Online Appendix for "Polarization, Valence, and Policy Competition"

Peter Buisseret* Richard Van Weelden†

October 22, 2021

Abstract

In this Supplementary Appendix we prove the results in Section 4: More Actions from the main text and provide results for the five-policy case. For Online Publication only.

A. Proof of Results from Section 4

Proof of Proposition 3. We start with $p < v$. The payoff matrix is displayed in Figure 1. The probabilities $\underline{\pi}$ and $\overline{\pi}$ are the same as in the two-action case (defined in expressions (2) and (3) of the main text respectively and evaluated at $\mu = 0$). The proof of Proposition 2 establishes that $\underline{\pi} < 1/2$ for all parameters and that $\overline{\pi} > 1/2$ if and only if $v < 1 - 2\mu = 1$.

If $y_L = -1$ and $y_R = 1$ a liberal voter supports candidate L if $y^i < (p-v)/4$ and a conservative supports L if $y^i < (-p-v)/4$. Note that if $v = 0$ the candidates are symmetric and so each candidate wins with probability $1/2$ but that L's vote share and win probability decreases in v. Thus L wins with probability $\hat{\pi} < 1/2$.

*Department of Government, Harvard University, *Email*: pbuisseret@fas.harvard.edu

†Department of Economics, University of Pittsburgh, *Email*: rmv22@pitt.edu

		R		
		-1	0	1
L	-1	$0,1$	$\underline{\pi},1-\underline{\pi}$	$\hat{\pi},1-\hat{\pi}$
	0	$\overline{\pi},1-\overline{\pi}$	$0,1$	$\overline{\pi},1-\overline{\pi}$
	1	$\hat{\pi},1-\hat{\pi}$	$\underline{\pi},1-\underline{\pi}$	$0,1$

Figure 1 – Low Polarization

Let $\sigma_i = \sigma_i(0)$ denote the probability of i choosing $y_i = 0$, so in any symmetric strategy $\sigma_i(-1) = \sigma_i(1) = (1-\sigma_i)/2$. Observation yields that $y_R = 0$ is the unique best response to any symmetric strategy $\sigma_L \in [0,1]$ if and only if $\hat{\pi} > 2\underline{\pi}$. Moreover, if R chooses $y_R = 0$, then $\sigma_L = 0$ is L's unique (symmetric) best response. We conclude that if $\hat{\pi} > 2\underline{\pi}$, $\sigma_R = 1$ and $\sigma_L = 0$ is the unique symmetric equilibrium. When $\hat{\pi} < 2\underline{\pi}$, we obtain a unique symmetric equilibrium in mixed strategies:

$$\sigma_R^{LP} = \frac{2\overline{\pi} - \hat{\pi}}{2(\overline{\pi} + \underline{\pi}) - \hat{\pi}} > \frac{2\underline{\pi} - \hat{\pi}}{2(\overline{\pi} + \underline{\pi}) - \hat{\pi}} = \sigma_L^{LP}.$$

In either case, R chooses policy 0 strictly more often than L.

Next, we consider $p > v$. In this case the payoff matrix is given in Figure 2. Recalling that $\overline{\pi} > 1/2$ when $v < 1$ and $\overline{\pi} < 1/2$ when $v > 1$, if $y_L = 0$ then R's best response is $\sigma_R = 1$ if $v < 1$ and $\sigma_R = 0$ if $v > 1$.

		R		
		-1	0	1
L	-1	$\frac{1}{2},\frac{1}{2}$	$\underline{\pi},1-\underline{\pi}$	$\hat{\pi},1-\hat{\pi}$
	0	$\overline{\pi},1-\overline{\pi}$	$\frac{1}{2},\frac{1}{2}$	$\overline{\pi},1-\overline{\pi}$
	1	$\hat{\pi},1-\hat{\pi}$	$\underline{\pi},1-\underline{\pi}$	$\frac{1}{2},\frac{1}{2}$

Figure 2 – High Polarization

Now consider L. For any $\sigma_R \in [0,1]$, L's net payoff from $y = 0$ versus $y \in \{-1, 1\}$ is

$$\Delta_L(\sigma_R) = \sigma_R \left(\frac{1}{2} - \underline{\pi}\right) + (1 - \sigma_R)\left(\overline{\pi} - \frac{1}{2}\left(\frac{1}{2} + \hat{\pi}\right)\right).$$

Note that $\Delta_L(1) > 0$ and that $\Delta_L(0) > 0$ if and only if

$$\overline{\pi} > \frac{1}{4} + \frac{\hat{\pi}}{2}.$$

Since $\hat{\pi} < 1/2 < \overline{\pi}$ when $v < 1$, it follows that L has a strictly best response of $y_L = 0$ for all σ_R when $v < 1$. Hence the unique equilibrium is the pure strategy equilibrium with median convergence: $\sigma_L = \sigma_R = 1$.

When $v > 1$ the equilibrium is in mixed strategies and there are two cases to consider. If $\overline{\pi} > 1/4 + \hat{\pi}/2$, any symmetric equilibrium must have $\sigma_L = 1$. Since R's best response is then $\sigma_R = 0$, the unique symmetric equilibrium is $\sigma_L = 1, \sigma_R = 0$. If $\overline{\pi} < 1/4 + \hat{\pi}/2$, any symmetric equilibrium involves $\sigma_L, \sigma_R \in (0,1)$. Using Figure 2, we obtain

$$\sigma_L^{HP} = \frac{\frac{1}{2}\left(\frac{1}{2} + \hat{\pi}\right) - \underline{\pi}}{\frac{1}{2}\left(\frac{1}{2} + \hat{\pi}\right) - \overline{\pi} + \frac{1}{2} - \underline{\pi}} > \frac{\frac{1}{2}\left(\frac{1}{2} + \hat{\pi}\right) - \overline{\pi}}{\frac{1}{2}\left(\frac{1}{2} + \hat{\pi}\right) - \overline{\pi} + \frac{1}{2} - \underline{\pi}} = \sigma_R^{HP}.$$

In either case L chooses policy strictly 0 more often than R. $\square$

Proof of Remark 1. The case $p < v$ is direct from the main text. We therefore focus on the case in which $p > v$. If $y_L = y_R = y$ then each candidate wins with probability $1/2$. We now show that R would have a profitable deviation for any such $y \in \mathbb{R}$. Without loss of generality consider $y \leq 0$, the expected median.

Let $0 < \varepsilon < \sqrt{v}$, and suppose R deviates to $y_R = y + \varepsilon$. A liberal voter then prefers R if and only if

$$y^i \geq y + \frac{1}{2}\left(\varepsilon + \frac{p - v}{\varepsilon}\right) := y^*$$

and a conservative prefers R if and only if

$$y^i \geq y + \frac{1}{2}\left(\varepsilon - \frac{p+v}{\varepsilon}\right) = y^* - \frac{p}{\varepsilon}.$$

Now define $\gamma^*(r)$ to solve

$$r\left(1 - B\left(y^* - \frac{p}{\varepsilon} - \gamma\right)\right) + (1-r)\left(1 - B(y^* - \gamma)\right) = \frac{1}{2}. \tag{A.1}$$

The LHS is continuous and strictly increasing in γ; it approaches one as $\gamma \to \infty$ and approaches zero as $\gamma \to -\infty$ so there is a unique solution $\gamma^*(r)$. Using the symmetry of $B(\cdot)$, it is easy to verify that

$$\gamma^*(1-r) = -\gamma^*(r) + 2y^* - \frac{p}{\varepsilon}. \tag{A.2}$$

Since R wins the election if and only if $\gamma > \gamma^*(r)$, by locating at $y+\varepsilon$ her probability of winning is

$$\int_0^1 (1 - G(\gamma^*(r)))\, dF(r).$$

Deviating from y to $y + \varepsilon$ is profitable if this is strictly greater than $1/2$, or equivalently if

$$\int_0^{1/2} (1 - G(\gamma^*(r))\, dF(r) + \int_{1/2}^1 (1 - G(\gamma^*(r))\, dF(r) > \frac{1}{2} = \int_{1/2}^1 dF(r).$$

Re-arranging and using the symmetry of $G(\cdot)$ this is equivalent to

$$\int_0^{1/2} G(-\gamma^*(r))\, dF(r) = \int_{1/2}^1 G(-\gamma^*(1-r))\, dF(r) > \int_{1/2}^1 G(\gamma^*(r))\, dF(r).$$

Using (A.2), R therefore has a profitable deviation if

$$\int_{1/2}^1 G\left(\gamma^*(r) - 2y^* + \frac{p}{\varepsilon}\right)\, dF(r) > \int_{1/2}^1 G(\gamma^*(r))\, dF(r),$$

which holds because when $y \leq 0$ and $0 < \varepsilon < \sqrt{v}$,

$$\begin{aligned}-2y^* + \frac{p}{\varepsilon} =& -2y - \varepsilon - \frac{p-v}{\varepsilon} + \frac{p}{\varepsilon} \\ =& -2y + \frac{v}{\varepsilon} - \varepsilon \\ >& \, 0.\end{aligned}$$

We conclude that R always has a profitable deviation from $y_L = y_R = y$. This implies that there is no pure strategy Nash equilibrium in which $y_L = y_R$.

Consider, instead, a pure strategy profile in which $y_L \neq y_R$. In that case, the candidates must win with probability one half: otherwise, the candidate who wins less often can profitably deviate to co-locate with the other candidate and win with probability one half. If the candidates win with probability one half, however, the previous argument shows that R has a profitable deviation to a policy in a neighborhood of y_L.

We conclude that there does not exist a pure strategy Nash equilibrium. □

B. Five Policies

In this section we analyze the model with five policies: $Y = \{-2, -1, 0, 1, 2\}$. To make the analysis tractable we assume that voter types, y_i, are uniformly distributed on the interval $[\gamma - \tau, \gamma + \tau]$,

$$B(x) = \begin{cases} 0 & \text{if } x < \gamma - \tau \\ \frac{x-(\gamma-\tau)}{2\tau} & \text{if } x \in [\gamma - \tau, \gamma + \tau], \\ 1 & \text{if } x > \gamma + \tau, \end{cases}$$

and that γ is uniformly distributed on $[-\kappa, \kappa]$,

$$G(x) = \begin{cases} 0 & \text{if } x < -\kappa \\ \frac{1}{2} + \frac{x}{2\kappa} & \text{if } x \in [-\kappa, \kappa]. \\ 1 & \text{if } x > \kappa. \end{cases}$$

Thus, the median position on the y issue is uniformly distributed as are voter bliss points around the median.

Our benchmark model assumes that $B(\cdot)$ and $G(\cdot)$ have unbounded support, but all results extend so long as these distributions' supports are 'large enough'. We assume $\kappa > \max\{v, 2\}$ so that $\gamma^*(y_L, y_R, r)$ defined in (B.3) lies in $[-\kappa, \kappa]$ for all $(y_L, y_R) \in Y^2$. Similarly, we assume that σ is large enough that for all (y_L, y_R) and any $\gamma \in [-\kappa, \kappa]$, the cut-off voter types defined in (B.1) and (B.2) below lie in $[\gamma - \tau, \gamma + \tau]$: $\sigma > \kappa + v + p + 2$.

Then, for any pair (y_L, y_R) such that $y_L \neq y_R$, the indifferent liberal type y_{lib} satisfies:

$$-(y_{\text{lib}} - y_L)^2 = -(y_{\text{lib}} - y_R)^2 - p + v \iff y_{\text{lib}} = \frac{y_L + y_R}{2} - \frac{v - p}{2(y_R - y_L)}. \tag{B.1}$$

Likewise, the indifferent conservative type y_{con} satisfies:

$$-(y_{\text{con}} - y_L)^2 = -(y_{\text{con}} - y_R)^2 + p + v \iff y_{\text{con}} = y_{\text{lib}} - \frac{p}{y_R - y_L}. \tag{B.2}$$

Thus, for any $y_L < y_R$, L wins if and only if

$$r\left(\frac{y_{\text{con}} - (\gamma - \sigma)}{2\sigma}\right) + (1 - r)\left(\frac{y_{\text{lib}} - (\gamma - \sigma)}{2\sigma}\right) \geq \frac{1}{2}$$
$$\iff \gamma \leq \gamma^*(y_L, y_R, r) = \frac{y_L + y_R}{2} - \frac{v + p(2r - 1)}{2(y_R - y_L)}. \tag{B.3}$$

The assumption that $B(\cdot)$ is uniform permits a closed-form expression for γ^*. Similarly, for any

$y_L > y_R$, L wins if and only if $\gamma \geq \gamma^*(y_L, y_R, r)$. Let $\pi_i(y_i, y_j)$ denote $i \in \{L, R\}$'s probability of winning the election when her platform is y_i and her opponent's is y_j. We have

$$\pi_L(y_L, y_R) = \begin{cases} \int_0^1 G\left(\gamma^*(y_L, y_R, r)\right) dF(r) & \text{if } y_L < y_R \\ 0 & \text{if } y_L = y_R \text{ and } p < v \\ \frac{1}{2} & \text{if } y_L = y_R \text{ and } p > v \\ 1 - \int_0^1 G\left(\gamma^*(y_L, y_R, r)\right) dF(r) & \text{if } y_L > y_R, \end{cases}$$

and R's probability of winning is $\pi_R(y_R, y_L) = 1 - \pi_L(y_L, y_R)$.

With these conditions we obtain the payoff matrix depicted in Figure 3 for the case of low polarization ($p < v$), and Figure 4 for high polarization ($p > v$). To economize on space, we display only $\pi_L(y_L, y_R)$ in each cell.

Strategies and Symmetric Equilibrium. Let $\sigma_i(y)$ denote the probability that candidate $i \in \{L, R\}$ plays action $y \in Y$. We say that a strategy $\sigma = (\sigma_L, \sigma_R)$ is *symmetric* if $\sigma_i(y) = \sigma_i(-y)$ for all $y \in Y$ and each candidate $i \in \{L, R\}$. We focus on symmetric equilibria, in which strategies are symmetric.

For any symmetric strategy σ_i, the distance of candidate i's platform from the expected me-

L \ R	-2	-1	0	1	2
-2	0	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{2} + \frac{3}{2}\right)\right)$	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{4} + 1\right)\right)$	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{6} + \frac{1}{2}\right)\right)$	$\frac{1}{2}\left(1 - \frac{v}{8\kappa}\right)$
-1	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{2} - \frac{3}{2}\right)\right)$	0	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{2} + \frac{1}{2}\right)\right)$	$\frac{1}{2}\left(1 - \frac{v}{4\kappa}\right)$	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{6} - \frac{1}{2}\right)\right)$
0	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{4} - 1\right)\right)$	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{2} - \frac{1}{2}\right)\right)$	0	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{2} - \frac{1}{2}\right)\right)$	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{4} - 1\right)\right)$
1	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{6} - \frac{1}{2}\right)\right)$	$\frac{1}{2}\left(1 - \frac{v}{4\kappa}\right)$	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{2} + \frac{1}{2}\right)\right)$	0	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{2} - \frac{3}{2}\right)\right)$
2	$\frac{1}{2}\left(1 - \frac{v}{8\kappa}\right)$	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{6} + \frac{1}{2}\right)\right)$	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{4} + 1\right)\right)$	$\frac{1}{2}\left(1 - \frac{1}{\kappa}\left(\frac{v}{2} + \frac{3}{2}\right)\right)$	0

Figure 3 – Low Polarization. Each cell contains L's probability of winning.

L \ R	−2	−1	0	1	2
−2	$\frac{1}{2}$	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{2}+\frac{3}{2}\right)\right)$	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{4}+1\right)\right)$	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{6}+\frac{1}{2}\right)\right)$	$\frac{1}{2}\left(1-\frac{v}{8\kappa}\right)$
−1	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{2}-\frac{3}{2}\right)\right)$	$\frac{1}{2}$	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{2}+\frac{1}{2}\right)\right)$	$\frac{1}{2}\left(1-\frac{v}{4\kappa}\right)$	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{6}-\frac{1}{2}\right)\right)$
0	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{4}-1\right)\right)$	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{2}-\frac{1}{2}\right)\right)$	$\frac{1}{2}$	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{2}-\frac{1}{2}\right)\right)$	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{4}-1\right)\right)$
1	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{6}-\frac{1}{2}\right)\right)$	$\frac{1}{2}\left(1-\frac{v}{4\kappa}\right)$	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{2}+\frac{1}{2}\right)\right)$	$\frac{1}{2}$	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{2}-\frac{3}{2}\right)\right)$
2	$\frac{1}{2}\left(1-\frac{v}{8\kappa}\right)$	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{6}+\frac{1}{2}\right)\right)$	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{4}+1\right)\right)$	$\frac{1}{2}\left(1-\frac{1}{\kappa}\left(\frac{v}{2}+\frac{3}{2}\right)\right)$	$\frac{1}{2}$

Figure 4 – High Polarization. Each cell contains L's probability of winning.

dian $|y_i|$ is a random variable. Notice that if $\sigma_i(0) \leq \sigma_j(0)$, and $\sigma_i(0) + 2\sigma_i(1) \leq \sigma_j(0) + 2\sigma_j(1)$, with at least one strict inequality, then $|y_i| \succ_{FOSD} |y_j|$. That is $|y_i|$ first order stochastic dominates $|y_j|$ and so i unambiguously locates further from the expected median policy of 0 than candidate j. We have the following result.

Proposition B.1. *A symmetric equilibrium always exists when $Y = \{-2, -1, 0, 1, 2\}$. Furthermore:*

1. *When $p < v$, in every symmetric equilibrium $|y_L| \succ_{FOSD} |y_R|$.*
2. *If $p > v$ then,*
   (a) if $v < 1$, in the unique symmetric equilibrium $\sigma_R^*(0) = \sigma_L^*(0) = 1$.
   (b) If $v > 1$, in every symmetric equilibrium: $|y_R| \succ_{FOSD} |y_L|$

To prove Proposition B.1 we first develop some notation. For $y \in \{1, 2\}$, we let $\varsigma_i(y) = \sigma_i(y) + \sigma_i(-y)$. That is: $\varsigma_i(y)$ is the frequency with which i plays *either* $+y$ or $-y$. Thus, $\sigma_i(0) + \varsigma_i(1) + \varsigma_i(2) = 1$, and a symmetric strategy satisfies $\sigma_i(y) = \sigma_i(-y) = \varsigma_i(y)/2$ for each $y \in \{1, 2\}$. Thus, $i \in \{L, R\}$'s symmetric strategy is fully described by $(\varsigma_i(1), \varsigma_i(2))$.

Let:

$$\Pi_L(0; \varsigma_R(1), \varsigma_R(2)) = \sigma_R(0)\pi_L(0, 0) + \varsigma_R(1)\pi_L(0, 1) + \varsigma_R(2)\pi_L(0, 2). \tag{B.4}$$

$\Pi_L(0; \varsigma_R(1), \varsigma_R(2))$ is L's payoff from choosing policy zero when R's symmetric strategy is $(\varsigma_R(1), \varsigma_R(2))$. Similarly, for $y \in \{1, 2\}$:

$$\Pi_L(y; \varsigma_R(1), \varsigma_R(2)) = \sigma_R(0)\pi_L(y, 0) + \frac{1}{4} \sum_{x \in \{1,2\}} \varsigma_R(x)[\pi_L(y, x) + \pi_L(y, -x) + \pi_L(-y, x) + \pi_L(-y, -x)]. \tag{B.5}$$

Here, $\Pi_L(y; \varsigma_R(1), \varsigma_R(2))$ is L's expected payoff from choosing policy $y \in \{1, 2\}$ with probability one half and $-y$ with probability one half when R's symmetric strategy is $(\varsigma_R(1), \varsigma_R(2))$. Notice that $\pi_L(y, 0) = \pi_L(-y, 0)$ for $y \in \{1, 2\}$. The corresponding payoff $\Pi_R(\cdot; \sigma_L(0), \sigma_L(1))$ for R is similarly defined.

We first establish that a symmetric equilibrium exists.

Lemma B.1. *A symmetric equilibrium exists.*

Proof. Call the original game Γ, and consider a modified game Γ' that differs from Γ in the following respects: (1) each player's action set is $Y(\Gamma') = \{0, 1, 2\}$, and (2) recalling $\Pi_i(y; \varsigma_i(1), \varsigma_i(2))$ that we defined in (B.5), i's payoff from action pair $(y_i, y_j) \in \{0, 1, 2\}^2$ is:

$$\tilde{\pi}_i(y_i, y_j) \equiv \begin{cases} \pi_i(0, y_j) & \text{if } y_i = 0, \\ \pi_i(y_i, 0) & \text{if } y_j = 0, \\ \Pi_i(y_i; 1, 0) & \text{if } y_i \in \{1, 2\},\, y_j = 1, \\ \Pi_i(y_i; 0, 1) & \text{if } y_i \in \{1, 2\},\, y_j = 2. \end{cases}$$

We note that if $\sigma^{\Gamma'} = (\sigma_L^{\Gamma'}, \sigma_R^{\Gamma'})$ is a Nash equilibrium of Γ', then the strategy profile σ^{Γ} satisfying:

$$\sigma_i^{\Gamma}(y) = \begin{cases} \sigma_i^{\Gamma'}(0) & \text{if } y = 0 \\ \frac{1}{2}\sigma_i^{\Gamma'}(1) & \text{if } y \in \{-1, 1\} \\ \frac{1}{2}\sigma_i^{\Gamma'}(2) & \text{if } y \in \{-2, 2\}, \end{cases}$$

is a symmetric Nash equilibrium of Γ. Since Γ' is a finite action game and thus possesses a Nash equilibrium, it follows that Γ possesses a symmetric equilibrium. □

Having established a symmetric equilibrium exists we prove the characterization result separately for the cases in which $p < v$ and when $p > v$, breaking each argument into multiple lemmas.

Part 1: When $p < v$ the payoff matrix appears in Figure 3—each cell identifies L's payoff $\pi_L(y_L, y_R)$, and R's is $\pi_R(y_R, y_L) \equiv 1 - \pi_L(y_L, y_R)$.

We first establish Lemma B.2 which shows that in every symmetric equilibrium, if $p < v$, there are no 'gaps' in the support of candidate R's strategy. Formally: candidate $i \in \{L, R\}$'s strategy σ_i has a gap at $y \in \{-1, 0, 1\}$ if $\sigma_i(y) = 0$ and there exist $z \in Y$ and $x \in Y$ such that $z < y < x$, $\sigma_i(z) > 0$, and $\sigma_i(x) > 0$. Candidate $i \in \{L, R\}$'s strategy has *no gaps* if it does not have a gap at any $y \in \{-1, 0, 1\}$.

Lemma B.2. *Let $p < v$. In every symmetric equilibrium, candidate R's strategy has no gaps.*

Proof. We proceed in two steps. The first step argues $\sigma_R(0) > 0$ in a symmetric equilibrium, when $p < v$. The second step argues that $\varsigma_R(2) > 0$ implies $\varsigma_R(1) > 0$. Observe that when $p < v$, we cannot have a symmetric equilibrium in which $\sigma_L(0) = 1$.

Step 1. First, we argue that when $p < v$, $\sigma_R(0) > 0$ in every symmetric equilibrium. Suppose, to the contrary, $\sigma_R(0) = 0$. $\kappa > v$ implies $\Pi_L(0; \varsigma_R(1), 1 - \varsigma_R(1)) > \Pi_L(1; \varsigma_R(1), 1 - \varsigma_R(1))$ for any $\varsigma_R(1) \in [0, 1]$. Since $\sigma_L(0) < 1$ in any equilibrium, $\sigma_L(0) = 1 - \varsigma_L(2) \in (0, 1)$. L's indifference requires $\varsigma_R(1) = \frac{24(\kappa+2)-9v}{24\kappa-v-24}$, and R's indifference requires $\sigma_L(0) = \frac{24\kappa-13v+48}{24\kappa-v-24}$. Observe $\sigma_L(0) \geq 0$ if and only if $\kappa \geq (48+13v)/24$, which further implies $\varsigma_R \leq 1$ if and only if $v \geq 9$. Finally $\Pi_R(2; 0, -\frac{12(v+2)}{-24\kappa+v+24}) \geq \Pi_R(0; 0, -\frac{12(v+2)}{-24\kappa+v+24})$ requires $\frac{24(\kappa-2)(2\kappa+1)+11v^2+(31-50\kappa)v}{4\kappa(-24\kappa+v+24)} \geq 0$, which fails whenever $9 \leq v \leq \kappa$. This contradicts $\sigma_R(0) = 0$.

Step 2. Second, we argue that when $p < v$, $\varsigma_R(2) > 0$ implies $\varsigma_R(1) > 0$ in every symmetric equilibrium. Suppose, to the contrary, $\varsigma_R(2) > 0$ and $\varsigma_R(1) = 0$. By the previous step, $\sigma_R(0) > 0$. Thus, $\varsigma_R(2) = 1 - \sigma_R(0)$.

(a) Suppose $\sigma_L(0) = 0$. If $\varsigma_L(1) > 0$ and $\varsigma_L(2) > 0$, L's indifference requires $\sigma_R(0) = \frac{24\kappa+48-13v}{24(\kappa+1)-5v}$, which implies $v \geq 3$. Likewise, R's indifference pins down $\varsigma_L(1) = 1 - \frac{8(9+v)}{24(\kappa+1)-v}$. Algebra yields $\Pi_R(1; \varsigma_L(1), 1-\varsigma_L(1)) - \Pi_L(2; \varsigma_L(1), 1-\varsigma_L(1)) > 0$, contradicting $\varsigma_R(2) > 0$.

(b) Suppose $\sigma_L(0) > 0$. Since $\Pi_R(1; 1-\sigma_L(0), 0) - \Pi_L(2; 1-\sigma_L(0), 0) > 0$ for all $\sigma_L(0) \in [0,1]$, $\varsigma_R(1) = 0$ implies that if $\sigma_L(0) > 0$, then $\varsigma_L(2) > 0$.

(i) If $\varsigma_L(2) = 1 - \sigma_L(0) > 0$, L's indifference requires $\sigma_R(0) = \frac{3v-8(\kappa+2)}{7v-24\kappa}$. Algebra yields that for all $\kappa > \max\{2, v\}$, $\Pi_L(1; 0, 1 - \frac{3v-8(\kappa+2)}{7v-24\kappa}) - \Pi_L(2; 0, 1 - \frac{3v-8(\kappa+2)}{7v-24\kappa}) > 0$, contradicting $\varsigma_R(1) = 0$.

(ii) If $\varsigma_L(1) > 0$, L's indifference conditions pin down $\sigma_R(0) = 1 - \frac{12(v-2)}{24(\kappa+1)-v} = \frac{v}{12\kappa-5v-6}$, which is interior only if $v \geq 2$. For any $v \geq 2$, we find that $1 - \frac{12(v-2)}{24(\kappa+1)-v} - \frac{v}{12\kappa-5v-6}$ strictly decreases in v; evaluating this difference at $v = \kappa$ yields $\frac{6(\kappa(9\kappa+41)-48)}{(7\kappa-6)(23\kappa+24)}$, which is strictly positive for all $\kappa \geq 2$. We conclude that there is no $\sigma_R(0) \in [0,1]$ that yields L's indifference between all her actions. □

While R's strategy cannot have a gap, L's could. Lemma B.3 and Lemma B.4 establish that for any location of the gap, the polarization of L's strategy first order stochastic dominates R's.

Lemma B.3. *Let $p < v$. In every symmetric equilibrium, $\varsigma_L(2) > 0$ and $\varsigma_L(1) = 0$ implies $\sigma_R(0) > \sigma_L(0)$.*

Proof. Let $\varsigma_L(2) > 0$ and $\varsigma_L(1) = 0$. If $\sigma_R(0) = 1$, or if $\varsigma_L(2) = 1$, the lemma trivially holds. Thus, we restrict subsequent attention to $\sigma_L(0) = 1 - \varsigma_L(2) > 0$ and $\sigma_R(0) < 1$; Lemma B.2 implies we may further restrict attention to symmetric no-gap strategies by R.

(a) Suppose $\varsigma_R(1) = 1 - \sigma_R(0)$. R's indifference requires $\sigma_L(0) = \frac{v}{12\kappa - 5v + 6}$ and L's indifference requires $\sigma_R(0) = \frac{2(v-9)}{-12\kappa+5v-6}$; $\kappa > v$ implies $\sigma_R(0) \in [0,1]$ if and only if $v < 9$. Moreover, $\frac{v}{12\kappa-5v+6} \geq \frac{2(9-v)}{12\kappa-5v+6}$ only if $v \geq 6$. However, algebra yields that for all $v \in [6,9]$, $\Pi_R(2; 0, 1 - \frac{v}{12\kappa-5v+6}) - \Pi_R(0; 0, 1 - \frac{v}{12\kappa-5v+6}) > 0$, contradicting $\varsigma_R(2) = 0$.

(b) Suppose R plays all actions with strictly positive probability. R's indifference conditions yield $\sigma_L(0) = \frac{v}{12\kappa-5v+6} = \frac{-24\kappa+13v+48}{-24\kappa+v+24}$. For all $\kappa > v$, $\frac{-24\kappa+13v+48}{-24\kappa+v+24}$ strictly decreases in v, and is strictly positive if and only if $v \leq (\kappa-2)24/13$. We further observe that $\frac{v}{12\kappa-5v+6}$ strictly increases in v. We conclude that there exists at most one v^* such that $\sigma_L(0) = \frac{v^*}{12\kappa-5v^*+6} = \frac{-24\kappa+13v^*+48}{-24\kappa+v^*+24} \equiv \sigma^*(v^*)$, and direct calculation yields that $v^* \leq \kappa$ if and only if $\kappa \leq (41 + \sqrt{3409})/18$, and also that $\sigma^*(v^*) < (71 - \sqrt{3409})/102 \approx .12366$. L's indifference between policies 0 and 2 requires $\sigma_R(0) = \frac{(\varsigma_R(2)+8)v - 24((\kappa-1)\varsigma_R(2)+3)}{4(-12\kappa+5v-6)}$. In conjunction with this value of $\sigma_R(0)$ and $\kappa \leq (41+\sqrt{3409})/18$, algebra yields that L's weak preference not to select policy 1 implies a (not tight) upper bound $\varsigma_R(2) \leq .25$, which implies $\sigma_R(0) + \varsigma_R(1) \geq .75$. Further algebra yields a lower bound $\sigma_R(0) \geq \frac{2(v-9)}{-12\kappa+5v-6}$ for all $\varsigma_R(2)$, and this lower bound itself weakly exceeds $\frac{1}{102}\left(23\sqrt{3409} - 1327\right) \approx .1558$. Since we showed $\sigma_L(0) < (71 - \sqrt{3409})/102 \approx .12366$, we conclude that $\sigma_R(0) > \sigma_L(0)$. □

Lemma B.4. *Let $p < v$. In a symmetric equilibrium, if $\sigma_L(0) = 0$ then $|y_L| \succ_{FOSD} |y_R|$.*

Proof. If $\varsigma_R(2) = 0$, the Lemma holds, trivially. If R randomizes over all actions, observe that since $\Pi_R(1; 1, 0) > \Pi_R(2; 1, 0)$, we must have $\varsigma_L(2) = 1 - \varsigma_L(1) > 0$. R's indifference requires $\sigma_L(1) = \frac{2v}{12+11v-12\kappa} = \frac{24\kappa-13v-48}{48\kappa-23v}$. Observe that $\frac{2v}{12+11v-12\kappa} - \frac{24\kappa-13v-48}{48\kappa-23v} \propto 288\left(\kappa^2 - 3\kappa + 2\right) + 97v^2 - 36(9\kappa - 19)v$, which is strictly positive when $v = 0$ for all $\kappa \geq 2$, and which strictly increases in v if $\kappa \leq 19/9$. If $\kappa > 19/9$, then the expression is positive for all $v \in [0, \kappa]$. □

Lemma B.2 shows that R's strategy cannot have a gap and Lemma B.3 and Lemma B.4 establish the first order stochastic dominance ranking if L's strategy has a gap. We can now complete the proof restricting attention to strategy profiles in which neither candidate's strategy has a gap.

Proof of Part 1. Focusing on no gap symmetric strategy profiles we can index the cases according to R's strategy. Recall that we must have $\sigma_L(0) < 1$ so if $\sigma_R(0) = 1$, the result is immediate.

Suppose $\varsigma_R(2) = 0$ and $\varsigma_R(1) > 0$. Since we must have $\sigma_L(0) < 1$, there are two possibilities. If $\varsigma_L(2) = 0$, indifference conditions yield mixtures such that $\sigma_R(0) > \sigma_L(0)$. If $\varsigma_L(2) > 0$, L's indifference conditions require $\sigma_R(0) = \frac{3v-4(\kappa+1)}{7v-12\kappa} = \frac{2(v-9)}{-12\kappa+5v-6}$, which requires $v \leq 9$. R's indifference requires that $\sigma_L(0) = -\frac{12\kappa\varsigma_L(1)-12\varsigma_L(1)-11\varsigma_L(1)v+2v}{2(-12\kappa+5v-6)}$, which is linear in $\varsigma_L(1) \in [0, 1-\sigma_L(0)]$. If $\varsigma_L(1) = 1 - \sigma_L(0)$, we obtain $\sigma_L(0) = \frac{4\kappa-3v-4}{12\kappa-7v} < \frac{4\kappa+4-3v}{12\kappa-7v} = \sigma_R(0)$. If $\varsigma_L(1) = 0$, we obtain $\sigma_L(0) = \frac{v}{12\kappa-5v+6} < \frac{2(9-v)}{12\kappa-5v+6}$ for all $v < 6$, while $\sigma_R(0) - \sigma_L(0) = \frac{3v-4(\kappa+1)}{7v-12\kappa} - \frac{v}{12\kappa+5v-6}$ strictly increases in κ for all $v \geq 6$. Setting $\kappa = v$, the difference $\frac{3v-4(\kappa+1)}{7v-12\kappa} - \frac{v}{12\kappa+5v-6}$ is strictly positive. Thus, $\sigma_R(0) > \sigma_L(0)$, and since $\sigma_R(0) + \varsigma_R(1) = 1$, we are done.

Suppose, instead, $\varsigma_R(2) > 0$ and $\varsigma_R(1) > 0$. Recall that we focus on symmetric no-gap strategies by L. If $\varsigma_L(1) = 1 - \sigma_L(0)$, R's indifference over all her actions requires $\sigma_L(0) = \frac{-4\kappa+3v+4}{7v-12\kappa} = \frac{6(v+2)}{12(\kappa+1)-11v} + 1$, but the the latter lies outside the unit interval. If both L and R randomize over all of their actions, solving indifference conditions yields strategies that satisfy the required properties. □

Part 2: We now turn to the high-polarization setting, in which $p > v$ and the payoff matrix is given in Figure 4. The steps proceed as in part 1 with the candidates reversed. We begin by ruling out gaps for the L candidate.

Lemma B.5. *Let $p > v$. In every symmetric equilibrium, candidate L's strategy has no gaps.*

Proof. The proof proceeds in two steps.

Step 1. First, we argue that if $p > v$, $\sigma_L(0) > 0$ in every symmetric equilibrium. Conjecture, to the contrary, that $\sigma_L(0) = 0$.

(a) Suppose $\varsigma_L(1) = 1$, which implies $\varsigma_L(2) = 0$. Straightforward calculation yields $\Pi_R(0; 1, 0) > \max\{\Pi_R(1; 1, 0), \Pi_R(2; 1, 0)\}$, so that $\sigma_R(0) = 1$ is R's unique symmetric best response to L's strategy. But since $\Pi_L(0; 0, 0) > \max\{\Pi_L(1; 0, 0), \Pi_L(2; 0, 0)\}$, we cannot have $\sigma_L(0) = 0$.

(b) Suppose $\varsigma_L(2) = 1$, which implies $\varsigma_L(1) = 0$. Straightforward calculation yields $\Pi_R(1; 0, 1) >$

$\max\{\Pi_R(0;0,1),\Pi_R(2;0,1)\}$, $\varsigma_R(1)=1$ is R's unique symmetric best response to L's strategy. But since $\Pi_L(1;1,0)>\Pi_L(2;1,0)$, L strictly prefers $\varsigma_L(1)=1$ to $\varsigma_L(2)=1$, a contradiction.

(c) Suppose $0<\varsigma_L(1)<1$.

(i) If $\varsigma_R(2)=0$, $\Pi_L(1;1,0)>\Pi_L(2;1,0)$ and $\varsigma_L(1)<1$ implies $\varsigma_R(1)<1$. If $\varsigma_R(1)=0$, $\sigma_L(0)=1$ is strictly preferred to any $\varsigma_L(1)\in(0,1)$. So, we must have $\sigma_R(0)=1-\varsigma_R(1)\in(0,1)$. Solving L's indifference condition yields $\sigma_R(0)=\frac{5v+24}{11v+12}$. Direct calculation yields $\Pi_L(1;1-\frac{5v+24}{11v+12},0)<\Pi_L(0;1-\frac{5v+24}{11v+12},0)$, contradicting $\varsigma_L(1)>0$.

(ii) If $\varsigma_R(1)=0$, L's indifference condition yields $\sigma_R(0)=1-\varsigma_R(2)=\frac{13v-48}{v-24}$, which is positive only if $v\leq 48/13$, in which case $\Pi_L(0;0,1-\frac{13v-48}{v-24})>\Pi_L(1;0,1-\frac{13v-48}{v-24})$, contradicting $\varsigma_L(1)>0$.

Steps (i) and (ii) yield that $\varsigma_R(1)>0$ and $\varsigma_R(2)>0$.

(iii) If $\sigma_R(0)=0$, then $\varsigma_R(2)=1-\varsigma_R(1)\in(0,1)$. Indifference for R requires $\varsigma_L(1)=\frac{48+13v}{23v}$. However, $\Pi_R(0;\frac{48+13v}{23v},1-\frac{48+13v}{23v})>\Pi_R(1;\frac{48+13v}{23v},1-\frac{48+13v}{23v})$, contradicting $\sigma_R(0)=0$.

(iv) If R randomizes chooses all her actions with strictly positive probability, her indifference between 0 and 1 requires $\varsigma_L(1)=\frac{2v}{12+11v}$. But, $\Pi_R(0;\frac{2v}{12+11v},1-\frac{2v}{12+11v})>\Pi_R(2;\frac{2v}{12+11v},1-\frac{2v}{12+11v})$, contradicting $\varsigma_R(2)>0$.

We conclude that $\sigma_L(0)>0$ in every symmetric equilibrium.

Step 2. We argue that if $p>v$, $\varsigma_L(1)=0$ implies $\varsigma_L(2)=0$ in every symmetric equilibrium. Conjecture, to the contrary, that $\varsigma_L(1)=0$ and $\varsigma_L(2)>0$. We already showed $\sigma_L(0)>0$, and thus $\varsigma_L(2)=1-\sigma_L(0)>0$. Moreover, $\Pi_R(1;0,1-\sigma_L(0))>\Pi_R(2;0,1-\sigma_L(0))$ for any $\sigma_L(0)\in[0,1]$, so $\varsigma_R(2)=0$. This implies $\varsigma_R(1)=1-\sigma_R(0)$. Since L plays both 0 and 2 with positive probability, her indifference requires $\Pi_L(0;1-\sigma_R(0),0)=\Pi_L(2;1-\sigma_R(0),0)$, which yields $\sigma_R(0)=\frac{2(v-9)}{5v-6}$, which implies $v\geq 9$. However, $\Pi_L(1;1-\frac{2(v-9)}{5v-6},0)>\Pi_L(0;1-\frac{2(v-9)}{5v-6},0)$, contradicting $\varsigma_L(1)=0$ and $\sigma_L(0)>0$.

Steps 1 and 2 yield that when $p>v$, L's strategy has no gaps in every symmetric equilibrium. □

Having established that L's strategy has no gaps the next two lemmas show that if R's strategy has a gap it must be at 0, and that in that case the first order stochastic dominance ranking holds.

Lemma B.6. *Let $p > v$. In every symmetric equilibrium, $\varsigma_R(2) = 0$ implies $\varsigma_R(1) = 0$.*

Proof. Suppose $\varsigma_R(2) > 0$, but $\varsigma_R(1) = 0$. For all $\sigma_R(0) = 1 - \varsigma_R(2) \in [0, 1]$, we have $\Pi_L(0; 0, 1 - \sigma_R(0)) > \Pi_L(1; 0, 1 - \sigma_R(0))$, which implies $\varsigma_L(1) = 0$. The previous Lemma implies that L's strategy has no gaps, so $\varsigma_L(2) = 0$. But $\Pi_R(1; 0, 0) > \Pi_R(2; 0, 0)$ contradicts $\varsigma_R(2) > 0$. □

Lemma B.7. *Let $p > v$. In a symmetric equilibrium, if $\sigma_R(0) = 0$ then $|y_R| \succ_{FOSD} |y_L|$.*

Proof. We need only consider possible equilibria in which L randomizes over all of her actions, and in which R's strategy has a gap at zero. L's indifference conditions pin down $\varsigma_R(1) = \frac{2v}{11v-12} = \frac{13v-48}{23v}$, which can only be satisfied if $v = \frac{6}{97}\left(57 + \sqrt{1697}\right)$. R's indifference between policies 1 and 2 requires $\varsigma_L(1) = \frac{24\varsigma_L(2)+\varsigma_L(2)v+12v+24}{2(11v-12)}$. When $v = \frac{6}{97}\left(57 + \sqrt{1697}\right)$, this implies however that for any $\varsigma_L(2) \in]0, 1]$, R strictly prefers to choose zero rather than 1. □

Lemma B.5, Lemma B.6 and Lemma B.7 imply that we may restrict our attention to symmetric equilibria in which neither candidate's strategy has a gap, which we use to complete the proof.

Proof of Part 2: We first consider $v < 1$ show that $\sigma_L(0) = \sigma_R(0) = 1$ is the unique symmetric equilibrium. This follows from the observation that when $v < 1$ then $\sigma_L(0) = 1$ is L's unique best response to any symmetric σ_R, and that $\Pi_R(0; 0, 0) > \max\{\Pi_R(1; 0, 0), \Pi_R(2; 0, 0)\}$ so R's unique best response to $\sigma_L(0) = 1$ is $\sigma_R(0) = 1$. This proves part (a).

Henceforth, we focus on $v > 1$ restricting attention to symmetric equilibria in which neither candidate's strategies have gaps. We index strategy profiles according to L's symmetric no-gap strategy.

(1) Suppose $\sigma_L(0) = 1$. Then, $v > 1$ implies $\Pi_R(0; 0, 0) < \max\{\Pi_R(1; 0, 0), \Pi_R(2; 0, 0)\}$, and thus $\sigma_R(0) = 0$. The proposition follows.

(2) Suppose $\sigma_L(0) < 1$ and $\varsigma_L(1) = 1 - \sigma_L(0)$. If $\sigma_R(0) = 0$, the proposition follows. If $\varsigma_R(1) = 1 - \sigma_R(0)$, indifference conditions yield $\sigma_R(0) = \frac{3}{7} - \frac{4}{7v}$ and $\sigma_L(0) = \frac{3}{7} + \frac{4}{7v}$, which satisfy the proposition. Finally, if R plays all actions with positive probability, then R's indifference requires $\sigma_L(0) = \frac{3v+4}{7v} = \frac{5v-24}{11v-12}$, which is possible if and only if $v \in \{\frac{12}{\sqrt{478}+22}, 2\left(\sqrt{478}+22\right)\}$. Similarly, L's indifference across actions 0 and 1 requires $\sigma_R(0) = \frac{-12\varsigma_R(1)+11\varsigma_R(1)v-2v}{2(5v+6)}$, which is strictly negative for all $\varsigma_R(1) \in [0,1]$ if $v = \frac{12}{\sqrt{478}+22}$. Thus, we must have $v = 2\left(\sqrt{478}+22\right)$. In that case, straightforward calculation yields that L weakly prefers not to play action 2 only if $\varsigma_R(1) \geq \frac{2911\sqrt{478}-38782}{46933}$. This yields that $\sigma_R(0) \leq \frac{820399-20827\sqrt{478}}{985593} < \frac{1}{21}\left(31-\sqrt{478}\right) = \frac{3v+4}{7v} = \sigma_L(0)$.

(3) Suppose L plays all of her actions with positive probability. We must have $\sigma_R(0) > 0$ since R's strategy has no gaps, and we must also have $\sigma_R(0) < 1$ if L does not play 0 with probability one. If $\varsigma_R(1) = 1 - \sigma_R(0) > 0$, L's indifference conditions require $\sigma_R(0) = \frac{3v-4}{7v} = \frac{5v+24}{11v+12}$, which holds for no $v > 0$. If, instead, R plays all actions with positive probability, then it is straightforward to solve indifference conditions and obtain strategies that satisfy the proposition.

Together, these three cases establish part (b). □