Online Appendix
Repeated Trading: Transparency and Market Structure

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Partial pooling equilibria in transparent markets with intra-period buyer competition

Proposition 2 in the main text constructs a fully separating equilibrium. In this section we construct a class of partial pooling equilibria. Similar to the fully separating equilibrium, conditional on high quality, these equilibria feature a positive amount of trade which is less than its efficient level. Further, high quality's trade takes place always at the same price. Let Q_H be the expected discounted frequency with which the high quality trades, and P_H be the price at which she trades. Unlike in the fully separating equilibrium, the low quality now pools with the high quality along the said path with positive probability. With the remaining probability, the low quality trades efficiently (with probability 1 each period) at price v_L.

We construct trading paths that cycle through several periods of trade with single-period pauses.[1] For this purpose, for each k define the frequency Q_k by

$$Q_k = \frac{\delta + \delta^2 + \cdots + \delta^k}{1 + \delta + \cdots + \delta^k},$$

and the price P_k by

$$v_L - c_L = Q_k(P_k - c_L).$$

We show that as long as $Q_k > (1-\delta)$ and $P_k \in [c_H, v_H]$, there exists an equilibrium where $Q_H = Q_k$ and $P_H = P_k$.

To construct such an equilibrium, define $\tau(h)$ to be the number of periods since the last pause of trade. Let $\tau(h) = \infty$ if every previous period involved trade or it is the null

[1] This construction is similar to the one-step separation equilibria constructed in Kaya and Roy (2022). That paper considers limited records of past trading, and thus it cannot appeal to belief punishments for unexpected trading. In the current paper, such punishments are possible, and this makes it possible to construct different trading cycles than those discussed here.

history, and naturally $\tau(h) = 0$ if the last period outcome was trade. We describe beliefs and strategies as functions of τ. We partition non-null histories into two groups:

- Case 1: There has been no previous streaks of trade exceeding k consecutive periods.
- Case 2: There has been at least one previous streak of trade exceeding k consecutive period.

Buyer strategies: In case 2, offer v_L. In case 1, if $\tau(h) = k$, offer v_L, otherwise offer P_k.

Seller strategies: The seller uses a type- and history-dependent reservation price. With an abuse of notation we write these reservation prices as functions of τ. They satisfy:

- Case 1: For $\tau < k$, $\theta = L, H$,

  $$(1-\delta)\left[(P_\theta(\tau) - c_\theta) + \delta(P_k - c_\theta) + \cdots + \delta^{k-\tau-1}(P_k - c_\theta)\right] + \delta^{k-\tau} Q_k(P_k - c_\theta) = Q_k(P_k - c_\theta).$$

  For this case, we note that $P_\theta < P_k$. To see this substitute $P_\theta(\tau) = P_k$ to yield

  $$(1-\delta)\left[(P_k - c_\theta) + \delta(P_k - c_\theta) + \cdots + \delta^{k-\tau-1}(P_k - c_\theta)\right] + \delta^{k-\tau} Q_k(P_k - c_\theta) = [(1-\delta^{k-\tau}) + \delta^{k-\tau} Q_k](P_k - c_\theta).$$

  on the left-hand-side, which is larger than the right-hand-side since $Q_k < 1$.

  <u>For $\tau = k$</u>:

  $$(1-\delta)(P_L(\tau) - c_L) + \delta(v_L - c_L) = Q_k(P_k - c_L)$$
  $$(1-\delta)(P_H(\tau) - c_H) = Q_k(P_k - c_H).$$

  We note that in this case by choice of Q_k, P_k, $P_L(\tau) = v_L$. Further, since $(1-\delta) < Q_k$, $P_H(\tau) > P_k$.

- Case 2: $P_\theta(\tau) = c_\theta$.
- At $t = 1$: the reservation prices are identical to the case where $\tau = k$.

At all histories, the high quality seller accepts all offers that weakly exceed his reservation price, and rejects others. At $t = 1$ the low quality seller accepts his reservation price v_L with probability β satisfying

$$\frac{\mu_0}{1-\mu_0} = \frac{\mu_k}{1-\mu_k}(1-\beta),$$

where μ_k is defined by

$$\mu_k(v_H - P_k) + (1-\mu_k)(v_L - P_k) = 0.$$

At all other histories in Case 1, the low quality seller rejects all offers weakly less than his reservation price and accepts those that are strictly higher. In Case 2, she accepts all offers that weakly exceeds her reservation price and rejects all others.

Beliefs: In Case 2, $\mu(h) = 0$, in Case 1, $\mu(h) = \mu_k$.

Optimality of buyer strategies:

- In case 2, all buyers offering v_L is a bidding equilibrium because the belief is 0.
- In case 1, when $\tau < k$, we have $P_L(\tau) < P_H(\tau) < P_k$ and the expected quality is P_k. Therefore, it is a bidding equilibrium for all buyers to offer P_k. When $\tau = k$, we have $P_L(\tau) = v_L < P_k$ and $P_H(\tau) > P_k$. Thus offering v_L is a bidding equilibrium.

Optimality of seller strategies: The reservation prices are calculated using buyer offer strategies. Thus the decisions based on these reservation prices are optimal.

Belief consistency: Follows trivially from Bayes rule, when possible.

Maximally pooling equilibria when $\mu_0 \leq \mu^*$.

In the partial pooling equilibria constructed above, the buyers are always making pure strategy offers, and the belief remain strictly above μ^* except in a potential knife-edge case where there exists k with Q_k equal to

$$\frac{v_L - c_L}{c_H - c_L} \equiv Q^*$$

Here, we construct an equilibrium in which the high quality seller trades only at price c_H and at an expected discounted frequency $Q^* \equiv \frac{v_L - c_L}{c_H - c_L}$. In addition to being of interest for comparisons, it can also serve as an alternative punishment equilibrium to support partial and full pooling equilibria discussed so far.

In this equilibrium, the low quality seller follows this path with probability β satisfying

$$\frac{\mu_0}{1-\mu_0} = \frac{\mu^*}{1-\mu^*}(1-\beta),$$

and trades efficiently otherwise. The construction is almost identical to the pure-offer partial pooling equilibria above with the following modifications.

Fix k and α such that

$$\frac{\delta + \cdots + \delta^k}{1 + \delta + \cdots + \delta^k} \geq \frac{v_L - c_L}{c_H - c_L} \geq \frac{\delta + \cdots + \delta^{k-1}}{1 + \delta + \cdots + \delta^{k-1}},$$

and

$$\frac{v_L - c_L}{c_H - c_L} = \frac{\delta + \cdots + \delta^{k-1} + \alpha\delta^k}{1 + \delta + \cdots + \delta^{k-1} + \alpha\delta^k}.$$

As above define $\tau(h)$ to be the number of periods since the last pause of trade. Let $\tau(h) = \infty$ if every previous period involved trade or it is the null history, and naturally $\tau(h) = 0$ if the last period outcome was trade.

Buyer strategies: Offer c_H if $\tau(h) < k$, offer v_L if $\tau(h) > k$, offer c_H with overall probability α if $\tau(h) = k$, and v_L otherwise.[2]

Seller strategies: As above, each type of the seller uses a reservation price strategy. Once again, we express reservation prices as functions of τ.

[2]Note that these strategies do not punish unexpected trade with a forever switch to low prices. Instead, after each pause of trade, the buyers offer c_H again for the next consecutive k or $k+1$ periods.

- $P_H(\tau) = c_H$ for any τ.
- $P_L(h)$ satisfies
  - If $\tau \geq k$
  $$(1-\delta)(P_L(\tau) - c_L) + \delta Q^*(c_H - c_L) = Q^*(c_H - c_L),$$
  therefore $P_L(h) = v_L$.
  - If $\tau < k$:
  $$(1-\delta)(P_L(\tau) - c_L) + \alpha \left\{\delta \left[1 + \delta + \cdots + \delta^{k-\tau}\right](1-\delta)(c_H - c_L) + \delta^{k-\tau+1} Q^*(c_H - c_L)\right\}$$
  $$+(1-\alpha)\left\{\delta \left[1 + \delta + \cdots + \delta^{k-\tau-1}\right](1-\delta)(c_H - c_L) + \delta^{k-\tau} Q^*(c_H - c_L)\right\}.$$
  In this case we note that $P_L(\tau) < c_H$. This is because, substituting c_H instead of $P_L(\tau)$ would yield the following left-hand-side:
  $$\left[(1 - \alpha\delta^{k-\tau+1} - (1-\alpha)\delta^{k-\tau}) + (\alpha\delta^{k-\tau+1} + (1-\alpha)\delta^{k-\tau})Q^*\right](c_H - c_L),$$
  which is larger than the right-hand-side since $Q^* < 1$.

The high quality seller accepts all offers weakly exceeding c_H. At $t = 1$, the low quality seller accepts his reservation price with probability β defined above. At $t \geq 2$, the low quality seller accepts his reservation price with probability 1 if $\tau = \infty$. Otherwise, he rejects his reservation price with probability 1.

Beliefs: If $\tau = \infty$, $\mu(h) = 0$. Otherwise, $\mu(h) = \mu^*$.

Optimality of buyer strategies: When $\tau = \infty$, the belief is 0, thus it is a bidding equilibrium for all buyers to offer v_L. When $k \leq \tau < \infty$, since the belief is μ^*, $P_L(h) = v_L$ and $P_H(h) = c_H$, all buyers offering c_H, all buyers offering v_L as well as buyers randomizing across c_H and v_L are bidding equilibria.

Optimality of seller strategies: Reservation prices are calculated using buyer offer strategies, and are therefore optimal.

Belief consistency: Follows trivially using Bayes rule from equilibrium strategies.

Accuracy of screening and gains from trade

Each of the partial pooling and the fully separating equilibria discussed so far are characterized by the price P_H at which the high quality trades and the expected discounted frequency Q_H with which she trades. In all these equilibria, the screening of the seller is completed in the first period, and thereafter, the belief is not updated on the equilibrium path. These equilibria can be ranked with respect to how accurate their screening is. In fact, take $P_H > P'_H$ and associated $Q_H < Q'_H$, a partial pooling equilibrium featuring (P_H, Q_H) is more informative in the sense of Blackwell than an equilibrium featuring (P'_H, Q'_H). The finer learning allows the high quality seller to trade at higher prices, but at lower frequency to ensure the credibility of learning. We note that in spite of this trade-off, the equilibria with more accurate learning feature higher gains from trade. To see this first note that in all these equilibria buyers' payoff is 0 and the low quality seller's payoff is $v_L - c_L$. Thus, the higher the high quality seller's payoff, the higher is the gains from trade (since the total gains from trade is equal to the sum of the payoffs of all players). The high quality seller's payoff can be expressed as

$$Q_H(P_h - c_H) = (v_L - c_L)\frac{P_H - c_H}{P_H - c_L},$$

because $Q_H = (v_L - c_L)/(P_H - c_L)$. It is easy to see that this expression increases in P_H.

Constructing self-generating sets of payoffs: A special case:

Our definition of self-generation requires that we specify payoff sets $\mathcal{U}_\mu$ for all possible μ. One may wonder if it is possible to construct some sets of enforceable payoffs for a subset of beliefs in isolation. For instance, it is trivial to see that at belief $\mu = 1$, $U = c_H - c_L$ is enforceable with respect to the set $\mathcal{U}_1 = \{c_H - c_L\}$ by choosing $\mu^A = \mu^R = 1$ and $U^A = U^R = c_H - c_L$.[3] Similarly, at belief $\mu = 0$, $U = 0$ is enforceable with respect to $\mathcal{U}_0 = \{0\}$. A more interesting case is when $\mu = \mu^*$. By Lemma 6 in the main text, starting from μ^*, on the equilibrium path, belief is never updated. Thus, it is natural to wonder if a subset of enforceable payoffs at belief μ^* can be characterized in isolation. Here, we demonstrate that this is possible for some parameter values but not others. This exercise, in addition to clarifying our method of construction, also highlights some challenges we encounter.

For reference, we replicate the equilibrium conditions from the main text, preserving their equation numbers:

- Low quality seller's reservation price relate to U^A, U^R as follows:

$$(1-\delta)(P_L - c_L) = \delta(U^R - U^A). \tag{6}$$

- The optimality of the buyer strategy requires that

$$\alpha = \begin{cases} 1 & \text{if } \tilde{\mu}(v_H - c_H) + (1-\tilde{\mu})(v_L - c_H) > (1-\tilde{\mu})(v_L - P_L)\beta \\ \in [0,1] & \text{if } \tilde{\mu}(v_H - c_H) + (1-\tilde{\mu})(v_L - c_H) = (1-\tilde{\mu})(v_L - P_L)\beta \\ 0 & \text{if } \tilde{\mu}(v_H - c_H) + (1-\tilde{\mu})(v_L - c_H) < (1-\tilde{\mu})(v_L - P_L)\beta \end{cases}. \tag{7}$$

- The L-seller's payoff can be calculated along the possibly off-equilibrium path where she rejects her reservation price when offered. Thus,

$$U = \alpha[(1-\delta)(c_H - c_L) + \delta U^A] + (1-\alpha)\delta U^R. \tag{8}$$

[3] Here, we are abusing terminology since our formal notion of enforceability is required to specify $\mathcal{U}_\mu$ for all possible μ.

- Belief updating

$$\mu^A = \frac{\mu\alpha}{\mu\alpha + (1-\mu)(\alpha + (1-\alpha)\beta)} \text{ and } \mu^R = \frac{\mu(1-\alpha)}{\mu(1-\alpha) + (1-\mu)(1-\alpha)(1-\beta)} \tag{9}$$

Claim: If $\delta(c_H - v_L) > v_L - c_L$, then any $U \in [v_L - c_L, c_H - v_L]$ is enforceable with respect to $\mathcal{U}_{\mu^*} \equiv [v_L - c_L, c_H - v_L]$ at belief μ^*.

Proof of claim: Since on the equilibrium path belief is never going to be updated, it is necessary that $\beta = 0$ so that both types of the seller trade if and only if c_H is offered. Consider U that can be enforced by choosing $P_L = v_L$ and $\mu^A = \mu^R = \mu^*$ together with some U^A, and $U^R = U^A + (v_L - c_L)\frac{1-\delta}{\delta}$. These choices satisfy (6) and (9). Further, given that $\mu = \mu^*, P_L = v_L$ and $\beta = 0$, any $\alpha \in [0, 1]$ satisfies (7). For U^A, U^R as specified to be in $\mathcal{U}_{\mu^*}$, it is necessary that $U^R \in \left[\frac{v_L - c_L}{\delta}, c_H - v_L\right]$, which is non-empty since by assumption $\delta(c_H - v_L) > v_L - c_L$. Then by (8), any U satisfying the following for some $\alpha \in [0, 1]$ and $U^R \in \left[\frac{v_L - c_L}{\delta}, c_H - v_L\right]$ can be enforced:

$$U = \alpha(1-\delta)(c_H - v_L) + \delta U^R.$$

Thus, $v_L - c_L$ is enforced by choosing $\alpha = 0$ and $U^R = (v_L - c_L)/\delta$, while $c_H - v_L$ is enforced by choosing $\alpha = 1$ and $U^R = (c_H - v_L)$. Since α and U^R can vary continuously over their respective ranges, all $\mathcal{U}_{\mu^*} = [v_L - c_L, c_H - v_L]$ is self-generating regardless of how $\mathcal{U}_\mu$ are satisfied for other μ.

If $\delta(c_H - v_L) > v_L - c_L$ does not hold, the above construction fails. Typically, it becomes impossible to characterize self-generating sets of payoffs at belief μ^* without characterizing $\mathcal{U}_\mu$ for other μ since off-path punishments and off-path rewards become necessary. In the main text, we construct $\mathcal{U}_\mu$ for each μ without imposing any restrictions on the parameters.

References

Kaya, Ayca, and Santanu Roy. 2022. "Market Screening with Limited Records." *Games and Economic Behavior*, 106–132.