Online Appendix
Global Innovation and Knowledge Diffusion

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O.1 Proof of Theorem 1

Proof. We first prove sufficiency. Let J be an integer and fix some $\ell_j \in \{1, \ldots, L\}$ and $t_j \in \mathbb{R}$ for each $j = 1, \ldots, J$. Under Assumption 1, the distribution of productivity satisfies

$$\begin{aligned}
\mathbb{P}\left[Z_{\ell_j}(t_j, v) \leq z_j, \forall j = 1, \ldots, J\right] &= \mathbb{P}\left[\max_{i=1,2,\ldots} Q_i(v) A_{i\ell_j}(t_j, v) \leq z_{\ell_j}, \forall j = 1, \ldots, J\right] \\
&= \mathbb{P}\left[Q_i(v) A_{i\ell_j}(t_j, v) \leq z_{\ell_j}, \forall j = 1, \ldots, J, \forall i = 1, 2, \ldots\right] \\
&= \mathbb{P}\left[Q_i(v) \leq \min_{j=1,\ldots,J} \frac{z_{\ell_j}}{A_{i\ell_j}(t_j, v)}, \forall i = 1, 2, \ldots\right] \\
&= \mathbb{P}\left[Q_i(v) > \min_{j=1,\ldots,J} \frac{z_{\ell_j}}{A_{i\ell_j}(t_j, v)}, \text{ for no } i = 1, 2, \ldots\right],
\end{aligned}$$

where we take $1/0 = \infty$. This last expression is a void probability. We can use the marking theorem for Poisson processes (see Kingman, 1992) to calculate this void probability. In particular, under Assumption 2 and Assumption 3 we can take $\{Q_i(v), t_i^*(v)\}_{i=1,2,\ldots}$ as a base Poisson process and take the stochastic process $\{A_{i\ell}(t, v)\}_{\ell=1,\ldots,L, t\in\mathbb{R}}$ as a mark of the i'th point. Then, by the marking

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theorem, the collection $\{Q_i(v), t_i^*(v), \{A_{i\ell}(t,v)\}_{\ell=1,\ldots,L,t\in\mathbb{R}}\}_{i=1,2,\ldots}$ is itself a Poisson process and

$$\begin{aligned}
&\mathbb{E}\sum_{i=1}^{\infty}\mathbf{1}\{Q_i(v) > \underline{q}, t_i^*(v) \le t, A_{i\ell_j}(t_j,v) \le a_j \ \ \forall j = 1,\ldots,J\} \\
&= \int_{\underline{q}}^{\infty}\int_{-\infty}^{\infty}\mathbb{P}\left[A_{i\ell_j}(t_j,v) \le a_j \ \ \forall j = 1,\ldots,J \mid t_i^*(v) = t^*\right]\theta q^{-\theta-1}\mathrm{d}q\Lambda(\mathrm{d}t^*) \\
&= \underline{q}^{-\theta}\int_{-\infty}^{\infty}\mathbb{P}\left[A_{i\ell_j}(t_j,v) \le a_j \ \ \forall j = 1,\ldots,J \mid t_i^*(v) = t^*\right]\Lambda(\mathrm{d}t^*).
\end{aligned}$$

Using this result for the mean measure,

$$\begin{aligned}
&\mathbb{P}\left[Q_i(v) > \min_{j=1,\ldots,J}\frac{z_{\ell_j}}{A_{i\ell_j}(t_j,v)}, \text{ for no } i = 1,2,\ldots\right] \\
&= \exp\left[-\int_{-\infty}^{\infty}\int_{\mathbb{R}_+^J}\int_{\min_{j=1,\ldots,J}\frac{z_j}{a_j}}^{\infty}\theta q^{-\theta-1}\mathrm{d}q\mathrm{d}\mathbb{P}\left[A_{i\ell_j}(t_j,v) \le a_j \ \ \forall j = 1,\ldots,J \mid t_i^*(v) = t^*\right]\Lambda(\mathrm{d}t^*)\right] \\
&= \exp\left[-\int_{-\infty}^{\infty}\int_{\mathbb{R}_+^J}\max_{j=1,\ldots,J}\left(\frac{a_j}{z_j}\right)^{\theta}\mathrm{d}\mathbb{P}\left[A_{i\ell_j}(t_j,v) \le a_j \ \ \forall j = 1,\ldots,J \mid t_i^*(v) = t^*\right]\Lambda(\mathrm{d}t^*)\right].
\end{aligned}$$

Now, let $v_j \ge 0$ for each $j = 1,\ldots,J$. The distribution of $\max_{j=1,\ldots,J} v_j Z_{\ell_j}(t_j,v)$ is

$$\begin{aligned}
&\mathbb{P}\left[\max_{j=1,\ldots,J} v_j Z_{\ell_j}(t_j,v) \le z\right] = \mathbb{P}\left[Z_{\ell_j}(t_j,v) \le z/v_j \ \ \forall j = 1,\ldots,J\right] \\
&= \exp\left[-\int_{-\infty}^{\infty}\int_{\mathbb{R}_+^J}\max_{j=1,\ldots,J}\left(\frac{v_j a_j}{z}\right)^{\theta}\mathrm{d}\mathbb{P}\left[A_{i\ell_j}(t_j,v) \le a_j \ \ \forall j = 1,\ldots,J \mid t_i^*(v) = t^*\right]\mathrm{d}\Lambda(t^*)\right] \\
&= \exp\left[-\int_{-\infty}^{\infty}\int_{\mathbb{R}_+^J}\max_{j=1,\ldots,J}(v_j a_j)^{\theta}\,\mathrm{d}\mathbb{P}\left[A_{i\ell_j}(t_j,v) \le a_j \ \ \forall j = 1,\ldots,J \mid t_i^*(v) = t^*\right]\Lambda(\mathrm{d}t^*)z^{-\theta}\right].
\end{aligned}$$

Therefore, $\max_{j=1,\ldots,J} v_j Z_{\ell_j}(t_j,v)$ is distributed Fréchet and productivity is a max-stable process.

Moreover, if we take $J = L$, $\ell_j = j$ and $t_j = t$ for each $j = 1,\ldots,L$, we have

$$\begin{aligned}
\mathbb{P}\left[Z_\ell(t,v) \le z_\ell, \forall \ell \in \mathbb{L}\right] &= \exp\left[-\int_{-\infty}^{\infty}\int_{\mathbb{R}_+^L}\max_{\ell\in\mathbb{L}}\left(\frac{a_\ell}{z_\ell}\right)^{\theta}\mathrm{d}\mathbb{P}\left[A_{i\ell}(t,v) \le a_\ell \ \ \forall \ell \in \mathbb{L} \mid t_i^*(v) = t^*\right]\mathrm{d}\Lambda(t^*)\right] \\
&= \exp\left[-\int_{-\infty}^{t}\int_{\mathbb{R}_+^L}\max_{\ell\in\mathbb{L}}\left(\frac{a_\ell}{z_\ell}\right)^{\theta}\mathrm{d}\mathbb{P}\left[A_{i\ell}(t,v) \le a_\ell \ \ \forall \ell \in \mathbb{L} \mid t_i^*(v) = t^*\right]\mathrm{d}\Lambda(t^*)\right] \\
&= \exp\left[-\int_{\mathbb{R}_+^L}\max_{\ell\in\mathbb{L}}\left(\frac{a_\ell}{z_\ell}\right)^{\theta}\mathrm{d}M(a_1,\ldots,a_L;t)\right],
\end{aligned}$$

where the second line uses the fact that applicability is zero at any time before an idea's discovery time, and the final line uses the definition of M. Therefore, at any moment in time t, the distribution of productivity across production locations is max-stable multivariate Fréchet with scale $T_\ell(t) \equiv \int a_\ell^\theta \mathrm{d}M(a_1,\ldots,a_L;t)$ and correlation function $G(x_1,\ldots,x_L;t) \equiv \int \max_{\ell=1,\ldots,N}\frac{a_\ell^\theta}{T_\ell(t)}x_\ell \mathrm{d}M(a_1,\ldots,a_L;t)$.

It remains to show that productivity is a measurable stochastic process. From Assumption 1, productivity satisfies $Z_\ell(t,v) = \max_{i=1,2,\ldots} Q_i(v)A_{i\ell}(t,v)$, and $t \mapsto A_{i\ell}(t,v)$ is measurable by Assumption 3. Since the maximum of a countable collection of measurable functions is measurable, productivity is a measurable stochastic process.

Necessity follows from Theorem 3.1 and Proposition 4.1 in Wang and Stoev (2010). The second result ensures that productivity is separable in probability, which, combined with first result, implies that a minimal spectral representation exists with respect to a standard Lebesgue space.

Let $\{Z_\ell(t,v)\}_{(\ell,t)\in\mathbb{L}\times\mathbb{R}}$ be a max-stable process that is independent and identically distributed across $v\in[0,1]$. Denote the background probability space by $(\Omega,\mathcal{F},\mathbb{P})$. Further assume that productivity is measurable—for each fixed $\omega\in\Omega$ the map $(\ell,t)\to Z_\ell(t,v)$ is (Borel) measurable. Then by Theorem 3.1 and Proposition 4.1 in Wang and Stoev (2010), and the equivalence of extremal integral spectral representations to De Haan (1984) spectral representations (see Stoev and Taqqu, 2005), there exists a $\theta>0$, a standard Lebesgue space $([0,1],\mathcal{B}([0,1]),\mu)$, measurable functions $s\mapsto A_\ell(t,s)$ for each $(\ell,t)\in\mathbb{L}\times\mathbb{R}$ with $\int_0^1 A_\ell(t,s)^\theta \mathrm{d}\mu(s)<\infty$, and a Poisson process $\{Q_i(v),s_i(v)\}_{i=1,2,\ldots}$ for each v with intensity $\theta q^{-\theta-1}\mathrm{d}q\mathrm{d}\mu(s)$ such that $Z_\ell(t,v)=\max_{i=1,2,\ldots}Q_i(v)A_\ell(t,s_i(v))$. Moreover, the mapping $(\ell,t,s)\to A_\ell(t,s)$ can be taken to be jointly $\mathcal{B}(\mathbb{L}\times\mathbb{R})\otimes\mathcal{B}([0,1])$-measurable.

Since $s\to A_\ell(t,s)$ is measurable, we can define a stochastic process $\{A_{i\ell}(t,v)\}_{(\ell,t)\in\mathbb{L}\times\mathbb{R}}$ for each i and v such that $A_{i\ell}(t,v)=A_\ell(t,s_i(v))$ for all ℓ and t which is independent of $Q_i(v)$ and independent and identically distribution across i (since $\{Q_i(v),s_i(v)\}_{i=1,2,\ldots}$ is Poisson with intensity $\theta q^{-\theta-1}\mathrm{d}q\mathrm{d}\mu(s)$). The joint measurability of $(\ell,t,s)\to A_\ell(t,s)$ then implies that $A_{i\ell}(t,v):\Omega\to\mathbb{R}$ is $\mathcal{B}(\mathbb{L}\times\mathbb{R})$-measurable for each $\omega\in\Omega$. In other words, $\{A_{i\ell}(t,v)\}_{(\ell,t)\in\mathbb{L}\times\mathbb{R}}$ is a measurable stochastic process for each $i=1,2,\ldots$ and $v\in[0,1]$.

Next, define $t_i^*(v)\equiv\min_{\ell\in\mathbb{L}}\inf\{t\in\mathbb{R}\mid A_{i\ell}(t,v)>0\}$, which is a hitting time. Since $\{A_{i\ell}(t,v)\}_{(\ell,t)\in\mathbb{L}\times\mathbb{R}}$ is measurable and adapted to its natural filtration, it has a progressively-measurable modification. Taking $\{A_{i\ell}(t,v)\}_{(\ell,t)\in\mathbb{L}\times\mathbb{R}}$ as this modification, by the debut theorem (Bass, 2010, 2011), $t_i^*(v)$ is then a stopping time and is therefore a well-defined random variable that is adapted to the natural filtration of $\{A_{i\ell}(t,v)\}_{(\ell,t)\in\mathbb{L}\times\mathbb{R}}$. As a result, the function $s\to\min_{\ell\in\mathbb{L}}\inf\{t\in\mathbb{R}\mid A_\ell(t,s)>0\}\equiv\tau(s)$ is measurable. Then by the mapping theorem for Poisson processes (see Klenke, 2013, Theorem 24.16), $\{Q_i(v),t_i^*(v)\}_{i=1,2,\ldots}$ is a Poisson process with intensity $\theta q^{-\theta-1}\mathrm{d}q\Lambda(\mathrm{d}t)$ where $\Lambda(B)\equiv\mu(\tau^{-1}(B))$ for each $B\in\mathcal{B}(\mathbb{R})$.

Finally, we get finite moments by applying Campbell's theorem (see Kingman, 1992):

$$\int_\infty^t \mathbb{E}\left[A_{i\ell}(t,v)^\theta \mid t_i^*(v)=t^*\right]\Lambda(\mathrm{d}t^*)=\mathbb{E}\sum_{i=1}^\infty \mathbf{1}\{Q_i(v)>1,t_i^*(v)\le t\}A_{i\ell}(t,v)^\theta$$
$$=\mathbb{E}\sum_{i=1}^\infty \mathbf{1}\{Q_i(v)>1\}A_{i\ell}(t,v)^\theta=\mathbb{E}\sum_{i=1}^\infty \mathbf{1}\{Q_i(v)>1\}A_\ell(t,s_i(v))^\theta=\int_0^1 A_\ell(t,s)^\theta\mathrm{d}\mu(s)<\infty.$$

□

O.2 Proof of Proposition 1

Proof. Using the definition of the correlation function G in (6), we calculate

$$
\begin{aligned}
&T_\ell(t)W_\ell(t)^{-\theta}G_\ell(T_1(t)W_1(t)^{-\theta},\ldots,T_L(t)W_L(t)^{-\theta};t)\\
&=\int \mathbf{1}\left\{\frac{W_\ell(t)}{a_\ell}\leq\frac{W_l(t)}{a_l}\ \ \forall l\neq\ell\right\}\left(\frac{W_\ell(t)}{a_\ell}\right)^{-\theta}\mathrm{d}M(a_1,\ldots,a_L;t)\\
&=\int \mathbf{1}\left\{a_l\leq\frac{W_l(t)}{W_\ell(t)}a_\ell\ \ \forall l\neq\ell\right\}\left(\frac{W_\ell(t)}{a_\ell}\right)^{-\theta}\mathrm{d}M(a_1,\ldots,a_L;t)\\
&=\int_0^\infty\left(\frac{W_\ell(t)}{a_\ell}\right)^{-\theta}M\left(\frac{W_1(t)}{W_\ell(t)}a_\ell,\ldots,\mathrm{d}a_\ell,\ldots,\frac{W_L(t)}{W_\ell(t)}a_\ell\right)\\
&=\int_0^\infty\left(\frac{W_\ell(t)}{a_\ell}\right)^{-\theta}M_\ell\left(\frac{W_1(t)}{W_\ell(t)}a_\ell,\ldots,a_\ell,\ldots,\frac{W_L(t)}{W_\ell(t)}a_\ell\right)\mathrm{d}a_\ell,
\end{aligned}
$$

with

$$
G(T_1(t)W_1(t)^{-\theta},\ldots,T_L(t)W_L(t)^{-\theta};t)=\sum_{\ell=1}^{L}T_\ell(t)W_\ell(t)^{-\theta}G_\ell(T_1(t)W_1(t)^{-\theta},\ldots,T_L(t)W_L(t)^{-\theta};t).
$$

Using (12), we have

$$
\pi_\ell(t)=\frac{\int_0^\infty\left(\frac{W_\ell(t)}{a_\ell}\right)^{-\theta}M_\ell\left(\frac{W_1(t)}{W_\ell(t)}a_\ell,\ldots,a_\ell,\ldots,\frac{W_L(t)}{W_\ell(t)}a_\ell\right)\mathrm{d}a_\ell}{\sum_{\ell'=1}^{L}\int_0^\infty\left(\frac{W_{\ell'}(t)}{a_{\ell'}}\right)^{-\theta}M_{\ell'}\left(\frac{W_1(t)}{W_{\ell'}(t)}a_{\ell'},\ldots,a_{\ell'},\ldots,\frac{W_L(t)}{W_{\ell'}(t)}a_{\ell'}\right)\mathrm{d}a_{\ell'}}.
$$

Then, for $\ell'\neq\ell$,

$$
\frac{\partial\pi_\ell(t)}{\partial\ln W_{\ell'}(t)}=\int_0^\infty\left(\frac{W_\ell}{a_\ell}\right)^{-\theta}\frac{W_{\ell'}(t)}{W_\ell(t)}a_\ell M_{\ell\ell'}\left(\frac{W_1(t)}{W_\ell(t)}a_\ell,\ldots,a_\ell,\ldots,\frac{W_L(t)}{W_\ell(t)}a_\ell\right)\mathrm{d}a_\ell.
$$

These semi-elasticities can be re-expressed as elasticities by dividing by $\pi_\ell(t)$. We then do a change of variables from a_ℓ to $q=W_\ell/a_\ell$.

□

O.3 Independent Max-Stable Fréchet Applicability

To operationalize the closed form for the productivity distribution in Theorem 1, we focus on the class of models where (conditional) applicability is distributed independent max-stable Fréchet with shape σ. In this case, the measure of ideas can be written as

$$\begin{aligned} M(a_1,\ldots,a_L;t) &= \mathbb{P}\left[A_{i1}(t,v)\le a_L,\ldots,A_{iL}(t,v)\le a_L \mid t_i^*(v)\le t\right]\Lambda(t) \\ &= \int \exp\left[-\sum_{\ell=1}^{L}\left(\frac{a_\ell}{\phi_\ell}\right)^{-\sigma}\right]\mathrm{d}\mathcal{F}(\sigma,\phi_1,\ldots,\phi_L;t)\Lambda(t), \end{aligned} \tag{O.1}$$

where $\mathcal{F}$ is a distribution function for each t, and ϕ_ℓ^σ is the scale of Fréchet applicability. Due to max stability, the conditional distribution of $\max_{\ell\in\mathbb{L}} A_{i\ell}(t,v)^\theta z_\ell^{-\theta}(t,v)$ is also max-stable Fréchet with shape σ/θ. As a consequence, we can smooth over the max operator in (3) to get

$$\mathbb{P}\left[Z_1(t,v)\le z_L,\ldots,Z_L(t,v)\le z_L\right] = \exp\left[-\Gamma(1-\theta/\sigma)\int\left[\sum_{\ell=1}^{L}\left(\frac{a_\ell}{\phi_\ell}\right)^{-\sigma}\right]^{\frac{\theta}{\sigma}}\mathrm{d}\mathcal{F}(\sigma,\phi_1,\ldots,\phi_L;t)\Lambda(t)\right].$$

Note that the sum in this expression converges to a max as $\sigma\to\infty$, undoing the smoothing.

This smoothed version of (3) is convenient because it implies the following closed form for expenditure.

$$\pi_\ell(t) = \int\frac{(W_\ell(t)/\phi)^{-\sigma}}{\sum_{\ell'=1}^{L}(W_{\ell'}(t)/\phi)^{-\sigma}}\left[\sum_{\ell'=1}^{L}(W_{\ell'}(t)/\phi)^{-\sigma}\right]^{\frac{\theta}{\sigma}}\mathrm{d}\mathcal{F}(\sigma,\phi_1,\ldots,\phi_L;t).$$

This demand system is a generalization of the mixed-CES demand system used in Adao et al. (2017), which arises as the limiting case as $\theta\to 0$.

The examples we use throughout the paper imply functional forms for the measure of ideas as in (O.1). For example, the productivity distribution implied by the case of ideas that are shared across all locations once they diffuse (all-or-nothing diffusion) corresponds to the case of

$$\begin{aligned} \mathcal{F}(\tilde\sigma,\phi_1,\ldots,\phi_L;t) = &\sum_{\ell^*=1}^{L}\mathbf{1}\{\tilde\sigma\le\sigma,\phi_{\ell^*}\le 1,\phi_\ell\le 0\ \ \forall\ell\ne\ell^*\}\frac{T_{\ell^*}^{ND}(t)}{\Gamma(1-\theta/\sigma)\Lambda(t)} \\ &+\mathbf{1}\{\tilde\sigma\le\sigma,\phi_\ell\le 1\ \ \forall\ell\in\mathbb{L}\}\frac{T^D(t)}{\Gamma(1-\theta/\sigma)\Lambda(t)}. \end{aligned}$$

Using these results, we can derive (9),

$$\begin{aligned} -\ln\mathbb{P}\left[Z_1(t)\le z_1,\ldots,Z_L(t)\le z_L\right] &= \int\max_\ell a_\ell^\theta z_\ell^{-\theta}\mathrm{d}\sum_{\ell=1}^{L}\int_{-\infty}^{t}\mathbb{P}[A_{i\ell}(t,v)\le a_\ell\mid\ell_i^*(v)=\ell,t_i^*(v)=s]\lambda_\ell(s)\mathrm{d}s \\ =\sum_{\ell=1}^{L}\left[\int a_\ell^\theta\int_{-\infty}^{t}\mathrm{d}F^*(a_\ell\mid\ell^*,s;t)\lambda_\ell(s)\mathrm{d}s\right]z_\ell^{-\theta} &= \sum_{\ell=1}^{L}\left[\int a_\ell^\theta\mathrm{d}M(a_1,\ldots,a_L;t)\right]z_\ell^{-\theta}\equiv\sum_{\ell=1}^{L}T_\ell(t)z_\ell^{-\theta}, \end{aligned}$$

and (11),

$$
\begin{aligned}
&-\ln \mathbb{P}\left[Z_1(t) \leq z_1, \ldots, Z_L(t) \leq z_L\right] = \int \max_{\ell} a_\ell^{\theta} z_\ell^{-\theta} \mathrm{d} M(a_1, \ldots, a_L) \\
&= \sum_{\ell=1}^{L} \int a_\ell^{\theta} z_\ell^{-\theta} \mathrm{d}\left[e^{-a_\ell^{-\sigma}}(1-\delta_\ell(t))\Lambda_\ell(t)\right] + \int \max_{\ell} a_\ell^{\theta} z^{-\theta} \mathrm{d}\left[\prod_{\ell'=1}^{L} e^{-a_{\ell'}^{-\sigma}} \sum_{\ell=1}^{L} \delta_\ell(t)\Lambda_\ell(t)\right] \\
&= \sum_{\ell=1}^{L} \Gamma(\rho)(1-\delta_\ell(t))\Lambda_\ell(t) z^{-\theta} + \left(\sum_{\ell} z_\ell^{-\frac{\theta}{1-\rho}}\right)^{1-\rho} \Gamma(\rho) \sum_{\ell=1}^{L} \delta_\ell(t)\Lambda_\ell(t).
\end{aligned}
$$

References

Adao, R., A. Costinot, and D. Donaldson (2017). Nonparametric counterfactual predictions in neoclassical models of international trade. *The American Economic Review 107*(3), 633–689.

Bass, R. (2010). The measurability of hitting times. *Electronic communications in Probability 15*, 99–105.

Bass, R. (2011). Correction to" the measurability of hitting times". *Electronic Communications in Probability 16*, 189–191.

De Haan, L. (1984). A spectral representation for max-stable processes. *The Annals of Probability 12*(4), 1194–1204.

Kingman, J. (1992). *Poisson Processes*. Oxford Studies in Probability. Clarendon Press.

Klenke, A. (2013). *Probability theory: a comprehensive course*. Springer Science & Business Media.

Stoev, S. A. and M. S. Taqqu (2005). Extremal stochastic integrals: a parallel between max-stable processes and α-stable processes. *Extremes 8*(4), 237–266.

Wang, Y. and S. A. Stoev (2010). On the structure and representations of max-stable processes. *Advances in Applied Probability 42*(3), 855–877.