Buying from a Group

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Online Appendix

B. Supporting analysis for Section 6

The following lemma shows that the share-weighted average of N independent types has a well-behaved distribution if each component does, and documents features of this distribution at the edges of its support.[25]

Lemma 6: *Let G denote the cumulative distribution function of $\mathbf{v} = \sigma \cdot \boldsymbol{\theta}$.*

- *The distribution G admits a continuous density g which is strictly positive on the interior of its support $[\sigma \cdot \underline{\theta}, \sigma \cdot \bar{\theta}]$.*
- *As $v \nearrow \sigma \cdot \bar{\theta}$, we have*
$$\frac{g(v)}{(\sigma \cdot \bar{\theta} - v)^N} \to \frac{1}{(N-1)!} \prod_{i \in N} \frac{f_i(\bar{\theta}_i)}{\sigma_i}.$$
- *As $v \searrow \sigma \cdot \underline{\theta}$, we have*
$$\frac{g(v)}{(v - \sigma \cdot \underline{\theta})^{N-1}} \to \frac{1}{(N-1)!} \prod_{i \in N} \frac{f_i(\underline{\theta}_i)}{\sigma_i}.$$
- *As $v \searrow \sigma \cdot \underline{\theta}$, we have*
$$\frac{G(v)}{(v - \sigma \cdot \underline{\theta})^N} \to \frac{1}{N} \cdot \frac{1}{(N-1)!} \prod_{i \in N} \frac{f_i(\underline{\theta}_i)}{\sigma_i} \text{ and } \frac{\Phi(v) - v}{v - \sigma \cdot \underline{\theta}} \to \frac{1}{N},$$
where $\Phi(v) := v + \frac{G(v)}{g(v)}$.

Proof. For each $n \in N$, let G_n denote the CDF of $\sum_{i=1}^n \sigma_i \boldsymbol{\theta}_i$. The support of G_n is $\left[\sum_{i=1}^n \sigma_i \underline{\theta}_i,\ \sum_{i=1}^n \sigma_i \bar{\theta}_i\right]$, and when $n > 1$ any z in this support has

$$G_n(z) \quad = \quad \int_{\underline{\theta}_n}^{\bar{\theta}_n} G_{n-1}(z - \sigma_n \theta_n) f_n(\theta_n) \, \mathrm{d}\theta_n.$$

Because $G_1(v_1) = F_1(\frac{v_1}{\sigma_1})$ for every v_1, it follows that G_1 is continuously differentiable on its support with derivative $g_1(v_1) = \frac{1}{\sigma_1} f_1(\frac{v_1}{\sigma_1})$. Then, by induction on n,

[25] Whereas previous analyses apply readily to the case in which f_i may fail to be continuous and strictly positive at the endpoints of its support (like Example 1), the analysis of this section makes use of the fact that $\lim_{\theta_i \searrow \underline{\theta}_i} f_i(\underline{\theta}_i)$ and $\lim_{\theta_i \nearrow \bar{\theta}_i} f_i(\bar{\theta}_i)$ are both in $(0, \infty)$. Nevertheless, our qualitative results can be adapted to the case of power distributions—with $F_i(\theta_i) = \theta_i^\alpha$ for $\alpha > 0$—albeit with the threshold $\frac{2}{N+1}$ being replaced with the threshold $\frac{\alpha+1}{N\alpha+1}$.

every $n \in N$ has G_n continuously differentiable on the interior of its support with the associated density at z in its support given by

$$g_n(z) = \int_{\underline{\theta}_n}^{\bar{\theta}_n} g_{n-1}(z - \sigma_n\theta_n) f_n(\theta_n) \, \mathrm{d}\theta_n.$$

Also by induction, g_n is strictly positive on the interior of its support because f_n is and (in the case of $n > 1$) g_{n-1} is. This establishes the first bullet.

To see the fourth bullet would follow from the third, note L'Hôpital's rule yields

$$\lim_{v \searrow \underline{\theta}_1} \tfrac{G(v)}{(v-\underline{\theta}_1)^N} = \lim_{v \searrow \underline{\theta}_1} \tfrac{g(v)}{N(v-\underline{\theta}_1)^{N-1}},$$

and note $\frac{\Phi(v)-v}{v-\underline{\theta}_1} = \frac{G(v)}{(v-\underline{\theta}_1)g(v)}$. So it remains to show the second and third bullets. Because the two are identical up to relabeling, we prove only the second bullet.

For any $\varepsilon > 0$ and any $n \in N$, let

$$h_n(\varepsilon) := (n-1)! \left[\prod_{i=1}^{n} \tfrac{\sigma_i}{f_i(\underline{\theta}_i)}\right] g_n\left(\varepsilon + \sum_{i=1}^{n} \sigma_i\underline{\theta}_i\right).$$

We need to show that $\frac{h_N(\varepsilon)}{\varepsilon^{N-1}} \to 1$ as $\varepsilon \searrow 0$. Let us show by induction that every $n \in N$ has $\frac{h_n(\varepsilon)}{\varepsilon^{n-1}} \to 1$ as $\varepsilon \searrow 0$, which will then deliver the lemma. For the base case, note that

$$\tfrac{h_1(\varepsilon)}{\varepsilon^0} = \tfrac{\sigma_1}{f_1(\underline{\theta}_1)} g_1\left(\varepsilon + \sigma_1\underline{\theta}_1\right) = \tfrac{f_1\left(\underline{\theta}_1 + \frac{\varepsilon}{\sigma_1}\right)}{f_1(\underline{\theta}_1)},$$

which converges to 1 as $\varepsilon \searrow 0$.

For the inductive step, suppose $n > 1$ and that the limit equation holds for $n-1$. Then, any small enough $\varepsilon > 0$ has

$$\begin{aligned} h_n(\varepsilon) &= (n-1)\tfrac{\sigma_n}{f_n(\underline{\theta}_n)} \int_{\underline{\theta}_n}^{\bar{\theta}_n} h_{n-1}\left(\varepsilon - \sigma_n\left[\theta_n - \underline{\theta}_n\right]\right) f_n(\theta_n) \, \mathrm{d}\theta_n \\ &= \tfrac{n-1}{f_n(\underline{\theta}_n)} \int_{\underline{\theta}_n}^{\underline{\theta}_n + \frac{\varepsilon}{\sigma_n}} h_{n-1}\left(\varepsilon - \sigma_n\left[\theta_n - \underline{\theta}_n\right]\right) f_n(\theta_n)\sigma_i \, \mathrm{d}\theta_n \\ &= \tfrac{n-1}{f_n(\underline{\theta}_n)} \int_0^{\varepsilon} h_{n-1}(\tilde{\varepsilon}) f_n(\underline{\theta}_n + \tfrac{\varepsilon-\tilde{\varepsilon}}{\sigma_n}) \, \mathrm{d}\tilde{\varepsilon} \\ \implies \tfrac{h_n(\varepsilon)}{\varepsilon^{n-1}} - 1 &= \tfrac{h_n(\varepsilon)}{\varepsilon^{n-1}} - \tfrac{1}{\varepsilon^{n-1}} \int_0^{\varepsilon} (n-1)\tilde{\varepsilon}^{n-2} \, \mathrm{d}\tilde{\varepsilon} \\ &= (n-1)\tfrac{1}{\varepsilon} \int_0^{\varepsilon} \left(\tfrac{\tilde{\varepsilon}}{\varepsilon}\right)^{n-2} \left[\tfrac{h_{n-1}(\tilde{\varepsilon})}{\tilde{\varepsilon}^{n-2}} \tfrac{f_n\left(\underline{\theta}_n + \frac{\varepsilon-\tilde{\varepsilon}}{\sigma_n}\right)}{f_n(\underline{\theta}_n)} - 1\right] \mathrm{d}\tilde{\varepsilon}, \end{aligned}$$

which converges to zero (because the integrand does uniformly) as $\varepsilon \searrow 0$, as required. □

The next lemma shows how virtual costs from our group setting and the single-agent analogue can be ranked for very high and very low types.

Lemma 7: *Suppose $\{F_i\}_{i\in N}$ all coincide (so $\omega = (\frac{1}{N}, \dots, \frac{1}{N})$ is optimal). Let G, g, and Φ be as defined in Lemma 6. Then:*

- *Every $v \in \Theta_1 \backslash \{\bar{\theta}_1\}$ close enough to $\bar{\theta}_1$ has $\Phi(v) > \varphi_1(\bar{\theta}_1)$.*
- *If $\sigma_i < \frac{2}{N+1}$ for every $i \in N$, then every $\theta \in \Theta \backslash \{\underline{\theta}\}$ close enough to $\underline{\theta}$ has $\omega \cdot \varphi(\theta) > \Phi(\sigma \cdot \theta)$.*
- *If $\sigma_i > \frac{2}{N+1}$ for some $i \in N$, then some $\eta \in \mathbb{R}^N_{++}$ exists such that every sufficiently small $\varepsilon > 0$ has $\omega \cdot \varphi(\underline{\theta} + \varepsilon\eta) < \Phi(\sigma \cdot (\underline{\theta} + \varepsilon\eta))$.*

Proof. All three parts follow from Lemma 6. First, as $v \nearrow \bar{\theta}_1$, that lemma tells us $g(v) \to 0$ so that $\Phi(v) \to \infty$. Meanwhile, that f_1 is continuous and strictly positive implies φ_1 is bounded. Hence, large enough $v \in [\underline{\theta}_1, \bar{\theta}_1)$ have $\Phi(v) > \varphi_1(\bar{\theta}_1)$.

Toward the second and third bullets, let us write $o(\theta - \underline{\theta})$ for any function of $\theta \in \Theta$ with $\frac{o(\theta-\underline{\theta})}{\|\theta-\underline{\theta}\|} \xrightarrow{\theta \searrow \underline{\theta}} 0$.[26] Lemma 6 tells us $\lim_{v \searrow \underline{\theta}_1} \frac{\Phi(v)-v}{v-\underline{\theta}_1} = \frac{1}{N}$, so that

$$\Phi(\sigma \cdot \theta) - \underline{\theta}_1 = \sigma \cdot \theta - \underline{\theta}_1 + \tfrac{1}{N}(\sigma \cdot \theta - \underline{\theta}_1) + o(\theta - \underline{\theta}) = \tfrac{N+1}{N}\sigma \cdot (\theta - \underline{\theta}) + o(\theta - \underline{\theta}).$$

Meanwhile, as $\theta_1 \searrow \underline{\theta}_1$, both $f_1(\theta_1)$ and $\frac{F_1(\theta_1)}{\theta_1 - \underline{\theta}_1}$ converge to $f_1(\underline{\theta}_1)$, so that

$$\tfrac{\varphi_1(\theta_1)-\underline{\theta}_1}{\theta_1-\underline{\theta}_1} = 1 + \tfrac{\varphi_1(\theta_1)-\theta_1}{\theta_1-\underline{\theta}_1} = 1 + \tfrac{F_1(\theta_1)}{(\theta_1-\underline{\theta}_1)f_1(\theta_1)} \to 2.$$

So $\varphi_i(\theta_i) - \underline{\theta}_i = 2(\theta_i - \underline{\theta}_i) + o(\theta - \underline{\theta})$, implying $\omega \cdot \varphi(\theta) - \underline{\theta}_1 = 2\omega \cdot (\theta - \underline{\theta}) + o(\theta - \underline{\theta})$. Therefore,

$$\begin{aligned} \omega \cdot \varphi(\theta) - \Phi(\sigma \cdot \theta) &= \left(2\omega - \tfrac{N+1}{N}\sigma\right) \cdot (\theta - \underline{\theta}) + o(\theta - \underline{\theta}) \\ &= \tfrac{N+1}{N}\left(\tfrac{2}{N+1}\mathbf{1}_N - \sigma\right) \cdot (\theta - \underline{\theta}) + o(\theta - \underline{\theta}). \end{aligned}$$

We now pursue the second bullet. If $\sigma_i < \frac{2}{N+1}$ for every $i \in N$, then the vector $\frac{N+1}{N}\left(\frac{2}{N+1}\mathbf{1}_N - \sigma\right)$ has strictly positive entries, so that $\omega \cdot \varphi(\theta) - \Phi(\sigma \cdot \theta) > 0$ for sufficiently small $\theta \in \Theta \backslash \{\underline{\theta}\}$.

Finally, to establish the third bullet, suppose some $i \in N$ has $\sigma_i > \frac{2}{N+1}$. Then, some $\gamma \in (0,1)$ exists such that

$$\gamma\left(\tfrac{2}{N+1} - \sigma_i\right) + (1-\gamma)\max_{j \in N}\left(\tfrac{2}{N+1} - \sigma_j\right) < 0.$$

Then $\eta \in \mathbb{R}^N_{++}$ with $\eta_i = \gamma$ and every other entry equal to $\frac{1-\gamma}{N-1}$ is as desired. □

Now, we introduce a notion of (utilitarian) efficiency ranking of allocation rules.

Definition 5: *Given an allocation rule x, the **surplus** generated by x in state $\theta \in \Theta$ is*

$$s_x(\theta) := x(\theta)(b - \sigma \cdot \theta).$$

*Given two allocation rules x and $\tilde{x}$, say x is **ex-ante more efficient** than $\tilde{x}$ if*

$$\mathbb{E}\left[s_x(\boldsymbol{\theta})\right] > \mathbb{E}\left[s_{\tilde{x}}(\boldsymbol{\theta})\right];$$

[26] Note this property is independent of the norm because any two norms on $\mathbb{R}^N$ have bounded ratio.

and say x is ***ex-post more efficient*** *than $\tilde{x}$ if*

$$\mathbb{P}\left\{s_x(\boldsymbol{\theta}) \geqslant s_{\tilde{x}}(\boldsymbol{\theta})\right\} = 1 \text{ and } \mathbb{P}\left\{s_x(\boldsymbol{\theta}) > s_{\tilde{x}}(\boldsymbol{\theta})\right\} > 0.$$

The next definition initializes language to discuss incentive properties and optimality of mechanisms in the single-agent benchmark.

Definition 6: *Say a mechanism (x, m) is* ***single-agent incentive compatible (SIC)*** *if*

$$\theta \in \text{argmax}_{\hat{\theta} \in \Theta} \left[m(\hat{\theta}) - \sigma \cdot \theta x(\hat{\theta})\right], \ \forall \theta \in \Theta,$$

that is, report $\hat{\theta} = \theta$ maximizes the expected payoff of type profile θ over all possible reports in Θ. Say the mechanism is ***single-agent individually rational (SIR)*** *if*

$$m(\theta) - \sigma \cdot \theta x(\theta) \geqslant 0, \ \forall \theta \in \Theta,$$

that is, the expected payoff of type profile θ, when reporting truthfully, is nonnegative. A ***single-agent-optimal mechanism*** *is an SIC and SIR mechanism that generates weakly higher buyer profit than any other SIC and SIR mechanism.*[27] *A* ***single-agent-optimal allocation rule*** *is any allocation rule x such that (x, m) is a single-agent-optimal mechanism for some m.*

Say an allocation rule x is ***single-agent implementable*** *if some transfer rule m exists such that the mechanism (x, m) is SIR; and say x is* ***single-agent monotone*** *if*

$$x(\sigma \cdot \theta) \leqslant x(\sigma \cdot \tilde{\theta}), \ \forall \theta, \tilde{\theta} \in \Theta \text{ with } \sigma \cdot \theta > \sigma \cdot \tilde{\theta}.$$

The next lemma shows that any single-agent-optimal mechanism is bounded between two cutoff mechanisms for the aggregated cost, in which the cutoffs solve a first-order condition equating the benefit of trade to single agent's virtual cost.

Lemma 8: *Some smallest and largest $\underline{p}_b, \bar{p}_b \in (\sigma \cdot \underline{\theta}, \ \sigma \cdot \bar{\theta})$ exist such that $\Phi(\underline{p}_b) = \Phi(\bar{p}_b) = b$, where Φ is as defined in Lemma 6. Moreover, some single-agent-optimal allocation rule exists, and any single-agent-optimal allocation rule x satisfies $\mathbb{1}_{\sigma \cdot \boldsymbol{\theta} \leqslant \underline{p}_b} \leqslant x(\boldsymbol{\theta}) \leqslant \mathbb{1}_{\sigma \cdot \boldsymbol{\theta} \leqslant \bar{p}_b}$ almost surely.*

Proof. Lemma 6 tells us Φ is continuous on $(\sigma \cdot \underline{\theta}, \ \sigma \cdot \bar{\theta})$, and that $\Phi(v)$ converges to $\sigma \cdot \underline{\theta}$ [resp. ∞] as $v \searrow \sigma \cdot \underline{\theta}$ [resp. $v \nearrow \sigma \cdot \bar{\theta}$]. Therefore, the set $\left\{p \in (\sigma \cdot \underline{\theta}, \ \sigma \cdot \bar{\theta}) : \ \Phi(p) = b\right\}$ is closed and bounded away from $\{\sigma \cdot \underline{\theta}, \ \sigma \cdot \bar{\theta}\}$ (hence compact), the set is nonempty by the intermediate value theorem, and every price $p \in (\sigma \cdot \underline{\theta}, \ \sigma \cdot \bar{\theta})$ strictly below [resp. above] this set has $\Phi(p) < b$ [resp. $\Phi(p) > b$]. In particular, this set of prices has a smallest and largest element, $\underline{p}_b$ and $\bar{p}_b$, respectively.

[27] Note, in this this short-hand, a single-agent-optimal mechanism/allocation is optimal *for the buyer* in the single-agent setting, and is not the preferred mechanism of the agent.

Now define the allocation rule x^* by

$$x^*(\theta) := \mathbb{1}_{\sigma\cdot\theta\leqslant \underline{p}_b} + x(\theta)\mathbb{1}_{\sigma\cdot\theta\in(\underline{p}_b,\bar{p}_b]}.$$

Because x is $[0,1]$-valued, a given $\theta \in \Theta$ has $\mathbb{1}_{\sigma\cdot\theta\leqslant \underline{p}_b} \leqslant x(\theta) \leqslant \mathbb{1}_{\sigma\cdot\theta\leqslant \bar{p}_b}$ if and only if $x(\theta) = x^*(\theta)$. It therefore remains to show $x(\boldsymbol{\theta}) = x^*(\boldsymbol{\theta})$ almost surely.

To show this equality, note that (straightforwardly adapting standard results from unidimensional mechanism design) a given allocation rule $\tilde{x}$ is single-agent implementable if and only if it is single-agent monotone, and that the maximum buyer value attainable by an SIC and SIR mechanism with allocation rule $\tilde{x}$ is $\mathbb{E}\left\{\tilde{x}(\boldsymbol{\theta})\left[b - \Phi(\sigma\cdot\boldsymbol{\theta})\right]\right\}$. Existence of an optimal allocation rule then follows from the observation that $\tilde{x} \mapsto \mathbb{E}\left\{\tilde{x}(\boldsymbol{\theta})\left[b - \Phi(\sigma\cdot\boldsymbol{\theta})\right]\right\}$ is a weak*-continuous function on the weak*-compact set $\tilde{\mathcal{X}}$.

Now, by construction (and since $\underline{p}_b \leqslant \bar{p}_b$), the allocation rule x^* is single-agent monotone because x is. Therefore, single-agent optimality of x tells us

$$\begin{aligned} 0 &\geqslant \mathbb{E}\left\{x^*(\boldsymbol{\theta})\left[b - \Phi(\sigma\cdot\boldsymbol{\theta})\right]\right\} - \mathbb{E}\left\{x(\boldsymbol{\theta})\left[b - \Phi(\sigma\cdot\boldsymbol{\theta})\right]\right\} \\ &= \mathbb{E}\left\{\left[1 - x(\boldsymbol{\theta})\right]\left[b - \Phi(\sigma\cdot\boldsymbol{\theta})\right]\mathbb{1}_{\sigma\cdot\boldsymbol{\theta}\leqslant \underline{p}_b} + x(\boldsymbol{\theta})\left[\Phi(\sigma\cdot\boldsymbol{\theta}) - b\right]\mathbb{1}_{\sigma\cdot\boldsymbol{\theta}>\bar{p}_b}\right\}. \end{aligned}$$

Because a nonnegative random variable can have nonpositive expectation only if said random variable is almost surely zero, it follows that the random variable $\left[1 - x(\boldsymbol{\theta})\right]\left[b - \Phi(\sigma\cdot\boldsymbol{\theta})\right]\mathbb{1}_{\sigma\cdot\boldsymbol{\theta}\leqslant \underline{p}_b} + x(\boldsymbol{\theta})\left[\Phi(\sigma\cdot\boldsymbol{\theta}) - b\right]\mathbb{1}_{\sigma\cdot\boldsymbol{\theta}>\bar{p}_b}$ is almost surely zero. Equivalently, $\left[1 - x(\boldsymbol{\theta})\right]\mathbb{1}_{\sigma\cdot\boldsymbol{\theta}\leqslant \underline{p}_b} + x(\boldsymbol{\theta})\mathbb{1}_{\sigma\cdot\boldsymbol{\theta}>\bar{p}_b}$ is almost surely zero. Thus, $x(\boldsymbol{\theta}) = x^*(\boldsymbol{\theta})$ almost surely, as required. □

Finally, we prove an efficiency ranking result

Proposition 3: *Suppose $\{F_i\}_{i\in N}$ all coincide.*

(i) If b is large enough, then any optimal allocation rule for our group setting is ex-post more efficient than any single-agent-optimal allocation rule.

(ii) If b is small enough, then any optimal allocation rule for our group setting is ex-ante less efficient than any single-agent-optimal allocation rule.

Moreover, this efficiency ranking is an ex-post ranking if $\sigma_i < \frac{2}{N+1}$ for every $i \in N$ (in particular, if σ is close enough to ω), and is not an ex-post ranking if $\sigma_i > \frac{2}{N+1}$ for some $i \in N$ (in particular if σ is close enough to δ_i).

Proof. Because $\{F_i\}_{i\in N}$ all coincide, it follows from the uniqueness part of Theorem 1 that every allocation rule in our model agrees almost everywhere with x_ω, where $\omega = (\frac{1}{N}, \ldots, \frac{1}{N})$. Moreover, because ex-ante and ex-post efficiency rankings are both invariant to probability-zero changes to an allocation rule, we can prove the result by comparing single-agent-optimal allocation rules to x_ω. In what follows, let Φ be as defined in Lemma 6; let $\underline{p}_b, \bar{p}_b$ be as defined in Lemma 8; and use the notation $\mathbf{y} >_{\mathbb{P}} \tilde{\mathbf{y}}$ to say that the random variables $\mathbf{y}$ and $\tilde{\mathbf{y}}$ have $\mathbf{y} \geqslant \tilde{\mathbf{y}}$ almost surely with $\mathbb{P}\{\mathbf{y} > \tilde{\mathbf{y}}\} > 0$.

First, let us show x_ω is ex-post more efficient than single-agent-optimal allocation rules when b is high enough. To that end, note Lemma 7 tells us every $v \in [\underline{\theta}_1, \bar{\theta}_1)$ in some neighborhood of $\bar{\theta}_1$ in Θ has $\Phi(v) > \varphi_1(\bar{\theta}_1)$. Because $\theta \mapsto \omega \cdot \varphi(\theta)$ is continuous and strictly increasing, some $b^* \in (\bar{\theta}_1, \varphi_1(\bar{\theta}_1))$ exists such that every $\theta \in \Theta \backslash \{\bar{\theta}\}$ with $\omega \cdot \varphi(\theta) > b^*$ is in said neighborhood. Now, take any $b \in [b^*, \varphi_1(\bar{\theta}_1))$ and any single-agent monotone allocation rule x; we want to show x is ex-post less efficient than x_ω. To see it, given any $\theta \in \Theta \backslash \{\bar{\theta}\}$ with $\omega \cdot \varphi(\theta) \geqslant b$, note that any $\tilde{\theta} \in \Theta$ with $\tilde{\theta} \geqslant \theta$ has $\omega \cdot \varphi(\tilde{\theta}) \geqslant b$ and so $\Phi(\sigma \cdot \tilde{\theta}) > \varphi_1(\bar{\theta}_1) > b$. Thus, any $\theta \in \Theta$ with $\omega \cdot \varphi(\theta) \geqslant b$ has $\sigma \cdot \theta > \bar{p}_b$, where $\bar{p}_b$ is as given by Lemma 8. Because φ is continuous, it follows that any $\theta \in \Theta$ with $\omega \cdot \varphi(\theta)$ close enough to b also has $\sigma \cdot \theta > \bar{p}_b$. Therefore, $x_\omega(\boldsymbol{\theta}) >_{\mathbb{P}} \mathbb{1}_{\sigma \cdot \boldsymbol{\theta} \leqslant \bar{p}_b}$. Lemma 8 then implies $x_\omega(\boldsymbol{\theta}) >_{\mathbb{P}} x(\boldsymbol{\theta})$. Finally, because $b > \bar{\theta}_1$, it follows that $s_{x_\omega}(\boldsymbol{\theta}) >_{\mathbb{P}} s_x(\boldsymbol{\theta})$. That is, x_ω is ex-post more efficient than x.

Next, specializing to the case in which each $i \in N$ has $\sigma_i < \frac{2}{N+1}$, let us show x_ω is ex-post less efficient than single-agent-optimal allocation rules when b is low enough. To that end, note Lemma 7 tells us every $\theta \in \Theta \backslash \{\underline{\theta}\}$ in some neighborhood of $\underline{\theta}$ has $\omega \cdot \varphi(\theta) > \Phi(\sigma \cdot \theta)$; let $b_* \in (\underline{\theta}_1, \bar{\theta}_1)$ be small enough that every $\theta \in \Theta \backslash \{\underline{\theta}\}$ with $\omega \cdot \theta \leqslant b_*$ or $\sigma \cdot \theta \leqslant b_*$ is in said neighborhood. Now, take any $b \in (\underline{\theta}_1, b_*]$ and any single-agent-optimal allocation rule x; we want to show x is ex-post less efficient than x_ω. To see it, given any $\theta \in \Theta \backslash \{\underline{\theta}\}$ with $\omega \cdot \varphi(\theta) \leqslant b$, note that any $\tilde{\theta} \in \Theta$ with $\tilde{\theta} \leqslant \theta$ has $\omega \cdot \tilde{\theta} \leqslant \omega \cdot \varphi(\tilde{\theta}) \leqslant b$ and so $\Phi(\sigma \cdot \tilde{\theta}) < \omega \cdot \varphi(\tilde{\theta}) \leqslant b$. Thus, any $\theta \in \Theta$ with $\omega \cdot \varphi(\theta) \leqslant b$ has $\omega \cdot \theta < \underline{p}_b$, where $\underline{p}_b$ is as given by Lemma 8. Because φ is continuous, it follows that any $\theta \in \Theta$ with $\omega \cdot \varphi(\theta)$ close enough to b also has $\omega \cdot \theta < \underline{p}_b$. Therefore, $x_\omega(\boldsymbol{\theta}) <_{\mathbb{P}} \mathbb{1}_{\omega \cdot \boldsymbol{\theta} \geqslant \underline{p}_b}$. Lemma 8 then implies $x_\omega(\boldsymbol{\theta}) <_{\mathbb{P}} x(\boldsymbol{\theta})$. Moreover, because $\underline{p}_b < \Phi(\underline{p}_b) = b$, it follows from Lemma 8 that, almost surely, either $x(\boldsymbol{\theta}) = 0$ or $\sigma \cdot \boldsymbol{\theta} < b$. It follows that $s_{x_\omega}(\boldsymbol{\theta}) <_{\mathbb{P}} s_x(\boldsymbol{\theta})$. That is, x is ex-post more efficient than x_ω.

Now, specializing to the case in which some $i \in N$ has $\sigma_i > \frac{2}{N+1}$, let us show x_ω is *not* ex-post more efficient than single-agent-optimal allocation rules when b is low enough. To that end, let note Lemma 7 delivers some $\eta \in \mathbb{R}^N_{++}$ such that $\omega \cdot \varphi(\underline{\theta} + \varepsilon\eta) < \Phi\left(\sigma \cdot (\underline{\theta} + \varepsilon\eta)\right)$ for all sufficiently small $\varepsilon > 0$. Let $\theta(\varepsilon) := \underline{\theta} + \varepsilon\eta$ for every ε, and for any $b \in (\underline{\theta}_1, \varphi_1(\bar{\theta}_1))$, let $\varepsilon_b := \frac{\bar{p}_b - \underline{\theta}_1}{\sigma \cdot \eta}$ so that $\sigma \cdot \theta(\varepsilon_b) = \bar{p}_b$. That $b = \Phi(\bar{p}_b) > \bar{p}_b$ then implies $\sigma \cdot \theta(\varepsilon_b) < b$ and $\bar{p}_b \searrow \underline{\theta}_1$ as b does, and so too does ε_b. Therefore, whenever $b \in (\underline{\theta}_1, \varphi_1(\bar{\theta}_1))$ is sufficiently small, we have that $\theta(\varepsilon_b)$ is interior in Θ and

$$\omega \cdot \varphi(\theta(\varepsilon_b)) < \Phi(\sigma \cdot \theta(\varepsilon_b)) = \Phi(\bar{p}_b) = b.$$

Let us fix such a small b and any single-agent allocation rule x, with a view to showing x is not ex-post more efficient than x_ω. Because φ is continuous, then, some $\hat{\theta} > \theta(\varepsilon_b)$ in the interior of Θ is close enough to $\theta(\varepsilon_b)$ to ensure $\omega \cdot \varphi(\hat{\theta}) < b$ and $\sigma \cdot \hat{\theta} < b$. Thus,

$$\bar{p}_b < \sigma \cdot \hat{\theta} < b \text{ and } \omega \cdot \varphi(\hat{\theta}) < b.$$

Again by continuity, every θ in some neighborhood of $\hat{\theta}$ satisfies the same three inequalities. Therefore, $\mathbb{P}\left\{\bar{p}_b < \sigma \cdot \boldsymbol{\theta} < b \text{ and } \omega \cdot \varphi(\boldsymbol{\theta}) < b\right\} > 0$. Lemma 8 then tells us $\mathbb{P}\left\{x(\boldsymbol{\theta}) = 0,\ x_\omega(\boldsymbol{\theta}) = 1, \text{ and } \omega \cdot \varphi(\boldsymbol{\theta}) < b\right\} > 0$. Thus, x is not ex-post more efficient than x_ω.

Finally, returning to the case of general σ, let us show x_ω is ex-ante less efficient than single-agent-optimal allocation rules when b is low enough. For any $b \in (\underline{\theta}_1, \varphi_1(\bar{\theta}_1))$, let $x_{\sigma,b}$ denote some single-agent-optimal allocation rule, and let

$$S(\sigma, b) := \mathbb{E}\{x_{\sigma,b}(\boldsymbol{\theta})(b - \sigma \cdot \boldsymbol{\theta})\}$$

denote the surplus it generates. We want to show that

$$S(\sigma, b) > \mathbb{E}\{x_\omega(\boldsymbol{\theta})(b - \sigma \cdot \boldsymbol{\theta})\}$$

when b is close enough to $\underline{\theta}_1$. Because $\{\boldsymbol{\theta}_i\}_{i \in N}$ are i.i.d., we know that $\mathbb{E}\{x_\omega(\boldsymbol{\theta})(b - \boldsymbol{\theta}_i)\}$ is the same for each $i \in N$, so that $\mathbb{E}\{x_\omega(\boldsymbol{\theta})(b - \sigma \cdot \boldsymbol{\theta})\} = \mathbb{E}\{x_\omega(\boldsymbol{\theta})(b - \omega \cdot \boldsymbol{\theta})\}$. Meanwhile, the ex-post efficiency ranking of this proof's second paragraph implies (by taking expectations) the ex-ante efficiency ranking $S(\omega, b) > \mathbb{E}\{x_\omega(\boldsymbol{\theta})(b - \omega \cdot \boldsymbol{\theta})\}$ for all sufficiently small b. The proposition will therefore follow if we can show $S(\sigma, b) \geqslant S(\omega, b)$ when $b \in (\underline{\theta}_1, \varphi_1(\bar{\theta}_1))$ is close enough to $\underline{\theta}_1$. We now pursue this ranking.

Let $\gamma := \frac{f_1(\underline{\theta}_1)^N}{(N-1)!} \prod_{i \in N} \frac{1}{\sigma_i} > 0$. In what follows, we use Lemma 6's calculations of the behavior of G, g, and Φ around $\underline{\theta}_1$. First, for $p \in (\underline{\theta}_1, \bar{\theta}_1)$, we have

$$\begin{aligned}
\tfrac{1}{(p-\underline{\theta}_1)^{N+1}} \mathbb{E}\left[\mathbb{1}_{\sigma \cdot \boldsymbol{\theta} \leqslant p}\left(\sigma \cdot \boldsymbol{\theta} - \underline{\theta}_1\right)\right] &= \tfrac{1}{(p-\underline{\theta}_1)^{N+1}} \int_{\underline{\theta}_1}^{p} (v - \bar{\theta}_1) g(v) \, \mathrm{d}v \\
&= \tfrac{\gamma}{(p-\underline{\theta}_1)^{N+1}} \int_{\underline{\theta}_1}^{p} (v - \bar{\theta}_1)^N \, \mathrm{d}v \\
&\quad + \tfrac{1}{p-\underline{\theta}_1} \int_{\underline{\theta}_1}^{p} \left(\tfrac{v-\underline{\theta}_1}{p-\underline{\theta}_1}\right)^N \left[\tfrac{g(v)}{(v-\underline{\theta}_1)^{N-1}} - \gamma\right] \mathrm{d}v \\
&= \tfrac{\gamma}{N+1} + \tfrac{1}{p-\underline{\theta}_1} \int_{\underline{\theta}_1}^{p} \left(\tfrac{v-\underline{\theta}_1}{p-\underline{\theta}_1}\right)^N \left[\tfrac{g(v)}{(v-\underline{\theta}_1)^{N-1}} - \gamma\right] \mathrm{d}v \\
&\xrightarrow{p \searrow \underline{\theta}_1} \tfrac{\gamma}{N+1}.
\end{aligned}$$

Moreover, we have

$$\tfrac{\Phi(p)-\underline{\theta}_1}{p-\underline{\theta}_1} = 1 + \tfrac{\Phi(p)-p}{p-\underline{\theta}_1} \xrightarrow{p \searrow \underline{\theta}_1} 1 + \tfrac{1}{N} = \tfrac{N+1}{N}.$$

Therefore,

$$\begin{aligned}
&\tfrac{1}{[\Phi(p)-\underline{\theta}_1]^{N+1}} \mathbb{E}\left\{\mathbb{1}_{\sigma \cdot \boldsymbol{\theta} \leqslant p}\left[\Phi(p) - \sigma \cdot \boldsymbol{\theta}\right]\right\} \\
= \; & \left[\tfrac{p-\underline{\theta}_1}{\Phi(p)-\underline{\theta}_1}\right]^N \tfrac{G(p)}{(p-\underline{\theta}_1)^N} - \left[\tfrac{p-\underline{\theta}_1}{\Phi(p)-\underline{\theta}_1}\right]^{N+1} \tfrac{1}{(p-\underline{\theta}_1)^{N+1}} \mathbb{E}\left[\mathbb{1}_{\sigma \cdot \boldsymbol{\theta} \leqslant p}\left(\sigma \cdot \boldsymbol{\theta} - \underline{\theta}_1\right)\right] \\
\xrightarrow{p \searrow \underline{\theta}_1} \; & \left(\tfrac{N}{N+1}\right)^N \tfrac{\gamma}{N} - \left(\tfrac{N}{N+1}\right)^{N+1} \tfrac{\gamma}{N+1} \\
= \; & \tfrac{N^{N-1}}{(N+1)^{N+2}(2N+1)} \gamma.
\end{aligned}$$

Meanwhile, any $b \in (\underline{\theta}_1, \bar{\theta}_1)$ has $\underline{\theta}_1 < \underline{p}_b \leqslant \bar{p}_b < \Phi(\bar{p}_b) = b$, so that $\underline{p}_b, \bar{p}_b \searrow \underline{\theta}_1$ as b does. We can therefore specialize the above calculation to deduce

$$\tfrac{1}{(b-\underline{\theta}_1)^{N+1}} \mathbb{E}\left[\mathbb{1}_{\sigma \cdot \boldsymbol{\theta} \leqslant \underline{p}_b}\left(b - \sigma \cdot \boldsymbol{\theta}\right)\right] \text{ and } \tfrac{1}{(b-\underline{\theta}_1)^{N+1}} \mathbb{E}\left[\mathbb{1}_{\sigma \cdot \boldsymbol{\theta} \leqslant \bar{p}_b}\left(b - \sigma \cdot \boldsymbol{\theta}\right)\right]$$

both converge to $\frac{N^{N-1}}{(N+1)^{N+2}(2N+1)}\gamma$ as $b \searrow \underline{\theta}_1$. Now, because every $v \leqslant \bar{p}$ has $v \leqslant \Phi(\bar{p}) = b$, Lemma 8 implies

$$\mathbb{1}_{\sigma\cdot\boldsymbol{\theta}\leqslant \underline{p}_b}\left(b - \sigma\cdot\boldsymbol{\theta}\right) \leqslant x_{\sigma,b}(\boldsymbol{\theta})\left(b - \sigma\cdot\boldsymbol{\theta}\right) \leqslant \mathbb{1}_{\sigma\cdot\boldsymbol{\theta}\leqslant \bar{p}_b}\left(b - \sigma\cdot\boldsymbol{\theta}\right),$$

so that

$$\begin{aligned}\tfrac{1}{(b-\underline{\theta}_1)^{N+1}}S(\sigma,b) &\xrightarrow{b\searrow\underline{\theta}_1} \tfrac{N^{N-1}}{(N+1)^{N+2}(2N+1)}\gamma \\ &= \tfrac{N^{N-1}f_1(\underline{\theta}_1)^N}{(N-1)!(N+1)^{N+2}(2N+1)}\prod_{i\in N}\frac{1}{\sigma_i} \\ &= \frac{\tilde{\gamma}}{\prod_{i\in N}\sigma_i}, \text{ where}\end{aligned}$$

where $\tilde{\gamma} := \frac{N^{N-1}f_1(\underline{\theta}_1)^N}{(N-1)!(N+1)^{N+2}(2N+1)} > 0$. Note that this calculation specializes to

$$\tfrac{1}{(b-\underline{\theta}_1)^{N+1}}S(\omega,b) \xrightarrow{b\searrow\underline{\theta}_1} \frac{\tilde{\gamma}}{\left(\frac{1}{N}\right)^N}.$$

Therefore,

$$\frac{S(\sigma,b)}{S(\omega,b)} \xrightarrow{b\searrow\underline{\theta}_1} \frac{\left(\frac{1}{N}\right)^N}{\prod_{i\in N}\sigma_i} = \left[\frac{\frac{1}{N}\sum_{i\in N}\sigma_i}{\left(\prod_{i\in N}\sigma_i\right)^{\frac{1}{N}}}\right]^N.$$

The inequality of arithmetic and geometric means (AM-GM) tells us that this limit ratio is strictly greater than 1 if $\sigma \neq \omega$, so that $S(\sigma,b) \leqslant S(\omega,b)$ when $b \in (\underline{\theta}_1, \bar{\theta}_1)$ is sufficiently small. The proposition follows. □

C. Supporting analysis for Section 7

C.1. Dominant strategies

In light of the revelation principle, we formalize more demanding incentive constraints through direct mechanisms below.

Definition 7: *Say a mechanism (x,m) is **dominant-strategy incentive compatible (DIC)** if*

$$\theta_i \in \operatorname{argmax}_{\hat{\theta}_i\in\Theta_i}\left\{m(\hat{\theta}_i,\theta_{-i}) - \theta_i x(\hat{\theta}_i,\theta_{-i})\right\}, \ \forall i\in N, \ \forall\theta\in\Theta; \qquad \text{(DIC)}$$

A mechanism is DIC if an agent finds truthful reporting dominant in the direct revelation game; that is, he would willingly report truthfully even if he knew others' reported types.

We showed in Lemma 1 that for a given allocation rule, interim montonicity is equivalent to BIC implementability. Said differently, we showed that being able to BIC-implement an allocation rule with agent-specific transfers is equivalent to being able to do so with only collective transfers. Moreover, Theorem 1 explicitly characterizes the allocation rule from optimal BIC and IR mechanisms, showing it stipulates trade if and only if the benefit to the buyer exceeds the player-weighted virtual cost. Notice, though, that this allocation rule is monotone in the agents' *profile* of types. If our seller could engage in agent-specific transfers, such monotonicity would render the same allocation rule DIC implementable too. Therefore, a natural conjecture is that (as in single-good auction settings) our seller can attain DIC at no additional cost.

The following result shows the above natural conjecture is false: the restriction to DIC mechanisms is with loss of optimality for the seller. Optimal mechanisms must leverage agents' uncertainty about others' realized types.

Proposition 4 (Dominance binds)**:** *If at least two $j \in N$ have $b < \bar{\theta}_j$, then no DIC mechanism is optimal.*

The proof of Proposition 4 leverages the fact that the essentially unique optimal allocation rule is bang-bang—every type profile leads to a deterministic trade outcome. The main thrust of our proof is a structural lemma that characterizes the full class of DIC bang-bang mechanisms, as summarized in two properties. The first property concerns the transfer: It can be decomposed into a price (p) that will be paid if and only if trade occurs and a subsidy that will be paid to the sellers whether or not trade occurs. The second property gives a representation of the allocation rule: trade is determined by the price and $\mathcal{J}$, a collection of subsets of N such that the good is sold if and only if, for some $J \in \mathcal{J}$, every agent in J agrees to the purchase at price p.

The proof of the structural lemma proceeds in two steps. First, we show the transfer rule is constant among type profiles leading to certain trade, and constant among type profiles leading to non-trade, which leads directly to the price/subsidy form. To prove this property, consider any two type profiles θ and θ' such that $x(\theta) = x(\theta')$; say this trade probability is equal to 1, the alternative case being analogous. Letting θ^* be a type profile that is coordinatewise higher than both θ and θ', we construct a finite sequence of type profiles such that the first type profile in the sequence is θ and the last is θ^*, the type profiles get coordinatewise higher as the sequence progresses, and consecutive entries in the sequence differ in only one agent's type. But then, because DIC (for the agent whose type is raised in a given increment of the sequence) implies x must be monotone, it follows that every type profile in the sequence generates probability 1 of trade. Hence, DIC (again, for the agent whose type is incremented) implies consecutive sequence members yield an identical transfer. A symmetric argument applies to θ', so that $m(\theta') = m(\theta^*) = m(\theta)$. Hence, any DIC-implementing transfer takes the given price-subsidy form. The second property that the structural lemma establishes is the structure on the allocation rule. Given that the mechanism is incentive-equivalent to a collective posted price of p, DIC implies (fixing a realization of others' types) the trade decision must be identical for all types of agent i below p and for all types of agent i above p. Hence, the allocation rule is essentially a decreasing $\{0, 1\}$-valued transformation of the vector-valued function $\theta \mapsto (\mathbb{1}_{\theta_j \geqslant p})_{j \in N}$. The "coalitional" property amounts to a more explicit description of such functions.

C.2. Ex-post participation

Let us formulate a notion of ex-post individual rationality. As usual, we do so for direct mechanisms—for convenience and without loss. Say a mechanism (x, m) is **ex-post indiviually rational (epIR)** if $m(\theta) - \theta_i x(\theta) \geqslant 0$ for every $\theta \in \Theta$ and $i \in N$.

The following lemma reduces IC-and-epIR implementability to the study of the allocation rule and agents' interim values.

Lemma 9: *Given allocation rule x and $\underline{U} \in \mathbb{R}^N$, the following are equivalent:*

(i) Some transfer rule m exists such that the mechanism (x, m) is IC and epIR and gives interim utility $\underline{U}_i$ to type $\bar{\theta}_i$ of each agent i.

(ii) The allocation rule x is interim monotone, the quantities $\{\underline{U}_i + \mathbb{E}[x(\boldsymbol{\theta})\boldsymbol{\varphi}_i]\}_{i \in N}$ all coincide, and every agent $i \in N$ and type $\theta_i \in \Theta_i$ have (letting $X_i := X_i^x$):

$$\int_{\theta_i}^{\bar{\theta}_i} X_i(\tilde{\theta}_i)\, \mathrm{d}\tilde{\theta}_i \geqslant \mathbb{E}\left[x(\theta_i, \boldsymbol{\theta}_{-i}) \left(\max_{j \in N \setminus \{i\}} \boldsymbol{\theta}_j - \theta_i\right)_+\right] - \underline{U}_i.$$

The above lemma shows how an epIR constraint can be formulated directly over allocation rules. The conditions given in the lemma amount to saying that, when the interim transfer rules are solved out from the allocation rule via the sellers' IC constraint and revenue equivalence, seller i's interim transfer is at least her interim expectation of the minimum transfer required to stop all sellers from walking away. This condition is trivially necessary, but we constructively show it to be sufficient too.

As a demonstration that the above characterization is useful, let us apply it to derive a sufficient condition for epIR to be without loss of optimality for our buyer.

Proposition 5 (Sufficient condition for epIR)**:** *Suppose $N = 2$ and $F_1 = F_2$. Then, some optimal mechanism is epIR if the virtual cost $\boldsymbol{\varphi}_1$ admits a nonincreasing density on its support.*

In particular, this proposition applies to the special case of Example 1 in which $\alpha_1 = \alpha_2 \leqslant 1$.

C.3. Pareto-optimal Mechanisms

Recall, a **Pareto-optimal mechanism**, is an IC and IR mechanism such that no alternative IC and IR mechanism delivers a weakly higher buyer profit, and a weakly higher agent i value for each agent i, with at least one of these $N + 1$ inequalities strict. Then, a **Pareto-optimal allocation** is any allocation rule x such that (x, m) is a Pareto-optimal mechanism for some m. In this subsection, we provide a characterization of which mechanisms are Pareto optimal, and explain the reasoning behind it.

Following standard arguments, one can show that any Pareto optimal mechanism can be represented as a solution to a program maximizing a weighted sum of values of the $N + 1$ individuals (N sellers and the buyer), and—because increasing the transfer by a constant preserves all constraints—the Pareto weight on the buyer (normalized to 1) is at least as high as the sum of weights $\{\lambda_i\}_{i \in N}$ on the agents. Conversely, we observe that any interim monotone allocation rule that maximizes such a weighted sum is Pareto optimal.[28]

We can therefore solve a family of programs much like the buyer's problem (BP), but with modified objective, to trace out the entire Pareto frontier. Vectors λ of Pareto weights are paired with endogenous allocation weights ω to describe the following class of allocation rules.

Definition 8: *Let $\Delta(2N)$ denote the set of all (λ, ω) with $\lambda, \omega \in \mathbb{R}^N_+$ and*

$$\sum_{i \in N} (\lambda_i + \omega_i) = 1.$$

*For any such (λ, ω), let the (λ, ω)-**allocation rule**, denoted by $x_{\lambda,\omega}$, be given by*

$$x_{\lambda,\omega}(\theta) := \mathbb{1}_{\lambda \cdot \theta + \omega \cdot \varphi(\theta) \leqslant b}.$$

We now state our main characterization theorem of this section. It characterizes Pareto-optimal allocation rules as those that weigh the benefit of trade against a weighted average of its *actual and virtual* costs.

Theorem 3 (Pareto-optimal allocations)**:** *The (λ, ω)-allocation rule is Pareto optimal for any $(\lambda, \omega) \in \Delta(2N)$ satisfying the following two equivalent conditions:*

1. $\omega \in \operatorname{argmin}_{\tilde{\omega}:\ (\lambda,\tilde{\omega}) \in \Delta N} \mathbb{E}[(b - \lambda \cdot \boldsymbol{\theta} - \tilde{\omega} \cdot \boldsymbol{\varphi})_+]$.
2. $\operatorname{supp}(\omega) \subseteq \operatorname{argmax}_{i \in N} \mathbb{E}[\boldsymbol{\varphi}_i \mid \lambda \cdot \boldsymbol{\theta} + \omega \cdot \boldsymbol{\varphi} \leqslant b]$.

Moreover, every Pareto-optimal allocation rule is essentially of this form.[29]

Finally, analogous to Proposition 2, it is natural to explore whether some Pareto optimal mechanism can be implemented via posted prices. That is, one can ask whether the suboptimality of collective posted prices was an artefact of our focus on the buyer-optimal mechanisms. However, as we show in Proposition 7, no Pareto optimal mechanism can be implemented as a collective posted price mechanism (if trade is neither unambiguously efficient nor inefficient). Therefore, this intuitive class of simple mechanisms is strictly suboptimal regardless of whether one favors the buyer or the seller.

C.4. Pre-market trade of land shares

Consider a game that extends our model by adding a pre-market phase in which the agents trade their shares. The buyer then observes the agents' shares and chooses a profit-maximizing mechanism. We study two different versions of this

[28] The latter observation would be obvious if all weights were strictly positive. We show it holds in our setting even with some zero weights, because the optimizer is essentially unique.

[29] The proof also establishes that, if the (λ, ω)- and $(\lambda, \tilde{\omega})$-allocation rules are both Pareto optimal, then they essentially coincide.

game: one in which agents must be paid proportionally to their shares, as we have required throughout this paper, and one without this constraint. We show that in the first regime, agents do not benefit from trade, but in the second regime they do.

We start with some notation and then define the game. Let $\Sigma = \{\sigma = (\sigma_1, \ldots, \sigma_N) \in (0,1)^N : \sum_{i \in N} \sigma_i = 1\}$ be the set of possible profiles of shares that the agents might have. Agents' initial shares $\sigma = (\sigma_1, \ldots, \sigma_N) \in \Sigma$ are fixed but unknown to the buyer. For the present analysis, we assume $\{\boldsymbol{\theta}_i\}_{i \in N}$ are i.i.d. with distribution $F_i = F_1$. A mechanism is a profile of functions $(x, m_1, \ldots, m_N)$ where $x : \Theta \to [0,1]$ is the allocation rule and $m_i : \Theta \to \mathbb{R}$ is the transfer rule of agent i. Notice that here we are considering a more general class of mechanisms than the one studied throughout this paper because here we impose no structure relating the transfers of different agents to each other. Play proceeds as follows.

1. Seller 1 proposes shares $\hat{\sigma} \in \Sigma$ and lump-sum net transfers $\hat{\tau} \in \mathbb{R}^N$ with $\mathbb{1} \cdot \tau = 0$, and then the other agents sequentially vote on whether to accept the proposal. Realized shares σ' and transfers τ are then equal to $\hat{\sigma}$ and $\hat{\tau}$ if all accept the proposal, and equal to σ and $\vec{0}$ if anyone rejects.[30]
2. The buyer observes σ' and chooses a mechanism (x, m).

   In the **discriminatory-pricing** regime, the buyer can choose any mechanism. In the **uniform-pricing** regime, the buyer can choose any uniform-pricing mechanism—that is a mechanism $(x, m_1, \ldots, m_N)$ in which agents are paid proportionally to their chosen shares, $m_i = \sigma_i' \sum_{j \in N} m_j$.
3. Each agent i privately learns his type $\boldsymbol{\theta}_i$ drawn independently from F, decides whether to participate in the mechanism, and if he participates, what type $\hat{\theta}_i$ to report.
4. The good is sold with probability $x(\hat{\theta})$ and each agent i is paid $m_i(\hat{\theta})$. The payoff of agent i is then $\tau_i + m_i(\hat{\theta}) - \sigma_i' \boldsymbol{\theta}_i x(\hat{\theta})$.[31]

Our solution concept, which we simply call equilibrium for brevity, is perfect Bayesian equilibrium in which:

- Players do not signal what they do not know—hence, play from stage 2 onward corresponds to the mechanism design problem with shares σ' and type distribution $\boldsymbol{\theta} \sim \bigotimes_{i \in N} F_1$;
- The buyer-optimal mechanism is offered, and sellers all participate and truthfully report their types, for any realized shares σ'.[32]

[30] The specific bargaining protocol is immaterial, though we fix one for concreteness. What matters for our analysis is that the realized shares σ' are set to maximize sellers' sum of payoffs.

[31] A more natural specification would be $\tau_i + m_i(\hat{\theta}) + \sigma_i' \boldsymbol{\theta}_i [1 - x(\hat{\theta})]$, which explicitly takes into account that seller i has value $\sigma_i' \boldsymbol{\theta}_i$ (which depends on σ') if he retains his land. Because we maintain i.i.d. types for the present analysis, though, the difference $\sum_{i \in N} \mathbb{E}[\sigma_i' \boldsymbol{\theta}_i]$ does not vary with σ', and so will not affect pre-market non-manipulability. We therefore maintain the payoff specification of our main model for ease of comparison.

[32] The latter feature simplifies the analysis, but is not necessary. If we removed this equilibrium refinement, but enriched the model to allow the buyer to pay sellers even when some sellers do not participate, our results would remain unchanged.

We say that the game is **pre-market non-manipulable** if some equilibrium exists in which the agents choose $\sigma' = \sigma$ in the first stage and the buyer attains her optimal value (among all IC and IR mechanisms for shares σ). The following result shows that a uniform-pricing regime generates such non-manipulability.

Proposition 6 (Uniform pricing avoids pre-market trade)**:** *Suppose sellers' types are i.i.d. Then, the uniform-pricing game is pre-market non-manipulable, but the discriminatory-pricing game need not be.*

The proof shows that sellers' total surplus is invariant to their shares under uniform pricing, and shows a numerical example (an instance of Example 1) in which the sellers increase their total surplus by making their shares more symmetric.

D. Proofs for Section C

D.1. Proofs for Section C.1

Lemma 10: *Suppose that (x, m) is a DIC mechanism and $\theta, \theta' \in \Theta$ have $x(\theta) = x(\theta') \in \{0, 1\}$. Then $m(\theta) = m(\theta')$.*

Proof. Define $\theta^* := \theta \vee \theta'$ if $x(\theta) = x(\theta') = 0$, and $\theta^* := \theta \wedge \theta'$ if $x(\theta) = x(\theta') = 1$. We will observe that $m(\theta) = m(\theta^*) = m(\theta')$; by symmetry, it suffices to show $m(\theta) = m(\theta^*)$. To show it, define the type profile

$$\theta^\ell := (\theta_i^* \mathbb{1}_{i \leqslant \ell} + \theta_i \mathbb{1}_{i > \ell})_{i \in N} \in \Theta \text{ for each } \ell \in \{0, \ldots, N\} = N \cup \{0\}.$$

Observe, either $\theta^0 \leqslant \cdots \leqslant \theta^N$ and $x(\theta^0) = 0$, or $\theta^0 \geqslant \cdots \geqslant \theta^N$ and $x(\theta^0) = 1$. In either case, because x is weakly decreasing (due to DIC) and can only take values in $[0, 1]$, it follows by induction that $x(\theta^0) = \cdots = x(\theta^N)$. For each $i \in N$, because θ^i and θ^{i-1} differ only in the i coordinate and $x(\theta^{i-1}) = x(\theta^i)$, it follows from DIC (for agent i) that $m(\theta^{i-1}) = m(\theta^i)$. Thus, $m(\theta) = m(\theta^0) = \cdots = m(\theta^N) = m(\theta^*)$, as desired. ☐

Definition 9: *Say a mechanism (x, m) or an allocation rule x is **bang-bang** if $x(\boldsymbol{\theta}) \in \{0, 1\}$ almost surely.*

Lemma 11: *Suppose (x, m) is a DIC bang-bang mechanism. Then, some $p, s \in \mathbb{R}$ and $\mathcal{J} \subseteq 2^N$ exist such that, almost surely:*

(i) $m(\boldsymbol{\theta}) = px(\boldsymbol{\theta}) + s$;

(ii) $x(\boldsymbol{\theta}) = \mathbb{1}_{\bigcup_{J \in \mathcal{J}} \bigcap_{j \in J} \{\boldsymbol{\theta}_j \leqslant p\}}$.

Moreover, we may assume without loss that no two members of $\mathcal{J}$ are nested, and that $\underline{\theta}_j < p < \bar{\theta}_j$ for each $j \in \bigcup \mathcal{J}$.

Proof. Fix a DIC mechanism (x, m) such that $x(\boldsymbol{\theta})$ almost surely in $\{0, 1\}$. By Lemma 10, some constants $m^L, m^H \in \mathbb{R}$ exists such that $m(\theta) = m^L$ [resp. m^H] for every $\theta \in \Theta$ with $x(\theta) = 0$ [resp. 1]. So, defining $p := m^H - m^L \geqslant 0$ and letting $s := m^L$, we have $m(\boldsymbol{\theta}) = px(\boldsymbol{\theta}) + s$ whenever $x(\boldsymbol{\theta}) \in \{0, 1\}$, an almost sure event.

Now, modifying x on an a null set, and similarly modifying the transfer rule to maintain $m = px + s$, we may assume without loss that x is (statewise) $\{0,1\}$-valued.[33] DIC of the modified mechanism follows from DIC of the original one.

Next, we show x has the desired structure. Given an agent $i \in N$ and type realization $\theta_i \in \Theta_i$, his payoff from a reported type profile of $\hat{\theta}$ is $(p - \theta_i)x(\hat{\theta}) - s$, which is strictly increasing [resp. decreasing] in $x(\hat{\theta})$ if $\theta_i < p$ [resp. $\theta_i > p$]. Hence, given $\theta_{-i} \in \Theta_{-i}$ DIC implies that one the following three possibilities holds: $x(\cdot, \theta_{-i}) = 1$ globally, $x(\cdot, \theta_{-i}) = 0$ globally, or $x(\theta_i, \theta_{-i}) = 1$ [resp. $x(\theta_i, \theta_{-i}) = 0$] for each $\theta_i \in \Theta_i$ with $\theta_i < p$ [resp. $\theta_i > p$]. Hence, letting $\tilde{\Theta} := \prod_{i \in N}[\Theta_i \backslash \{p\}]$, some $y : \{0,1\}^N \to \{0,1\}$ exists such that every $\theta \in \tilde{\Theta}$ has $x(\theta) = y\left((\mathbb{1}_{\theta_i \leqslant p})_{i \in N}\right)$. Moreover, we may assume without loss that y is constant in its i coordinate if $p \leqslant \underline{\theta}_i$ or $p \geqslant \bar{\theta}_i$ for $i \in N$. Then, monotonicity of x implies y is monotone too. If we let $\tilde{\mathcal{J}} := \{J \subseteq N : \ y(\mathbb{1}_J) = 1\}$, then, $x(\boldsymbol{\theta}) = \mathbb{1}_{\bigcup_{\tilde{J} \in \tilde{\mathcal{J}}} \bigcap_{j \in \tilde{J}} \{\boldsymbol{\theta}_j \geqslant p\}}$ almost surely.

Define $\hat{\mathcal{J}} := \left\{\{j \in \tilde{J} : \ \underline{\theta}_j < p\} : \ \tilde{J} \in \tilde{\mathcal{J}} \text{ with } \bar{\theta}_j > p \ \forall j \in \tilde{J}\right\}$. Then, $x(\boldsymbol{\theta}) = \mathbb{1}_{\bigcup_{\hat{J} \in \hat{\mathcal{J}}} \bigcap_{j \in \hat{J}} \{\boldsymbol{\theta}_j \leqslant p\}}$ almost surely, and $\underline{\theta}_j < p < \bar{\theta}_j$ for each $j \in \bigcup \hat{\mathcal{J}}$. Finally, let $\mathcal{J} := \{J \in \hat{\mathcal{J}} : \ \nexists \hat{J} \in \hat{\mathcal{J}} \text{ with } \hat{J} \subsetneq J\}$. Then, $x(\boldsymbol{\theta}) = \mathbb{1}_{\bigcup_{J \in \mathcal{J}} \bigcap_{j \in J} \{\boldsymbol{\theta}_j \leqslant p\}}$ almost surely, $\underline{\theta}_j < p < \bar{\theta}_j$ for each $j \in \bigcup \mathcal{J}$, and no two members of $\mathcal{J}$ are nested. Thus, $(p, s, \mathcal{J})$ is as required. □

Proof of Proposition 4. Let x be any DIC-implementable allocation rule. First, let $(p, s, \mathcal{J})$ be as delivered by Lemma 11 (with $\mathcal{J}$ chosen so that the "moreover" part of the lemma holds).

Let us show x it cannot be optimal. First, if $\mathcal{J}$ is either $\varnothing$ or $\{\varnothing\}$, then $\mathbb{E}[x(\boldsymbol{\theta})] \in \{0,1\}$, and so Theorem 1 says (given that $\underline{\theta}_i < b < \varphi_i(\bar{\theta}_i)$ for each $i \in N$) that x is not optimal. Second, if $i \in J \in \mathcal{J}$, then X_i^x is discontinuous at $p \in (\underline{\theta}_i, \bar{\theta}_i)$, implying (by Lemma 5 and since the last assertion of Theorem 1 tells us ω is nontrivial) that x is not an optimal allocation rule. □

D.2. Proofs for Section C.2

Proof of Lemma 9. First, define the transfer rule $\underline{m}$ by letting $\underline{m}(\theta) := \max_{i \in N} \theta_i x(\theta)$. Note that a transfer rule m is such that (x, m) is epIR if and only if $m \geqslant \underline{m}$.

Now, for each agent i, let M_i^* be as defined in the proof of Lemma 1. As explained in that proof, given a transfer rule m, the mechanism (x, m) is IC and gives high-type utility $\underline{U}_i$ to each agent i if and only if x is interim monotone and $M_i^m = M_i^* + \underline{U}_i$ for each agent i. So condition (i) holds if and only if x is interim monotone and some transfer rule m exists such that $m \geqslant \underline{m}$ and $M_i^m = M_i^* + \underline{U}_i$ for each agent i. Observe, the last condition also implies that $\{\underline{U}_i + \mathbb{E}\left[x(\boldsymbol{\theta})\boldsymbol{\varphi}_i\right]\}_{i \in N}$ all coincide because (as noted in the proof of Lemma 1) each $i \in N$ has $\mathbb{E}\left[M_i^*(\boldsymbol{\theta})\right] = \mathbb{E}\left[x(\boldsymbol{\theta})\boldsymbol{\varphi}_i\right]$.

To prove the lemma, it therefore suffices to show the following: Given a profile $(M_i)_{i \in N}$ of interim transfer rules such that $\{\mathbb{E}\left[M_i(\boldsymbol{\theta}_i)\right]\}_{i \in N}$ all coincide, the

[33] For instance, if x is almost-surely constant, we can modify it to be constant; and otherwise, we can replace x with $\theta \mapsto \mathbb{1}_{x(\theta) > 0}$.

following are equivalent:

- Some transfer rule $m \geqslant \underline{m}$ has $M_i^m = M_i$ for each agent i;
- Each agent i has $M_i \geqslant M_i^{\underline{m}}$.

To see this equivalence delivers the lemma, note that the the inequality $M_i(\theta_i) \geqslant M_i^{\underline{m}}(\theta_i)$ rearranges to exactly the inequality in the lemma's statement.

The first bullet immediately implies the second, because integration is monotone. To pursue the converse, suppose the second bullet holds, that is, $M_i \geqslant M_i^{\underline{m}}$ for each agent i. Let $\mathtt{m} := \mathbb{E}\left[M_i(\boldsymbol{\theta}_i) - M_i^{\underline{m}}(\boldsymbol{\theta}_i)\right]$, which is the same nonnegative quantity for every agent i. If $\mathtt{m} > 0$, then the transfer rule m given by

$$m(\theta) := \underline{m}(\theta) + \mathtt{m}^{-(N-1)} \prod_{i \in N} \left[M_i(\theta_i) - M_i^{\underline{m}}(\theta_i)\right]$$

is as desired; and if $\mathtt{m} = 0$, then the transfer rule m given by

$$m(\theta) := \underline{m}(\theta) + \max_{i \in N} \left[M_i(\theta_i) - M_i^{\underline{m}}(\theta_i)\right]$$

is as desired. Indeed, in both cases, $m \geqslant \underline{m}$ by construction; in the $\mathtt{m} > 0$ case, $M_i^m = M_i$ because agents' types are independent; and in the $\mathtt{m} = 0$ case each $M_i^m = M_i$ because types are independent and each agent $j \neq i$ has $M_j^m(\boldsymbol{\theta}_j) = M_j^{\underline{m}}(\boldsymbol{\theta}_j)$ almost surely. □

The following lemma simplifies the characterization of the previous lemma to understand when epIR is without loss of optimality in our buyer's problem, in the two-agent symmetric case.

Lemma 12: *Suppose $N = 2$ and $F_1 = F_2$. Let $\bar{z} := \boldsymbol{\varphi}_1(\bar{\theta}_1)$, let G denote the CDF of $\boldsymbol{\varphi}_1$, let $\lambda := \varphi_1^{-1} : [\underline{\theta}_1, \bar{z}] \to \Theta_1$ extended to be constant above $\bar{z}$, and let $\hat{(\cdot)} : \mathbb{R} \to \mathbb{R}$ be given by $\hat{y} := 2b - y$ (the reflection across b).*

Then, some optimal mechanism is epIR if and only if every $z \in [\underline{\theta}_1, b]$ has

$$G(z)\left[\lambda(\hat{z}) - \lambda(z)\right] + \int_z^{\hat{z}} \left[\lambda(\hat{y}) - \lambda(y)\right] \,\mathrm{d}G(y) + \int_{\hat{z}}^{\bar{z}} G(\hat{y}) \,\mathrm{d}\lambda(y) \geqslant 0.$$

Proof. Let x denote the $(\frac{1}{2}, \frac{1}{2})$-weighted allocation rule. From Theorem 1, we know that an optimal mechanism exists with allocation rule x and IR binding for both agents. Letting $X_1 := X_1^x$, define the function $\eta : \Theta_1 \to \mathbb{R}$ via

$$\eta(\theta_1) := \int_{\theta_1}^{\bar{\theta}_1} X_1(\tilde{\theta}_1) \,\mathrm{d}\tilde{\theta}_1 - \mathbb{E}\left[x(\theta_1, \boldsymbol{\theta}_2)\,(\boldsymbol{\theta}_2 - \theta_1)_+\right]$$

for each $\theta_1 \in \Theta_1$. Given symmetry and given Lemma 9, we know some optimal mechanism is epIR if the function η is globally nonnegative. Conversely, because (given Theorem 1) any optimal mechanism has binding IR and has an allocation rule that agrees with x almost surely, and because (as will be clear from our analysis below) η is continuous, it follows that nonnegativity of the function η is also necessary for some optimal mechanism to be epIR. The lemma will then follow if we show the inequalities in the lemma's statement characterize global nonnegativity of η.

Observe now that $G, \lambda, \hat{(\cdot)}$ are all continuous and monotone, and λ is strictly increasing. To see when $\eta(\theta_1) \geqslant 0$ for every $\theta_1 \in \Theta_1$, we equivalently characterize when $\eta(\lambda(z)) \geqslant 0$ for every $z \in [\underline{\theta}_1, \bar{z}]$.

Now, let us compute η more explicitly. Any $z \in [\underline{\theta}_1, \bar{z}]$ has

$$x(\lambda(z), \boldsymbol{\theta}_2) = \mathbb{1}_{\frac{1}{2}\varphi_1(\lambda(z))+\frac{1}{2}\boldsymbol{\varphi}_2 \leqslant b} = \mathbb{1}_{\boldsymbol{\varphi}_2 \leqslant \hat{z}},$$

and so (extending λ to equal $\bar{\theta}_1$ above $\bar{z}$),

$$\begin{aligned}
\eta(\lambda(z)) &= \int_{\lambda(z)}^{\lambda(\bar{z})} X_1(\tilde{\theta}_1)\ \mathrm{d}\tilde{\theta}_1 - \mathbb{E}\left\{\mathbb{1}_{\boldsymbol{\varphi}_2 \leqslant \hat{z}} \left[\lambda(\boldsymbol{\varphi}_2) - \lambda(z)\right] \mathbb{1}_{\lambda(\boldsymbol{\varphi}_2) > \lambda(z)}\right\} \\
&= \int_z^{\bar{z}} X_1(\lambda(y))\ \mathrm{d}\lambda(y) - \mathbb{E}\left\{\mathbb{1}_{z<\boldsymbol{\varphi}_2 \leqslant \hat{z}} \left[\lambda(\boldsymbol{\varphi}_2) - \lambda(z)\right]\right\} \\
&= \int_z^{\bar{z}} G(\hat{y})\ \mathrm{d}\lambda(y) - \mathbb{1}_{z<\hat{z}} \int_z^{\hat{z}} \left[\lambda(y) - \lambda(z)\right]\ \mathrm{d}G(y).
\end{aligned}$$

Now, observe $\mathbb{1}_{z<\hat{z}} = \mathbb{1}_{z<b}$. Thus, if $z \geqslant b$, we have $\eta(\lambda(z)) = \int_z^{\bar{z}} G(\hat{y})\ \mathrm{d}\lambda(y) \geqslant 0$.

So let us focus on the remaining case of $z < b$. In this case, note that $\hat{(\cdot)}$ is a continuous decreasing bijection on $[z, \hat{z}]$, and so

$$\begin{aligned}
\int_z^{\hat{z}} G(\hat{y})\ \mathrm{d}\lambda(y) &= \int_{\hat{z}}^{z} G(y)\ \mathrm{d}[\lambda \circ \hat{(\cdot)}](y) \\
&= \left[G(y)\lambda(\hat{y})\right]_{y=\hat{z}}^{z} - \int_{\hat{z}}^{z} \lambda(\hat{y})\ \mathrm{d}G(y) \\
&= \left[G(z)\lambda(\hat{z}) - G(\hat{z})\lambda(z)\right] + \int_z^{\hat{z}} \lambda(\hat{y})\ \mathrm{d}G(y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\eta(\lambda(z)) &= \int_z^{\bar{z}} G(\hat{y})\ \mathrm{d}\lambda(y) - \int_z^{\hat{z}} \left[\lambda(y) - \lambda(z)\right]\ \mathrm{d}G(y) \\
&= \int_z^{\hat{z}} G(\hat{y})\ \mathrm{d}\lambda(y) + \int_{\hat{z}}^{\bar{z}} G(\hat{y})\ \mathrm{d}\lambda(y) + \int_z^{\hat{z}} \lambda(z)\ \mathrm{d}G(y) - \int_z^{\hat{z}} \lambda(y)\ \mathrm{d}G(y) \\
&= \left\{\left[G(z)\lambda(\hat{z}) - G(\hat{z})\lambda(z)\right] + \int_z^{\hat{z}} \lambda(\hat{y})\ \mathrm{d}G(y)\right\} \\
&\quad + \int_{\hat{z}}^{\bar{z}} G(\hat{y})\ \mathrm{d}\lambda(y) + \lambda(z)\left[G(\hat{z}) - G(z)\right] - \int_z^{\hat{z}} \lambda(y)\ \mathrm{d}G(y) \\
&= G(z)\left[\lambda(\hat{z}) - \lambda(z)\right] + \int_z^{\hat{z}} \left[\lambda(\hat{y}) - \lambda(y)\right]\ \mathrm{d}G(y) + \int_{\hat{z}}^{\bar{z}} G(\hat{y})\ \mathrm{d}\lambda(y).
\end{aligned}$$

Thus, η is globally nonnegative if and only if the last expression is globally nonnegative for each $z \in [\underline{\theta}_1, b)$, as required. □

Proof of Proposition 5. Let $\bar{z}, G, \lambda, \hat{(\cdot)}$ be as defined in the previous lemma. In light of that lemma, we need to show each $z \in [\underline{\theta}_1, b)$ has

$$G(z)\left[\lambda(\hat{z}) - \lambda(z)\right] + \int_z^{\hat{z}} \left[\lambda(\hat{y}) - \lambda(y)\right] \, \mathrm{d}G(y) + \int_{\hat{z}}^{\bar{z}} G(\hat{y}) \, \mathrm{d}\lambda(y) \geqslant 0.$$

By hypothesis G admits a nonincreasing density g on its support. In this case, any $z \in [\underline{\theta}_1, b)$ has

$$\begin{aligned}
& G(z)\left[\lambda(\hat{z}) - \lambda(z)\right] + \int_z^{\hat{z}} \left[\lambda(\hat{y}) - \lambda(y)\right] \, \mathrm{d}G(y) + \int_{\hat{z}}^{\bar{z}} G(\hat{y}) \, \mathrm{d}\lambda(y) \\
\geqslant \; & \int_z^{\hat{z}} \left[\lambda(\hat{y}) - \lambda(y)\right] \, \mathrm{d}G(y) \\
= \; & \int_z^{b} \left[\lambda(\hat{y}) - \lambda(y)\right] g(y) \, \mathrm{d}y + \int_b^{\hat{z}} \left[\lambda(\hat{y}) - \lambda(y)\right] g(y) \, \mathrm{d}y \\
= \; & \int_z^{b} \left[\lambda(\hat{y}) - \lambda(y)\right] \left[g(y) - g(\hat{y})\right] \, \mathrm{d}y \\
\geqslant \; & 0,
\end{aligned}$$

as required. □

D.3. Proofs for Section C.3

In this section, we prove our characterization of the Pareto frontier. We also extend some previous results for buyer-optimal mechanisms to the entire Pareto frontier.

D.3.1. Proof of Theorem 3

To simplify our algebra in what follows, let $\vec{y}$ denote the vector $y\mathbb{1}_N \in \mathbb{R}^N$ for any scalar $y \in \mathbb{R}$.

Definition 10: *Let Λ denote the set of all vectors $\lambda \in \mathbb{R}^N_+$ such that $\vec{1} \cdot \lambda \leqslant 1$. Given $\lambda \in \Lambda$, a λ-**compatible** vector is any ω such that $(\lambda, \omega) \in \Delta(2N)$.*

The following lemma studies a program in which an allocation rule is chosen to maximize a weighted sum of utilities, the monotonicity property required by IC is ignored, the payment formula is assumed, and the constant on the payment formula is chosen to make IR bind for some agent. To state the lemma, for any $x \in \mathcal{X}$ or $\tilde{\mathcal{X}}$, define the profit level

$$\pi(x) := \min_{i \in N} \mathbb{E}\left[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)\right].$$

In light of Lemma 1, if $x \in \mathcal{X}$ is implementable, this profit level is the highest one consistent with IC and IR mechanisms that use allocation rule x.

Lemma 13: *Given any $\lambda \in \Lambda$, a unique solution exists to program*

$$\max_{x \in \tilde{\mathcal{X}}} \left\{ (1 - \vec{1} \cdot \lambda)\pi(x) + \lambda \cdot \mathbb{E}\left[x(\boldsymbol{\theta})(\vec{b} - \boldsymbol{\theta}) \right] \right\}.$$

This solution is given by the (λ, ω)-allocation rule, where ω is any λ-compatible vector satisfying the following two equivalent conditions:

(i) $\omega \in \operatorname{argmin}_{\tilde{\omega}:\ (\lambda,\tilde{\omega}) \in \Delta N} \mathbb{E}[(b - \lambda \cdot \boldsymbol{\theta} - \tilde{\omega} \cdot \boldsymbol{\varphi})_+]$.

(ii) $\operatorname{supp}(\omega) \subseteq \operatorname{argmax}_{i \in N} \mathbb{E}[\boldsymbol{\varphi}_i \mid \lambda \cdot \boldsymbol{\theta} + \omega \cdot \boldsymbol{\varphi} \leqslant b]$.

Proof. Substituting the definition of $\pi(x)$ and rearranging, the program's objective can be rewritten as

$$\min_{i \in N} \mathbb{E}\left\{ x(\boldsymbol{\theta}) \left[b - \lambda \cdot \boldsymbol{\theta} - (1 - \vec{1} \cdot \lambda)\boldsymbol{\varphi}_i \right] \right\}.$$

We can therefore follow the proof of Lemma 2 by modifying the two-player zero-sum game. Specifically, have Minimizer choose from the altered strategy space $(1 - \vec{1} \cdot \lambda)\Delta N$ of λ-compatible vectors, and change the objective to

$$\mathcal{G}_\lambda(x, \omega) := \mathbb{E}\left[x(\boldsymbol{\theta}) \left(b - \lambda \cdot \boldsymbol{\theta} - \omega \cdot \boldsymbol{\varphi} \right) \right].$$

Following exactly the proof of Lemma 2, mutatis mutandis, delivers the result. □

Motivated by the above lemma, we will say a vector ω is **λ-optimal** if it is λ-compatible and satisfies the numbered conditions in Lemma 13.

In what follows, let $Z \subseteq \mathbb{R} \times \mathbb{R}^N$ denote the set

$$Z = \left\{ \left(\pi,\ \mathbb{E}\left[x(\boldsymbol{\theta})(\vec{b} - \boldsymbol{\theta}) \right] - \vec{\pi} \right) :\ x \in \mathcal{X},\ \pi \in \mathbb{R},\ \mathbb{E}[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)] \geqslant \pi\ \forall i \in N \right\},$$

which is the set of payoff vectors induced by all mechanisms when the payment formula and IR are imposed. Given any $\tilde{Z} \subseteq \mathbb{R} \times \mathbb{R}^N$, say a point (π, u) is **Pareto optimal in** $\tilde{Z}$ if $(\pi, u) \in \tilde{Z}$ and no $(\tilde{\pi}, \tilde{u}) \in \tilde{Z} \backslash \{(\pi, u)\}$ exists with $(\tilde{\pi}, \tilde{u}) \geqslant (\pi, u)$.

The following lemma establishes a useful technical property of the payoff set Z and its Pareto frontier.

Lemma 14: *Every $z \in Z$ admits some $\tilde{z} \geqslant z$ that is Pareto optimal in Z.*

Proof. We begin with useful preliminary claim: The set $\{z \in Z :\ z \geqslant \underline{z}\}$ is compact for any $\underline{z} \in \mathbb{R} \times \mathbb{R}^N$. To show this fact, write $\underline{z} = (\underline{\pi}, \underline{u})$. Letting $\bar{\pi} := \min_{i \in N} \mathbb{E}[(b - \boldsymbol{\varphi}_i)_+]$, note that no $x \in \mathcal{X}$ and $\pi > \bar{\pi}$ can satisfy $\mathbb{E}[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)] \geqslant \pi\ \forall i \in N$. Because $\tilde{\mathcal{X}}$ is weak* compact (by Banach Alaoglu), the set

$$\left\{ \left(\pi,\ \mathbb{E}\left[x(\boldsymbol{\theta})(\vec{b} - \boldsymbol{\theta}) \right] - \vec{\pi} \right) :\ x \in \tilde{\mathcal{X}},\ \pi \in [\underline{\pi}, \bar{\pi}],\ \mathbb{E}[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)] \geqslant \pi\ \forall i \in N \right\}$$

is a continuous image of a compact space. Therefore, $\{z \in Z :\ z \geqslant \underline{z}\}$ is the intersection of the closed set $\mathbb{R} \times (\underline{u} + \mathbb{R}^N_+)$ with a compact set, and so is compact.

With the compactness claim in hand, we now establish the lemma. View Z as a subset of $\mathbb{R}^{\{0,\ldots,N\}}$, and let $z^{-1} := z \in Z$. For each $\tilde{z} \in Z$, let $Z(\tilde{z}) := \{\hat{z} \in Z : \hat{z} \geqslant \tilde{z}\}$, a nonempty (containing $\tilde{z}$) and compact subset of Z. Recursively for each

$j \in \{0, \dots, N\}$, we can therefore take $z^j \in \text{argmax}_{\tilde{z} \in Z(z^{j-1})} \tilde{z}_j$. By construction, $z \leqslant z^0 \leqslant \cdots \leqslant z^N$. Let us observe $\tilde{z} := z^N$ is Pareto optimal in Z. To that end, let $\hat{z} \in Z(\tilde{z})$; we want to show $\hat{z} \leqslant \tilde{z}$. And indeed, every $j \in \{0, \dots, N\}$ has $\hat{z} \in Z(z^{j-1})$, so that $\hat{z}_j \leqslant z^j_j \leqslant \tilde{z}_j$. Therefore, $\hat{z} = \tilde{z}$, delivering the lemma. □

The following lemma links Pareto optimality in the value set Z to the cutoff rule form.

Lemma 15: *Take any $\pi^* \in \mathbb{R}$ and $x^* \in \mathcal{X}$, and let $u^* := \mathbb{E}\left[x^*(\boldsymbol{\theta})(\vec{b} - \boldsymbol{\theta})\right] - \vec{\pi}^*$. The vector (π^*, u^*) is Pareto optimal in Z if and only if some $\lambda \in \Lambda$ and λ-optimal ω exist such that: $x^*(\boldsymbol{\theta}) = x_{\lambda,\omega}(\boldsymbol{\theta})$ almost surely; and $\pi^* \leqslant \pi(x_{\lambda,\omega})$, with equality if $\omega \neq \vec{0}$.*

Proof. Let us prove the following three conditions are equivalent:

(a) Payoff vector (π^*, u^*) is Pareto optimal in Z.

(b) Some $\lambda \in \Lambda$ exists such that

$$\begin{aligned}(\pi^*, x^*) \in \quad & \text{argmax}_{(\pi,x) \in \mathbb{R} \times \mathcal{X}} && \left\{\pi + \lambda \cdot \left(\mathbb{E}\left[x(\boldsymbol{\theta})(\vec{b} - \boldsymbol{\theta})\right] - \vec{\pi}\right)\right\} \\ & \text{s.t.} && \mathbb{E}\left[x(\boldsymbol{\theta})(\vec{b} - \boldsymbol{\varphi}_i)\right] \geqslant \pi \ \forall i \in N.\end{aligned}$$

(c) Some $\lambda \in \Lambda$ and λ-optimal ω exist such that: $x^*(\boldsymbol{\theta}) = x_{\lambda,\omega}(\boldsymbol{\theta})$ almost surely; and $\pi^* \leqslant \pi(x_{\lambda,\omega})$, with equality if $\omega \neq \vec{0}$.

First, let us see that conditions (b) and (c) are equivalent. To that end, fix $\lambda \in \Lambda$, and consider the program in condition (b), which can be rewritten as

$$\max_{(\pi,x) \in \mathbb{R} \times \mathcal{X}} \left\{(1 - \vec{1} \cdot \lambda)\pi + \lambda \cdot \mathbb{E}\left[x(\boldsymbol{\theta})(\vec{b} - \boldsymbol{\theta})\right]\right\} \text{ s.t. } \pi \leqslant \pi(x).$$

For any given $x \in \mathcal{X}$, the optimization for π is trivial to solve. The objective is weakly increasing in π (because $\vec{1} \cdot \lambda \leqslant 1$), strictly so if $\vec{1} \cdot \lambda < 1$. Therefore, condition (b) is satisfied if and only if:

- $\pi^* \leqslant \pi(x^*)$, with equality if $\vec{1} \cdot \lambda < 1$;
- $x^* \in \text{argmax}_{x \in \mathcal{X}} \left\{(1 - \vec{1} \cdot \lambda)\pi(x) + \lambda \cdot \mathbb{E}\left[x(\boldsymbol{\theta})(\vec{b} - \boldsymbol{\theta})\right]\right\}$.

The equivalence then follows directly from Lemma 13.

Now, we establish condition (b) implies condition (a). To that end, suppose condition (b) holds, and take any $(\pi, u) \in Z$ with $(\pi, u) \geqslant (\pi^*, u^*)$; we want to show $(\pi, u) = (\pi^*, u^*)$. First, by definition of Z, some allocation rule x exists such that $x = \mathbb{E}\left[x(\boldsymbol{\theta})(\vec{b} - \boldsymbol{\theta})\right] - \vec{\pi}$. Then, that $(\pi, u) \geqslant (\pi^*, u^*)$ implies—because $(\pi, u) \mapsto \pi + \lambda \cdot (u - \vec{\pi})$ is weakly increasing—that (π, x) is also an optimal solution to the program in condition (b). Hence, x is an optimal solution to the program in Lemma 13. The uniqueness part of Lemma 13 therefore tells us $x(\boldsymbol{\theta}) = x^*(\boldsymbol{\theta})$ almost surely. By revenue equivalence (Myerson, 1981; Myerson and Satterthwaite, 1983), then, $u - u^* = (\pi - \pi^*)\vec{1}$. Hence,

$$(\pi - \pi^*)(1,\ -\vec{1}) = (\pi, u) - (\pi^*, u^*) \geqslant 0,$$

implying $\pi - \pi^* = 0$, and so $u = u^*$, as required.

Finally, let us show condition (a) implies condition (b). Supposing (π^*, u^*) is Pareto optimal in Z, we want to show some $\lambda \in \Lambda$ exists such that $(\pi^*, u^*) \in \text{argmax}_{(\pi,u)\in Z} [\pi + \lambda \cdot u]$. First note, Z is the linear image of a set defined by linear inequality constraints on a convex domain; hence it is convex, and so too is $Z_- := Z - (\mathbb{R}_+ \times \mathbb{R}^N_+)$. Now, because (π^*, u^*) is Pareto optimal in Z, it is also Pareto optimal in Z_-, hence on the boundary of the latter. By the supporting hyperplane theorem, some nonzero $(\gamma, \lambda) \in \mathbb{R} \times \mathbb{R}^N$ exists such that

$$(\pi^*, u^*) \in \text{argmax}_{(\pi,u)\in Z_-} [\gamma\pi + \lambda \cdot u] .$$

Because Z_- is downward comprehensive, the separation property implies $(\gamma, \lambda) \geqslant 0$. Scaling the nonzero nonnegative vector (γ, λ) by a strictly positive constant, we may assume without loss that $\max \{\gamma, \ \max_{i\in N} \lambda_i\} = 1$. Finally, the definition of Z implies $(\pi^* - 1, u^* + \vec{1}) \in Z$ too, so that $(\pi^*, u^*) \in \text{argmax}_{(\pi,u)\in Z} [\gamma\pi + \lambda \cdot u]$ requires $\gamma \geqslant \vec{1} \cdot \lambda$. Thus, $\gamma = 1$, and λ is as desired. □

We now prove the characterization theorem.

Proof of Theorem 3. Lemma 13 says the two numbered conditions on ω are equivalent, so we need only show x^* is Pareto optimal if and only if some $\lambda \in \Lambda$ and λ-optimal ω exist such that $x^*(\boldsymbol{\theta}) = x_{\lambda,\omega}(\boldsymbol{\theta})$ almost surely.

First, suppose $\lambda \in \Lambda$, the vector ω is λ-optimal, and $x^*(\boldsymbol{\theta}) = x_{\lambda,\omega}(\boldsymbol{\theta})$ almost surely. By Lemma 15, then, the vector (π^*, u^*) is Pareto optimal in Z, where $\pi^* := \pi(x^*)$ and $u^* := \mathbb{E}\left[x^*(\boldsymbol{\theta})(\vec{b} - \boldsymbol{\theta})\right] - \vec{\pi}^*$ Lemma 1 then implies that some transfer rule m^* exists such that (x^*, m^*) is IC and $\Pi(x^*, m^*) = \pi^*$; that $U(x^*, m^*) = u^*$ and (x^*, m^*) is IR; and (given revenue equivalence) that every alternative IC and IR mechanism (x, m) has $(\Pi(x, m), U(x, m)) \in Z$. Hence, Pareto optimality of the mechanism (x^*, m^*) follows from Pareto optimality of (π^*, u^*) in Z.

Conversely, suppose (x^*, m^*) is a Pareto optimal mechanism for some transfer rule m^*. Letting $\pi^* := \Pi(x^*, m^*)$ and $u^* := U(x^*, m^*)$, Lemma 1 and revenue equivalence tell us $(\pi^*, u^*) \in Z$. Lemma 14 therefore delivers some $(\tilde{\pi}, \tilde{u}) \geqslant (\pi^*, u^*)$ that is Pareto optimal in Z. By definition of Z, some allocation rule $\tilde{x}$ exists such that $\tilde{u} = \mathbb{E}\left[\tilde{x}(\boldsymbol{\theta})(\vec{b} - \boldsymbol{\theta})\right] - \tilde{\pi}\vec{1}$. Given Lemma 15, we may assume without loss that $\tilde{x} = x_{\lambda,\omega}$ for some $\lambda \in \Lambda$ and λ-optimal, and $\tilde{\pi} \leqslant \pi(\tilde{x})$. Because $\tilde{x}$ is monotone (hence interim monotone) and shifting transfers by a constant preserves IC, Lemma 1 tells us some transfer rule $\tilde{m}$ exists such that $(\tilde{x}, \tilde{m})$ is IC and generates $\Pi(x_{\lambda,\omega}, \tilde{m}) = \tilde{\pi}$, and that $(\tilde{x}, \tilde{m})$ is IR because $\tilde{\pi} \leqslant \pi(\tilde{x})$ and $U(\tilde{x}, \tilde{m}) = \tilde{u}$. Hence, the IC and IR mechanism $(x_{\lambda,\omega}, \tilde{m})$ generates a payoff vector $(\tilde{\pi}, \tilde{u}) \geqslant (\pi^*, u^*)$. Because the mechanism (x^*, m^*) is Pareto optimal, it follows that $(\tilde{\pi}, \tilde{u}) = (\pi^*, u^*)$. Finally, the uniqueness statement in Lemma 13 implies $x^*(\boldsymbol{\theta}) = \tilde{x}(\boldsymbol{\theta})$ almost surely, delivering the theorem. □

D.3.2. Generalizing other results to Pareto-optimal mechanisms

Now, we establish that the main result of Section 5 applies more generally to the entire Pareto frontier, as does the main result reported in Section C.1.

Proposition 7 (Simple mechanisms Pareto dominated)**:** *If $b < \bar{\theta}_j$ for every $j \in N$, then no collective posted-price mechanism is Pareto optimal, and no DIC mechanism is Pareto optimal.*

Proof. First, Theorem 3 tells us any Pareto-optimal allocation rule x is essentially identical to the (λ, ω)-allocation rule for some λ and ω. Because $\underline{\theta}_i < b < \bar{\theta}_i < \varphi_i(\bar{\theta}_i)$ for every $i \in N$, it follows that $0 < \mathbb{E}[x(\boldsymbol{\theta})] < 1$. Now, observe that x generates interim allocation rules X_i that are continuous on $(\underline{\theta}_i, \bar{\theta}_i)$ for every $i \in N$ with $\lambda_i + \omega_i < 1$, and nonconstant on $(\underline{\theta}_i, \bar{\theta}_i)$ if the optimal weights (λ, ω) have $\lambda_i + \omega_i > 0$. Indeed, the proof is identical to the proof of Lemma 5, but with Theorem 3 playing the role of Theorem 1, and $\lambda_j + \omega_j$ playing the role of ω_j and $\lambda_j \boldsymbol{\theta}_j + \omega_j \boldsymbol{\varphi}_j$ playing the role of $\omega_j \boldsymbol{\varphi}_j$ for each $j \in N$. To see that some agent i has X_i being both non-constant and continuous on $(\underline{\theta}_i, \bar{\theta}_i)$, it suffices to show no agent i has $\lambda_i + \omega_i = 1$; assume otherwise for a contradiction. Note that $\lambda \cdot \theta + \omega \cdot \varphi(\theta) = \omega_i \varphi(\theta_i) + (1 - \omega_i)\theta_i$ is a strictly increasing transformation of $\theta_i \in \Theta_i$ that lies between θ_i and $\varphi_i(\theta_i)$. Therefore, some cutoff $p \in [\varphi_i^{-1}(b), b]$ exists such that $\lambda \cdot \boldsymbol{\theta} + \omega \cdot \boldsymbol{\varphi} \leqslant b$ if and only if $\boldsymbol{\theta}_i \leqslant p$. Hence,

$$\mathbb{E}[\boldsymbol{\varphi}_i \mid \lambda \cdot \boldsymbol{\theta} + \omega \cdot \boldsymbol{\varphi} \leqslant b] = \mathbb{E}[\boldsymbol{\varphi}_i \mid \boldsymbol{\theta}_i \leqslant p] = p,$$

where the last equality holds because a posted price of p (with agent i alone choosing whether to buy) generates the allocation rule $x_{\lambda,\omega}$ with binding IR for agent i. Therefore,

$$\mathbb{E}[\boldsymbol{\varphi}_i \mid \lambda \cdot \boldsymbol{\theta} + \omega \cdot \boldsymbol{\varphi} \leqslant b] \leqslant b < \bar{\theta}_j = \mathbb{E}[\boldsymbol{\varphi}_j] = \mathbb{E}[\boldsymbol{\varphi}_j \mid \lambda \cdot \boldsymbol{\theta} + \omega \cdot \boldsymbol{\varphi} \leqslant b],$$

as desired.

Next, we observe that no agent i exists such that $X_i^x|_{(\underline{\theta}_i, \bar{\theta}_i)}$ is both continuous and nonconstant, if (x, m) is either an IC collective posted-price mechanism or a bang-bang DIC mechanism—which will deliver the proposition. Given that we have seen $0 < \mathbb{E}[x(\boldsymbol{\theta})] < 1$, and that Theorem 3 tells us all Pareto-optimal mechanisms are bang-bang, the result follows directly from Lemma 11 for the case of bang-bang DIC mechanisms. So let us focus on showing it for the case of an IC posted-price mechanism. Let (x, m) be an IC collective posted price mechanism with price $p \in \mathbb{R}$. Below, we show no agent i exists such that $X_i^x|_{(\underline{\theta}_i, \bar{\theta}_i)}$ is both continuous and nonconstant. By the previous paragraph, it will follow that x is not Pareto optimal. Consider any agent i. Every $\theta_i, \hat{\theta}_i \in \Theta_i$ have

$$M_i^m(\hat{\theta}_i) - \theta_i X_i^x(\hat{\theta}_i) = (p - \theta_i) X_i^x(\hat{\theta}_i),$$

and so IC implies $X_i^x(\theta_i) = \max_{\hat{\theta}_i \in \Theta_i} X_i^x(\hat{\theta}_i)$ for any $\theta_i \in [\underline{\theta}_i, p)$ and $X_i^x(\theta_i) = \min_{\hat{\theta}_i \in \Theta_i} X_i^x(\hat{\theta}_i)$ for any $\theta_i \in (p, \bar{\theta}_i]$. In particular, X_i^x is constant both on $[\underline{\theta}_i, p)$ and on $(p, \bar{\theta}_i]$. Therefore, X_i^x is either constant on $(\underline{\theta}_i, \bar{\theta}_i)$ or discontinuous at $p \in (\underline{\theta}_i, \bar{\theta}_i)$. □

D.4. Proof for Section C.4

Proof of Proposition 6. First, consider the uniform-pricing regime. Following the agents' choice of shares σ', the buyer's optimal mechanism is characterized by Theorem 1. An optimal mechanism (x^*, m^*) is independent of the shares and depends only on the agents' distributions of types.[34] Because agents have identical distributions, the weights in the optimal mechanism are all $\omega_i = \frac{1}{N}$ and the optimal mechanism is symmetric. Hence, some $u \in \mathbb{R}_+$ exists such that a seller with share σ_i' gets payoff $\tau_i + \sigma_i' u$ if the buyer chooses this optimal mechanism and all sellers participate and truthfully report. The following play thus describes an equilibrium:

- The first seller proposes shares split σ and zero upfront transfers.
- Any other seller accepts a proposal $(\tilde{\sigma}, \tilde{\tau})$ if and only if $\tilde{\tau}_i + \tilde{\sigma}_i u > \sigma_i u$.
- For any realized shares σ', the buyer proposes the mechanism (x^*, m^*).
- If the mechanism (x^*, m^*) is proposed, then all sellers participate and truthfully report their types.
- If a mechanism other than (x^*, m^*) is proposed, then all sellers decline to participate.

This equilibrium has $\sigma' = \sigma$ and yields the buyer her optimal value, as required. Thus, the uniform-pricing game is pre-market non-manipulable.

Now, consider the discriminatory-pricing regime. Given any realized shares σ', seller i's cost of parting with his land is $\sigma_i' \boldsymbol{\theta}_i$, and so (a straightforward computation shows) his virtual cost is $\sigma_i' \boldsymbol{\varphi}_i$. Following Proposition 4.3 in Güth and Hellwig (1986), the essentially unique buyer-optimal mechanism for realized shares σ' has allocation rule given by $x(\theta) = \mathbb{1}_{\sigma' \cdot \boldsymbol{\varphi} \leqslant b}$, and transfers set so that IR binds for each agent.[35] By Proposition 4.2 of the same paper, seller i's expected payoff (gross of τ_i) is then $\mathbb{E}\left[\mathbb{1}_{\sigma' \cdot \boldsymbol{\varphi} \leqslant b}\ \sigma_i'(\boldsymbol{\varphi}_i - \boldsymbol{\theta}_i)\right]$. Therefore, the sum of the sellers' payoffs is

$$U(\sigma') := \mathbb{E}\left[\mathbb{1}_{\sigma' \cdot \boldsymbol{\varphi} \leqslant b}\ \sigma' \cdot (\boldsymbol{\varphi} - \boldsymbol{\theta})\right].$$

if a buyer-optimal mechanism is played—that is, if the buyer proposes it and all sellers participate and truthfully report—following share choice σ'.

To complete the proof of the proposition, we show by example that some specification of the model has $U(\tilde{\sigma}) > U(\sigma)$ for some $\tilde{\sigma} \in \Sigma$. The proposition will then follow, because any equilibrium would involve a successful proposal away from shares σ—for otherwise, the first seller could propose shares $\tilde{\sigma}$ together with lump-sum transfers to make every seller better off.

Consider the case with two sellers, each of whom has $\boldsymbol{\theta}_i$ uniform on $[0, 1]$, and

[34] The proof of Theorem 1 establishes that the optimal allocation rule is essentially unique, and Lemma 1 then implies all optimal mechanisms have the same interim transfer rules. In particular, all mechanisms yield the same per-share payoffs to all agents.

[35] Güth and Hellwig (1986) characterize profit-maximizing mechanisms for a seller who provides a public good to a group of agents and is allowed to use agent-specific transfers. A straightforward relabelling turns their model into one with a buyer who buys a public good from a group of sellers, so their analysis gives a characterization of buyer-optimal mechanisms. Güth and Hellwig (1986) impose a stronger regularity assumption (equivalent to assuming $\varphi_i(\theta_i) - \theta_i$ is increasing in our setting), but their proof applies identically under our weaker regularity assumption that φ_i is strictly increasing. Finally, they do not state essential uniqueness, but their proof establishes it because the allocation rule that solves their relaxed program is essentially unique.

a benefit $b = 1$. Then, any $\sigma' \in \Sigma$ has

$$U(\sigma') = \mathbb{E}\left[\mathbb{1}_{\sigma' \cdot \boldsymbol{\varphi} \leqslant b}\ \sigma' \cdot (\boldsymbol{\varphi} - \boldsymbol{\theta})\right] = \mathbb{E}\left[\mathbb{1}_{\sigma' \cdot \boldsymbol{\varphi} \leqslant 1}\ \sigma' \cdot \boldsymbol{\theta}\right].$$

In particular, the uniform share vector $\tilde{\sigma} = (\frac{1}{2}, \frac{1}{2})$ has

$$\begin{aligned} U(\tilde{\sigma}) &= \tfrac{1}{2}\mathbb{E}\left[\mathbb{1}_{\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 \leqslant 1}\ (\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2)\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 \leqslant 1}\ \boldsymbol{\theta}_1\right] \\ &= \int_0^1 \int_0^{1-\theta_1} \theta_1 \ \mathrm{d}\theta_2 \ \mathrm{d}\theta_1 = \int_0^1 (1 - \theta_1)\theta_1 \ \mathrm{d}\theta_1 = \left[\tfrac{1}{2}\theta_1^2 - \tfrac{1}{3}\theta_1^3\right]_{\theta_1=0}^1 \\ &= \frac{1}{6}. \end{aligned}$$

Meanwhile, as $\sigma \to (1, 0)$, the quantity $U(\sigma)$ converges (by the dominated convergence theorem) to

$$\mathbb{E}\left[\mathbb{1}_{\boldsymbol{\theta}_1 \leqslant \frac{1}{2}}\ \boldsymbol{\theta}_1\right] = \int_0^{\frac{1}{2}} \theta_1 \ \mathrm{d}\theta_1 = \left[\tfrac{1}{2}\theta_1^2\right]_{\theta_1=0}^{\frac{1}{2}} = \frac{1}{8} < \frac{1}{6}.$$

In particular, when the initial shares σ are sufficiently asymmetric, we have $U(\tilde{\sigma}) > U(\sigma)$, as required. □