Online Appendix for Monotone Function Intervals: Theory and Applications

Kai Hao Yang[*] Alexander K. Zentefis[†]

May 1, 2024

OA.1 Proof of Theorem 2

To show that $\mathcal{H}_\tau \subseteq \mathcal{I}(F_R^\tau, F_L^\tau)$, consider any $H \in \mathcal{H}_\tau$. Let $\mu \in \mathcal{M}$ and any $r \in \mathcal{R}_\tau$ be a signal and a selection rule, respectively, such that $H^\tau(\cdot|\mu, r) = H$. By the definition of $H^\tau(\cdot|\mu, r)$, it must be that, for all $x \in \mathbb{R}$,

$$H^\tau(x|\mu, r) \leq \mu(\{G \in \mathcal{F}_0 | G^{-1}(\tau) \leq x\}) = \mu(\{G \in \mathcal{F}_0 | G(x) \geq \tau\}).$$

Now consider any $x \in \mathbb{R}$. Clearly, $\mu(\{G \in \mathcal{F}_0 | G(x) \geq \tau\}) \leq 1$, since μ is a probability measure. Moreover, let $M_x^+(q) := \mu(\{G \in \mathcal{F}_0 | G(x) \geq q\})$ for all $q \in [0, 1]$. From (1), it follows that the left-limit of $1 - M_x^+$ is a CDF and a mean-preserving spread of a Dirac measure at $F(x)$. Therefore, whenever $\tau \geq F(x)$, then $M_x^+(\tau)$ can be at most $F(x)/\tau$ to have a mean of $F(x)$.[1] Together, this implies that $\mu(\{G \in \mathcal{F}_0 | G(x) \geq \tau\}) \leq F_L^\tau(x)$ for all $x \in \mathbb{R}$.

At the same time, by the definition of $H^\tau(\cdot|\mu, r)$, it must be that, for all $x \in \mathbb{R}$,

$$H^\tau(x^-|\mu, r) \geq \mu(\{G \in \mathcal{F}_0 | G^{-1}(\tau^+) < x\}) = \mu(\{G \in \mathcal{F}_0 | G(x) > \tau\}).$$

Consider any $x \in \mathbb{R}$. Since μ is a probability measure, it must be that $\mu(\{G \in \mathcal{F}_0 | G(x) > \tau\}) \geq 0$. Furthermore, let $M_x^-(q) := \mu(\{G \in \mathcal{F}_0 | G(x) > q\})$ for all $q \in [0, 1]$. From (1), it follows that $1 - M_x^-$ is a CDF and a mean-preserving spread of a Dirac measure at $F(x)$. Therefore, whenever $\tau \leq F(x)$, then $M_x^-(\tau)$ must be at least $(F(x) - \tau)/(1 - \tau)$ to have a

[*]Yale School of Management, Email: kaihao.yang@yale.edu

[†]Yale School of Management, Email: alexander.zentefis@yale.edu

[1]More specifically, to maximize the probability at τ, a mean-preserving spread of $F(x)$ must assign probability $F(x)/\tau$ at τ, and probability $1 - F(x)/\tau$ at 0.

mean of $F(x)$.[2] Together, this implies that $\mu(\{G \in \mathcal{F}_0 | G(x) > \tau\}) \geq F_R^\tau$ for all $x \in \mathbb{R}$, which, in turn, implies that $F_R^\tau(x) \leq H^\tau(x^-|\mu, r) \leq H^\tau(x|\mu, r) \leq F_L^\tau(x)$ for all $x \in \mathbb{R}$, as desired.

To prove that $\mathcal{I}(F_R^\tau, F_L^\tau) \subseteq \mathcal{H}_\tau$, we first show that for any extreme point H of $\mathcal{I}(F_R^\tau, F_L^\tau)$, there exists a signal $\mu \in \mathcal{M}$ and a selection rule $r \in \mathcal{R}_\tau$ such that $H(x) = H^\tau(x|\mu, r)$ for all $x \in \mathbb{R}$. Consider any extreme point H of $\mathcal{I}(F_R^\tau, F_L^\tau)$. By Theorem 1, there exists a countable collection of intervals $\{(\underline{x}_n, \overline{x}_n)\}_{n=1}^\infty$ such that H satisfies 1 and 2. Since $(1 - F_L^\tau(x))F_R^\tau(x) = 0$ for all $x \notin [F^{-1}(\tau), F^{-1}(\tau^+)]$, there exists at most one $n \in \mathbb{N}$ such that $0 < H(\underline{x}_n) = F_L^\tau(\underline{x}_n) = F_R^\tau(\overline{x}_n^-) = H(\overline{x}_n^-) < 1$. Therefore, for $\underline{x}$ and $\overline{x}$ defined as

$$\underline{x} := \sup\{\underline{x}_n | n \in \mathbb{N},\, H(\underline{x}_n) = F_L^\tau(\underline{x}_n)\},$$

and

$$\overline{x} := \inf\{\overline{x}_n | n \in \mathbb{N},\, H(\overline{x}_n^-) = F_R^\tau(\overline{x}_n^-)\},$$

respectively, it must be that $\overline{x} \geq \underline{x}$, and that for all $n \in \mathbb{N}$, either $\overline{x}_n \leq \underline{x}$ and $H(\underline{x}_n) = F_L^\tau(\underline{x}_n)$; or $\underline{x}_n \geq \overline{x}$ and $H(\overline{x}_n^-) = F_R^\tau(\overline{x}_n^-)$. Henceforth, let $\mathbb{N}_1$ be the collection of $n \in \mathbb{N}$ such that $\overline{x}_n \leq \overline{x}$ and $H(\underline{x}_n) = F_L^\tau(\underline{x}_n)$, and let $\mathbb{N}_2$ be the collection of $n \in \mathbb{N}$ such that $\underline{x}_n \geq \underline{x}$ and $H(\overline{x}_n^-) = F_R^\tau(\overline{x}_n^-)$. Note that $\mathbb{N}_1 \cup \mathbb{N}_2 = \mathbb{N}$ and that $|\mathbb{N}_1 \cap \mathbb{N}_2| \leq 1$, with $\underline{x}_n = \underline{x}$ and $\overline{x}_n = \overline{x}$ whenever $n \in \mathbb{N}_1 \cap \mathbb{N}_2$.

We now construct a signal $\mu \in \mathcal{M}$ and a selection rule $r \in \mathcal{R}_\tau$ such that $H^\tau(\cdot|\mu, r) = H$. To this end, let $\eta := H(\overline{x}^-) - H(\underline{x})$ and let $\hat{x} := \inf\{x \in [\underline{x}, \overline{x}] | H(x) = H(\overline{x}^-)\}$. Note that by the definition of $\underline{x}$ and $\overline{x}$, if $\eta > 0$, then $\hat{x} \in (\underline{x}, \overline{x})$ and $H(x) = H(\underline{x})$ for all $x \in [\underline{x}, \hat{x})$, while $H(x) = H(\overline{x}^-)$ for all $x \in [\hat{x}, \overline{x})$. In particular, $F_L^\tau(\hat{x}) \geq H(\hat{x}) = F_L^\tau(\underline{x}) + \eta$, and hence $F(\hat{x}) - \tau\eta \geq F(\underline{x})$. Likewise, $F(\hat{x}) + (1-\tau)\eta \leq F(\overline{x}^-)$. Let

$$\underline{y} := F^{-1}([F(\hat{x}) - \tau\eta]^+), \quad \text{and} \quad \overline{y} := F^{-1}(F(\hat{x}) + (1-\tau)\eta)).$$

It then follows that $\underline{x} \leq \underline{y} \leq \hat{x} \leq \overline{y} \leq \overline{x}$, with at least one inequality being strict if $\eta > 0$. Next, define $\widehat{F}$ as follows: $\widehat{F} \equiv 0$ if $\eta = 0$; and

$$\widehat{F}(x) := \begin{cases} 0, & \text{if } x < \underline{y} \\ \frac{F(x) - (F(\hat{x}) - \tau\eta)}{\eta}, & \text{if } x \in [\underline{y}, \overline{y}) \\ 1, & \text{if } x \geq \overline{y} \end{cases},$$

if $\eta > 0$. Clearly $\widehat{F} \in \mathcal{F}_0$ if $\eta > 0$, and $\hat{x} \in [\widehat{F}^{-1}(\tau), \widehat{F}^{-1}(\tau^+)]$. Moreover, for all $x \in \mathbb{R}$, let

$$\widetilde{F}(x) := \frac{F(x) - \eta\widehat{F}(x)}{1 - \eta}.$$

[2]More specifically, to minimize the probability at τ, a mean-preserving spread of $F_0(x)$ must assign probability $(F(x) - \tau)/(1-\tau)$ at 1, and probability $1 - (F(x) - \tau)/(1-\tau)$ at 0.

By construction, $\eta\widehat{F} + (1-\eta)\widetilde{F} = F$. From the definition of $\underline{y}$ and $\overline{y}$, it can be shown that $\widetilde{F} \in \mathcal{F}_0$ as well. Furthermore,

$$\widetilde{F}(\overline{x}^-) - \widetilde{F}(\underline{x}) = \frac{F(\overline{x}^-) - F(\underline{x}) - \eta}{1-\eta} = \frac{1}{1-\eta}\left[\frac{\tau}{1-\tau}(1 - F(\overline{x}^-)) + \frac{1-\tau}{\tau}F(\underline{x})\right].$$

Next, define $\widetilde{F}_1$ and $\widetilde{F}_2$ as follows:

$$\widetilde{F}_1(x) := \begin{cases} \frac{F(x)}{F(\underline{x})+\alpha(F(\overline{x}^-)-F(\underline{x})-\eta)}, & \text{if } x < \underline{x} \\ \frac{F(\underline{x})\alpha(F(x)-F(\underline{x})-\eta)}{F(\underline{x})+\alpha(F(\overline{x}^-)-F(\underline{x})-\eta)}, & \text{if } x \in [\underline{x}, \overline{x}) \\ 1, & \text{if } x \geq \overline{x} \end{cases};$$

and

$$\widetilde{F}_2(x) := \begin{cases} 0, & \text{if } x < \underline{x} \\ \frac{(1-\alpha)(F(x)-F(\underline{x})-\eta)}{1-F(\overline{x}^-)+(1-\alpha)(F(\overline{x}^-)-F(\underline{x})-\eta)}, & \text{if } x \in [\underline{x}, \overline{x}) \\ \frac{F(x)-F(\underline{x})+(1-\alpha)(F(\overline{x}^-)-F(\underline{x})-\eta)}{1-F(\overline{x}^-)+(1-\alpha)(\widetilde{F}(\overline{x}^-)-\widetilde{F}(\underline{x})-\eta)}, & \text{if } x \geq \overline{x} \end{cases},$$

where

$$\alpha := \frac{\frac{1-\tau}{\tau}F(\underline{x})}{\frac{\tau}{1-\tau}(1 - F(\overline{x}^-)) + \frac{1-\tau}{\tau}F(\underline{x})}.$$

By construction, $\widetilde{\alpha}\widetilde{F}_1 + (1-\widetilde{\alpha})\widetilde{F}_2 = \widetilde{F}$, where $\widetilde{\alpha} \in (0,1)$ is given by $\widetilde{\alpha} := [F(\underline{x}) + \alpha(F(\overline{x}^-) - F(\underline{x}) - \eta)]/(1-\eta)$. Moreover, $\widetilde{F}_1(\underline{x}) \geq \tau$, and $\widetilde{F}_2(\overline{x}^-) \leq \tau$.

Now define two classes of distributions, $\{\widetilde{F}_1^x\}_{x \leq \underline{x}}$ and $\{\widetilde{F}_2^x\}_{x \geq \overline{x}}$, as follows:

$$\widetilde{F}_1^x(z) := \begin{cases} 0, & \text{if } z < x \\ \widetilde{F}(\underline{x}), & \text{if } z \in [x, \underline{x}) \\ \widetilde{F}(z), & \text{if } z \geq \underline{x} \end{cases}; \text{ and } \widetilde{F}_2^x(z) := \begin{cases} \widetilde{F}(z), & \text{if } z < \overline{x} \\ \widetilde{F}(\overline{x}^-), & \text{if } z \in [\overline{x}, x) \\ 1, & \text{if } z \geq x \end{cases}.$$

Note that since $\widetilde{F}_1(\underline{x}) \geq \tau$ and $\widetilde{F}_2(\overline{x}^-) \leq \tau$, $x \in [(\widetilde{F}_1^x)^{-1}(\tau), (\widetilde{F}_1^x)^{-1}(\tau^+)]$ for all $x \leq \underline{x}$ and $x \in [(\widetilde{F}_2^x)^{-1}(\tau), (\widetilde{F}_2^x)^{-1}(\tau^+)]$ for all $x \geq \overline{x}$. Moreover, for any $n \in \mathbb{N}_1$ and for any $m \in \mathbb{N}_2$, let

$$\widetilde{F}_1^n(z) := \frac{1}{\widetilde{F}(\overline{x}_n) - \widetilde{F}(\underline{x}_n)} \int_{\underline{x}_n}^{\overline{x}_n} \widetilde{F}_1^x(z)\widetilde{F}(\mathrm{d}x),$$

and

$$\widetilde{F}_2^m(z) := \frac{1}{\widetilde{F}(\overline{x}_m) - \widetilde{F}(\underline{x}_m)} \int_{\underline{x}_m}^{\overline{x}_m} \widetilde{F}_2^x(z)\,\mathrm{d}\widetilde{F}(\mathrm{d}x),$$

for all $z \in \mathbb{R}$. By construction, $\widetilde{F}_1^n, \widetilde{F}_2^m \in \mathcal{F}_0$ and $\overline{x}_n \in [(\widetilde{F}_1^n)^{-1}(\tau), (\widetilde{F}_1^n)^{-1}(\tau^+)]$, $\underline{x}_m \in [(\widetilde{F}_2^m)^{-1}(\tau), (\widetilde{F}_2^m)^{-1}(\tau^+)]$ for all $n \in \mathbb{N}_1$ and $m \in \mathbb{N}_2$.

Next, for any $x \in \mathbb{R}$, let $\widetilde{G}^x \in \mathcal{F}_0$ be defined as

$$\widetilde{G}^x(z) := \begin{cases} \widetilde{F}_1^x(z), & \text{if } x \in (-\infty, \overline{x}] \setminus \cup_{n \in \mathbb{N}_1} [\underline{x}_n, \overline{x}_n) \\ \widetilde{F}_1^n(z), & \text{if } x \in [\underline{x}_n, \overline{x}_n),\ n \in \mathbb{N}_1 \\ \widetilde{F}_2^x(z), & \text{if } x \in [\overline{x}, \infty) \setminus \cup_{m \in \mathbb{N}_2} [\underline{x}_m, \overline{x}_m) \\ \widetilde{F}_2^m(z), & \text{if } x \in [\underline{x}_m, \overline{x}_m),\ m \in \mathbb{N}_2 \end{cases},$$

for all $z \in \mathbb{R}$. Let

$$\widetilde{H}(x) := \begin{cases} \frac{H(x)}{1-\eta}, & \text{if } x < \underline{x} \\ \frac{H(\underline{x})}{1-\eta}, & \text{if } x \in [\underline{x}, \overline{x}) \\ \frac{H(x)-\eta}{1-\eta}, & \text{if } x \geq \overline{x} \end{cases},$$

and define $\tilde{\mu}$ as

$$\tilde{\mu}(\{\widetilde{G}^x \in \mathcal{F}_0 | x \leq z\}) := \widetilde{H}(z),$$

for all $z \in \mathbb{R}$. Then, by construction, for any $z \in \mathbb{R}$,

$$\int_{\mathcal{F}} F(z)\tilde{\mu}(\mathrm{d}F) = \int_{\mathbb{R}} \widetilde{G}^x(z)\widetilde{H}(\mathrm{d}x) = \widetilde{F}(z). \tag{OA.1}$$

Moreover, let $\tilde{r} : \mathcal{F}_0 \to \Delta(\mathbb{R})$ be defined as

$$\tilde{r}(G) := \begin{cases} \delta_{\{G^{-1}(\tau^+)\}}, & \text{if } G = \widetilde{G}^x,\ x \geq \overline{x} \\ \delta_{\{G^{-1}(\tau)\}}, & \text{otherwise} \end{cases},$$

for all $G \in \mathcal{F}_0$. It then follows that $H^\tau(x|\tilde{\mu}, \tilde{r}) = \widetilde{H}(x)$ for all $x \in \mathbb{R}$. Next, let $\mu \in \Delta(\mathcal{F}_0), r \in \mathcal{R}_\tau$ together be defined as

$$\mu := (1-\eta)\tilde{\mu} + \eta\delta_{\{\widehat{F}\}},$$

and

$$r(G) := \begin{cases} \delta_{\{\hat{x}\}}, & \text{if } G = \widehat{F} \\ \tilde{r}(G), & \text{otherwise} \end{cases},$$

for all $G \in \mathcal{F}_0$. Since $F = \eta\widehat{F} + (1-\eta)\widetilde{F}$, together with (OA.1), we have $\mu \in \mathcal{M}$. Moreover, since $H^\tau(\cdot|\tilde{\mu}, \tilde{r}) = \widetilde{H}$, we have $H^\tau(x|\mu, r) = H(x)$ for all $x \in \mathbb{R}$.

Lastly, let Γ be a collection of probability measures $\gamma \in \Delta(\mathbb{R} \times \mathcal{F}_0)$ such that $\gamma(\{(x, G) \in \mathbb{R} \times \mathcal{F}_0 | x \in [G^{-1}(\tau), G^{-1}(\tau^+)]\}) = 1$ and

$$\int_{\mathbb{R} \times \mathcal{F}_0} G(x)\gamma(\mathrm{d}x, \mathrm{d}G) = F(x),$$

for all $x \in \mathbb{R}$. Define a linear functional $\Xi : \Gamma \to \mathcal{F}_0$ as

$$\Xi(\gamma)[x] := \gamma((-\infty, x], \mathcal{F}_0),$$

for all $\gamma \in \Gamma$ and for all $x \in \mathbb{R}$. Then, since for any $\widehat{H}$ in the set of extreme points $\text{ext}(\mathcal{I}(F_R^\tau, F_L^\tau))$ of $\mathcal{I}(F_R^\tau, F_L^\tau)$, there exists $\hat{\mu} \in \mathcal{M}$ and $\hat{r} \in \mathcal{R}_\tau$ such that $H^\tau(x|\hat{\mu}, \hat{r}) = \widehat{H}(x)$ for all $x \in \mathbb{R}$, it must be that $\text{ext}(\mathcal{I}(F_R^\tau, F_L^\tau)) \subseteq \Xi(\Gamma)$.

Now consider any $H \in \mathcal{I}(F_R^\tau, F_L^\tau)$. Since $\mathcal{I}(F_R^\tau, F_L^\tau)$ is a compact and convex set of a metrizable, locally convex topological space,[3] Choquet's theorem implies that there exists a probability measure $\Lambda_H \in \Delta(\mathcal{I}(F_R^\tau, F_L^\tau))$ with $\Lambda_H(\text{ext}(\mathcal{I}(F_R^\tau, F_L^\tau))) = 1$ such that

$$\int_{\mathcal{I}(F_R^\tau, F_L^\tau)} \widehat{H}(x) \Lambda_H(\mathrm{d}\widehat{H}) = H(x),$$

for all $x \in \mathbb{R}$. Define a measure $\widetilde{\Lambda}_H$ by

$$\widetilde{\Lambda}_H(A) := \Lambda_H(\{\Xi(\gamma) | \gamma \in A\}),$$

for all measurable $A \subseteq \Gamma$. Since $\Lambda_H(\text{ext}(\mathcal{I}(F_R^\tau, F_L^\tau))) = 1$ and $\text{ext}(\mathcal{I}(F_R^\tau, F_L^\tau)) \subseteq \Xi(\Gamma)$, $\widetilde{\Lambda}_H$ is a probability measure on Γ. For any $x \in \mathbb{R}$ and for any measurable $A \subseteq \mathcal{F}_0$, let

$$\gamma((-\infty, x], A) := \int_\Gamma \tilde{\gamma}((-\infty, x], A) \widetilde{\Lambda}_H(\mathrm{d}\tilde{\gamma}),$$

and let $\mu(A) := \gamma(\mathbb{R}, A)$. By construction, for all $x \in \mathbb{R}$,

$$\int_{\mathcal{F}} G(x) \mu(\mathrm{d}G) = \int_\Gamma \left(\int_{\mathbb{R} \times \mathcal{F}_0} G(x) \tilde{\gamma}(\mathrm{d}\tilde{x}, \mathrm{d}G) \right) \widetilde{\Lambda}_H(\mathrm{d}\tilde{\gamma}) = F(x),$$

and hence $\mu \in \mathcal{M}$. Furthermore, by the disintegration theorem (c.f., Çinlar 2010, theorem 2.18), there exists a transition probability $r : \mathcal{F}_0 \to \Delta(\mathbb{R})$ such that $\gamma(\mathrm{d}x, \mathrm{d}G) = r(\mathrm{d}x|G)\mu(\mathrm{d}G)$. Since $\widetilde{\Lambda}_H(\Gamma) = 1$, and since r is a transition probability, we have $r \in \mathcal{R}_\tau$.

[3]To see this, recall that for any sequence $\{H_n\} \subseteq \mathcal{I}(F_R^\tau, F_L^\tau)$, Helly's selection theorem implies that there exists a subsequence $\{H_{n_k}\} \subseteq \{H_n\}$ that converges pointwise (and hence, in weak-*) to some $H \in \mathcal{I}(F_R^\tau, F_L^\tau)$.

Finally, for any $x \in \mathbb{R}$, since Ξ is affine,

$$\begin{aligned} H^\tau(x|\mu, r) = \gamma((-\infty, x], \mathcal{F}_0) =& \Xi(\gamma)[x] \\ =& \int_\Gamma \Xi(\tilde{\gamma})[x] \widetilde{\Lambda}_H(\mathrm{d}\tilde{\gamma}) \\ =& \int_{\mathrm{ext}(\mathcal{I}(F_R^\tau, F_L^\tau))} \widehat{H}(x) \Lambda_H(\mathrm{d}\widehat{H}) \\ =& H(x), \end{aligned}$$

as desired. This completes the proof. ■

OA.2 Proof of Theorem 3

By Theorem 2,

$$\widetilde{\mathcal{H}}_\tau \subseteq \mathcal{H}_\tau = \mathcal{I}(F_R^\tau, F_L^\tau).$$

It remains to show that

$$\bigcup_{\varepsilon > 0} \mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon}) \subseteq \widetilde{\mathcal{H}}_\tau.$$

To this end, let $\widetilde{\mathcal{M}}_\tau$ be the collection of $\mu \in \mathcal{M}$ such that $\mu(\{G \in \mathcal{F}_0 | G^{-1}(\tau) < G^{-1}(\tau^+)\}) = 0$. Consider any $\varepsilon > 0$ and any extreme point H of $\mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon})$. By Theorem 1, there exists a countable collection of intervals $\{(\underline{x}_n, \overline{x}_n)\}_{n=1}^\infty$ such that H satisfies 1 and 2. Since $(1 - F_R^{\tau,\varepsilon}(x))F_L^{\tau,\varepsilon}(x) = 0$ for all $x \neq F_0^{-1}(\tau)$, there exists at most one $n \in \mathbb{N}$ such that $0 < H(\underline{x}_n) = F_R^{\tau,\varepsilon}(\underline{x}_n) = F_L^{\tau,\varepsilon}(\overline{x}_n^-) = H(\overline{x}_n^-) < 1$. Therefore, for $\underline{x}$ and $\overline{x}$ defined as

$$\underline{x} := \sup\{\underline{x}_n | n \in \mathbb{N},\, H(\underline{x}_n) = F_R^{\tau,\varepsilon}(\underline{x}_n)\} \quad \text{and} \quad \overline{x} := \inf\{\overline{x}_n | n \in \mathbb{N},\, H(\overline{x}_n^-) = F_L^{\tau,\varepsilon}(\overline{x}_n^-)\},$$

respectively, it must be that $\overline{x} \geq \underline{x}$, and that, for all $n \in \mathbb{N}$, either $\overline{x}_n \leq \underline{x}$ and $H(\underline{x}_n) = F_L^{\tau,\varepsilon}(\underline{x}_n)$, or $\underline{x}_n \geq \overline{x}$ and $H(\overline{x}_n^-) = F_R^{\tau,\varepsilon}(\overline{x}_n^-)$. Henceforth, let $\mathbb{N}_1$ be the collection of $n \in \mathbb{N}$ such that $\overline{x}_n \leq \overline{x}$ and $H(\underline{x}_n) = F_L^{\tau,\varepsilon}(\underline{x}_n)$, and let $\mathbb{N}_2$ be the collection of $n \in \mathbb{N}$ such that $\underline{x}_n \geq \underline{x}$ and $H(\overline{x}_n^-) = F_R^{\tau,\varepsilon}(\overline{x}_n^-)$. Note that $\mathbb{N}_1 \cup \mathbb{N}_2 = \mathbb{N}$ and that $|\mathbb{N}_1 \cap \mathbb{N}_2| \leq 1$, with $\underline{x}_n = \underline{x}$ and $\overline{x}_n = \overline{x}$ whenever $n \in \mathbb{N}_1 \cap \mathbb{N}_2$.

We now construct a signal $\mu \in \widetilde{\mathcal{M}}_\tau$ such that $H^\tau(\cdot|\mu) = H$. First, let $\eta := H(\overline{x}^-) - H(\underline{x})$ and let $\hat{x} := \inf\{x \in [\underline{x}, \overline{x}] | H(x) = H(\overline{x}^-)\}$. Note that, by the definition of $\underline{x}$ and $\overline{x}$, if $\eta > 0$, then $\hat{x} \in (\underline{x}, \overline{x})$ and $H(x) = H(\underline{x})$ for all $x \in [\underline{x}, \hat{x})$, while $H(x) = H(\overline{x}^-)$ for all $x \in [\hat{x}, \overline{x})$. In particular, $F_L^{\tau,\varepsilon}(\hat{x}) \geq H(\hat{x}) = F_L^{\tau,\varepsilon}(\underline{x}) + \eta$, and hence $F(\hat{x}) - (\tau + \varepsilon)\eta \geq F(\underline{x})$. Likewise, $F(\hat{x}) + (1 - \tau + \varepsilon)\eta \leq F(\overline{x}^-)$. Now let

$$\underline{y} := F^{-1}(F(\hat{x}) - (\tau + \varepsilon)\eta), \quad \text{and} \quad \overline{y} := F^{-1}(F(\hat{x}) + (1 - \tau + \varepsilon)\eta).$$

It then follows that $\underline{x} \le \underline{y} \le \hat{x} \le \overline{y} \le \overline{x}$, with at least one inequality being strict if $\eta > 0$. Next, define $\widehat{F}$ as follows: $\widehat{F} \equiv 0$ if $\eta = 0$; and

$$\widehat{F}(x) := \begin{cases} 0, & \text{if } x < \underline{y} \\ \frac{F(x)-(F(\hat{x})-(\tau+\varepsilon)\eta)}{\eta}, & \text{if } x \in [\underline{y}, \overline{y}) \\ 1, & \text{if } x \ge \overline{y} \end{cases},$$

if $\eta > 0$. Clearly $\widehat{F} \in \mathcal{F}_0$ if $\eta > 0$, and $\hat{x} = \widehat{F}^{-1}(\tau)$. Moreover, for all $x \in \mathbb{R}$, let

$$\widetilde{F}(x) := \frac{F(x) - \eta \widehat{F}(x)}{1-\eta}.$$

By construction, $\eta\widehat{F} + (1-\eta)\widetilde{F} = F$. From the definition of $\underline{y}$ and $\overline{y}$, it can be shown that $\widetilde{F} \in \mathcal{F}_0$ as well. Furthermore,

$$\widetilde{F}(\overline{x}^-) - \widetilde{F}(\underline{x}) = \frac{F(\overline{x}^-) - F(\underline{x}) - \eta}{1-\eta} = \frac{1}{1-\eta}\left[\frac{\tau-\varepsilon}{1-(\tau-\varepsilon)}(1-F(\overline{x}^-)) + \frac{1-(\tau+\varepsilon)}{\tau+\varepsilon}F(\underline{x})\right].$$

Next, define $\widetilde{F}_1$ and $\widetilde{F}_2$ as follows:

$$\widetilde{F}_1(x) := \begin{cases} \frac{F(x)}{F(\underline{x})+\alpha(F(\overline{x}^-)-F(\underline{x})-\eta)}, & \text{if } x < \underline{x} \\ \frac{F(\underline{x})\alpha(F(x)-F(\underline{x})-\eta)}{F(\underline{x})+\alpha(F(\overline{x}^-)-F(\underline{x})-\eta)}, & \text{if } x \in [\underline{x}, \overline{x}) \\ 1, & \text{if } x \ge \overline{x} \end{cases};$$

and

$$\widetilde{F}_2(x) := \begin{cases} 0, & \text{if } x < \underline{x} \\ \frac{(1-\alpha)(F(x)-F(\underline{x})-\eta)}{1-F(\overline{x}^-)+(1-\alpha)(F(\overline{x}^-)-F(\underline{x})-\eta)}, & \text{if } x \in [\underline{x}, \overline{x}) \\ \frac{F(x)-F(\underline{x})+(1-\alpha)(F(\overline{x}^-)-F(\underline{x})-\eta)}{1-F(\overline{x}^-)+(1-\alpha)(\widetilde{F}(\overline{x}^-)-\widetilde{F}(\underline{x})-\eta)}, & \text{if } x \ge \overline{x} \end{cases},$$

where

$$\alpha := \frac{\frac{1-(\tau+\varepsilon)}{\tau+\varepsilon}F(\underline{x})}{\frac{\tau-\varepsilon}{1-(\tau-\varepsilon)}(1-F(\overline{x}^-)) + \frac{1-(\tau+\varepsilon)}{\tau+\varepsilon}F(\underline{x})}.$$

By construction, $\widetilde{\alpha}\widetilde{F}_1 + (1-\widetilde{\alpha})\widetilde{F}_2 = \widetilde{F}$, where $\widetilde{\alpha} \in (0,1)$ is given by $\widetilde{\alpha} := [F(\underline{x}) + \alpha(F(\overline{x}^-) - F(\underline{x}) - \eta)]/(1-\eta)$. Moreover, $\widetilde{F}_1(\underline{x}) = \tau + \varepsilon > \tau$, and $\widetilde{F}_2(\overline{x}^-) = \tau - \varepsilon < \tau$.

Now define two classes of distributions, $\{\widetilde{F}_1^x\}_{x \le \underline{x}}$ and $\{\widetilde{F}_2^x\}_{x \ge \overline{x}}$, as follows:

$$\widetilde{F}_1^x(z) := \begin{cases} 0, & \text{if } z < x \\ \widetilde{F}(\underline{x}), & \text{if } z \in [x, \underline{x}) \\ \widetilde{F}(z), & \text{if } z \ge \underline{x} \end{cases}; \text{ and } \widetilde{F}_2^x(z) := \begin{cases} \widetilde{F}(z), & \text{if } z < \overline{x} \\ \widetilde{F}(\overline{x}^-), & \text{if } z \in [\overline{x}, x) \\ 1, & \text{if } z \ge x \end{cases}.$$

Note that since $\widetilde{F}_1(\underline{x}) > \tau$ and $\widetilde{F}_2(\overline{x}^-) < \tau$, $x = (\widetilde{F}_1^x)^{-1}(\tau) = (\widetilde{F}_1^x)^{-1}(\tau^+)$ for all $x \leq \underline{x}$ and $x = (\widetilde{F}_2^x)^{-1}(\tau) = (\widetilde{F}_2^x)^{-1}(\tau^+)$ for all $x \geq \overline{x}$. Moreover, for any $n \in \mathbb{N}_1$ and for any $m \in \mathbb{N}_2$, let

$$\widetilde{F}_1^n(z) := \frac{1}{\widetilde{F}(\overline{x}_n) - \widetilde{F}(\underline{x}_n)} \int_{\underline{x}_n}^{\overline{x}_n} \widetilde{F}_1^x(z)\widetilde{F}(\mathrm{d}x),$$

and

$$\widetilde{F}_2^m(z) := \frac{1}{\widetilde{F}(\overline{x}_m) - \widetilde{F}(\underline{x}_m)} \int_{\underline{x}_m}^{\overline{x}_m} \widetilde{F}_2^x(z)\widetilde{F}(\mathrm{d}x),$$

for all $z \in \mathbb{R}$. By construction, $\widetilde{F}_1^n, \widetilde{F}_2^m \in \mathcal{F}_0$ and $\overline{x}_n = (\widetilde{F}_1^n)^{-1}(\tau) = (\widetilde{F}_1^n)^{-1}(\tau^+)$, $\underline{x}_m = (\widetilde{F}_2^m)^{-1}(\tau) = (\widetilde{F}_2^m)^{-1}(\tau^+)$ for all $n \in \mathbb{N}_1$ and $m \in \mathbb{N}_2$. Next, for any $x \in \mathbb{R}$, let $\widetilde{G}^x \in \mathcal{F}_0$ be defined as

$$\widetilde{G}^x(z) := \begin{cases} \widetilde{F}_1^x(z), & \text{if } x \in (-\infty, \overline{x}] \setminus \cup_{n \in \mathbb{N}_1} [\underline{x}_n, \overline{x}_n) \\ \widetilde{F}_1^n(z), & \text{if } x \in [\underline{x}_n, \overline{x}_n),\ n \in \mathbb{N}_1 \\ \widetilde{F}_2^x(z), & \text{if } x \in [\overline{x}, \infty) \setminus \cup_{m \in \mathbb{N}_2} [\underline{x}_m, \overline{x}_m) \\ \widetilde{F}_2^m(z), & \text{if } x \in [\underline{x}_m, \overline{x}_m),\ m \in \mathbb{N}_2 \end{cases},$$

for all $z \in \mathbb{R}$. Let

$$\widetilde{H}(x) := \begin{cases} \frac{H(x)}{1-\eta}, & \text{if } x < \underline{x} \\ \frac{H(\underline{x})}{1-\eta}, & \text{if } x \in [\underline{x}, \overline{x}) \\ \frac{H(x)-\eta}{1-\eta}, & \text{if } x \geq \overline{x} \end{cases},$$

and define $\tilde{\mu}$ as

$$\tilde{\mu}(\{\widetilde{G}^x \in \mathcal{F}_0 | x \leq z\}) := \widetilde{H}(z),$$

for all $z \in \mathbb{R}$. Then, by construction, for any $z \in \mathbb{R}$,

$$\int_{\mathcal{F}_0} G(z)\tilde{\mu}(\mathrm{d}G) = \int_{\mathbb{R}} \widetilde{G}^x(z)\widetilde{H}(\mathrm{d}x) = \widetilde{F}(z). \tag{OA.2}$$

Furthermore, $H^\tau(x|\tilde{\mu}) = \widetilde{H}(x)$ for all $x \in \mathbb{R}$. As a result, from (OA.2), for $\mu \in \Delta(\mathcal{F}_0)$ defined as

$$\mu := (1-\eta)\tilde{\mu} + \eta\delta_{\{\widehat{F}\}},$$

since $F = \eta\widehat{F} + (1-\eta)\widetilde{F}$, it must be that $\mu \in \widetilde{\mathcal{M}}_\tau$. Moreover, since $H^\tau(\cdot|\tilde{\mu}) = \widetilde{H}$, we have $H^\tau(x|\mu) = H(x)$ for all $x \in \mathbb{R}$.

Lastly, consider any $H \in \mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon})$. Since $\mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon})$ is a convex and compact set in a metrizable space, Choquet's theorem implies that there exists a probability measure $\Lambda_H \in \Delta(\mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon}))$ that assigns probability 1 to $\mathrm{ext}(\mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon}))$ such that

$$H(x) = \int_{\mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon})} \widetilde{H}(x)\Lambda_H(\mathrm{d}\widetilde{H}).$$

Meanwhile, define the linear functional $\Xi : \widetilde{\mathcal{M}}_\tau \to \mathcal{F}_0$ as

$$\Xi(\tilde{\mu})[x] := \tilde{\mu}(\{G \in \mathcal{F}_0 | G^{-1}(\tau) \leq x\}),$$

for all $\tilde{\mu} \in \widetilde{\mathcal{M}}_\tau$ and for all $x \in \mathbb{R}$. Now, define a probability measure $\widetilde{\Lambda}$ on $\widetilde{\mathcal{M}}_\tau$ by

$$\widetilde{\Lambda}_H(A) := \Lambda_H(\{\Xi(\tilde{\mu}) | \tilde{\mu} \in A\}),$$

for all $A \subseteq \widetilde{\mathcal{M}}_\tau$. Then, since $\Lambda_H(\text{ext}(\mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon}))) = 1$ and since, for any $\widetilde{H} \in \text{ext}(\mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon}))$, there exists $\tilde{\mu} \in \widetilde{\mathcal{M}}_\tau$ such that $H(x) = H^\tau(x|\tilde{\mu})$, it must be that $\widetilde{\Lambda}_H(\widetilde{\mathcal{M}}_\tau) = 1$, and hence $\widetilde{\Lambda}_H$ is a probability measure on $\widetilde{\mathcal{M}}_\tau$. Let $\tilde{\mu} \in \widetilde{\mathcal{M}}_\tau$ be defined as

$$\tilde{\mu}(A) := \int_{\widetilde{\mathcal{M}}_\tau} \mu(A) \widetilde{\Lambda}_H(\mathrm{d}\mu),$$

for all measurable $A \subseteq \mathcal{F}_0$. Then, since Ξ is linear, it follows that

$$\begin{aligned} H(x) = \int_{\mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon})} \widetilde{H}(x) \Lambda_H(\mathrm{d}\widetilde{H}) = &\int_{\widetilde{\mathcal{M}}_\tau} \Xi(\mu)[x] \widetilde{\Lambda}_H(\mathrm{d}\mu) \\ =& \Xi(\tilde{\mu})[x] \\ =& H^\tau(x|\tilde{\mu}), \end{aligned}$$

and therefore, $H \in \widetilde{\mathcal{H}}_\tau$. Together, for any $\varepsilon > 0$, any $H \in \mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon})$ must be in $\widetilde{\mathcal{H}}_\tau$. In other words,

$$\bigcup_{\varepsilon > 0} \mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon}) \subseteq \widetilde{\mathcal{H}}_\tau.$$

This completes the proof. ■

OA.3 Proof of Corollary 1

For 1, consider any $H \in \mathcal{H}_q$. By Theorem 2, $H \in \mathcal{I}(F_R^q, F_L^q)$. Thus, $(F_L^q)^{-1}(\tau) \leq H^{-1}(\tau) \leq H^{-1}(\tau^+) \leq (F_R^q)^{-1}(\tau^+)$, and therefore $[H^{-1}(\tau), H^{-1}(\tau^+)] \subseteq [(F_L^q)^{-1}(\tau), (F_R^q)^{-1}(\tau^+)]$. Conversely, consider any interval $Q = [\underline{x}, \overline{x}] \subseteq [(F_L^q)^{-1}(\tau), (F_R^q)^{-1}(\tau^+)]$. Then, let $\widehat{H}$ be defined as

$$\widehat{H}(x) := \begin{cases} 0, & \text{if } x < \underline{x} \\ \tau, & \text{if } x \in [\underline{x}, \overline{x}) \\ 1, & \text{if } x \geq \overline{x} \end{cases},$$

for all $x \in \mathbb{R}$. Then $\widehat{H} \in \mathcal{I}(F_L^q, F_R^q)$ and $Q = [H^{-1}(\tau), H^{-1}(\tau^+)]$. Moreover, by Theorem 2, $\widehat{H} \in \mathcal{H}_q$, as desired.

For 2, consider any $H \in \widetilde{\mathcal{H}}_q$. By Theorem 2, $H \in \mathcal{I}(F_R^q, F_L^q)$. Thus, it must be that

$[H^{-1}(\tau), H^{-1}(\tau^+)] \subseteq [(F_L^q)^{-1}(\tau), (F_R^q)^{-1}(\tau)]$. Conversely, for any $\hat{x} \in ((F_L^q)^{-1}(\tau), (F_R^q)^{-1}(\tau^+))$, note that since $\hat{x} > (F_L^q)^{-1}(\tau)$ and since F is continuous, we have $F(\hat{x})/\tau > q$. Similarly, we also have $(F(\hat{x})-\tau)/(1-\tau) < q$. Let $\varepsilon := \min\{F(\hat{x})/\tau - q, q-(F(\hat{x})-\tau)/(1-\tau)\}$. Then, either $\hat{x} = (F_L^{q,\varepsilon})^{-1}(\tau)$ or $\hat{x} = (F_R^{q,\varepsilon})^{-1}(\tau)$. Since both $F_L^{q,\varepsilon}$ and $F_R^{q,\varepsilon}$ are in $\mathcal{I}(F_R^{q,\varepsilon}, F_L^{q,\varepsilon})$, Theorem 3 implies that $\hat{x} = H^{-1}(\tau)$ for some $H \in \widetilde{H}_q$. Lastly, note that under a signal $\mu \in \mathcal{M}$ such that μ assigns probability τ to F_L^τ and probability $1-\tau$ to F_R^τ, we have $\mu \in \widetilde{M}_q$ and $H^q(x|\mu) = \tau$ for all $x \in [(F_L^q)^{-1}(\tau), (F_R^q)^{-1}(\tau)]$. Hence, $[(F_L^q)^{-1}(\tau), (F_R^q)^{-1}(\tau)] \subseteq [H^{-1}(\tau), H^{-1}(\tau^+)]$ for some $H \in \widetilde{H}_q$, as desired. ■

OA.4 Proof of Corollary 4

(i) and *(ii)* follow immediately from the fact that any $H \in \mathcal{I}(F_R^\tau, F_L^\tau)$ is dominated by F_R^τ and dominates F_L^τ, and that $v_S(x)$ is increasing in x for all $x \leq a$ and is decreasing in x for all $x > a$.

For *(iii)*, suppose that for any $\underline{a} \leq \overline{a}$, $H_{\underline{a},\overline{a}}^C$ is not optimal. Then, since at least one extreme point of $\mathcal{I}(F_R^\tau, F_L^\tau)$ must be the solution of (3), consider any such extreme point and denote it by H. By Theorem 1, there exists a countable collection of intervals $\{[\underline{x}_n, \overline{x}_n)\}_{n=1}^\infty$ such that conditions 1 and 2 of Theorem 1 hold. Since $H_{\underline{a},\overline{a}}^C$ is not optimal for any $\underline{a} \leq \overline{a}$, $H \neq H_{\underline{a},\overline{a}}^C$ for all $\underline{a} \leq \overline{a}$. In particular, there must exist $n \in \mathbb{N}$ such that $\underline{x}_n < \overline{x}_n$ and either $H(\overline{x_n}^-) > F_R^\tau(\overline{x}_n)$ or $H(\underline{x}_n) < F_L^\tau(\underline{x}_n)$. Let a be the minimizer of v_S and suppose that $a \leq F^{-1}(\tau)$. Suppose that $H(\overline{x}_n^-) > F_R^\tau(\overline{x}_n)$. Then it must be that $H(\underline{x}_n) = F_L^\tau(\underline{x}_n)$. Moreover, since $H(\overline{x}_n^-) > F_R^\tau(\overline{x}_n)$, $H(\overline{x}_n) > F_R^\tau(\overline{x}_n)$ as well. If $\overline{x}_n \leq a$, then by replacing $H(x)$ with $\min\{F_L^\tau(x), H(\overline{x}_n)\}$ for all $x \in [\underline{x}_n, \overline{x}_n)$ and otherwise leaving H unchanged, the resulting distribution $\widehat{H}$ must still be in $\mathcal{I}(F_R^\tau, F_L^\tau)$. Since v_S is strictly decreasing on $[\underline{x}_n, \overline{x}_n)$, $\widehat{H}$ must give a higher value, a contradiction. If, on the other hand, $\overline{x}_n > a$, then since $H(\overline{x}_n) > F_R^\tau(\overline{x}_n)$ and since F is continuous, there exists $y > \overline{x}_n$ such that $H(\overline{x}_n^-) > F_R^\tau(y)$. Moreover, since H satisfies conditions 1 and 2, $H(x) > H(\overline{x}_n^-)$ for all $x \in [\overline{x}_n, y)$. Therefore, by replacing $H(x)$ with $H(\overline{x}_n^-)$ for all $x \in [\overline{x}, y)$ and leaving H unchanged otherwise, the resulting $\widehat{H}$ must still be in $\mathcal{I}(F_R^\tau, F_L^\tau)$. Since v_S is strictly increasing on $[\overline{x}_n, y)$, $\widehat{H}$ must give a higher value, a contradiction. Analogous arguments also lead to a contradiction for the case of $H(\underline{x}_n) < F_L^\tau(\underline{x}_n)$, as well as $a > F^{-1}(\tau)$. Therefore, $H_{\underline{a}.\overline{a}}^C$ must be optimal for some $\underline{a} \leq \overline{a}$.

For *(iv)*, note that F is not an extreme point of $\mathcal{I}(F_R^\tau, F_L^\tau)$ according to Theorem 1. Therefore, it is never the unique solution of (3). ■

OA.5 Proof of Proposition 2

Let $\bar{v}(G) := \sup_{x \in [G^{-1}(\tau), G^{-1}(\tau^+)]} v_S(x)$ for all $G \in \mathcal{F}_0$. Then, by Theorem 2,

$$\text{cav}(\hat{v})[F] \leq \text{cav}(\bar{v})[F] = \sup_{H \in \mathcal{I}(F_R^\tau, F_L^\tau)} \int_{\mathbb{R}} v_S(x) H(\mathrm{d}x).$$

Meanwhile, by Theorem 3,

$$\sup_{H \in \cup_{\varepsilon > 0} \mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon})} \int_{\mathbb{R}} v_S(x) H(\mathrm{d}x) \leq \text{cav}(\hat{v})[F].$$

Together, since $\text{cl}(\{\mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon})\}) = \mathcal{I}(F_R^\tau, F_L^\tau)$, (3) then follows. ■

OA.6 Proof of Corollary 5

For necessity, consider any $H \in \widetilde{\mathcal{H}}_\tau$ such that $H(z_k^-) - H(z_{k-1}^-) = \theta_k$ for all $k \in \{1, \dots, K\}$. Then for any $k \in \{1, \dots, K\}$, $\sum_{i=1}^k \theta_i = H(z_k^-)$. Since $H \in \widetilde{\mathcal{H}}_\tau$, there exists a signal $\mu \in \mathcal{M}$ for which μ-almost all posteriors have a unique τ-quantile and $H(z_k^-) = \mu(\{G \in \mathcal{F}_0 | G^{-1}(\tau) < z_k\}) = \mu(\{G \in \mathcal{F}_0 | \tau < G(z_k)\})$. Since $\mu \in \mathcal{M}$, $G(z_k)$ is a mean-preserving spread of $F(z_k)$ when $G \sim \mu$. Thus, $\mu(\{G \in \mathcal{F}_0 | \tau < G(z_k)\}) < F(z_k)/\tau$, and hence (4) holds. Analogous arguments can be applied to show that (5) holds as well.

For sufficiency, consider any prediction dataset $(\theta_k)_{k=1}^K$ such that (4) and (5) hold. Let H be the distribution that assigns probability θ_k at $(z_k + z_{k-1})/2$. Then, there exists $\varepsilon > 0$ such that $H \in \mathcal{I}(F_R^{\tau,\varepsilon}, F_L^{\tau,\varepsilon})$. By Theorem 3, there exists a signal μ with $\mu(\{G \in \mathcal{F}_0 | G^{-1}(\tau) < G^{-1}(\tau^+)\}) = 0$ such that $H(x) = H^\tau(x|\mu)$ for all $x \in \mathbb{R}$, which in turn implies that μ τ-quantile-rationalizes $(\theta_k)_{k=1}^K$, as desired. ■

OA.7 Proofs of Proposition 3 and Proposition 5

We prove the following result that leads to Proposition 3 and Proposition 5 immediately.[4]

Theorem OA.1. *Let $\overline{F}(x) := x$ and $\underline{F}(x) := 0$ for all $x \in [0, 1]$. For any $J \in \mathbb{N}$, for any collection of bounded linear functionals $\{\Gamma_j\}_{j=1}^J$ on $L^1([0, 1])$ and for any collection $\{\gamma_j\}_{j=1}^J \subseteq$*

[4] Rolewicz (1984) characterizes the extreme points of bounded Lipschitz functions defined on the unit interval that vanish at zero, and he shows that a function is an extreme point of the unit ball of this set if and only if the absolute value of its derivative equals 1 almost everywhere (see also Rolewicz 1986; Farmer 1994; Smarzewski 1997). The convex set of interest here is different. First, functions in $\mathcal{I}(\underline{F}, \overline{F})$ are subject to an additional monotonicity constraint. Second, these functions are bounded by $\underline{F}$ and $\overline{F}$ under the pointwise dominance order, rather than the Lipschitz (semi) norm. In particular, functions in $\mathcal{I}(\underline{F}, \overline{F})$ may have unbounded derivatives, whenever well-defined. Lastly, Theorem OA.1 below characterizes the extreme points of this set subject to finitely many other linear constraints, which are not present in the characterization of Rolewicz (1984).

$\mathbb{R}$, *let* $\mathcal{I}^c$ *be a convex subset of* $\mathcal{I}(\underline{F}, \overline{F})$ *defined as*

$$\mathcal{I}^c := \left\{ H \in \mathcal{I}(\underline{F}, \overline{F}) | \Gamma_j(H) \geq \gamma_j, \, \forall j \in \{1, \dots, J\} \right\}.$$

Suppose that $H \in \mathcal{I}^c$ *is an extreme point of* $\mathcal{I}^c$*. Then there exists countably many intervals* $\{[\underline{x}_n, \overline{x}_n)\}_{n=1}^{\infty}$ *such that:*

1. $H(x) = x$ *for all* $x \notin \cup_{n=1}^{\infty} [\underline{x}_n, \overline{x}_n)$*.*

2. *For all* $n, m \in \mathbb{N}$*, with* $n \neq m$*,* H *is constant on* $[\underline{x}_n, \overline{x}_n)$ *and* $H(\underline{x}_n) \neq H(\underline{x}_m)$*.*

3. *For all but at most* J *many* $n \in \mathbb{N}$*,* $H(\underline{x}_n) = \underline{x}_n$*.*

Proof. Consider any extreme point H of $\mathcal{I}^c$. We first claim that for any $x \in (0, 1)$, it must be either $H(x) = x$ or $H(y) = H(x)$ for all $y \in (x, x + \delta)$ for some $\delta > 0$. To see this, note that since $\mathcal{I}^c$ is a subset of $\mathcal{I}(\underline{F}, \overline{F})$ defined by J linear constraints, Proposition 2.1 of Winkler (1988) implies that there exists $\{H_j\}_{j=1}^{J+1} \subseteq \text{ext}(\mathcal{I}(\underline{F}, \overline{F}))$ and $\{\lambda_j\}_{j=1}^{J+1} \subseteq [0, 1]$ such that $H(x) = \sum_{j=1}^{J+1} \lambda_j H_j(x)$ for all $x \in [0, 1]$ and $\sum_{j=1}^{J+1} \lambda_j = 1$. Now suppose that $H(x) < x$ for some $x \in (0, 1)$. Then there must exist a nonempty subset $\mathcal{J} \subseteq \{1, \dots, J + 1\}$ such that $H_j(x) < x$ for all $j \in \mathcal{J}$ and that $H_j(x) = x$ for all $j \in \{1, \dots, J + 1\} \backslash \mathcal{J}$. Since H_j is an extreme point of $\mathcal{I}(\underline{F}, \overline{F})$ for all $j \in \mathcal{J}$, Theorem 1 implies that for each $j \in \mathcal{J}$, there exists an interval $[\underline{x}^j, \overline{x}^j)$ containing x on which H_j is constant. Let $(\underline{x}, \overline{x})$ be the interior of the intersection of $\{[\underline{x}^j, \overline{x}^j)\}_{j \in \mathcal{J}}$. Then it must be that

$$H(y) = \alpha y + (1 - \alpha)\eta$$

for all $y \in (\underline{x}, \overline{x})$, for some $\eta < x$, and $\alpha \in (0, 1)$. Now suppose that for any $\delta > 0$, there exists $y \in (x, x+\delta)$ such that $H(x) < H(y)$. Take any $\hat{\delta} \in (0, \min\{(1-\alpha)(x-\eta)/(1+\alpha), x-\underline{x}, \overline{x}-x\})$ and let $x_* := x - \hat{\delta}$ and $x^* := x + \hat{\delta}$. Then it must be that $H(y) < x$ for any $y \in [x_*, x^*]$ and that $H(x^*) < x_*$. Moreover, the function $h : [x_*, x^*] \to [H(x_*), H(x^*)]$ defined as $h(y) := H(y)$ for all $y \in [x_*, x^*]$ must not be a step function, since otherwise, as h is right-continuous on (x_*, x^*), there must be some $\delta > 0$ such that $H(y) = h(y) = h(x) = H(x)$ for all $y \in [x, x + \delta)$, a contradiction. Meanwhile, since each functional $\Gamma_j : L^1([0, 1]) \to \mathbb{R}$ is bounded, Riesz's representation implies that there must exist $\Phi_j \in L^\infty([0, 1])$ such that

$$\Gamma_j(\widetilde{H}) = \int_0^1 \widetilde{H}(x)\Phi_j(x) \, \mathrm{d}x,$$

for all $\widetilde{H} \in \mathcal{I}(\underline{F}, \overline{F})$. Therefore, since any extreme point of the collection of nondecreasing, right-continuous functions $\tilde{h}$ from $[x_*, x^*]$ to $[H(x_*), H(x^*)]$ such that

$$\int_{x_*}^{x^*} \tilde{h}(x)\Phi_j(x)\,\mathrm{d}x \geq \gamma_j$$

for all $j \in \{1, \ldots, J\}$ is a step function with at most $J+1$ steps, as implied by Proposition 2.1 of Winkler (1988), the function h is not an extreme point of this collection. Thus, there exists two distinct functions $h_1, h_2 : [x_*, x^*] \to [H(x_*), H(x^*)]$ and $\lambda \in (0,1)$ such that $h(y) = \lambda h_1(y) + (1-\lambda)h_2(y)$ for all $y \in [x_*, x^*]$ and that

$$\int_{x_*}^{x^*} h_l(x)\Phi_j(x)\,\mathrm{d}x = \int_{x_*}^{x^*} H(x)\Phi_j(x)\,\mathrm{d}x, \tag{OA.3}$$

for all $j \in \{1, \ldots, J\}$ and for all $l \in \{1, 2\}$. Now let H_1, H_2 be defined as

$$H_1(y) := \begin{cases} H(y), & \text{if } y \notin [x_*, x^*] \\ h_1(y), & \text{if } y \in [x_*, x^*] \end{cases} ; \quad H_2(y) := \begin{cases} H(y), & \text{if } y \notin [x_*, x^*] \\ h_2(y), & \text{if } y \in [x_*, x^*] \end{cases} .$$

Then, $H = \lambda H_1 + (1-\lambda)H_2$ and $H_1 \neq H_2$. Moreover, since $h_1(y), h_2(y) \leq H(x^*) < x_*$ for all $y \in [x_*, x^*]$, and since $H \in \mathcal{I}(\underline{F}, \overline{F})$, it must be that both H_1 and H_2 are in $\mathcal{I}(\underline{F}, \overline{F})$. Furthermore, by (OA.3), it must be that

$$\begin{aligned}
\Gamma_j(H_l) = \int_0^1 H_l(x)\Phi_j(x)\,\mathrm{d}x &= \int_{[0,1]\setminus[x_*,x^*]} H(x)\Phi_j(x)\,\mathrm{d}x + \int_{x_*}^{x^*} h_l(x)\Phi_j(x)\,\mathrm{d}x \\
&= \int_{[0,1]\setminus[x_*,x^*]} H(x)\Phi_j(x)\,\mathrm{d}x + \int_{x_*}^{x^*} H(x)\Phi_j(x)\,\mathrm{d}x \\
&= \int_0^1 H(x)\Phi_j(x)\,\mathrm{d}x \\
&\geq \gamma_j,
\end{aligned}$$

for all $j \in \{1, \ldots, J\}$ and for all $l \in \{1, 2\}$. Thus, $H_1, H_2 \in \mathcal{I}^c$, a contradiction. Together, for any $x \in (0,1)$, it must be either $H(x) = x$ or $H(y) = H(x)$ for all $y \in (x, x+\delta)$ for some $\delta > 0$.

Let $X \subseteq [0,1]$ be the collection of $x \in [0,1]$ such that $H(x) = x$. For any $x \notin X$, let $\overline{\delta}_x := \sup\{y \in [0,1] | H(y) = H(x)\}$ and $\underline{\delta}_x := \inf\{y \in [0,1] | H(y) = H(x)\}$. Then it must be $\underline{\delta}_x < \overline{\delta}_x$ for all $x \notin X$. Moreover, for any $x, y \in [0,1]\setminus X$ with $x < y$, $H(x) < H(y)$ if and only if $\overline{\delta}_x < \underline{\delta}_y$. Therefore, $[0,1]\setminus X$ can be expressed as a union of a collection I of disjoint intervals. Since I is a collection of disjoint intervals on $[0,1]$, each element of I must uniquely contain at least one rational number. Thus, there exists an injective map from the collection I to a subset of rational numbers in $[0,1]$, and therefore the collection I must be countable.

Enumerate I as $\{[\underline{x}_n, \overline{x}_n)\}_{n=1}^{\infty}$ and suppose that there is a subset $\mathcal{N}$ of these intervals, with $|\mathcal{N}| > J$, such that $H(\underline{x}_n) < \underline{x}_n$. For each $n \in \mathcal{N}$, since $H(\underline{x}_n) < \underline{x}_n$ and since $H(x) = x$ for all $x \notin \cup_{n=1}^{\infty}[\underline{x}_n, \overline{x}_n)$, H must be discontinuous at $\underline{x}_n$. Let $\eta_n := H(\underline{x}_n) - H(\underline{x}_n^-)$ for all $n \in \mathcal{N}$, and let $\eta := \min\{\eta_n\}_{n\in\mathcal{N}}$. Furthermore, let $\phi_j^n \in \mathbb{R}$ be defined as

$$\phi_j^n := \int_{\underline{x}_n}^{\overline{x}_n} \Phi_j(x)\,\mathrm{d}x,$$

for all $n \in \mathcal{N}$ and for all $j \in \{1, \dots, J\}$. Then the $|\mathcal{N}| \times J$ matrix $\Phi := (\phi_j^n)_{j\in\{1,\dots,J\}}^{n\in\mathcal{N}}$ is a linear map from $\mathbb{R}^{|\mathcal{N}|}$ to $\mathbb{R}^J$. Since $|\mathcal{N}| > J$, $\dim(\mathrm{null}(\Phi)) \geq 1$, and thus there must exist a nonzero vector $\{\hat{h}_n\}_{n\in\mathcal{N}}$ such that

$$\sum_{n\in\mathcal{N}} \phi_j^n \hat{h}_n = 0, \tag{OA.4}$$

for all $j \in \{1, \dots, J\}$. Let $\varepsilon := \min\{\eta/4|\hat{h}_n|, (\underline{x}_n - H(\underline{x}_n))/4|\hat{h}_n|\}_{n\in\mathcal{N}}$, and let $\widehat{H}$ be defined as

$$\widehat{H}(x) := \begin{cases} 0, & \text{if } x \notin \cup_{n\in\mathcal{N}}[\underline{x}_n, \overline{x}_n) \\ \varepsilon\hat{h}_n, & \text{if } x \in [\underline{x}_n, \overline{x}_n),\ n \in \mathcal{N} \end{cases}.$$

Then, since $\{\hat{h}_n\}_{n\in\mathcal{N}}$ is a nonzero vector in $\mathbb{R}^{|\mathcal{N}|}$ and since $\varepsilon > 0$, $\hat{H} \neq 0$. Moreover, since $\varepsilon < \eta/4|\hat{h}_n|$ for all $n \in \mathcal{N}$, $H(x) - |\widehat{H}(x)| = H(\underline{x}_n) - \varepsilon|\hat{h}_n| > H(\underline{x}_n) - \eta/2 > H(\underline{x}_n^-) + \eta/4 > H(x) + |\widehat{H}(x)|$ for all $x < \underline{x}_n$ and for all $n \in \mathcal{N}$. Therefore, both $H + \widehat{H}$ and $H - \widehat{H}$ are nondecreasing. Meanwhile, since for any $n \in \mathcal{N}$ and for any $x \in [\underline{x}_n, \overline{x}_n)$, $H(x) + |\widehat{H}(x)| = H(\underline{x}_n) + \varepsilon|\hat{h}_n| < \underline{x}_n$, both $H + \widehat{H}$ and $H - \widehat{H}$ are in $\mathcal{I}(\underline{F}, \overline{F})$. In addition, by (OA.4), for any $j \in \{1, \dots, J\}$,

$$\begin{aligned}\int_0^1 (H(x) + \widehat{H}(x))\Phi_j(x)\,\mathrm{d}x &= \int_{[0,1]\setminus\cup_{n\in\mathcal{N}}[\underline{x}_n,\overline{x}_n)} H(x)\Phi_j(x)\,\mathrm{d}x + \int_{\cup_{n\in\mathcal{N}}[\underline{x}_n,\overline{x}_n)} H(x)\Phi_j(x)\,\mathrm{d}x + \varepsilon\sum_{n\in\mathcal{N}}\hat{h}_n\phi_j^n \\ &= \int_0^1 H(x)\Phi_j(x)\,\mathrm{d}x \\ &\geq \gamma_j,\end{aligned}$$

and

$$\begin{aligned}\int_0^1 (H(x) - \widehat{H}(x))\Phi_j(x)\,\mathrm{d}x &= \int_{[0,1]\setminus\cup_{n\in\mathcal{N}}[\underline{x}_n,\overline{x}_n)} H(x)\Phi_j(x)\,\mathrm{d}x + \int_{\cup_{n\in\mathcal{N}}[\underline{x}_n,\overline{x}_n)} H(x)\Phi_j(x)\,\mathrm{d}x - \varepsilon\sum_{n\in\mathcal{N}}\hat{h}_n\phi_j^n \\ &= \int_0^1 H(x)\Phi_j(x)\,\mathrm{d}x \\ &\geq \gamma_j.\end{aligned}$$

Together, both $H + \widehat{H}$ and $H - \widehat{H}$ are in $\mathcal{I}^c$, and hence, H is not an extreme point, a contradiction. This completes the proof. ■

Proofs of Proposition 3 and Proposition 5. Note that since $|\phi_e(x|e)|$ is dominated by an integrable function on $[0, 1]$, one can apply the dominated convergence theorem to show that the objective function of both (7) and (10) are continuous in (H, e) and $(H, \underline{z})$, respectively. Similarly, the constraint set can be shown to be closed. Therefore, both (7) and (10) admit a solution.

Consequently, since for any fixed e and $\underline{z}$, the objective is continuous in H and the feasible set is compact and convex in (7) and (10), respectively, Proposition 3 and the first part of Proposition 5 follow immediately from Theorem OA.1, with $J = 2$ and $J = 1$, respectively. This is because any H satisfying conditions 1 through 3 correspond to a contingent debt contract with at most J non-defaultable face values. The uniqueness part of Proposition 5 further follows from the fact that the objective of (10) is strictly convex in H when $\Phi(\cdot|s)$ has full support for all $s \in S$, and hence, every solution must be an extreme point of the feasible set. ■

OA.8 Proof of Proposition 4

Let $\Pi^*(e)$ be the value of the entrepreneur's problem (7) for a fixed $e \in [0, \bar{e}]$. We first show that there exists Lagrange multipliers $\lambda_1^* \neq 0$ and $\lambda_2^* \geq 0$ such that

$$\Pi^*(e) = \sup_{H \in \mathcal{I}(\underline{F}, \overline{F})} \Bigg[\int_0^1 (x - H(x))\phi(x|e)\,\mathrm{d}x + \lambda_1^* \left(\int_0^1 (x - H(x))\phi_e(x|e)\,\mathrm{d}x - C'(e) \right) + \lambda_2^* \left(\int_0^1 H(x)\phi(x|e)\,\mathrm{d}x - (1+r)I \right) \Bigg]. \tag{OA.5}$$

To this end, we adopt a similar argument as Nikzad (2023). For any fixed $e \in [0, \bar{e}]$ and for any $\gamma \in \mathbb{R}$, let $M_e(\gamma)$ be the value of

$$\begin{aligned} \sup_{H \in \mathcal{I}(\underline{F}, \overline{F})} & \left[\int_0^1 [x - H(x)]\phi(x|e)\,\mathrm{d}x - C(e) \right] \\ \text{s.t.} & \int_0^1 [x - H(x)]\phi_e(x|e)\,\mathrm{d}x = C'(e) \\ & \int_0^1 H(x)\phi(x|e)\,\mathrm{d}x \geq \gamma. \end{aligned} \tag{OA.6}$$

Note that

$$M_e((1+r)I) = \Pi^*(e) = \int_0^1 (x - H^*(x))\phi(x|e)\,\mathrm{d}x - C(e), \tag{OA.7}$$

where H^* is a solution of (7) with a fixed e. Moreover, M_e is nonincreasing and concave in γ. Indeed, monotonicity follows from the ordered structure of the feasible set as γ increases. For concavity, consider any $\gamma_1, \gamma_2 \in \mathbb{R}$ and let $\gamma^\lambda := \lambda\gamma_1 + (1-\lambda)\gamma_2$ for any $\lambda \in (0,1)$. Since (OA.6) admits a solution, there exists $H_1, H_2 \in \mathcal{I}(\underline{F}, \overline{F})$ such that

$$\int_0^1 (x - H_1(x))\phi(x|e)\,\mathrm{d}x - C(e) = M(\gamma_1); \quad \int_0^1 (x - H_2(x))\phi(x|e)\,\mathrm{d}x - C(e) = M(\gamma_2).$$

Furthermore,

$$\int_0^1 (x - H_i(x))\phi_e(x|e)\,\mathrm{d}x = C'(e)$$
$$\int_0^1 H_i(x)\phi(x|e)\,\mathrm{d}x \geq \gamma_i$$

for $i \in \{1,2\}$. Let $H^\lambda := \lambda H_1 + (1-\lambda)H_2$, we must have $H^\lambda \in \mathcal{I}(\underline{F}, \overline{F})$ and

$$\int_0^1 (x - H^\lambda(x))\phi_e(x|e)\,\mathrm{d}x = C'(e)$$
$$\int_0^1 H^\lambda(x)\phi(x|e)\,\mathrm{d}x \geq \gamma^\lambda.$$

Thus,

$$\begin{aligned}
M_e(\gamma^\lambda) \geq & \int_0^1 (x - H^\lambda(x))\phi(x|e)\,\mathrm{d}x - C(e) \\
= & \lambda \int_0^1 (x - H_1(x))\phi(x|e)\,\mathrm{d}x + (1-\lambda)\int_0^1 (x - H_2(x))\phi(x|e)\,\mathrm{d}x \\
= & \lambda M_e(\gamma_1) + (1-\lambda)M_e(\gamma_2).
\end{aligned}$$

Since M_e is nonincreasing and concave, and since $(1+r)I$ is an interior of the set

$$\left\{ \int_0^1 H(x)\phi(x|e)\,\mathrm{d}x \,\middle|\, H \in \mathcal{I}(\underline{F}, \overline{F}),\ \int_0^1 (x - H(x))\phi_e(x|e)\,\mathrm{d}x = C'(e) \right\},$$

there exists $\lambda_2^* \geq 0$ such that

$$M_e(\gamma) \leq M_e((1+r)I) - \lambda_2^*(\gamma - (1+r)I)$$

for all $\gamma \in \mathbb{R}$. Meanwhile, for any $H \in \mathcal{I}(\underline{F}, \overline{F})$ such that

$$\int_0^1 (x - H(x))\phi_e(x|e)\,\mathrm{d}x = C'(e), \tag{OA.8}$$

it must be that

$$M_e\left(\int_0^1 H(x)\phi(x|e)\,\mathrm{d}x\right) \geq \int_0^1 (x - H(x))\phi(x|e)\,\mathrm{d}x - C(e),$$

by the definition of M_e. Together with (OA.7), we have

$$\begin{aligned} M_e((1+r)I) &= \int_0^1 (x - H^*(x))\phi(x|e)\,\mathrm{d}x - C(e) \\ &\geq \int_0^1 (x - H(x))\phi(x|e)\,\mathrm{d}x - C(e) + \lambda_2^*\left(\int_0^1 H(x)\phi(x|e)\,\mathrm{d}x - (1+r)I\right), \end{aligned} \tag{OA.9}$$

for all $H \in \mathcal{I}(\underline{F}, \overline{F})$ such that (OA.8) holds. Since H^* is feasible for (7) with the fixed e, (OA.9) implies

$$\begin{aligned} &\int_0^1 (x - H^*(x))\phi(x|e)\,\mathrm{d}x + \lambda_2^*\left(\int_0^1 H^*(x)\phi(x|e)\,\mathrm{d}x - (1+r)I\right) \\ &\geq \int_0^1 (x - H(x))\phi(x|e)\,\mathrm{d}x + \lambda_2^*\left(\int_0^1 H(x)\phi(x|e)\,\mathrm{d}x - (1+r)I\right), \end{aligned} \tag{OA.10}$$

for all $H \in \mathcal{I}(\underline{F}, \overline{F})$ satisfying (OA.8). Now let

$$\mathcal{L}_e(H;\lambda) := \int_0^1 (x - H(x))\phi(x|e)\,\mathrm{d}x - C(e) + \lambda\left(\int_0^1 H(x)\phi(x|e)\,\mathrm{d}x - (1+r)I\right),$$

and let $\mathcal{L}_e(\lambda)$ be the value of

$$\begin{aligned} &\sup_{H \in \mathcal{I}(\underline{F}, \overline{F})} \mathcal{L}_e(H;\lambda) \\ &\quad \text{s.t.} \int_0^1 (x - H(x))\phi_e(x|e)\,\mathrm{d}x = C'(e). \end{aligned} \tag{OA.11}$$

Then, (OA.10) implies that H^* solves (OA.11) with $\lambda = \lambda_2^*$ and

$$\mathcal{L}_e(\lambda_2^*) = \int_0^1 (x - H^*(x))\phi(x|e)\,\mathrm{d}x - C(e).$$

Meanwhile, by the definition of $\mathcal{L}_e(\lambda)$,

$$\mathcal{L}_e(\lambda) \geq \int_0^1 (x - H(x))\phi(x|e)\,\mathrm{d}x - C(e)$$

for all feasible H of (7) with fixed e. Finally, since the constraint in (OA.11) is an equality, standard results (see, e.g., Theorem 3.12 of Anderson and Nash 1987) implies that there exits $\lambda_1 \neq 0$ such that (OA.5) holds.

For any fixed $e \in [0, \bar{e}]$, since the primal problem (7) is convex for any fixed $e \in [0, \bar{e}]$, there exists an extreme point H^* of the feasible set that attains $\Pi^*(e)$. By Theorem OA.1, there exists a countable collection of intervals $\{[\underline{x}_n, \overline{x}_n)\}_{n=1}^{\infty}$ such that H^* satisfies conditions 1 through 3 for $J = 2$. Meanwhile, as established above, H^* must also solve the dual problem (OA.5) of (7) for this fixed e. Note that the dual problem can be written as

$$\sup_{H \in \mathcal{I}(\underline{F}, \overline{F})} \left[\int_0^1 H(x)[(1+\lambda_2^*)\phi(x|e) - \lambda_1^* \phi_e(x|e)] \, \mathrm{d}x + \kappa \right],$$

with $\kappa \in \mathbb{R}$ being a constant that does not depend on H. Moreover,

$$(1+\lambda_2^*)\phi(x|e) - \lambda_1^* \phi_e(x|e) \geq 0 \iff \frac{\phi_e(x|e)}{\phi(x|e)} \leq \frac{1+\lambda_2^*}{\lambda_1^*}.$$

Since $\phi_e(\cdot|e)/\phi(\cdot|e)$ is at most N-peaked, there must be a finite interval partition $\{I_k\}_{k=1}^K$ of $[0,1]$ with $K \leq 2N$ such that $\phi_e(x|e)/\phi(x|e) - (1+\lambda_2^*)/\lambda_1^*$ takes the same sign for all $x \in I_k$.

Therefore, if there are more than $N+1$ intervals on which H^* is constant, then either there are at least two of them contained in a single interval I_k with $\phi_e(x|e)/\phi(x|e) < (1+\lambda_2^*)/\lambda_1^*$ for all $x \in I_k$, or there is at least one of them contained in an interval I_j with $\phi_e(x|e)/\phi(x|e) > (1+\lambda_2^*)/\lambda_1^*$ for all $x \in I_j$. If there are two intervals $[\underline{x}_n, \overline{x}_n)$, $[\underline{x}_m, \overline{x}_m)$, with $\overline{x}_n \leq \underline{x}_m$, that are contained in some I_k with $\phi_e(x|e)/\phi(x|e) < (1+\lambda_2^*)/\lambda_1^*$ for all $x \in I_k$, then, since by condition 2 of Theorem OA.1, $H^*(\underline{x}_n) < H^*(\underline{x}_m)$, for H^{**} defined as

$$H^{**}(x) := \begin{cases} H^*(x), & \text{if } x \notin [\underline{x}_n, \overline{x}_m) \\ H^*(\underline{x}_n), & \text{if } x \in [\underline{x}_n, \overline{x}_m) \end{cases},$$

for all $x \in [0,1]$, $H^{**} \in \mathcal{I}(\underline{F}, \overline{F})$ and yields a higher value to the objective of (OA.5) than H^*. Likewise, if there is at least one interval on which H^* is constant that is contained in some I_j such that $\phi_e(x|e)/\phi(x|e) < (1+\lambda_2^*)/\lambda_1^*$ for all $x \in I_j$, then, since $H^*(x) < x$ for all $x \in (\underline{x}_n, \overline{x}_n)$, for H^{**} defined as

$$H^{**}(x) := \begin{cases} H^*(x), & \text{if } x \notin I_j \\ \max\{x, H^*(\overline{x}_n)\}, & \text{if } x \in I_j \end{cases},$$

for all $x \in [0,1]$, $H^{**} \in \mathcal{I}(\underline{F}, \overline{F})$ and yields a higher value to the objective of (OA.5) than H^*. Thus, H^* cannot be a solution of the dual problem (OA.5) for this fixed e, a contradiction. Consequently, the solution H^* to the primal problem (7) for any fixed $e \in [0, \bar{e}]$ cannot admit more than $N+1$ intervals where H^* is constant. As a result, H^* is a contingent debt contract with at most $N+1$ face values. Since $e \in [0, \bar{e}]$ is arbitrary, this completes the proof. ■

References

Anderson, Edward J. and Peter Nash (1987) *Linear Programming in Infinite-Dimensional Space*: Wiley.

Çinlar, Erhan (2010) *Probability and Stochastics*: Springer.

Farmer, Jeff D. (1994) "Extreme Points of the Unit Ball of the Space of Lipschitz Functions," *Proceedings of the American Mathematical Society*, 121 (3), 807–813.

Nikzad, Afshin (2023) "Constrained Majorization: Applications in Mechanism Design," Working Paper.

Rolewicz, Stefan (1984) "On Optimal Observability of Lipschitz Systems.," in Hammer, G. and Diethard Pallaschke eds. *Selected Topics in Operations Research and Mathematical Economics: Proceedings of the 8th Symposium on Operations Research*, 152–158, Berlin: Springer.

——— (1986) "On Extremal Points of the Unit Ball in the Banach Space of Lipschitz Continuous Functions," *Journal of the Australian Mathematical Society*, 41 (1), 95–98.

Smarzewski, Ryszard (1997) "Extreme Points of Unit Balls in Lipschitz Function Spaces," *Proceedings of the American Mathematical Society*, 125 (5), 1391–1397.

Winkler, Gerhard (1988) "Extreme Points of Moment Sets," *Mathematics of Operations Research*, 13 (4), 581–587.