S^= = S$, $N^+ \cup N^= = N$, and, in particular, $S^+ \neq \emptyset$. Moreover, since $\boldsymbol{R}' \in \mathcal{T}^*(\boldsymbol{R}, (\boldsymbol{\rho}, \sigma))$, the set N^+ does not depend on whether it is defined with respect to $\boldsymbol{R}'$ or $\boldsymbol{R}$. Formally,

$$N^+ = \{i \in N : (\boldsymbol{\gamma}, \tau(i)) \; P_i \; (\boldsymbol{\rho}, \sigma(i))\} = \{i \in N : (\boldsymbol{\gamma}, \tau(i)) \; P_i' \; (\boldsymbol{\rho}, \sigma(i))\}.$$

A similar statement holds for $N^=$.

Let Γ be the weak-envy graph of $(\boldsymbol{\rho}, \sigma)$ at $\boldsymbol{R}$. Let

$$\Gamma^* = \Gamma \setminus \{t \xrightarrow{i} u \in \Gamma : \; \rho_u = b_u, \exists t' \xrightarrow{j} u \in \Gamma, j \succ_u i\}.$$

Thus Γ^* is composed of all existing arcs into schools not at capacity and only top-ranked arcs into schools at capacity. If Γ^* has any cycles, then by construction, each of these can be executed without violating fairness (though not necessarily all of them). So if any cycle contains an arc that represents strict preference, then executing the cycle generates a *fair* allocation that Pareto-dominates $(\boldsymbol{\rho}, \sigma)$ for $\boldsymbol{R}$—contradicting the student-optimality of RCE (Proposition 3). We shall show that when $S^+ \neq \emptyset$, then such a cycle exists.

Claim 7. *Assume $i \in N^=$ has $\sigma(i) = s \neq \tau(i)$. Then there is j with $\sigma(j) = t \neq s = \tau(j)$. Moreover, if $s \notin S^*$, then $s \in S^=$, $j \in N^=$ and $t \xrightarrow{j} s \in \Gamma^*$.*

Proof of claim. To arrive at a contradiction, assume $s \in S^+$. Then

$$(\boldsymbol{\gamma}, s) \; P_i \; (\boldsymbol{\rho}, s) = (\boldsymbol{\rho}, \sigma(i)) \; I_i \; (\boldsymbol{\gamma}, \tau(i)),$$

where the indifference is because $i \in N^=$. Then since $R'_i \in \mathcal{T}^*(R_i, (\boldsymbol{\rho}, \sigma))$, this yields $(\boldsymbol{\gamma}, s)\ P'_i\ (\boldsymbol{\gamma}, \tau(i))$, which violates the *fairness* of $(\boldsymbol{\gamma}, \tau)$—being an RCE—since $\gamma_s > \rho_s \geq b_s$. Thus, $s \notin S^+$.

Recalling that $\boldsymbol{\gamma} \geq \boldsymbol{\rho}$ we have that $\gamma_s = \rho_s$. Since $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$ are both RCEs for $\boldsymbol{R}'$, we can invoke Theorem 2, as $\sigma(i) = s \neq \tau(i)$, to conclude that there is j with $\sigma(j) = t \neq s = \tau(j)$. Moreover, as $(\boldsymbol{\gamma}, \tau)$ is student-optimal for $\boldsymbol{R}'$,

$$(\rho_s, s) = (\gamma_s, s) = (\boldsymbol{\gamma}, \tau(j))\ R'_j\ (\boldsymbol{\rho}, \sigma(j))\,.$$

As $R'_j \in \mathcal{T}^*(R_j, (\boldsymbol{\rho}, \sigma))$, this implies $(\boldsymbol{\rho}, s)\ R_j\ (\boldsymbol{\rho}, \sigma(j))$, and so $t \xrightarrow{j} s \in \Gamma$.

If $s \notin S^*$, then since $s \notin S^+$, this leaves $s \in S^=$. Then $t \xrightarrow{j} s \in \Gamma^*$. Moreover, recall that from Claim 3 in the proof of Proposition 5 we have $N^+ \subseteq \tau[S^* \cup S^+]$. As $\tau(j) = s \in S^=$, $j \in N \setminus N^+ = N^=$. □

Claim 8. *For each $s \in S^+$, there are $t \in S^*$ and an indifference path $t \rightsquigarrow s \subseteq \Gamma^*$ such that $\tau[t] \neq \sigma[t]$.*

Proof of claim. By Theorem 2, $|\tau[s]| = |\sigma[s]|$, and combined with $\gamma_s > \rho_s$ we have that s is not totally exhausted at $(\boldsymbol{\rho}, \sigma)$. By Lemma 3, there are a totally exhausted school t and an indifference-only path $t \rightsquigarrow s \subseteq \Gamma$, where t is the *only* totally exhausted school on the path. It follows that $t \rightsquigarrow s \subseteq \Gamma^*$. Let $w \rightsquigarrow s \subseteq t \rightsquigarrow s$ be the maximal (in inclusion) subpath that touches only S^+ schools. It may be that s is the only school on the path in S^+, and so $w \rightsquigarrow s$ has no arcs. Since S^+ schools cannot be totally exhausted at $(\boldsymbol{\rho}, \sigma)$, and so $t \notin S^+$, $w \rightsquigarrow s \subsetneq t \rightsquigarrow s$. Thus, we have $\{u \xrightarrow{i} w \rightsquigarrow s\} \subseteq t \rightsquigarrow s$, with $u \notin S^+$.

Since $w \in S^+$ and $(\boldsymbol{\gamma}, \tau)$ is *fair* for $\boldsymbol{R}'$ we have $\gamma_w > \rho_w \geq b_w$ and $(\boldsymbol{\gamma}, \tau(i))\ R'_i\ (\boldsymbol{\gamma}, w)\ P'_i\ (\boldsymbol{\rho}, w)$. Since $u \xrightarrow{i} w \in t \rightsquigarrow s$ is an indifference arc and $R'_i \in \mathcal{T}^*(R_i, (\boldsymbol{\rho}, \sigma))$, we have $(\boldsymbol{\rho}, w)\ I'_i\ (\boldsymbol{\rho}, u)$. Since $u \notin S^+$, $\rho_u = \gamma_u$, which by substitution into the previous yields $(\boldsymbol{\rho}, w)\ I'_i\ (\boldsymbol{\gamma}, u)$. It follows that $(\boldsymbol{\gamma}, \tau(i))\ P'_i\ (\boldsymbol{\gamma}, u) = (\boldsymbol{\gamma}, \sigma(i))$, yielding $\tau(i) \neq u = \sigma(i)$. If $u \in S^*$, then $u \rightsquigarrow s$ is the path required in the Claim, so assume henceforth that $u \notin S^*$.

Since $u \notin S^* \cup S^+$, $u \in S^=$. Then since $\tau(i) \neq u = \sigma(i)$, by Theorem 2, there is $j \in \tau[u] \setminus \sigma[u]$. Again invoking Claim 3 in the proof of Proposition 5 gives $N^+ \subseteq \tau[S^* \cup S^+]$. Then since $\tau(j) = u \in S^=$, $j \in N^=$. In sum, we have $j \in N^=$ such that $\sigma(j) \neq \tau(j) = u$, which yields

$$(\rho_u, u) = (\gamma_u, u) = (\boldsymbol{\gamma}, \tau(j))\ I_j\ (\boldsymbol{\rho}, \sigma(j))$$

and so $v \xrightarrow{j} u \in \Gamma^*$, with $v = \sigma(j)$, is an indifference arc. If $v \in S^*$, then $v \xrightarrow{j} u \xrightarrow{i} w \rightsquigarrow s \subseteq \Gamma^*$ is the path required by the claim. Otherwise, if $v \notin$

S^*, then apply Claim 7 to agent j to get $k \in N^=$ with $\sigma(k) = v' \neq v = \tau(k)$ and $v' \xrightarrow{k} v \xrightarrow{j} u \xrightarrow{i} w \rightsquigarrow s \subseteq \Gamma^*$. If $v' \notin S^*$ then Claim 7 is triggered again, and we repeat until we arrive at path of indifferences

$$t' \xrightarrow{k'} v' \rightsquigarrow u \xrightarrow{i} w \rightsquigarrow s \subseteq \Gamma^*, \tag{6}$$

where $t' \in S^*$, $v' \rightsquigarrow u$ touches only $S^=$ schools (by Claim 7), $w \rightsquigarrow s$ touches only S^+ schools, and finally $\sigma(k') = t' \neq v' = \tau(k')$. □

Claim 9. *For each $s \in S^*$ with $\tau[s] \neq \sigma[s]$, there is $t \xrightarrow{i} s \in \Gamma^*$. Moreover, if $t \notin S^+$, then $\tau[t] \neq \sigma[t]$ and one of the following are true:*

(1) $t \in S^*$
(2) $t \in S^=$ *and there is* $k \in N^=$ *with* $\sigma(k) \neq t = \tau(k)$.

Proof of claim. Fix $s \in S^*$ with $\tau[s] \neq \sigma[s]$. Note that by Theorem 2, this implies $\sigma[s] \neq \emptyset$. Moreover, there is $j \in \tau[s] \setminus \sigma[s]$. Since $N = N^+ \cup N^=$ we have $(\boldsymbol{\gamma}, \tau(j)) \ R_j \ (\boldsymbol{\rho}, \sigma(j))$. Thus, as $\tau(j) = s \in S^*$ and so $\rho_s = \gamma_s$ we have an arc $\sigma(j) \xrightarrow{j} s \in \Gamma$. It follows that there is $t \xrightarrow{i} s \in \Gamma^*$. By statement (1) of Lemma 4, $(\boldsymbol{\rho}, s) \ P_i \ (\boldsymbol{\rho}, t) = (\boldsymbol{\rho}, \sigma(i))$. Since $s \in S^*$, $(\boldsymbol{\gamma}, s) = (\boldsymbol{\rho}, s) \ P_i \ (\boldsymbol{\rho}, \sigma(i))$.

We first show that $i \in N^+$. Since $t \xrightarrow{i} s \in \Gamma^*$ and $\sigma(j) \xrightarrow{j} s \in \Gamma$, either $i \succ_s j$ or $i = j$. Thus since $\tau(j) = s$, by the *fairness* of $(\boldsymbol{\gamma}, \tau)$ for $\boldsymbol{R}'$, $(\boldsymbol{\gamma}, \tau(i)) \ R_i' \ (\boldsymbol{\gamma}, s)$. We concluded the previous paragraph with $(\boldsymbol{\gamma}, s) \ P_i$ $(\boldsymbol{\rho}, \sigma(i))$, and so since $R_i' \in \mathcal{T}^*(R_i, (\boldsymbol{\rho}, \sigma))$, we have $(\boldsymbol{\gamma}, s) \ P_i' \ (\boldsymbol{\rho}, \sigma(i))$. In sum, $(\boldsymbol{\gamma}, \tau(i)) \ P_i' \ (\boldsymbol{\rho}, \sigma(i))$, as desired.

Observe that if $\tau(i) = \sigma(i)$, then $i \in N^+$ and preference monotonicity imply $\sigma(i) = t \in S^+$. Thus, $t \notin S^+$ implies $\tau(i) \neq \sigma(i) = t$, and so of course $\tau[t] \neq \sigma[t]$. Recalling that $S = S^* \cup S^+ \cup S^=$, the only thing left to prove is statement (2), so assume $t \in S^=$. By Theorem 2, there is k with $\sigma(k) \neq t = \tau(k)$. Since $t \in S^=$ and $k \in N = N^+ \cup N^=$,

$$(\rho_t, t) = (\boldsymbol{\gamma}, \tau(k)) \ R_k \ (\boldsymbol{\rho}, \sigma(k)) \, .$$

As $(\boldsymbol{\rho}, \sigma)$ is *fair* for $\boldsymbol{R}$, and $t \in S^=$ implies $\rho_t > b_t$, we have $(\boldsymbol{\rho}, \sigma(k)) \ R_k$ $(\boldsymbol{\rho}, t)$, yielding $(\boldsymbol{\rho}, \sigma(k)) \ I_k \ (\boldsymbol{\rho}, t) = (\gamma_t, \tau(k))$ as desired. □

We now combine the above claims into a single induction step and conclude the proof. Assume $t \in S^*$ has $\tau[t] \neq \sigma[t]$. By Claim 9, we find $s \to t \subseteq \Gamma^*$. If $s \in S^+$, we invoke Claim 8 to construct path

$$t' \rightsquigarrow s \to t \subseteq \Gamma^* \text{ with } t' \in S^* \text{ and } \tau[t'] \neq \sigma[t']. \tag{7}$$

If $s \notin S^+$, then Claim 9 also yields $\tau[s] \neq \sigma[s]$ and its two further statements. If $s \in S^*$ and so statement (1) prevails, then we have the path in line (7) in a trivial way, with $t' = s$. If $s \in S^=$ and so statement (2) prevails, then there is $k \in N^=$ with $\sigma(k) \neq s = \tau(k)$. This allows us to invoke Claim 7, repeatedly if necessary, to construct the path in line (7), just as we did for line (6) in the proof of Claim 8. In sum, if there is $t \in S^*$ with $\tau[t] \neq \sigma[t]$, then there are $t' \in S^*$ with $\tau[t'] \neq \sigma[t']$ and a non-empty path $t' \rightsquigarrow t \in \Gamma^*$. By the finiteness of S, we can proceed inductively until $t' = t$. It thus only remains to show the existence of such t. However, since we assumed there is $s \in S^+$, then Claim 8 yields t as desired.

We have found that if $S^+ \neq \emptyset$, then there is a path in Γ^* that contains a strict preference arc and a repeated vertex. Thus, Γ^* contains a cycle we can execute to find a *fair* improvement over $(\boldsymbol{\rho}, \sigma)$ for $\boldsymbol{R}$, a contradiction. Thus, $S^+ = \emptyset$ and since $\boldsymbol{\gamma} \geq \boldsymbol{\rho}$, $\boldsymbol{\gamma} = \boldsymbol{\rho}$. □

We are now prepared to prove the incentive compatibility of φ.

Proof of Theorem 4. Let $\boldsymbol{R}' = (R'_i, R_{-i}) \in \mathcal{D}$. Suppose by contradiction that $\varphi_i(\boldsymbol{R}')\ P_i\ \varphi_i(\boldsymbol{R})$. Let $R''_i \in \mathcal{T}^*(R_i, \varphi_i(\boldsymbol{R}))$ be such that (1) $\boldsymbol{R}'' = (R''_i, \boldsymbol{R_{-i}}) \in \mathcal{D}$, and (2) for each $s \in S$, if $(\rho^*(\boldsymbol{R}'), s) \neq \varphi_i(\boldsymbol{R}')$ then $\varphi_i(\boldsymbol{R}')\ P''_i\ (1, s)$. Thus, $\varphi_i(\boldsymbol{R}')\ P_i\ \varphi_i(\boldsymbol{R})$ implies

$$\varphi_i(\boldsymbol{R}')\ P''_i\ \varphi_i(\boldsymbol{R}). \tag{8}$$

Since $R''_i \in \mathcal{T}(R_i, \varphi_i(\boldsymbol{R}))$ and all other students have the same preferences, it can be verified that $\varphi(\boldsymbol{R}')$ is an RCE for $\boldsymbol{R}''$. By definition of a maximal RCE at $\boldsymbol{R}''$ (Proposition 3), we have $\varphi_i(\boldsymbol{R}'')\ R''_i\ \varphi_i(\boldsymbol{R}')$. Together with line (8), it holds that

$$\varphi_i(\boldsymbol{R}'')\ R''_i\ \varphi_i(\boldsymbol{R}')\ P''_i\ \varphi_i(\boldsymbol{R}).$$

By $R''_i \in \mathcal{T}^*(R_i, \varphi_i(\boldsymbol{R}))$, we have $\varphi_i(\boldsymbol{R}'')\ P_i\ \varphi_i(\boldsymbol{R})$. Thus, if i can manipulate φ at $\boldsymbol{R}$ via R'_i, then i can manipulate via R''_i. Without loss of generality, we assume henceforth that $R'_i = R''_i$ and so $\boldsymbol{R}' = \boldsymbol{R}''$. By Theorem 7 (Locality), $\rho^*(\boldsymbol{R}') = \rho^*(\boldsymbol{R}) =: \boldsymbol{\rho}$, so let $(\boldsymbol{\rho}, \sigma) = \varphi(\boldsymbol{R})$ and $(\boldsymbol{\rho}, \tau) = \varphi(\boldsymbol{R}')$.

In the following, we shall therefore construct a classical school choice problem from these and derive a contradiction to the *strategy-proofness* of the student-optimal stable rule (Roth and Sotomayor [1990]) in this context. Let $S^* = \{s \in S : \sigma[s] \neq \emptyset,\ \rho_s = b_s\}$. We collapse all the schools in $S \setminus S^*$ into one classical school, $\grave{\mathsf{s}}$. Let the set of classical schools be $\mathsf{S} = S^* \cup \{\grave{\mathsf{s}}\}$ with generic element s. The capacity for each $s \in S^*$ is $c_s = \frac{1}{b_s}$, and the capacity for $\grave{\mathsf{s}}$ is $c_{\grave{\mathsf{s}}} = \sum_{s \in S \setminus S^*} |\sigma[s]|$. Since $\varphi(\boldsymbol{R})$ and $\varphi(\boldsymbol{R}')$ are both RCEs for $\boldsymbol{R}'$, by Theorem 2, each school is matched with the same number of students at both allocations. This implies that $S^* = \{s \in S : \tau[s] \neq \emptyset,\ \rho_s = b_s\}$ and $c_{\grave{\mathsf{s}}} = \sum_{s \in S \setminus S^*} |\tau[s]|$. Matchings σ and τ in our model map to matchings $\bar{\sigma}$ and $\bar{\tau}$ in the classical model in the obvious way: $\bar{\sigma}(i) = \sigma(i)$ if $\sigma(i) \in S^*$, $\bar{\sigma}(i) = \grave{\mathsf{s}}$ if otherwise, and similarly for $\bar{\tau}$ and τ. School priorities in the classical model will be denoted $\triangleright$. For $s \in S^*$, which maps to $\mathsf{s} \in \mathsf{S}$, set $\rhd_{\mathsf{s}} = \succ_s$. For $\grave{\mathsf{s}}$, set $k \rhd_{\grave{\mathsf{s}}} j$ if $k \in \tau[S \setminus S^*]$ and $j \in \tau[S^*]$. We shall not need to further specify $\rhd_{\grave{\mathsf{s}}}$.

Next we construct student preferences over S. We begin by deciding that $\grave{\mathsf{s}}$ shall inherit the rank of the highest-ranked school in $S \setminus S^*$. That is, let $j \in N$ and choose $\grave{s}_j \in S \setminus S^*$ such that for each $t \in S \setminus S^*$, $(\boldsymbol{\rho}, \grave{s}_j)\ R_j\ (\boldsymbol{\rho}, t)$. Then we can define R_j over S so that $\mathsf{s}\ \mathsf{R}_j\ \mathsf{t}$ if and only if $(\boldsymbol{\rho}, s)\ R_j\ (\boldsymbol{\rho}, t)$, where $\grave{\mathsf{s}} \in \mathsf{S}$ maps to $\grave{s}_j$ for the purposes of constructing j's preferences (each j may have a different $\grave{s}_j$). As usual, denote by P_j the strict part of R_j. By NCBI, R_j is strict on $\mathsf{S} \setminus \{\grave{\mathsf{s}}\}$, and so equals P_j. It follows that R_j has at most one non-singleton indifference class, of the form $\{\grave{\mathsf{s}}, \mathsf{t}\}$. Assuming $\grave{\mathsf{s}} \neq \mathsf{t}$, we break this tie using the following rules:

Rule 1: If $\sigma(j) \in S \setminus S^*$, then $\grave{\mathsf{s}}\ \mathsf{P}_j\ \mathsf{t}$.
Rule 2: Otherwise, $\mathsf{t}\ \mathsf{P}_j\ \grave{\mathsf{s}}$.

Note that in the case of Rule 1, since $\varphi(\boldsymbol{R})$ is an RCE, by *fairness*, $\sigma(j)$ maximizes j's welfare across schools in $S \setminus S^*$. By construction, $\grave{s}_j$ also maximizes j's welfare across schools in $S \setminus S^*$. Therefore, we have $(\boldsymbol{\rho}, \sigma(j))\ I_j\ (\boldsymbol{\rho}, \grave{s}_j)$.

We therefore have a linear order P_j on S where, for each $\mathsf{s}, \mathsf{t} \in \mathsf{S}$, $\mathsf{s}\ \mathsf{P}_j\ \mathsf{t}$ implies $(\boldsymbol{\rho}, s)\ R_j\ (\boldsymbol{\rho}, t)$. The converse need not be true; however, we have a partial converse when $s = \sigma(j)$. To see why, let $(\boldsymbol{\rho}, \sigma(j))\ R_j\ (\boldsymbol{\rho}, t)$. If the relation is strict then $\bar{\sigma}(j)\ \mathsf{P}_j\ \mathsf{t}$ by construction. In the case of indifference, by statement (2) of Lemma 4, we have $t \in S \setminus S^*$, and so $\mathsf{t} = \grave{\mathsf{s}}$. If $\sigma(j) \in S \setminus S^*$, then $\bar{\sigma}(j) = \grave{\mathsf{s}} = \mathsf{t}$. If $\sigma(j) \in S^*$, then by Rule 2, $\bar{\sigma}(j)\ \mathsf{P}_j\ \mathsf{t}$. In sum,

$$(\boldsymbol{\rho}, \sigma(j))\ R_j\ (\boldsymbol{\rho}, t) \implies \Big(\bar{\sigma}(j)\ \mathsf{P}_j\ \mathsf{t} \quad \text{or} \quad \mathsf{t} = \bar{\sigma}(j) = \grave{\mathsf{s}}\Big). \tag{9}$$

We now show that $\bar{\sigma}$ is stable (*no justified envy* and *non-wastefulness*) for the classical school choice problem $(\mathsf{S}, N, \mathbf{P}, \boldsymbol{c}, \rhd)$. Suppose $\mathsf{t}\ \mathsf{P}_j\ \bar{\sigma}(j)$, so $(\boldsymbol{\rho}, t)\ R_j\ (\boldsymbol{\rho}, \sigma(j))$. We claim that $(\boldsymbol{\rho}, t)\ P_j\ (\boldsymbol{\rho}, \sigma(j))$. Suppose by contradiction that $(\boldsymbol{\rho}, \sigma(j))\ R_j\ (\boldsymbol{\rho}, t)$. Applying line (9) and the assumption that $\mathsf{t}\ \mathsf{P}_j\ \bar{\sigma}(j)$, we get $\mathsf{t} = \bar{\sigma}(j) = \grave{\mathsf{s}}$. This further implies $\bar{\sigma}(j) = \mathsf{t}\ \mathsf{P}_j\ \bar{\sigma}(j)$, a contradiction in terms. Conclude that $(\boldsymbol{\rho}, t)\ P_j\ (\boldsymbol{\rho}, \sigma(j))$. Since $(\boldsymbol{\rho}, \sigma)$ is an RCE for $\boldsymbol{R}$, by *inferior empty schools*, $\sigma[t] \neq \emptyset$, so $\bar{\sigma}[t] \neq \emptyset$. Furthermore, (1) by *fairness*, $t \in S^*$, and so for each $k \in \sigma[t]$, $k \succ_t j$, and (2) by *fairness* and *exhaustiveness*, we have $\rho_t = b_t$, and $\rho_t^{-1} = b_t^{-1} = |\sigma[t]| = |\bar{\sigma}[t]| = c_t$. Thus, by (1) and (2), $\bar{\sigma}$ satisfies *no justified envy* and *non-wastefulness*, respectively.

We now show that $\bar{\tau}$ is stable for $(\mathsf{S}, N, (\mathsf{P}'_i, \mathbf{P}_{-i}), \boldsymbol{c}, \rhd)$. Recall that, by the assumptions on R'_i, student i prefers $\varphi_i(\boldsymbol{R}')$ to each other school s at $\rho_s = 1$. Thus, by construction, P'_i top ranks $\bar{\tau}(i)$, and so we may restrict attention to $j \neq i$. Suppose $\mathsf{t}\ \mathsf{P}_j\ \bar{\tau}(j)$, so $(\boldsymbol{\rho}, t)\ R_j\ (\boldsymbol{\rho}, \tau(j))$. Recall that $(\boldsymbol{\rho}, \tau)$ is an RCE for $\boldsymbol{R}'$. By similar arguments as above, we have that $\bar{\tau}[t] \neq \emptyset$.

Moreover, if $(\boldsymbol{\rho}, t)\ P_j\ (\boldsymbol{\rho}, \tau(j))$, then we have (1) $t \in S^*$ implying $\rhd_t = \succ_t$, (2) for each $k \in \tau[t]$, $k \succ_t j$, and (3) $\rho_t^{-1} = b_t^{-1} = |\tau[t]| = |\bar{\tau}[t]| = c_t$. Thus, we have that $\bar{\tau}$ is stable. If $(\boldsymbol{\rho}, t)\ I_j\ (\boldsymbol{\rho}, \tau(j))$, then by statement (2) of Lemma 4, $t \in S \setminus S^*$, so $\mathsf{t} = \grave{\mathsf{s}}$. Since $\grave{\mathsf{s}} = \mathsf{t}\ \mathsf{P}_j\ \bar{\tau}(j)$, it holds that $\bar{\tau}(j) \neq \grave{\mathsf{s}}$, so $\tau(j) \notin S \setminus S^*$ and $\tau(j) \in S^*$. For each $k \in \bar{\tau}[\grave{\mathsf{s}}]$, we have $k \in \tau[S \setminus S^*]$ and so by construction $k \rhd_{\grave{\mathsf{s}}} j$. Recall that $c_{\grave{\mathsf{s}}} = \sum_{t \in S \setminus S^*} |\tau[t]| = |\bar{\tau}[\grave{\mathsf{s}}]|$. Thus, we reach the desired conclusion.

Now we show that $\bar{\sigma}$ is the student-optimal stable match for $(\mathsf{S}, N, \mathbf{P}, \boldsymbol{c}, \rhd)$. Suppose by contradiction that $\bar{\mu} \neq \bar{\sigma}$ is a stable match that is at least as good as $\bar{\sigma}$ in the Pareto sense. Since $(\boldsymbol{\rho}, \sigma)$ is an RCE, for each $j \in N$, $(\boldsymbol{\rho}, \sigma(j))\ R_j\ (\boldsymbol{\rho}, \grave{s}_j)$, so recalling line (9), either $\bar{\sigma}(j) = \grave{\mathsf{s}}$ or $\bar{\sigma}(j)\ \mathsf{P}_j\ \grave{\mathsf{s}}$. Therefore, since $\bar{\mu}$ is Pareto at-least-as-good as $\bar{\sigma}$, going from $\bar{\sigma}$ to $\bar{\mu}$ cannot involve moving students into $\grave{\mathsf{s}}$ who are not already there. Then by distribution feasibility, with reference to the *exhaustiveness* of $(\boldsymbol{\rho}, \sigma)$, no students can move out of $\grave{\mathsf{s}}$. Thus, $\bar{\mu}[\grave{\mathsf{s}}] = \bar{\sigma}[\grave{\mathsf{s}}]$. In other words, $\bar{\sigma}$ to $\bar{\mu}$ involves only reassignment of the students at schools in S^*. Let μ be a matching in the school choice problem with crowding that coincides with $\bar{\mu}$ on S^* and with σ otherwise. Since $\bar{\mu} \neq \bar{\sigma}$ is at least as good as $\bar{\sigma}$, each reassigned j has $\bar{\mu}(j)\ \mathsf{P}_j\ \bar{\sigma}(j)$, and since both schools are in S^*, by NCBI, $(\boldsymbol{\rho}, \mu(j))\ P_j\ (\boldsymbol{\rho}, \sigma(j))$. Thus, $(\boldsymbol{\rho}, \mu)$ Pareto dominates $(\boldsymbol{\rho}, \sigma)$, and as such cannot be *fair* for $\boldsymbol{R}$. As the distribution is unchanged and welfare not decreased, no violations of *fairness* can be introduced with schools in $S \setminus S^*$. There are then $j, k \in N$ with $\bar{\mu}(j) = \mu(j) = s \in S^*$, $(\boldsymbol{\rho}, s)\ P_k\ (\boldsymbol{\rho}, \mu(k))$ and $k \succ_s j$. If $\mu(k) \in S^*$, then $\mu(k) = \bar{\mu}(k)$. If $\mu(k) \in S \setminus S^*$, then $\mu(k) = \sigma(k)$ and $\bar{\sigma}(k) = \grave{\mathsf{s}}$. Since $\bar{\mu}[\grave{\mathsf{s}}] = \bar{\sigma}[\grave{\mathsf{s}}]$, we have $k \in \bar{\mu}[\grave{\mathsf{s}}]$ and so $\bar{\mu}(k) = \grave{\mathsf{s}}$. In either case, as no tie-breaking is necessary, $\mathsf{s}\ \mathsf{P}_k\ \bar{\mu}(k)$, and as $s \in S^*$, $\rhd_{\mathsf{s}} = \succ_s$, contradicting that $\bar{\mu}$ satisfies *no justified envy*. We conclude that there are no reassigned students within S^*, and thus S, and $\bar{\mu} = \bar{\sigma}$ as desired.

Recall that $\bar{\tau}$ is stable for $(\mathsf{S}, N, (\mathsf{P}'_i, \mathbf{P}_{-\boldsymbol{i}}), \boldsymbol{c}, \rhd)$. Since $\bar{\tau}(i)$ is the top-ranked school for P'_i, the student-optimal stable match at this problem assigns i to $\bar{\tau}(i)$. By assumption, $(\boldsymbol{\rho}, \tau(i))\ P_i\ (\boldsymbol{\rho}, \sigma(i))$, so by construction $\bar{\tau}(i)\ \mathsf{P}_i\ \bar{\sigma}(i)$, contradicting that i is not able to manipulate the student-optimal stable rule at the original problem $(\mathsf{S}, N, \mathbf{P}, \boldsymbol{c}, \rhd)$. $\square$

Appendix C. The Algorithm

C.1. **An Instance of the Algorithm.** Let $S = \{s_1, s_2\}$ and $N = \{1, 2, 3, 4\}$. Students have linear preferences and their valuations are as follows:

$$v_1(s_1) = \sqrt{11} \text{ and } v_1(s_2) = \sqrt{3}$$
$$v_2(s_1) = \sqrt{61} \text{ and } v_2(s_2) = \sqrt{13}$$
$$v_3(s_1) = \sqrt{7} \text{ and } v_3(s_2) = \sqrt{5}$$
$$v_4(s_1) = \sqrt{2} \text{ and } v_4(s_2) = \sqrt{17}$$

The lower bounds are $b_{s_1} = b_{s_2} = \frac{1}{2}$. School s_1 has the priority order $1 \succ_{s_1} 2 \succ_{s_1} 3 \succ_{s_1} 4$. School s_2 has the priority order $4 \succ_{s_2} 3 \succ_{s_2} 1 \succ_{s_2} 2$.

Time begins at $z_0 = 1$. The distribution is initialized at $\boldsymbol{\rho}_{z_0} = (\rho_{s_1 z_0}, \rho_{s_2 z_0}) = (1, 1)$ and $\mathbf{Q}_{z_0}$ is such that for each $i \in N$, we have $Q_{iz_0} = S$. The algorithm begins in the subroutine PAUSE.

At time z_0, since for each $i \neq 4$, $D_{iz_0} = \{s_1\}$ and $D_{4z_0} = \{s_2\}$, we have $E^*_{z_0} = \{s_1\}$ (at $\boldsymbol{\rho}_{z_0}$, each school can accept at most one student). Since $\rho_{s_1 z_0} = 1 > b_{s_1} = \frac{1}{2}$, go to subroutine DECREMENTING.

Advance time continuously from $z_0 = 1$ to $z_1 = \sqrt{7/5}$ in the DECREMENTING subroutine. We have $\rho_{s_1 z_1} = \frac{\rho_{s_1 z_0} \cdot z_0}{z_1} = \sqrt{5/7}$, $\rho_{s_2 z_1} = \rho_{s_2 z_0} = 1$, and for each $i \in N$, $Q_{iz_1} = Q_{iz_0} = S$. Note that for each $z \in [1, z_1)$, and each $i \neq 4$, $D_{iz} = D_{iz_0}$. Since at time z_1, $D_{3z_1} = \{s_1, s_2\} \neq D_{3z_0} = \{s_1\}$, exit condition (2) holds. Exit at z_1 and go to subroutine PAUSE.

At time z_1, $E^*_{z_1} = \{s_1, s_2\} = S$ (at $\boldsymbol{\rho}_{z_1}$, each school can accept at most one student). Since each school $s \in S$ has $\rho_{sz_1} > b_s = \frac{1}{2}$, go to subroutine DECREMENTING.

Advance time continuously from $z_1 = \sqrt{7/5}$ to $z_2 = 2$ in the DECREMENTING subroutine. We have $\rho_{s_1 z_2} = \frac{\rho_{s_1 z_1} \cdot z_1}{z_2} = \frac{\sqrt{5/7} \cdot \sqrt{7/5}}{2} = \frac{1}{2}$, $\rho_{s_2 z_2} = \frac{1}{2} \cdot \sqrt{7/5}$, and for each $i \in N$, $Q_{iz_2} = Q_{iz_1} = S$. Note that for each $z \in [z_1, z_2)$, and each $i \in N$, $D_{iz} = D_{iz_1}$. Since at time z_2, $\rho_{s_1 z_2} = \frac{1}{2}$, exit condition (1) holds. Exit at z_2 and go to subroutine PAUSE.

At time z_2, $E^*_{z_2} = \{s_1, s_2\} = S$ (at $\boldsymbol{\rho}_{z_2}$, s_1 can accept two students while s_2 can accept at most one student). Since $\rho_{s_1 z_2} = b_{s_1} = \frac{1}{2}$, go to subroutine REJECTION.

For s_1, student 3 has the lowest priority among those such that $s_1 \in D_{iz_2}$, namely students $i = 1, 2, 3$. Thus, s_1 rejects student 3, together with those who have a priority lower than student 3, i.e., student 4. We advance one unit of time and let $z_3 = z_2 + 1 = 3$. Then, $\boldsymbol{\rho}_{z_2} = \boldsymbol{\rho}_{z_3}$ and $Q_{1z_3} = Q_{2z_3} = S$ while $Q_{3z_3} = Q_{4z_3} = \{s_2\}$. Go to subroutine PAUSE.

At time z_3, since $D_{1z_3} = D_{2z_3} = \{s_1\}$ and $D_{3z_3} = D_{4z_3} = \{s_2\}$, we have that $E^*_{z_3} = \{s_2\}$ (at $\boldsymbol{\rho}_{z_3}$, s_1 can accept two students while s_2 can only accept one). Since $\rho_{s_2z_2} = \rho_{s_2z_3} = \frac{1}{2} \cdot \sqrt{7/5} > \frac{1}{2}$. We go to subroutine DECREMENTING.

Advance time continuously from $z_3 = 3$ to $z_4 = 3 \cdot \sqrt{7/5}$ in the DECREMENTING subroutine. We have $\rho_{s_1z_4} = \rho_{s_1z_3} = \frac{1}{2}$, $\rho_{s_2z_4} = \frac{1}{2}$, and $\boldsymbol{Q}_{z_4} = \boldsymbol{Q}_{z_3}$. Note that for each $z \in [z_3, z_4)$, and each $i = 3, 4$, $D_{iz} = D_{iz_3}$. Since at time z_4, $\rho_{s_2z_4} = \frac{1}{2} = b_{s_2}$, exit condition (1) holds. Exit at z_4 and go to subroutine PAUSE.

At time $z_4 = 3 \cdot \sqrt{7/5}$, $E^*_{z_4}$ is empty (at ρ_{z_4}, each school can accept two students). Thus the algorithm terminates at $(\frac{1}{2}, \frac{1}{2})$, students 1 and 2 enter s_1, and students 3 and 4 enter s_2, which is indeed a maximal RCE.

C.2. **Proof of Theorem 5.** Consider a slightly modified but equivalent algorithm. The modified algorithm unpacks the REJECTION subroutine into some extraneous steps that make the proof simpler to show. Doing so allows us to apply the concept of *threshold equilibrium*, which is a way of generalizing price equilibrium so that more abstract objects can play the role of the price.

Augment N with a null element ϕ having the property that, for each $s \in S$ and each $i \in N$, $i \succ_s \phi$. Denote this new set N_0. For each $s \in S$, let

$$\mathcal{T}_s = \{(r, i) \in [b_s, 1] \times N_0 : i \neq \phi \implies r = b_s\}.$$

A *threshold* $(r, i) \in \mathcal{T}_s$ *at school* s is simply a resource ratio and cutoff for s. Namely, the second argument i will later be used to specify that all students $\succ_s$-higher than i are allowed to demand the school, while i and below cannot. Define linear order $\sqsupset_s$ on $\mathcal{T}_s$ so that $(r, i) \sqsupset_s (q, j)$ if either $r < q$ or ($r = q = b_s$ and $i \succ_s j$). Let $\sqsupseteq_s$ denote the reflexive enlargement of $\sqsupset_s$. With abuse of notation, write $(\boldsymbol{\rho}, \boldsymbol{a})$ to indicate the *threshold* (vector) $((\rho_s, a_s))_{s \in S}$, where each $(\rho_s, a_s) \in \mathcal{T}_s$. Let $(\boldsymbol{\rho}, \boldsymbol{a}) \sqsupseteq (\boldsymbol{\gamma}, \boldsymbol{b})$ if, for each $s \in S$, $(\rho_s, a_s) \sqsupseteq_s (\gamma_s, b_s)$.

Given a threshold, student i's *constrained demand* is

$$C_i(\boldsymbol{\rho}, \boldsymbol{a}) = \{s \in S : i \succ_s a_s \text{ and } \forall t \in S, (i \succ_t a_t \implies (\boldsymbol{\rho}, s)\ R_i\ (\boldsymbol{\rho}, t))\}.$$

A threshold equilibrium $(\boldsymbol{\rho}, \boldsymbol{a}, \sigma)$ is a threshold $(\boldsymbol{\rho}, \boldsymbol{a})$ and a matching σ such that $(\boldsymbol{\rho}, \sigma)$ is an allocation and, for each $i \in N$, $\sigma(i) \in C_i(\boldsymbol{\rho}, \boldsymbol{a})$.

If allocation $(\boldsymbol{\rho}, \sigma)$ is *fair*, then for each $s \in S$, set $a_s = \phi$ if $\rho_s > b_s$ and $a_s = \max_{\succ_s}(\{i \in N : (\boldsymbol{\rho}, s)\ P_i\ (\boldsymbol{\rho}, \sigma(i))\} \cup \{\phi\})$ if $\rho_s = b_s$. Then $(\boldsymbol{\rho}, \boldsymbol{a}, \sigma)$ is a threshold equilibrium. On the other hand, if $(\boldsymbol{\rho}, \boldsymbol{a}, \sigma)$ is a threshold equilibrium and $(\rho_s, s)\ P_i\ (\rho_{\sigma(i)}, \sigma(i))$, then $s \notin C_i(\boldsymbol{\rho}, \boldsymbol{a})$, meaning $a_s \succ_s i$ or $a_s = i$, and so $\rho_s = b_s$ by the definition of $\mathcal{T}_s$. Then for each

$j \in \sigma[s]$, $j \succ_s a_s$ and so $j \succ_s i$—indicating (ρ, σ) is *fair*. Thus, threshold equilibria and *fair* allocations are equivalent. A maximal RCE is therefore also a student-optimal threshold equilibrium allocation (statement 3 of Proposition 3). This implies that if $(\boldsymbol{\rho}, \boldsymbol{a}, \sigma)$ is a student-optimal threshold equilibrium at which empty schools have resource ratio 1, then $\boldsymbol{\rho} = \boldsymbol{\rho}^*(\boldsymbol{R})$, which is the distribution at maximal RCE, which by statement 1 of Proposition 3, is unique. The cutoff component of the threshold at a student-optimal threshold equilibrium is not necessarily unique.[31] However, setting a_s equal to the highest-$\succ_s$-ranked student j who has $(\boldsymbol{\rho}^*(\boldsymbol{R}), s)\ P_j\ (\boldsymbol{\rho}^*(\boldsymbol{R}), \sigma(j))$ will $\sqsupset_s$-minimize the cutoff and therefore the threshold. Since the welfare at maximal RCE—and therefore at student-optimal threshold equilibrium—is unique, the $\sqsupset_s$-minimal threshold is independent of the particular student-optimal threshold equilibrium in place (Proposition 3). In sum, the *minimal* equilibrium threshold $(\boldsymbol{\rho}^*(\boldsymbol{R}), \boldsymbol{a}^*(\boldsymbol{R}))$ is well-defined. As $\boldsymbol{R}$ will remain constant in what follows, we suppress the arguments of $\boldsymbol{\rho}^*$ and $\boldsymbol{a}^*$.

Given these concepts, we make a slight modification of the algorithm in the main text that executes the same operations but with some further elaborated steps. Initialize the modified algorithm at $(\boldsymbol{\rho}_1, \boldsymbol{a}_1) = ((1, \ldots, 1), (\phi, \ldots, \phi))$. Note that this will ensure that empty schools always have threshold $(1, \phi)$. The PAUSE and DECREMENTING subroutines remain the same, with *constrained demand* replacing *rationed demand* in the calculation of excess demand. REJECTION, however, is modified so that, with each iteration, a_{sz} is incremented one step. Thus, $a_{s,z+1} = \min_{\succ_s}\{i \in N_0 : i \succ_s a_{sz}\}$. Let i be the student identified in the REJECTION subroutine at time z in the original algorithm. Then, in that algorithm, excluding s from $Q_{j,z+1}$ where j is i or any student $\succ_s$-lower ranked than i, is the same as repeating the modified REJECTION subroutine until $a_{s,z+k} = i$, for some k. We may assume that the modified algorithm selects schools at REJECTION subroutines so that this is precisely what happens.

The algorithm uses regimes of continuous and discrete time, and the beginning and end of these regimes will vary from problem to problem. Fix a preference profile R, and let Z be set of these regimes. We have the following:

Claim 10. *The set Z is the non-empty union of at-most-countably many closed intervals and finitely many isolated points. It therefore has an at-most-countable set $z_1 < z_2 < \ldots$ of boundary points. Finally, if $s \notin E^*_{z_k}$ then $(\rho_{s,z_{k+1}}, a_{s,z_{k+1}}) = (\rho_{s,z_k}, a_{s,z_k})$, and if $[z_k, z_{k+1}] \subseteq Z$, then $z' \mapsto E^*_{z'}$ is constant on $[z_k, z_{k+1})$.*

[31]Consider changing a threshold by moving a_s down one spot in $\succ_s$, allowing exactly one new student to demand a school. If the student does not demand s, then the new threshold and original matching still constitute a threshold equilibrium.

Proof of claim. Let $z \in \mathbb{R}$, and calculate E^*_z with respect to some threshold $(\boldsymbol{\rho}_z, \boldsymbol{a}_z)$. Assume $E^*_z \neq \emptyset$. If E^*_z contains a school t with $\rho_t = b_t$, then the modified algorithm would invoke the REJECTION subroutine at z and advance time to $z+1$, exiting to PAUSE. Each $s \notin E^*_z$ has $(\rho_{s,z+1}, a_{s,z+1}) = (\rho_{sz}, a_{sz})$. Otherwise, the modified algorithm invokes the DECREMENTING subroutine. Let $N' = \{i \in N : C_i(\boldsymbol{\rho}_z, \boldsymbol{a}_z) \subseteq E^*_z\}$. By line (1) and the construction of the price path in the DECREMENTING subroutine, there is $\epsilon > 0$ such that for each $z' > z$ with $z' - z < \epsilon$ and each $i \in N'$, $C_i(\boldsymbol{\rho}_{z'}, \boldsymbol{a}_{z'}) = C_i(\boldsymbol{\rho}_z, \boldsymbol{a}_z)$. Thus, $E^*_{z'} = E^*_z$, and again note that for $s \notin E^*_z$, $(\rho_{sz'}, a_{sz'}) = (\rho_{sz}, a_{sz})$. Let

$$Z_z = \bigcup \{[z, \tilde{z}) : \forall z' \in [z, \tilde{z}), \forall i \in N', C_i(\boldsymbol{\rho}_{z'}, \boldsymbol{a}_{z'}) = C_i(\boldsymbol{\rho}_z, \boldsymbol{a}_z)\}.$$

We have shown this is not empty. It is clearly also the limit of an increasing (in inclusion) sequence of half-open intervals, and so there is $z'' > z$ such that $Z_z = [z, z'')$. Moreover, $E^*_{z'}$ is constant on Z_z and the modified algorithm would exit DECREMENTING and enter PAUSE at time z'' to recalculate $E^*_{z''}$, which may equal E^*_z. We have thus shown that DECREMENTING lasts a closed interval of non-zero length, and for each $s \notin E^*_z$, $(\rho_{s,z''}, a_{s,z''}) = (\rho_{s,z}, a_{s,z})$. Obviously, there are at-most-countably many invocations of DECREMENTING. As each school s can accept b_s^{-1} students, it need increment a_s at most $|N| - b_s^{-1}$ many times. So there are finitely many invocations of the REJECTION subroutine, and thus finitely many isolated points. In sum, there are at-most-countably many subroutine invocations in total, and therefore Z has a countable set $z_1 < z_2 < \dots$ of boundary points. Finally, we have shown in each case that, for each such z_k, if $s \notin E^*_{z_k}$, then $(\rho_{s,z_{k+1}}, a_{s,z_{k+1}}) = (\rho_{s,z_k}, a_{s,z_k})$. □

Claim 11. *Given $z \in [z_k, z_{k+1})$, assume $(\boldsymbol{\rho}^*, \boldsymbol{a}^*) \sqsupseteq (\boldsymbol{\rho}_{z_k}, \boldsymbol{a}_{z_k})$ and for some $s \in S$, $(\rho_{sz}, a_{sz}) = (\rho^*_s, a^*_s)$. Then $(\rho_{sz_{k+1}}, a_{sz_{k+1}}) = (\rho_{sz}, a_{sz})$.*

Proof of claim. Since E^*_z is constant on $[z_k, z_{k+1})$ by Claim 10, if $s \notin E^*_z$, then $s \notin E^*_{z_k}$ and we are done. Assume, therefore, that $s \in E^*_z$. Let $\Omega_z \subseteq E^*_z$ be a minimal (with respect to subset inclusion) subset of E^*_z such that $s \in \Omega_z$ and for each $t \in \Omega_z$, if there are $t' \in S$ and $i \in N$ with $\{t, t'\} \subseteq C_i(\boldsymbol{\rho}_z, \boldsymbol{a}_z) \subseteq E^*_z$, then $t' \in \Omega_z$. Let $N(\Omega_z) = \{i \in N :\ C_i(\boldsymbol{\rho}_z, \boldsymbol{a}_z) \cap \Omega_z \neq \emptyset\}$. By construction of Ω_z, a student i is in $N(\Omega_z)$ if and only if $C_i(\boldsymbol{\rho}_z, \boldsymbol{a}_z) \subseteq \Omega_z$. Thus, recalling condition (2) of the definition of excess demand, we have

$$|\{i \in N(\Omega_z) :\ C_i(\boldsymbol{\rho}_z, \boldsymbol{a}_z) \subseteq \Omega_z\}| > \sum_{t \in \Omega_z} \lfloor \rho^{-1}_{tz} \rfloor. \tag{10}$$

By construction, for any pair $\{t_1, t_K\} \subseteq \Omega_z$, there is a chain

$$(\rho_{t_1}, t_1)\ I_{i_1}\ (\rho_{t_2}, t_2)\ I_{i_2}\ \dots (\rho_{t_{K-1}}, t_{K-1})\ I_{i_{K-1}}\ (\rho_{t_K}, t_K),$$

such that each $t_k \in \Omega_z$ and each $i_k \in N(\Omega_z)$. Thus, by NCBI, Ω_z contains at most one $t \in S$ such that $\rho_{tz} = b_t$. If there is a school $t \in \Omega_z$ such that $\rho_{tz} = b_t$ and $s \neq t$, then we have $\rho_{sz} > b_s$. In such a case, the modified algorithm is in the REJECTION subroutine, and therefore $[z, z+1] \cap Z = \{z, z+1\}$ and $(\rho_{s,z+1}, a_{s,z+1}) = (\rho_{s,z}, a_{s,z})$, as desired.

The remaining cases are that either (1) $\rho_{sz} = b_s$ or (2) for each $t \in \Omega_z$, $\rho_{tz} > b_t$. We will show that either leads to contradiction, and thus the previous paragraph is in fact the only case, and the Claim follows. Let $\Omega_z^* = \{t \in \Omega_z : (\rho_{tz}, a_{tz}) = (\rho_t^*, a_t^*)\}$ so $s \in \Omega_z^*$. The foregoing yields that for each $t \in \Omega_z \setminus \Omega_z^*$, $\rho_{tz} > \rho_t^*$.

By line (10) and Hall's Theorem (Hall [1935]), there is no threshold equilibrium matching of $N(\Omega_z)$ to Ω_z. Let μ instead be a maximal matching from $N(\Omega_z)$ to Ω_z such that (1) each matched student gets a school in their constrained demand set, and (2) each school $s \in \Omega_z$ is matched to $\lfloor \rho_{sz}^{-1} \rfloor$ students. It follows then that, for each $i \in \mu[\Omega_z^*]$, $C_i(\boldsymbol{\rho}_z, \boldsymbol{a}_z) \cap \Omega_z^* \neq \emptyset$. Recalling condition (2) in the definition of excess demand for Ω_z and $\Omega_z^* \subseteq \Omega_z$, Hall's theorem then implies there is a student $i \in N(\Omega_z) \setminus \mu[\Omega_z^*]$ with $C_i(\boldsymbol{\rho}_z, \boldsymbol{a}_z) \cap \Omega_z^* \neq \emptyset$. Recall also that for each $t \in \Omega_z \setminus \Omega_z^*$, $\rho_{tz} > \rho_t^*$, and thresholds are incremented monotonically. Thus, by the definition of $N(\Omega_z)$, *any* $j \in N(\Omega_z)$ with $C_j(\boldsymbol{\rho}_z, \boldsymbol{a}_z) \cap \Omega_z^* \neq \emptyset$ satisfies $C_j(\boldsymbol{\rho}^*, \boldsymbol{a}^*) \subseteq \Omega_z^*$. In sum, we have that for each $j \in \{i\} \cup \mu[\Omega_z^*]$, $C_j(\boldsymbol{\rho}^*, \boldsymbol{a}^*) \subseteq \Omega_z^*$. Then the students whose constrained demand sets at $(\boldsymbol{\rho}^*, \boldsymbol{a}^*)$ are a subset of Ω_z^* must include $\{i\} \cup \mu[\Omega_z^*]$. However,

$$|\{i\} \cup \mu[\Omega_z^*]| = 1 + |\mu[\Omega_z^*]| = 1 + \sum_{s \in \Omega_z^*} \lfloor \rho_{sz}^{-1} \rfloor = 1 + \sum_{s \in \Omega_z^*} \lfloor \rho_s^{*-1} \rfloor > \sum_{s \in \Omega_z^*} \lfloor \rho_s^{*-1} \rfloor,$$

where the second equality is because μ is a maximal feasible matching satisfying (1) and (2) from above. The last inequality implies that we cannot feasibly assign students in $\{i\} \cup \mu[\Omega_z^*]$ (whose constrained demand sets lie in Ω_z^*) to Ω_z^* at $(\boldsymbol{\rho}^*, \boldsymbol{a}^*)$. This contradicts the definition of a threshhold equilibrium. □

We now conclude the proof, arguing that the modified algorithm converges to a maximal RCE, implying that the original algorithm does as well. First observe that in order for the set of boundary points of Z to be infinite, there would have to be infinitely many invocations of REJECTION, as only these separate DECREMENTING intervals. As the number of REJECTION invocations is bounded by $|S||N|$, it follows that DECREMENTING is defined on *finitely* many closed intervals of non-zero length. Moreover, if one of these were unbounded, then by construction, there is $s \in S$ such that $\rho_{sz} \to 0$. But then, for large enough z, $\rho_{sz}^{-1} > |N|$, and so s could accomodate all students, and therefore would not be in any set in excess demand. Thus, the set of boundary points of Z is of the form $\{z_1, \ldots, z_T\}$, where $z_T = \max Z$ (recalling that the endpoint of DECREMENTING is always a PAUSE instance). In sum, DECREMENTING is defined on finitely many disjoint, closed intervals, and the path $\boldsymbol{\rho}_z$ is continuous (though not differentiable). Moreover, the modified algorithm terminates at z_T, and so $(\boldsymbol{\rho}_{z_T}, \boldsymbol{a}_{z_T})$ has no excess demand and therefore admits a threshold equilibrium.

Clearly, $(\boldsymbol{\rho}^*, \boldsymbol{a}^*) \sqsupseteq (\boldsymbol{\rho}_1, \boldsymbol{a}_1)$. Since each ρ_{sz} is continous, it will not pass ρ^*_s without first equaling it. Similarly, a_{sz} is incremented one step at a time, and so will not pass a^*_s without equaling it. By Claim 11 and induction, we find that for all $z \in Z$, $(\boldsymbol{\rho}^*, \boldsymbol{a}^*) \sqsupseteq (\boldsymbol{\rho}_z, \boldsymbol{a}_z)$. Recalling that $(\boldsymbol{\rho}^*, \boldsymbol{a}^*)$ is the unique minimal threshold that induces threshold equilibrium, and since the algorithm stops only at a threshold equilibrium (where $(\boldsymbol{\rho}_{z_T}, \boldsymbol{a}_{z_T}) \sqsupseteq (\boldsymbol{\rho}^*, \boldsymbol{a}^*)$), conclude that $(\boldsymbol{\rho}^*, \boldsymbol{a}^*) = (\boldsymbol{\rho}_{z_T}, \boldsymbol{a}_{z_T})$. Let σ be any matching satisfying constrained demands at $(\boldsymbol{\rho}^*, \boldsymbol{a}^*)$. Recall that the thresholds were initialized at $(1, \phi)$, and so schools empty at σ have threshold $(1, \phi)$. As discussed after the definition of threshold equilibrium, $(\boldsymbol{\rho}^*, \boldsymbol{a}^*, \sigma)$ is a student-optimal threshold equilibrium, and so $(\boldsymbol{\rho}^*, \sigma)$ is a maximal RCE. Finally, recall that our modified algorithm is the same as the original, except with the REJECTION subroutines elaborated in sub-steps. It follows that our original algorithm also converges to a maximal RCE.

Appendix D. When Students Can Be Unmatched

We have defined and analyzed a class of models for school choice with crowding. We have shown that a subclass of these models—the class that satisfies NCBI—preserves the desirable properties of the standard model. We now introduce a further subclass that naturally exhibits a possibility for students to remain unmatched.

Consider a school choice with crowding problem with the following special properties. The set of schools can be partitioned into two subsets: the set of "real" schools $S^\diamond$, and the set of dummy schools $S^\phi = \{\phi_i : i \in N\}$. Priorities and lower bounds in this subclass are restricted: for each $i \in N$, $b_{\phi_i} = 1$, and for $j \neq i$, $i \succ_{\phi_i} j$. Preferences in this subclass are also restricted: each i has, for each $j \neq i$, $(0, \phi_i)\ P_i\ (1, \phi_j)$. We shall show that, in this class, $(1, \phi_i)$ is the outside option for student i.

Suppose that, at some RCE $(\boldsymbol{\rho}, \sigma)$, $\sigma(i) = \phi_j$ for $j \neq i$. Since

$$(1, \phi_i)\ P_i\ (0, \phi_i)\ P_i\ (1, \phi_j)\ R_i\ (\boldsymbol{\rho}, \sigma(i)),$$

by *inferior empty schools*, there must be a student k assigned to ϕ_i. Moreover, by *respect of capacity*, $\rho_{\phi_i} = b_{\phi_i} = 1$. However, since $i \succ_{\phi_i} k$, this violates i's priority. Therefore, in this subclass, no RCE ever assigns student i to ϕ_j for $j \neq i$.

Thus, for each $i \in N$, either $\sigma(i) = \phi_i$ or ϕ_i is empty. In both cases, *respect of capacity* implies $\rho_{\phi_i} = 1$, and so the assignment $(1, \phi_i)$ is an outside option for i.

For each problem in this subclass, we may map it to a problem with a single "null school" ϕ with unlimited capacity, as is standard in school choice, by simply collapsing all the dummy schools to one point, without any resource. Furthermore, since any non-manipulable rule on a large domain remains non-manipulable on a smaller domain, restricting any maximal RCE mechanism to the subclass just defined results in a *strategy-proof* rule. Of course, the existence, structural results, and algorithm carry over as well.