Background Risk and Small-Stakes Risk Aversion

Xiaosheng Mu Luciano Pomatto Philipp Strack Omer Tamuz

Online Appendix

A Background Risk for Cumulative Prospect Theory Preferences

Table 2 shows the levels of background risk needed to make a decision maker with cumulative prospect theory (CPT) preferences to accept various gambles. The specific CPT preference we consider has gain/loss probability weighting functions $w^+(p) = \frac{p^\gamma}{(p^\gamma+(1-p)^\gamma)^{1/\gamma}}$, $w^-(p) = \frac{p^\delta}{(p^\delta+(1-p)^\delta)^{1/\delta}}$ with $\gamma = 0.61, \delta = 0.69$, loss aversion parameter $\lambda = 2.25$ and value function $v(x) = x^{0.88}$ for $x \geq 0$ and $v(x) = -\lambda(-x)^{0.88}$ for $x < 0$. These parameter values are taken from Tversky and Kahneman (1992, pages 309–312). Given this choice of parameters, the table is constructed by computing numerically the utility of each gamble as a function of the standard deviation of the background risk.

Gamble	**StDeviation of Background Risk:** σ		
Gain/Loss	Laplace	Logistic	Normal
\$11/\$10	$\sigma \geq \$61$	$\sigma \geq \$46$	$\sigma \geq \$43$
\$55/\$50	$\sigma \geq \$306$	$\sigma \geq \$230$	$\sigma \geq \$217$
\$110/\$100	$\sigma \geq \$611$	$\sigma \geq \$460$	$\sigma \geq \$434$
\$550/\$500	$\sigma \geq \$3057$	$\sigma \geq \$2299$	$\sigma \geq \$2169$
\$1000/\$1100	$\sigma \geq \$6114$	$\sigma \geq \$4598$	$\sigma \geq \$4338$

Table 2: Standard deviation of background risk sufficient for a CPT decision maker to accept various fifty-fifty gambles under different distributional assumptions on the background risk.

B Choice Between Two Gambles

In this section, we extend the analysis to situations where the decision maker faces a choice between two bounded gambles X and Y that have distinct distributions F_X and F_Y. We say it is *dominant to choose* X *over* Y under background risk W, if $W + X$ first-order stochastically dominates $W + Y$. A result similar to Theorem 1 can be obtained if we consider background risks with heavy tails both on the left and on the right. For this we define the *two-sided exponential size*

$$\mathrm{S}^*(W) = \left(\sup_a \left| \frac{g'(a)}{g(a)} \right| \right)^{-1},$$

which is equal to $\min\{\mathrm{S}(W), \mathrm{S}(-W)\}$. Then we have:

Theorem 2. *The following are equivalent:*

(i) $\mathbb{E}[X] > \mathbb{E}[Y]$*;*

(ii) there exists $s \in (0, \infty)$ *such that under any background risk* W *with* $\mathrm{S}^*(W) \geq s$*, choosing* X *over* Y *is dominant.*

Proof of Theorem 2. We first show (ii) implies (i). Given any finite s, we can choose W to have a Laplace distribution with sufficiently large variance. Then W satisfies $\mathrm{S}^*(W) \geq s$, and by assumption $W + X$ must first-order stochastically dominate $W + Y$. Since such a W has finite expectation, we have $\mathbb{E}[W + X] \geq \mathbb{E}[W + Y]$, which implies $\mathbb{E}[X] \geq \mathbb{E}[Y]$. The inequality is in fact strict, for otherwise $W + X$ would have the same distribution as $W + Y$, and X would have the same distribution as Y. This last claim can be proved by considering the moment generating function in a neighborhood of 0. Since $\mathbb{E}[\mathrm{e}^{tW}]$ is finite for t close to 0, both $\mathbb{E}[\mathrm{e}^{t(W+X)}]$ and $\mathbb{E}[\mathrm{e}^{t(W+Y)}]$ are finite and are equal. It follows that $\mathbb{E}[\mathrm{e}^{tX}] = \mathbb{E}[\mathrm{e}^{tY}]$ for t in a neighborhood of 0, which implies X and Y have the same distribution.

To prove (i) implies (ii), we assume $\mathbb{E}[X] > \mathbb{E}[Y]$ and take s to be a large positive number (to be determined later). Consider any background risk W with $\mathrm{S}^*(W) \geq s$, i.e. the density g satisfies $|g'(a)/g(a)| \leq 1/s$ for all a. Let $h(a) = \ln g(a)$, then we can rewrite the condition as

$$|h'(a)| \leq \frac{1}{s} \text{ for all } a \in \mathbb{R}.$$

We now use this to show $\mathbb{P}[W + Y \leq a] \geq \mathbb{P}[W + X \leq a]$ for all a. Since W is independent from both X and Y, integration by parts shows this comparison is equivalent to

$$\int_{-M}^{M} g(a - z) \cdot F_Y(z)\,\mathrm{d}z \geq \int_{-M}^{M} g(a - z) \cdot F_X(z)\,\mathrm{d}z,$$

where M is a large number such that $[-M, M]$ contains the support of both X and Y. This in turn is equivalent to

$$\int_{-M}^{M} \mathrm{e}^{h(a-z)} \cdot (F_Y(z) - F_X(z))\, \mathrm{d}z \geq 0.$$

Dividing both sides by $\mathrm{e}^{h(a)}$, we just need to show that for all a

$$\int_{-M}^{M} \mathrm{e}^{h(a-z)-h(a)} \cdot (F_Y(z) - F_X(z))\, \mathrm{d}z \geq 0.$$

Observe that since $|h'|$ is bounded above by $1/s$, we have $|h(a-z) - h(a)| \leq M/s$ for all $a \in \mathbb{R}$ and all $z \in [-M, M]$. Thus if s is chosen to be sufficiently large, then the above integral converges, uniformly across a, to the integral $\int_{-M}^{M} (F_Y(z) - F_X(z))\, \mathrm{d}z$. Since this limit integral evaluates, by integration by parts, to $\mathbb{E}[X] - \mathbb{E}[Y] > 0$, the result follows. $\square$

If we only know that the background risk has a heavy left tail (as in Theorem 1), then the condition $\mathbb{E}[X] > \mathbb{E}[Y]$ is no longer sufficient to guarantee the dominance of X. Below we derive the suitable condition in this case. We say that X *strongly dominates* Y *in the convex order*, if $\max[X] > \max[Y]$ and

$$\int_{a}^{\infty} (F_Y(z) - F_X(z))\, \mathrm{d}z > 0 \ \text{ for all } a < \max[X]. \tag{7}$$

In particular, this requires $\mathbb{E}[X] > \mathbb{E}[Y]$ in the limit $a \to -\infty$.

To interpret this condition, note that X dominates Y in the convex order if and only if $-Y$ dominates $-X$ in second-order stochastic dominance. In other terms, X can be obtained from Y by a combination of mean-preserving spreads and right-ward mass shifts. Conversely, if X is obtained from Y by replacing *each* realization y of Y by a gamble with expectation *strictly greater* than y, then X *strongly* dominates Y in the convex order. This is a natural generalization of the case studied in the main text, where Y is a constant and X is any gamble with a higher expectation.

Theorem 3. *Suppose* $\max[X] \neq \max[Y]$. *Then the following are equivalent:*

(i) X *strongly dominates* Y *in the convex order;*

(ii) there exists $s \in (0, \infty)$ *such that under any background risk* W *with* $\mathrm{S}(W) \geq s$, *choosing* X *over* Y *is dominant.*

Proof of Theorem 3. As in the proof of Theorem 1, choosing X over Y is dominant if and only if

$$\mathbb{E}[G(a-X)] \leq \mathbb{E}[G(a-Y)] \text{ for all } a \in \mathbb{R}\,.$$

Since we want this to hold for all background risks G with exponential size $\geq s$, and since the exponential size is translation-invariant, it is without loss to restrict to the case of $a = 0$. That is, we seek to understand the conditions under which

$$\mathbb{E}[G(-X)] \leq \mathbb{E}[G(-Y)] \text{ for all } G \text{ with exponential size } \geq s.$$

As before, let $U(a) = \mathrm{e}^{\frac{a}{s}}$ denote a risk-loving CARA utility function. Then G has exponential size at least s if and only if $G(a) = \phi(U(a))$ for some increasing concave function ϕ.[11] Thus, the above comparison can be rewritten as

$$\mathbb{E}\left[\phi\left(\mathrm{e}^{\frac{-X}{s}}\right)\right] \leq \mathbb{E}\left[\phi\left(\mathrm{e}^{\frac{-Y}{s}}\right)\right] \text{ for all increasing concave functions } \phi.$$

In other terms, the random variable $\tilde{Y} = \mathrm{e}^{\frac{-Y}{s}}$ should dominate $\tilde{X} = \mathrm{e}^{\frac{-X}{s}}$ with respect to second-order stochastic dominance.

Let $\tilde{F}_X$ and $\tilde{F}_Y$ denote the c.d.f. of $\tilde{X}$ and $\tilde{Y}$, respectively. Then second-order stochastic dominance holds if and only if (noting that $\tilde{X}$ and $\tilde{Y}$ are both supported on $\mathbb{R}_+$):

$$\int_0^c (\tilde{F}_X(t) - \tilde{F}_Y(t))\,\mathrm{d}t \geq 0 \text{ for all } c > 0.$$

If we write $t = \mathrm{e}^{-\frac{z}{s}}$, then $\tilde{F}_X(t) = 1 - F_X(z)$, $\tilde{F}_Y(t) = 1 - F_Y(z)$. Changing variables in the above integral, and denoting $a = -s\ln(c)$, we obtain the following equivalent condition (modulo a factor of $1/s$):

$$\int_a^\infty (F_Y(z) - F_X(z)) \cdot \mathrm{e}^{-\frac{z}{s}}\,\mathrm{d}z \geq 0 \text{ for all } a \in \mathbb{R}. \tag{8}$$

Below we show that when the maxima of X and Y are different, the above condition holds for some positive s if and only if X strongly dominates Y in the convex order.

In one direction, suppose $\max[X] > \max[Y]$ and (7) holds. Then intuitively (8) would also

[11] To be fully rigorous, we also need $g(a) = \phi'(\mathrm{e}^{a/s}) \cdot \frac{1}{s}\mathrm{e}^{a/s}$ to be strictly positive, continuously differentiable, and eventually decreasing. These additional restrictions on ϕ do not affect the subsequent analysis because on any compact domain, any increasing concave function can be uniformly approximated by another increasing concave function with these additional properties.

hold if s is large, in which case the integrand $(F_Y(z) - F_X(z)) \cdot \mathrm{e}^{-\frac{z}{s}}$ is close to $F_Y(z) - F_X(z)$. This can be formalized by observing that we only need to prove (8) for a in the compact interval $\min[X] \le a \le \max[Y]$. As $s \to \infty$ the integral $\int_a^\infty (F_Y(z) - F_X(z)) \cdot \mathrm{e}^{-\frac{z}{s}} \, \mathrm{d}z$ converges uniformly to $\int_a^\infty (F_Y(z) - F_X(z)) \, \mathrm{d}z$ on this interval. Since this limit is a continuous function in a and strictly positive on this interval, it is bounded away from 0. Thus by uniform convergence, there exists some large s such that (8) holds.

For the converse, suppose (8) holds for some s. Then there cannot exist some a with $F_Y(a) < 1 = F_X(a)$, since otherwise (8) fails at this point a. It follows that $\max[X] \ge \max[Y]$, and the inequality is in fact strict by the assumption that $\max[X] \ne \max[Y]$. As a result, $F_Y(z) - F_X(z)$ is strictly positive for $z \in [\max[Y], \ \max[X])$, and (8) holds with strict inequality for a in the same interval. We now use this to prove (7). Observe that

$$\begin{aligned}
&\int_a^\infty (F_Y(z) - F_X(z)) \, \mathrm{d}z \\
&= \mathrm{e}^{\frac{a}{s}} \int_a^\infty (F_Y(z) - F_X(z)) \cdot \mathrm{e}^{-\frac{z}{s}} \, \mathrm{d}z + \int_a^\infty \left(\frac{\mathrm{e}^{\frac{c}{s}}}{s} \cdot \int_c^\infty (F_Y(z) - F_X(z)) \cdot \mathrm{e}^{-\frac{z}{s}} \, \mathrm{d}z \right) \mathrm{d}c.
\end{aligned}$$

So from (8), we must have $\int_a^\infty (F_Y(z) - F_X(z)) \, \mathrm{d}z \ge 0$. Moreover, the inequality is strict because in the double integral on the RHS above, the term $\int_c^\infty (F_Y(z) - F_X(z)) \cdot \mathrm{e}^{-\frac{z}{s}} \, \mathrm{d}z$ is strictly positive for any $c \in [\max[Y], \ \max[X])$. For any $a < \max[X]$, the mass of such $c > a$ is strictly positive. Hence (7) holds with strict inequality, completing the proof. □

C Second-Order Stochastic Dominance

Our analysis can also be extended to the smaller class of risk-averse preferences. We say that accepting X is *dominant for a risk-averse decision maker* if $W + X$ dominates W with respect to second-order stochastic dominance. We also introduce a modified version of the exponential size: for any background risk W with c.d.f. G, let

$$\mathrm{S}_2(W) = \left(\sup_{a \in \mathbb{R}} \frac{g(a)}{G(a)} \right)^{-1}.$$

It is easy to show that $\mathrm{S}_2(W) \ge \mathrm{S}(W)$.[12]

[12] If $\mathrm{S}(W) = 0$ then the result is trivial. If instead $\mathrm{S}(W) > 0$, then we have the inequality $g(x) \ge g(y) \cdot \mathrm{e}^{\frac{x-y}{\mathrm{S}(W)}}$. Note that $G(y) = \int_{-\infty}^y g(x) \, \mathrm{d}x \to 0$ as $y \to -\infty$. Using the previous inequality, we deduce that $g(y) \to 0$ as $y \to -\infty$. Hence, for each a, it holds that $\frac{g(a)}{G(a)} = \frac{\int_{-\infty}^a g'(x) \, \mathrm{d}x}{\int_{-\infty}^a g(x) \, \mathrm{d}x} \le \sup_x \frac{g'(x)}{g(x)} = \frac{1}{\mathrm{S}(W)}$. As a result,

Theorem 4. *Under any given background risk W with finite expectation, it is dominant for a risk-averse decision maker to accept every gamble X with positive expectation and riskiness* $\mathrm{R}(X) \leq \mathrm{S}_2(W)$.

Proof of Theorem 4. Let $s = \mathrm{S}_2(W)$ and without loss focus on $s > 0$. By a well-known characterization of second-order stochastic dominance, it is dominant to accept X if and only if

$$\int_{-\infty}^{a} \mathbb{P}\left[W + X \leq t\right] \mathrm{d}t \leq \int_{-\infty}^{a} \mathbb{P}\left[W \leq t\right] \mathrm{d}t \text{ for all } a \in \mathbb{R}. \tag{9}$$

That the integrals in (9) are finite follows from the fact that W and $W + X$ have finite expectations. By Tonelli's Theorem, the quantity $\int_{-\infty}^{a} \mathbb{P}\left[W + X \leq t\right] \mathrm{d}t$ is equal to

$$\int_{-\infty}^{a} \mathbb{E}\left[G(t - X)\right] \mathrm{d}t = \mathbb{E}\left[\int_{-\infty}^{a} G(t - X)\, \mathrm{d}t\right] = \mathbb{E}\left[\int_{-\infty}^{a-X} G(t)\, \mathrm{d}t\right].$$

Hence, it is second-order dominant to accept a gamble X if and only if for every $a \in \mathbb{R}$

$$\mathbb{E}\left[u_G(a - X)\right] \leq u_G(a),$$

where $u_G(a) = \int_{-\infty}^{a} G(t)\, \mathrm{d}t$. Therefore, as in the proof of Theorem 1, we obtain that accepting X is dominant if

$$\mathbb{E}\left[\mathrm{e}^{-\frac{1}{s}X}\right] \leq 1 \implies \mathbb{E}\left[u_G(a - X)\right] \leq u_G(a) \text{ for all } a \in \mathbb{R}. \tag{10}$$

Equation (10) holds whenever u_G is globally more risk-averse than the CARA utility function $U(a) = \mathrm{e}^{\frac{a}{s}}$. The Arrow-Pratt index for u_G is $-g(a)/G(a)$, which by assumption is weakly larger than $-1/s$, the Arrow-Pratt index for U. Thus u_G is indeed more risk-averse than U, concluding the proof. □

D Additional Results

Proposition 2. *For any gamble X that is supported on $[-M, M]$ and has expectation $\epsilon > 0$, its riskiness index satisfies* $\mathrm{R}(X) \leq \frac{M^2}{\epsilon}$.

Proof of Proposition 2. Let $\lambda = \frac{\epsilon}{M^2}$. We first show that $\mathbb{E}[\mathrm{e}^{-\lambda X}] \leq 1$. Indeed, since $\epsilon = \mathbb{E}[X] \leq M$, we have $\lambda \leq \frac{1}{M}$. As $X \in [-M, M]$ with probability one, we have $-\lambda X \in [-1, 1]$.

$\mathrm{S}_2(W) \geq \mathrm{S}(W)$ again holds.

In this range, it always holds that $e^{-\lambda X} \leq 1 - \lambda X + (\lambda X)^2$. Hence $\mathbb{E}[e^{-\lambda X}] \leq 1 - \lambda \mathbb{E}[X] + \lambda^2 \mathbb{E}[X^2] \leq 1 - \lambda\epsilon + \lambda^2 M^2 = 1$.

Now consider the function $f(a) = \mathbb{E}[e^{-aX}]$, defined for $a \geq 0$. It is easy to see that $f(0) = 1$ and f is strictly convex. Thus, $\frac{1}{\mathrm{R}(X)}$ is the unique number $c > 0$ such that $f(c) = 1$. Since we just proved that $f(\lambda) \leq 1$, convexity implies $c \geq \lambda$. In other words $\frac{1}{\mathrm{R}(X)} \geq \frac{\epsilon}{M^2}$. $\square$