On the Equilibrium Properties of Spatial Models: Online Appendix

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A Additional Applications

We apply Theorem 1 to twelve additional applications in the fields of spatial networks, production networks, social networks, and demand estimation.

A.1 An urban model with spatial spillovers

Here we consider another variant of the urban spatial model based on the seminal work of Ahlfeldt, Redding, Sturm, and Wolf (2015) presented in Section 3, where we include productivity and amenity spillovers that depend flexibly on the distribution of workers and residents, respectively, across the entire city.

A.1.1 The Model

We first describe the model and derive its equilibrium conditions.

Setup Consider a city comprised of $i \in \{1, ..., N\} \equiv \mathcal{N}$ blocks inhabited by agents with measure $\bar{L}$. Each agent ν chooses where to live $i \in \mathcal{N}$ and where to work $j \in \mathcal{N}$ in order to maximize her utility:

$$U_{ij}(\nu) = \frac{u_i w_j}{\mu_{ij}} \varepsilon_{ij}(\nu), \tag{16}$$

where u_i and w_j are the value of living at block i and working at block j, respectively, common to all agents, $\mu_{ij} \geq 1$ is the commuting cost, and $\varepsilon_{ij}(\nu)$ is the idiosyncratic preference of agent ν over location pairs, which we assume is extreme value (Frechet) distributed with shape parameter $\theta > 0$.

Commuting flows The number of agents who choose to live in location i and work in location j can be written as:

$$L_{ij} = \left(\frac{u_i w_j}{\mu_{ij}}\right)^{\theta} \lambda, \tag{17}$$

where $\lambda \equiv \bar{L} W^{-\theta_h}$ and $W \equiv \left(\sum_{(i,j)\in\mathcal{N}^2} \left(\frac{u_i w_j}{\mu_{ij}}\right)^{\theta}\right)^{\frac{1}{\theta}} = \mathbb{E}\left(\max_{(i,j)\in\mathcal{N}^2} U_{ij}(\nu)\right)$ is the expected welfare of agents.

Spatial Spillovers Suppose that an agent working in block j produces a costlessly traded numeraire good, for which they are paid their marginal product A_j, which is the only value they derive from their work, i.e. $w_j = A_j$. Suppose that their productivity depends both on the innate productivity of block j, $\bar{A}_j$, and the entire distribution of populations of workers throughout the city as follows:

$$A_i = \bar{A}_i \left(\sum_{j\in\mathcal{N}} F^A_{ij} L^W_j\right)^{\alpha}, \tag{18}$$

where $F^A_{ij} > 0$ governs the effect of the number of workers in $j \in \mathcal{N}$ on the productivity of a worker in $i \in \mathcal{N}$ and α governs the overall strength of the productivity spillover.

Similarly, suppose that an agent residing in block i receives a value of living there that depends both on the innate amenity of block i, $\bar{u}_i$, and the entire distribution of populations of residents throughout the city as follows:

$$u_i = \bar{u}_i \left(\sum_{j \in \mathcal{N}} F_{ij}^u L_j^R \right)^{\beta}, \tag{19}$$

where $F_{ij}^u > 0$ governs the effect of the number of residents in $j \in \mathcal{N}$ on the amenity of a worker in $i \in \mathcal{N}$ and β governs the overall strength of the amenity spillover.[8]

Equilibrium For any geography $\left\{ \left\{ \mu_{ij}, F_{ij}^A, F_{ij}^u \right\}_{(i,j) \in \mathcal{N}^2}, \left\{ \bar{A}_i, \bar{u}_i \right\}_{i \in \mathcal{N}} \right\}$, measure of agents $\bar{L}$, and model elasticities $\{\theta, \alpha, \beta\}$, equilibrium is a set of workplace and residential populations $\left\{ L_i^W, L_i^R \right\}_{i \in \mathcal{N}}$ such that:

1. The measure of workers employed in block $i \in \mathcal{N}$ is equal to the total number of agents commuting to that location:
$$L_i^W = \sum_{j \in \mathcal{N}} L_{ji} \tag{20}$$

2. The measure of residents residing in block $i \in \mathcal{N}$ is equal to the total number of agents commuting from that location:
$$L_i^R = \sum_{j \in \mathcal{N}} L_{ij} \tag{21}$$

As in Section 3 (and unlike Ahlfeldt, Redding, Sturm, and Wolf (2015)) we do not impose that rental rates of residential and commercial floor spaces are equalized.

A.1.2 Applying Theorem 1

Substituting the commuting equation (17) into the equilibrium conditions (20) and (21) and rearranging equations (18) and (19) yields:

$$L_i^W A_i^{-\theta} = \lambda \sum_{j \in \mathcal{N}} \mu_{ji}^{-\theta} u_j^{\theta}$$

$$L_i^R u_i^{-\theta} = \lambda \sum_{j \in \mathcal{N}} \mu_{ij}^{-\theta} A_j^{\theta},$$

$$A_i^{\frac{1}{\alpha}} = \bar{A}_i^{\frac{1}{\alpha}} \sum_{j \in \mathcal{N}} F_{ij}^A L_j^W$$

$$u_i^{\frac{1}{\beta}} = \bar{u}_i \sum_{j \in \mathcal{N}} F_{ij}^u L_j^R,$$

which together comprise our equilibrium system. It is immediately evident that this system of $4N$ equations in $4N$ unknowns takes the form of equation (8), which is a special case of equation

[8] Assuming alternative spillover functions $A_i = \bar{A}_i \sum_{j \in \mathcal{N}} F_{ij}^A \left(L_j^W \right)^{\alpha}$ and $u_i = \bar{u}_i \sum_{j \in \mathcal{N}} F_{ij}^u \left(L_j^R \right)^{\beta}$ result in an elasticity matrix with the same spectral radius as the one below, i.e. the conclusions of Theorem 1 below are unchanged.

(1), so by applying Remarks 4 and 5, it is sufficient to characterize the spectral radius of matrix $\mathbf{A} \equiv \left|\mathbf{B}\mathbf{\Gamma}^{-1}\right|$,where:

$$\mathbf{B} \equiv \begin{pmatrix} 0 & 0 & 0 & \theta \\ 0 & 0 & \theta & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ \mathbf{\Gamma} \equiv \begin{pmatrix} 1 & 0 & -\theta & 0 \\ 0 & 1 & 0 & -\theta \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & \frac{1}{\beta} \end{pmatrix},$$

so that:

$$\mathbf{A} \equiv \begin{pmatrix} 0 & 0 & 0 & |\beta\theta| \\ 0 & 0 & |\alpha\theta| & 0 \\ 1 & 0 & |\alpha\theta| & 0 \\ 0 & 1 & 0 & |\beta\theta| \end{pmatrix}$$

From the Collatz–Wielandt Formula, a sufficient condition for uniqueness is hence $|\alpha|\,\theta \leq \frac{1}{2}$ and $|\beta|\,\theta \leq \frac{1}{2}$, i.e. both the productivity and amenity agglomeration forces must be no stronger than the dispersion forces arising from the heterogeneity in agent preferences governed by θ. Note these conditions are identical to the $H = 1$ case of the example presented in Section 3, i.e. the presence of spatial spillovers does not affect the uniqueness condition.

We remark that while the full model presented in Ahlfeldt, Redding, Sturm, and Wolf (2015) included spatial spillovers, that paper only offered conditions for uniqueness in the absence of such spillovers; as a result, to our knowledge this is the first proof of uniqueness of an urban model in the presence of spatial spillovers. A similar methodology can be applied to incorporate spatial spillovers in other spatial settings—but with very different implications for the properties of the model—as we illustrate in the following economic geography example.

A.2 An economic geography model with spatial spillovers

We now extend the economic geography framework of Allen and Arkolakis (2014) to incorporate spatial productivity and amenity spillovers. It turns out that any spatial productivity or amenity spillovers can result in multiple equilibria—a very different conclusion from the urban model—highlighting the importance of Theorem 1 part (iii).

A.2.1 The model

Setup There are N locations, each of which produces a differentiated variety of a good. Agents in location $i \in \{1, ..., N\} \equiv \mathcal{N}$ have constant elasticity of substitution preferences over the differentiated varieties so that their welfare W_i is:

$$W_i = \left(\sum_{j \in \mathcal{N}} q_{ji}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} u_i,$$

where q_{ji} is the quantity of goods produced in $j \in \mathcal{N}$ and consumed in i, $\sigma \geq 1$ is the elasticity of substitution, and u_i is the local amenity. Agents are perfectly mobile and earn wage w_i by supplying their unit labor inelastically. Labor is the only factor of production; let A_i be the productivity of an agent in location $i \in \mathcal{N}$. Finally, the transportation of goods are subject to iceberg transportation costs, where $T_{ij} \geq 1$ indicates the number of goods needed to be sent from $i \in \mathcal{N}$ in order for one unit to arrive in $j \in \mathcal{N}$.

Spatial spillovers We suppose that productivities and amenities depend on the distribution of labor across all locations through spatial spillovers as follows:

$$A_i = \bar{A}_i \left(\sum_{j \in \mathcal{N}} F^A_{ij} L_j \right)^{\alpha} \tag{22}$$

$$u_i = \bar{u}_i \left(\sum_{j \in \mathcal{N}} F^u_{ij} L_j \right)^{\beta} \tag{23}$$

where $\bar{A}_i$ and $\bar{u}_i$ are the exogenous productivity and amenity, respectively, of location $i \in \mathcal{N}$; $F^A_{ij} > 0$ and $F^u_{ij} > 0$ capture how the population in location $j \in \mathcal{N}$ affects the productivity and amenity, respectively in location $i \in \mathcal{N}$, and α and β are the productivity and amenity spillover elasticities, respectively common to all locations.[9]

Equilibrium For any geography $\left\{ \{T_{ij}\}_{(i,j)\in\mathcal{N}^2}, \{\bar{A}_i, \bar{u}_i\}_{i\in\mathcal{N}}, \left\{F^A_{ij}\right\}_{(i,j)\in N^2} \right\}$ equilibrium is a set of populations, wages, productivities, and amenities $\{L_i, w_i, A_i, u_i\}_{i\in\mathcal{N}}$ such that:

1. Markets clear, i.e. income in a location $i \in \mathcal{N}$ is equal to the value of all goods sold in all other locations:
$$w_i L_i = \sum_{j \in \mathcal{N}} X_{ij},$$
where $X_{ij} = \frac{T_{ij}^{1-\sigma}(w_i/A_i)^{1-\sigma}}{\sum_{k=1}^N T_{kj}^{1-\sigma}(w_k/A_k)^{1-\sigma}} w_j L_j$ is the bilateral flow of goods from $i \in \mathcal{N}$ to $j \in \mathcal{N}$.

2. Trade is balanced, i.e. income in a location $i \in \mathcal{N}$ is equal to the value of all goods purchased from all other locations:
$$w_i L_i = \sum_{j \in \mathcal{N}} X_{ji}$$

3. Welfare is equalized, i.e. there exists a scalar $W > 0$ such that for all $i \in \mathcal{N}, W_i \leq W$, with the equality strict if $L_i > 0$.

4. Productivities and amenities are given by equations (22) and (23).

A.2.2 Applying Theorem 1

Combining the first three equilibrium conditions (see equations (10) and (11) of Allen and Arkolakis (2014)) and re-arranging equations (22) and (23) yields the following system of $4N$ equilibrium

[9] Assuming alternative spillover functions $A_i = \bar{A}_i \sum_{j\in\mathcal{N}} F^A_{ij} L_j^{\alpha}$ and $u_i = \bar{u}_i \sum_{j\in\mathcal{N}} F^u_{ij} L_j^{\beta}$ result in an elasticity matrix with the same spectral radius as the one below, i.e. the conclusions of Theorem 1 below are unchanged.

conditions in $4N$ unknowns:

$$L_i w_i^{\sigma} A_i^{1-\sigma} = W^{1-\sigma} \sum_{j=1}^{N} T_{ij}^{1-\sigma} L_j w_j^{\sigma} u_j^{\sigma-1}$$
$$w_i^{1-\sigma} u_i^{1-\sigma} = W^{1-\sigma} \sum_{j=1}^{N} T_{ji}^{1-\sigma} w_j^{1-\sigma} A_j^{\sigma-1}$$
$$A_i^{\frac{1}{\alpha}} = \bar{A}_i^{\frac{1}{\alpha}} \sum_{j\in\mathcal{N}} F_{ij}^{A} L_j$$
$$u_i^{\frac{1}{\beta}} = \bar{u}_i^{\frac{1}{\beta}} \sum_{j\in\mathcal{N}} F_{ij}^{u} L_j$$

which together comprise our equilibrium system. It is immediately evident that this system takes the form of equation (8), which is a special case of equation (1), so by applying Remarks 4 and 5, it is sufficient to characterize the spectral radius of matrix $\mathbf{A} \equiv \left|\mathbf{B}\mathbf{\Gamma}^{-1}\right|$,where:

$$\mathbf{B} \equiv \begin{pmatrix} 1 & \sigma & 0 & \sigma-1 \\ 0 & 1-\sigma & \sigma-1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{\Gamma} \equiv \begin{pmatrix} 1 & \sigma & 1-\sigma & 0 \\ 0 & 1-\sigma & 0 & 1-\sigma \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & \frac{1}{\beta} \end{pmatrix},$$

so that:

$$\mathbf{A} \equiv \begin{pmatrix} 1 & 0 & |\alpha|(\sigma-1) & |\beta|(\sigma-1) \\ 0 & 1 & |\alpha|(\sigma-1) & |\beta|(\sigma-1) \\ 1 & \frac{\sigma}{\sigma-1} & |\alpha|(\sigma-1) & |\beta|\sigma \\ 1 & \frac{\sigma}{\sigma-1} & |\alpha|(\sigma-1) & |\beta|\sigma \end{pmatrix}.$$

It can be shown that $\rho(\mathbf{A}) \leq 1$ only if $\alpha = \beta = 0$, i.e. only if there are no spatial spillovers. Note that this is a substantial departure from Allen and Arkolakis (2014) and Allen, Arkolakis, and Takahashi (2020), who show that uniqueness is guaranteed in an economic geography model with *local* spillovers as long as the dispersion forces are stronger than agglomeration forces; in contrast, Theorem 1 part (iii) says that there will be geographies for which there are multiple equilibria for in the presence of any *spatial* spillover, i.e. for any non-zero α and β. Note too that this is also a major qualitative difference with the urban example above, where the conditions for uniqueness were the same for local and spatial spillovers.

A simple example suffices to provide intuition for the possibility of multiple equilibria. Consider a world of two identical locations (i.e. $\bar{A}_i = \bar{u}_i = 1$ for $i, j \in \{1, 2\}$) separated by trade costs $\tau > 1$. Suppose there are only productivity spillovers (i.e. $\beta = 0$); the case with amenity spillovers is similar. For any $\alpha > 0$ and $F_{ij}^{A} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ — i.e. a case where the spillovers are positive and depend only on one's own population—there exists a $\tau > 1$ such that there are three equilibria: one in which both locations have an equal population and one in which one of the two locations has a greater concentration of population (to take advantage of the agglomeration forces). But for any $\alpha < 0$ and $F_{ij}^{A} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$ — i.e. a case where the spillovers are negative and depend only on the other location's population—there exists a $\tau > 1$ such that there are again three equilibria: one in which both locations have an equal population and one in which one of the two locations has a greater concentration of population (to take advantage of the fact that the smaller population in the neighbor increases productivity spillovers). That is, with spatial spillovers, a dispersion force from population elsewhere acts like a local agglomeration force.

To our knowledge, this is the first characterization of uniqueness in an economic geography model with spatial spillovers.

A.3 A trade model with intermediate goods and tariffs

We now consider a Ricardian model based on the seminal work of Eaton and Kortum (2002) but augmented to include tariffs and an input-output network as in Alvarez and Lucas (2007).

A.3.1 The model

Setup

There are N locations, each of which produces 3 sets of goods: a continuum of tradeables $q_i(u)$ where $u \in [0,1]$, a aggregate intermediate good a_i, and a non-tradeable final good c_i. Agents in the economy derive their utility from the non-tradeable final good c_i. This final good c_i is produced in a Cobb-Douglas manner using the intermediate good a_i and labor i.e. $c_i = s_{fi}^{\alpha} a_{fi}^{1-\alpha}$ where s_{fi} and a_{fi} are the labor and intermediate inputs in final good production, respectively. The intermediate good a_i is a Spence-Dixit-Stiglitz aggregate of all varieties of tradeables:

$$a_i = \left[\int_0^1 (q_{i^*}(u))^{1-1/\eta} \, du\right]^{\frac{\eta}{\eta-1}},$$

where $i^* \equiv \arg\min_{j \in \mathcal{N}} p_{ji}(u)$, i.e. each variety of tradeable is sourced from the lowest cost location. Tradeables in turn are produced using the composite intermediate good I_i as input, along with labor as:

$$q_i(u) = x_i(u)^{-\theta} s_i(u)^{\beta} a_i(u)^{1-\beta}$$

where $x_i(u)^{-\theta}$ is the total factor productivity, $a_i(u)$ is the quantity of the intermediate good used in the production of tradeable variety u and $s_i(u)$ is the labor input. Following Alvarez and Lucas (2007), we assume $x_i(u)$ follows an exponential distribution with parameter λ_i and its draws are independent across u (and across countries), allowing us to rewrite the above equations in terms of x. Each country $i \in \{1, 2, ...N\} \equiv \mathcal{N}$ is endowed with immobile labor L_i. Transportation costs between countries are iceberg in nature, where to keep the notation similar to Alvarez and Lucas (2007), we denote by $\kappa_{ij} \leq 1$ as the fraction arriving in location $j \in \mathcal{N}$ if one unit is set from location $i \in \mathcal{N}$. Tariffs ω_{ij} are defined as the proportion of revenue received by producer in country j for a unit of its tradeable good sold in country i. In addition, we define Y_{mi} as the revenue of the tradeables sector and I_i as the expenditure on tradeables in country i.

Equilibrium

The equilibrium can be characterized by three sets of equations. The first one corresponds to equation 3.8 in Alvarez and Lucas (2007):

$$p_{mi}^{-1/\theta} = \sum_{j \in \mathcal{N}} \lambda_j \left(\frac{1}{\kappa_{ij}} \frac{AB}{\omega_{ij}}\right)^{-1/\theta} \left(w_j^{\beta} p_{mj}^{1-\beta}\right)^{-1/\theta}. \tag{24}$$

Now we derive the other two, which are different from those in Alvarez and Lucas (2007) and convenient for the exercise here. Let L_{fi} and L_{mi} be the numbers of labor used in country i's production of the final and intermediate goods. We have $\alpha Y_{fi} = L_{fi} w_i$ and $\beta Y_{mi} = L_{mi} w_i$. Adding both sides of the two equations, we get

$$\alpha Y_{fi} + \beta Y_{mi} = L_i w_i.$$

Also let T_i be the total tariffs collected by country i. Notice that the residents' total income in country i is $L_i w_i + T_i$ and all used to buy the final goods. That is $Y_{fi} = L_i w_i + T_i$. Substitute the expression into the above displayed equation. We can solve

$$L_i w_i = \frac{\alpha}{1-\alpha} T_i + \frac{\beta}{1-\alpha} Y_{mi}. \tag{25}$$

Let I_j be the total expenditure on intermediate goods in country j. Then $D_{ji} I_j$ is the amount spent on intermediate goods from country i, of which $(1-\omega_{ji}) D_{ji} I_j$ is tariff and goes to the government and $\omega_{ji} D_{ji} I_j$ goes to the producer. Thus, we have $T_i = \sum_{j\in\mathcal{N}} (1-\omega_{ji}) D_{ji} I_j$ and $Y_{mi} = \sum_{j\in\mathcal{N}} \omega_{ji} D_{ji} I_j$. Insert them into equation (25). Then we get our second equilibrium equation

$$L_i w_i = \sum_{j\in\mathcal{N}} \frac{\alpha (1-\omega_{ji}) + \beta \omega_{ji}}{1-\alpha} D_{ji} I_j. \tag{26}$$

Furthermore, notice that producers' total expenditure $I_i + L_i w_i$ must be equal to their total income $Y_{mi} + Y_{fi}$ i.e. $I_i + L_i w_i = Y_{mi} + Y_{fi}$. Since $L_i w_i + T_i = Y_{fi}$, I_i must be equal to $Y_{mi} + T_i$. Substituting the expression of Y_{mi} and T_i, we then have

$$I_i = \sum_{j\in\mathcal{N}} D_{ji} I_j. \tag{27}$$

Although the above equilibrium equations can be simply transformed the one in Theorem 1, unfortunately, the corresponding spectral radius we get is larger than 1.[10] We move to impose a quasi-symmetry condition like Allen and Arkolakis (2014) that can allow us to reduce the three sets of equilibrium equations into two. Specifically, we assume $\kappa_{ij}\omega_{ij} = \tau_{ij} c_i r_j$ for some τ_{ij}, c_i, and r_j where τ is symmetric i.e. for any i, j, $\tau_{ij} = \tau_{ji}$.

Notice that $\sum_{j\in\mathcal{N}} D_{ij} = 1$. Multiplying it with both sides of equation (27), we get $\sum_{j\in\mathcal{N}} D_{ij} I_i = \sum_{j\in\mathcal{N}} D_{ji} I_j$. Substitute into the expression of equation $D_{ij} = \lambda_j \left(\frac{w_j^\beta p_{mj}^{1-\beta}}{p_{mi}}\right)^{-1/\theta} \left(\frac{AB}{\kappa_{ij}\omega_{ij}}\right)^{-1/\theta}$ and $\kappa_{ij}\omega_{ij} = \tau_{ij} c_i r_j$, then:

$$\sum_{j\in\mathcal{N}} \lambda_j \left(\frac{w_j^\beta p_{mj}^{1-\beta}}{p_{mi}}\right)^{-1/\theta} \left(\frac{AB}{\tau_{ij} c_i r_j}\right)^{-1/\theta} I_i = \sum_{j\in\mathcal{N}} \lambda_i \left(\frac{w_i^\beta p_{mi}^{1-\beta}}{p_{mj}}\right)^{-1/\theta} \left(\frac{AB}{\tau_{ji} c_j r_i}\right)^{-1/\theta} I_j.$$

On the left side of this equation, keep all the i-related variables (c_i, p_{mi}, and I_i) and move the rest (the summation) to the right side; similarly, on the right side of this equation, keep all the

[10] This does not necessarily imply multiplicity of solutions since $\{K_{ijh} > 0\}_{i,j\in\mathcal{N},h\in\mathcal{H}}$ are correlated unlike Part (iii). For example, K_{ijh} in equations (26) and (27) both depend on tariffs and thus are correlated with each other.

i-related variables (λ_i, r_i, w_i, and p_{mi},) and move the rest (the summation) to the left side. We have:

$$\frac{c_i^{1/\theta} p_{mi}^{1/\theta} I_i}{\sum_{j\in\mathcal{N}} \left(\frac{AB}{\tau_{ji}}\right)^{-1/\theta} c_j^{1/\theta} p_{mj}^{1/\theta} I_j} = \frac{\lambda_i r_i^{1/\theta} \left(w_i^{\beta} p_{mi}^{1-\beta}\right)^{-1/\theta}}{\sum_{j\in\mathcal{N}} \left(\frac{AB}{\tau_{ij}}\right)^{-1/\theta} \lambda_j r_j^{1/\theta} \left(w_j^{\beta} p_{mj}^{1-\beta}\right)^{-1/\theta}}.$$

Denote the numerators, $c_i^{1/\theta} p_{mi}^{1/\theta} I_i$ and $\lambda_i r_i^{1/\theta} \left(w_i^{\beta} p_{mi}^{1-\beta}\right)^{-1/\theta}$, as $\tilde{I}_i$ and $\tilde{w}_i$ respectively. Furthermore, denote $\left(\frac{AB}{\tau_{ij}}\right)^{-1/\theta}$ as $\tilde{\tau}_{ij}$. Notice that $\tilde{\tau}_{ij} = \tilde{\tau}_{ji}$. Then we can write the above equation as $\frac{\tilde{I}_i}{\sum_{j\in\mathcal{N}} \tilde{\tau}_{ij} \tilde{I}_j} = \frac{\tilde{w}_i}{\sum_{j\in\mathcal{N}} \tilde{\tau}_{ij} \tilde{w}_j}$, of which the value we denote as γ_i. Then we can write this equation as two equations:

$$\tilde{I}_i = \sum_{j\in\mathcal{N}} \gamma_i \tilde{\tau}_{ij} \tilde{I}_j;$$
$$\tilde{w}_i = \sum_{j\in\mathcal{N}} \gamma_i \tilde{\tau}_{ij} \tilde{w}_j.$$

Thus $\tilde{I}$ and $\tilde{w}$ can be viewed as two solutions of equation $x_i = \sum_{j\in\mathcal{N}} \gamma_i \tilde{\tau}_{ij} x_j$. According to Perron–Frobenius theorem, $\tilde{I}$ and $\tilde{w}$ is different at most up to scale i.e. there exists some constant a such that $\tilde{I}_i = a\tilde{w}_i$. Furthermore, substitute into the expression of $\tilde{I}_i$ and $\tilde{w}_i$ and we get

$$I_i = a\lambda_i r_i^{1/\theta} c_i^{-1/\theta} \left(w_i^{\beta} p_{mi}^{2-\beta}\right)^{-1/\theta}. \tag{28}$$

Notice that in this expression, I_i, w_i, and p_{mi} are nominal variables and we can scale them arbitrarily and get the corresponding a. Therefore, a simply reflects the normalization of nominal variables and without loss of generality, we set $a = 1$.

Substitute equation (28) into equation (26). Then, the equilibrium can be characterized by equations (24) and (26) where the endogenous variables are: p_{mi}, the price index of tradeables in country i; and w_i, country i's wage.

Applying Theorem 1

As in the previous example, the equilibrium of this system can be expressed in the special form of equation (1) presented in equation (8) in Remark 5. Now we show how to transform the equilibrium equations into the form of equation (8). To see this, denote $\lambda_j \left(\frac{1}{\kappa_{ij}} \frac{AB}{\omega_{ij}}\right)^{-1/\theta}$ in equation (24) as K_{ij}^1, so that it becomes

$$p_{mi}^{-1/\theta} = \sum_{j\in\mathcal{N}} K_{ij}^1 \left(w_j^{\beta} p_{mj}^{1-\beta}\right)^{-1/\theta}. \tag{29}$$

Second, substitute the expression of $D_{ij} = \lambda_j \left(\frac{w_j^{\beta} p_{mj}^{1-\beta}}{p_{mi}}\right)^{-1/\theta} \left(\frac{AB}{\kappa_{ij}\omega_{ij}}\right)^{-1/\theta}$ and $I_i = a\lambda_i r_i^{1/\theta} c_i^{-1/\theta} \left(w_i^{\beta} p_{mi}^{2-\beta}\right)^{-1/\theta}$ into equation (26), multiply both sides by $\left(w_i^{\beta} p_{mi}^{1-\beta}\right)^{1/\theta} L_i^{-1}$, so that equation (26) becomes:

$$w_i^{1+\beta/\theta} p_{mi}^{(1-\beta)/\theta} = \sum_{j\in\mathcal{N}} K_{ij}^2 \left(w_j^{\beta} p_{mj}^{1-\beta} \right)^{-1/\theta}, \tag{30}$$

where $K_{ij}^2 \equiv \frac{\alpha(1-\omega_{ji})+\beta\omega_{ji}}{1-\alpha} \left(\frac{AB}{\kappa_{ji}\omega_{ji}}\right)^{-1/\theta} \lambda_i \lambda_j r_j^{1/\theta} c_j^{-1/\theta} L_i^{-1}$.

Now we have transformed the equilibrium equations into the form (8) and with two set of endogenous variables $\{p_{mi}, w_i\}_{i=1,2,...,n}$. Notice that K_{ij}^1 and K_{ij}^2, defined above are positive when $\alpha, \beta, \theta > 0$ and $0 < \omega_{ij} \leq 1$.

Then we have the corresponding parameter matrices

$$\Gamma = \begin{pmatrix} -1/\theta & 0 \\ (1-\beta)/\theta & 1+\beta/\theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -(1-\beta)/\theta & -\beta/\theta \\ -(1-\beta)/\theta & -\beta/\theta \end{pmatrix}$$

Clearly, Γ is always invertible as long as $\theta > 0$. Therefore, we have

$$\left|\mathbf{B\Gamma^{-1}}\right| = \begin{pmatrix} \frac{(1-\beta)\theta}{\beta+\theta} & \frac{\beta}{\beta+\theta} \\ \frac{(1-\beta)\theta}{\beta+\theta} & \frac{\beta}{\beta+\theta} \end{pmatrix}$$

Here $\rho\left(\left|\mathbf{B\Gamma^{-1}}\right|\right) = \frac{\beta+\theta-\beta\theta}{\beta+\theta} < 1$ i.e. we always have (up-to-scale) uniqueness with quasi-symmetry trade costs κ_{ij} and tariffs ω_{ij}. In comparison, the conditions for uniqueness in Alvarez and Lucas (2007) (see their Theorem 3) are:

$$\text{(i)} \left(\min_{i,j\in\mathcal{N}^2}\{\kappa_{ij}\} \min_{i,j\in\mathcal{N}^2}\{\omega_{ij}\}\right)^{\frac{2}{\theta}} \geq 1-\beta; \text{(ii)}\ \alpha \geq \beta; \text{(iii)}\ 1 - \min_{i,j\in\mathcal{N}^2}\{\omega_{ij}\} \leq \frac{\theta}{\alpha-\beta},$$

although these conditions are derived only for the special case of uniform tariffs (i.e. $\omega_{ij} = \omega_i$ for all $j \in \mathcal{N}$).

A.4 A production network with multiple intermediates goods

We extend the many firm production network in the seminal paper by Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012) to include (1) a constant elasticity of substitution (CES) aggregator across labor and intermediates (as discussed in Carvalho and Tahbaz-Salehi (2019)), (2) a constant elasticity of substitution between intermediate goods (as discussed in Carvalho, Nirei, Saito, and Tahbaz-Salehi (2021)), and (3) multiple types of intermediates goods.

A.4.1 The model

Setup There are N different competitive firms, each of which produce H distinct products using as intermediate goods the output of all other firms. The quantity of product H by firm $i \in \{1, ..., N\} \equiv \mathcal{N}$, Q_{ih}, is determined by a constant elasticity of substitution production function combining labor and a composite bundle of intermediate goods as follows:

$$Q_{ih} = \left[(1-\mu_{ih})^{\frac{1}{\zeta_h}} (A_{ih} L_{ih})^{\frac{\zeta_h - 1}{\zeta_h}} + \mu_{ih}^{\frac{1}{\zeta_h}} M_{ih}^{\frac{\zeta_h-1}{\zeta_h}} \right]^{\frac{\zeta_h}{\zeta_h - 1}}$$

where μ_{ih} is between 0 and 1 and governs the relative importance of labor and intermediate goods, L_{ih} is the amount of labor, A_{ih} is the (exogenous) labor productivity, ζ_h is the elasticity of substitution between labor and intermediates, and the intermediate input bundle M_{ih} is a CES aggregate of inputs purchased from other firms:

$$M_{ih} = \prod_{h' \in \mathcal{H}} \left(\left(\sum_{j \in \mathcal{N}} a_{jih'h}^{\frac{1}{\sigma_{h'}}} q_{jih'h}^{\frac{\sigma_{h'}-1}{\sigma_{h'}}} \right)^{\frac{\sigma_{h'}}{\sigma_{h'}-1}} \right)^{\beta_{h'h}},$$

where $\sigma_{h'}$ is the elasticity of substitution between different intermediate goods, $a_{jih'h}$ represents the production network of firms j supplying intermediates h' in firm i's production of product h, $q_{jih'h}$ is the quantity supplied, and $\{\beta_{h'h}\}_{h' \in \mathcal{H}}$ is the intermediates share satisfying $\sum_{h'} \beta_{h'h} = 1$ for all h.

Equilibrium Solving the cost minimization problem of the firm results in the following system of equations for equilibrium firm prices p_{ih}:

$$p_{ih}^{1-\zeta_h} = (1-\mu_{ih})\,(w/A_{ih})^{1-\zeta_h} + \mu_{ih} \prod_{h' \in \mathcal{H}} \beta_{h'h}^{-\beta_{h'h}} \left(\sum_{j \in \mathcal{N}} a_{jih'h} p_{jh'}^{1-\sigma_{h'}} \right)^{\frac{\beta_{h'h}}{1-\sigma_{h'}}(1-\zeta_h)} \tag{31}$$

where w is the (exogenous) market wage.

A.4.2 Applying Theorem 1

Take both sides of equation (31) to the power of $\frac{1}{1-\zeta_h}$. Then, due to Remark 1 it takes the form of equation (1). Denote its right side as $F_{ih}(\cdot)$. We can directly bound its elasticity as follows:

$$\sum_{j \in \mathcal{N}} \left| \frac{\partial \ln F_{ih}}{\partial \ln p_{jh'}} \right| < \beta_{h'h}.$$

Since $\sum_{h'} \beta_{h'h} = 1$, according to Remark 1, we have $\rho(\beta)$=1. Thus by Theorem 1 (part ii.a) and Remark 1, there exists at most one equilibrium. To our knowledge, this is the first proof of uniqueness of an equilibrium in a many firm production network with multiple types of intermediates goods and constant elasticity of substitution between different types of intermediate goods and between the intermediate goods bundle and labor.

A.5 Identification of productivities in a production network model with many locations and sectors

We next consider input-output production networks with many locations and sectors as in the work of Caliendo and Parro (2015). The purpose of this is two-fold: first, it demonstrates how Theorem 1 can be applied to establish identification results; second, it demonstrates the ubiquity of economic situations where $\rho(\mathbf{A}) = 1$, highlighting the importance of part (ii) of Theorem 1.

A.5.1 The Model

Setup Consider an economy comprised of $i \in \{1, ..., N\} \equiv \mathcal{N}$ locations and $h \in \{1, ..., H\} \equiv \mathcal{H}$ sectors. Each sector h in location i produces a differentiated intermediate good (denoted as good (i,h)) by combining local labor with a Cobb-Douglas combination of a CES composite of intermediates from all locations according to the following production function:

$$Q_{ih} = A_{ih} L_{ih}^{\alpha_h} \prod_{h' \in \mathcal{H}} \left(\left(\sum_{j \in \mathcal{N}} q_{jih'h}^{\frac{\sigma_{h'}-1}{\sigma_{h'}}} \right)^{\frac{\sigma_{h'}}{\sigma_{h'}-1}} \right)^{\beta_{h'h}},$$

where $q_{jih'h}$ is the quantity of the good (j,h') used as an intermediate good in the production of good (i,h), $\{\sigma_h\}_{h\in\mathcal{H}}$ are the sector elasticities of substitution across locations, $\{\alpha_h\}_{h\in\mathcal{H}}$ are the sector labor shares, and $\mathbf{B} \equiv [\beta_{h'h}]$ is an $H \times H$ input-output matrix of intermediate inputs, and $\{A_{ih}\}_{i\in\mathcal{N}}^{h\in\mathcal{H}}$ are the productivities of each sector-location. The shipment of good (j,h') from $j \in \mathcal{N}$ to $i \in \mathcal{N}$ incurs an iceberg trade cost $\tau_{ijh'} \geq 1$.

Suppose that each location $i \in \mathcal{N}$ is endowed with L_i agents, each of whom is perfectly mobile across sectors and earns (equilibrium) wage w_i for inelastically supplying one unit of labor. Agents use their wages to consume a non-traded final good produced by combining intermediate goods with the production function $C_i = \prod_{h\in\mathcal{H}} M_{ih}^{\gamma_h}$, where $\sum_{h\in\mathcal{H}} \gamma_h = 1$ are the consumption shares of each sector, $M_{ih} = \left(\sum_{j\in\mathcal{N}} m_{jh}^{\frac{\sigma_h-1}{\sigma_h}} \right)^{\frac{\sigma_h}{\sigma_h-1}}$, and m_{jh} is the quantity of the good (j,h) used in the production of final good.

Equilibrium From the cost minimization, the equilibrium price of the intermediate good produced by sector $h \in \mathcal{H}$ in location $i \in \mathcal{N}$ is:

$$p_{ih} = c_h \frac{1}{A_{ih}} w_i^{\alpha_h} \prod_{h'\in\mathcal{H}} P_{ih'}^{\beta_{h'h}}, \tag{32}$$

where

$$P_{ih}^{1-\sigma_h} = \sum_{j\in\mathcal{N}} \tau_{jih}^{1-\sigma_h} p_{jh}^{1-\sigma_h} \tag{33}$$

is a sector-location price index of intermediate goods purchased in all locations, and $c_h > 0$ is a exogenous constant.[11]

Let $Y_{ih} \equiv p_{ih} Q_{ih}$ denote the income of sector $h \in \mathcal{H}$ in location $i \in \mathcal{N}$, which in equilibrium is equal to its total sales to all locations and sectors:

$$Y_{ih} = \sum_{j\in\mathcal{N}} \tau_{ijh}^{1-\sigma_h} p_{ih}^{1-\sigma_h} P_{jh}^{\sigma_h-1} \sum_{h'\in\mathcal{H}} (\beta_{hh'} + \gamma_h \alpha_{h'}) Y_{jh'}, \tag{34}$$

where the two terms in the last summation captures how much spending in sector (j,h') translates to spending in sector (i,h) through intermediate production and final good purchases by consumers, respectively.

Identification The question we are interested in is the following. Suppose one observes (1) the sales of each sector $h \in \mathcal{H}$ in each location $i \in \mathcal{N}$, i.e. $\{Y_{ih}\}_{i\in\mathcal{N}}^{h\in\mathcal{H}}$; (2) the labor endowment $\{L_i\}_{i\in\mathcal{N}}$; (3) the sector elasticities $\{\sigma_h\}_{h\in\mathcal{H}}$; (4) the sector production function labor shares $\{\alpha_h\}_{h\in\mathcal{H}}$ and input-output matrix $\mathbf{B} \equiv [\beta_{h'h}]$; (5) the final good production shares $\{\gamma_h\}_{h\in\mathcal{H}}$;and (6) the sector-

[11] In particular, $c_h \equiv \alpha_h^{-\alpha_h} \prod_{h'\in\mathcal{H}} \beta_{h'h}^{-\beta_{h'h}}$.

specific bilateral trade costs $\{\tau_{ijh}\}_{i,j\in\mathcal{N}}^{h\in\mathcal{H}}$. Is it possible to identify the productivity of each sector $h\in\mathcal{H}$ in each location $i\in\mathcal{H}$, $\{A_{ih}\}_{i\in\mathcal{N}}^{h\in\mathcal{H}}$? One could imagine many instances where recovering the underlying productivities of different sectors in different locations from observed sales data is useful and important: e.g. in the study of comparative advantage, structural change, technological innovations, etc.

A.5.2 Applying Theorem 1

We begin by remarking that since wages can be inferred directly from the observed labor share of income and labor endowment, given knowledge of prices $\{p_{ih}\}_{i\in\mathcal{N}}^{h\in\mathcal{H}}$ and price indices $\{P_{ih}\}_{i\in\mathcal{N}}^{h\in\mathcal{H}}$, one can immediately recover productivities $\{A_{ih}\}_{i\in\mathcal{N}}^{h\in\mathcal{H}}$ from equation (32). Hence, it is sufficient to focus on the question of identification of prices and price indices.

Define the $2H\times 1$ endogenous vector $\boldsymbol{x}_i=\left[\left\{P_{ih}^{1-\sigma_h}\right\}_{h=1}^{H},\left\{p_{ih}^{\sigma_h-1}\right\}_{h=1}^{H}\right]$ so that equations (33) and (34) can be written as:

$$x_{ih}=\begin{cases}\sum_j K_{ij,h}x_{j,h+H}^{-1} & \text{if } h\in\{1,...,H\}\\ \sum_j F_{ij,h}x_{j,h-H}^{-1} & \text{if } h\in\{H+1,...,2H\}\end{cases}$$

where $K_{ij,h}\equiv\tau_{jih}^{1-\sigma_h}$ for $h\in\{1,...,H\}$ and $F_{ij,h}\equiv\tau_{ijh}^{1-\sigma_h}\left(\frac{\sum_{h'\in\mathcal{H}}(\beta_{hh'}+\gamma_h\alpha_{h'})Y_{jh'}}{Y_{ih}}\right)$ for $h\in\{H+1,...,2H\}$. As a result, we can define the $2H\times 2H$ matrix of elasticity bounds as $\mathbf{A}\equiv\begin{pmatrix}\mathbf{0} & \mathbf{I}_H\\ \mathbf{I}_H & \mathbf{0}\end{pmatrix}$,where $\mathbf{I}_H$ is the $H\times H$ identity matrix. Regardless of the particular input output structure (or the values of labor shares, final goods shares, or sector elasticities) we have $\rho(\mathbf{A})=1$, and so from Theorem 1 part (ii) there is at most one set of (column-wise to scale unique) prices $\{p_{ih}\}_{i\in\mathcal{N}}^{h\in\mathcal{H}}$ and price indices $\{P_{ih}\}_{i\in\mathcal{N}}^{h\in\mathcal{H}}$ consistent with equations (33) and (34). Thus, there is at most one (column-wise up to scale) unique set of productivities $\{A_{ih}\}$ consistent with observed sales data.[12]

A.6 A forward-looking migration model with agglomeration spillovers

We next consider a dynamic migration framework. The model is based on the work of Artuç, Chaudhuri, and McLaren (2010), extended into general equilibrium as in Caliendo, Dvorkin, and Parro (2019) with productivity and amenity spillovers as in Allen and Donaldson (2020). Here we consider a version of the framework where all locations produce a homogeneous and costlessly traded numeraire good.

Setup There are $i\in\{1,...,N\}\equiv\mathcal{N}$ locations inhabited by identical agents that are mobile across space. Time $t\in\{0,1,...,T\}\equiv\mathcal{T}$ is discrete and finite.[13] In each period $t\in\mathcal{T}$, agent v derives a period utility $\ln(u_{i,t}A_{i,t})$ from living at location $i\in\mathcal{N}$, where $u_{i,t}$ and $A_{i,t}$ refer to the amenity and productivity at location i, respectively. The agent discounts the future at a rate $\delta<1$.

We denote the value of living at location i at period t as $v_{i,t}$. For period $t=T$, this value is simply the period utility i.e. $v_{i,T}=\ln(u_{i,T}A_{i,T})$. For any period $t\leq T-1$, this value depends

[12] The column-wise up to scale uniqueness implies that the relative productivity within sector across locations can be identified from sales data, but the relative productivity across sectors cannot; intuitively, if the productivity of sector h doubles in all locations, given the unit price elasticity from the presumed Cobb-Douglas production function, its price will half, leaving its sales unchanged.

[13] Extending the results below to an infinite T is an interesting avenue for future research.

on both her period utility at location i and her highest utility of moving to another location next period:

$$v_{it} = \ln\left(u_{i,t} A_{i,t}\right) + \delta \mathbb{E}_t \left[\max_{j \in \mathcal{N}} \left(v_{j,t+1} - \mu_{ij,t+1} + \epsilon_{ij,t+1}\right)\right],$$

where $\mu_{ij,t+1}$ represents the migration cost from i to j, $\epsilon_{ij,t+1}(v)$ is the idiosyncratic utility shock and $\mathbb{E}_t$ is the expectation operator. Assuming that $\epsilon_{ij,t+1}(v)$ follows a Type-I Extreme Value distribution with zero mean and shape parameter θ, we obtain:

$$V_{it} \equiv \exp\left(v_{i,t}\right) = u_{i,t} A_{i,t} \left(\sum_j T_{ij,t+1}^{-\theta} V_{j,t+1}^{\theta}\right)^{\frac{\delta}{\theta}} \text{ for all } t \leq T-1, \tag{35}$$

where $T_{ijt} \equiv \exp\left(-\mu_{ij,t}\right)$ and

$$V_{i,T} = u_{i,T} A_{i,T}, \text{ for } t = T. \tag{36}$$

Spillovers We suppose that agents' location choices can endogenously affect the productivity A_{jt} and amenity u_{jt} of residing in a location. Specifically, these depend both on the innate productivity (amenity) of block i, $\bar{A}_{it}$ ($\bar{u}_{it}$), and the number of each type of agents working (residing) in that location:

$$A_{i,t} = \bar{A}_{i,t} L_{i,t}^{\alpha}; u_{i,t} = \bar{u}_{i,t} L_{i,t}^{\beta}. \tag{37}$$

Equilibrium For any given geography $\left\{\mu_{ij,t}, \bar{A}_{i,t}, \bar{u}_{i,t}\right\}_{i \in \mathcal{N}}^{t \in \mathcal{T}}$, initial population distribution $\left\{L_{i,0}\right\}_{i \in \mathcal{N}}$ and model parameters $\{\alpha, \beta, \theta, \delta\}$, an equilibrium is a pair of sequences of populations and values $\{L_{i,t}, V_{i,t}\}$ such that equations (36), (37) hold, equilibrium condition (35) holds, and the choice probability of agent in i to be in location j in period t, $\pi_{ij,t}$, is obtained from the value function (35), resulting in the following equilibrium law of motion of labor:

$$L_{it} = \sum_{j \in \mathcal{N}} \pi_{ij,t} L_{j,t-1} = \sum_{j \in \mathcal{N}} \frac{T_{ji,t}^{\theta} V_{i,t}^{\theta}}{\sum_{k \in \mathcal{N}} T_{jk,t}^{\theta} V_{k,t}^{\theta}} L_{j,t-1}, \tag{38}$$

which holds for all $i \in \mathcal{N}$ and $t \in \mathcal{T}$.

Theorem 1, parts (i) and (ii) We now apply Theorem 1 to characterize the equilibrium of the model. We proceed in three steps.

Step 1: Re-write the equilibrium in the form of equation (1) We first define $G_{i,t} \equiv \left(\sum_j T_{ij,t}^{\theta} V_{j,t}^{\theta}\right)^{\frac{1}{\theta}}$, so that equation (35) becomes $V_{i,t} = u_{i,t} A_{i,t} G_{i,t+1}^{\delta}$. Substituting this expression and equation (37) back into the definition of G_{it} yields:

$$G_{it}^{\theta} = \sum_{j \in \mathcal{N}} K_{ij,t} L_{j,t}^{\gamma\theta} G_{j,t+1}^{\delta\theta},$$

where $\gamma \equiv \alpha + \beta$ and $K_{ij,t} = T_{ij,t}^{\theta} \bar{A}_{j,t}^{\theta} \bar{u}_{j,t}^{\theta}$. Similar substitutions in equation (38) yields:

$$L_{i,t}^{1-\gamma\theta} G_{i,t+1}^{-\theta\delta} = \sum_{j \in \mathcal{N}} K_{ji,t} G_{j,t}^{-\theta} L_{j,t-1}.$$

We then apply the logic of Remark 5 and further rewrite equilibrium equations (35) and (38) in the format of equation (8), by setting $x_{i,t} \equiv G_{i,t}^{\theta}$ and $y_{i,t} \equiv L_{i,t}^{1-\gamma} G_{i,t+1}^{-\theta\delta}$ so that we obtain for all $t < T$:

$$x_{i,t} = \sum_{j \in \mathcal{N}} K_{ij,t} y_{j,t}^{\frac{\gamma\theta}{1-\gamma\theta}} x_{j,t+1}^{\frac{\delta}{1-\gamma\theta}}, \tag{39}$$

$$y_{i,t} = \sum_{j \in \mathcal{N}} K_{ji,t} x_{j,t}^{\frac{\delta-1+\gamma\theta}{1-\gamma\theta}} y_{j,t-1}^{\frac{1}{1-\gamma\theta}}. \tag{40}$$

Following this process $t = T$ yields:

$$x_{i,T} = \sum_{j \in \mathcal{N}} K_{ij,T} y_{j,T}^{\frac{\gamma\theta}{1-\gamma\theta}}; \tag{41}$$

$$y_{i,T} = \sum_{j \in \mathcal{N}} K_{ji,T} x_{j,T}^{\frac{\delta-1+\gamma\theta}{1-\gamma\theta}} y_{j,T-1}^{\frac{1}{1-\gamma\theta}}. \tag{42}$$

Equations (39)–(42) comprise an $N \times 2T$ system of equations in the form of equation (1) (with $H = 2T$), as required.

Step 2: Construct the matrix of elasticity bounds A We now construct the matrix of elasticity bounds $\mathbf{A}$. Because the elasticities are constant, the bounds are simply the elasticities themselves. Moreover, because the system of equations only depends on the endogenous outcomes in the prior period, current period, and subsequent period, we can write the matrix $\mathbf{A}$ solely as a function of the following three matrices corresponding to elasticities of x_{it}, y_{it} with respect to each of these two variables at the same time period t, at time period $t+1$, and at time period $t-1$:

$$\boldsymbol{M_D} \equiv \begin{pmatrix} 0 & \left|\frac{\gamma\theta}{1-\gamma\theta}\right| \\ \left|\frac{\delta-1+\gamma\theta}{1-\gamma\theta}\right| & 0 \end{pmatrix}, \; \boldsymbol{M_U} \equiv \begin{pmatrix} \left|\frac{\delta}{1-\gamma\theta}\right| & 0 \\ 0 & 0 \end{pmatrix}, \; \boldsymbol{M_L} \equiv \begin{pmatrix} 0 & 0 \\ 0 & \left|\frac{1}{1-\gamma\theta}\right| \end{pmatrix},$$

so that:

$$\mathbf{A} \equiv \begin{pmatrix} \boldsymbol{M_D} & \boldsymbol{M_U} & 0 & \cdots & 0 \\ \boldsymbol{M_L} & \boldsymbol{M_D} & \boldsymbol{M_U} & \cdots & 0 \\ 0 & \boldsymbol{M_L} & \boldsymbol{M_D} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \boldsymbol{M_U} \\ 0 & 0 & 0 & \boldsymbol{M_L} & \boldsymbol{M_D} \end{pmatrix}.$$

From parts (i) and (ii.b) of Theorem 1, the equilibrium system is unique if $\rho(\mathbf{A}) \leq 1$. The next (optional) step helps to provide additional economic intuition.

Step 3: Simplify the condition on the spectral radius Suppose that $\gamma \equiv \alpha + \beta \leq 0$, i.e. the spillovers are net dispersive. The Collatz–Wielandt Formula implies that the spectral radius can be bounded above by the maximum of the sum of the absolute value of the elements of each columns of the matrix; as a result, if the sum of every column of $\mathbf{A}$ is less than or equal to one, then uniqueness is assured. Consider the first column. Since $\delta < 1$ and $\gamma \leq 0$, we have:

$$\left|\frac{\delta - 1 + \gamma\theta}{1 - \gamma\theta}\right| = \frac{1 - \gamma\theta - \delta}{1 - \gamma\theta} < 1.$$

Similarly, it is straightforward to show that the sum of any other odd column or any even column is equal to one. Hence, as long as $\gamma \leq 0$, then for any $\delta < 1$, we have $\rho(|\mathbf{A}|) \leq 1$, and the equilibrium system is unique.

Theorem 1, part (iii): Multiplicity Part (iii) of Theorem 1 implies that for any choice of model parameters such that $\rho(\mathbf{A}) > 1$, there will exist a geography such that there are multiple equilibria. We illustrate this in a simple economy with two identical locations with symmetric migration costs ($N = 2$, $\bar{A}_i = \bar{u}_i = 1$, $\mu_{ijt} = \mu$ if $i \neq j$, and $\mu_{ijt} = 0$ if $i = j$) initially inhabited by an equal number of agents with preferences defined by a discount parameter of $\delta = 0.99$ and migration elasticity of $\theta = 2$. When $\gamma > 0$ (i.e. the spillovers are net agglomerative), as long as the migration costs μ are sufficiently large, there exist three possible equilibria: an (unstable) equilibrium where both locations remain equally populated, and another type of equilibrium where economic activity becomes concentrated in one of the two locations to take advantage of the agglomeration economies.

Comparison to previously known results To our knowledge, little is known about the equilibrium properties of a dynamic economic geography model. Kleinman, Liu, and Redding (2023) consider a log-linearized version of a dynamic economic geography model but do not characterize the non-linear system. Allen and Donaldson (2020) provide conditions for uniqueness, but those conditions themselves are written in terms of properties of corresponding second-order linear difference equations. Bilal (2023) provides sufficient conditions for local uniqueness around the steady state(s) for infinite lived agents in continuous time but does not consider the global uniqueness of the economy. In contrast, the results here—albeit in a setting with only migration costs and no trade frictions—are simple and straightforward: if spillovers are net dispersive, i.e. as long as agents would prefer to reside apart from each other, uniqueness is assured. It is worthwhile noting that this condition is identical to the one given by Allen and Arkolakis (2014) for a static setting with trade costs and perfect labor mobility.

A.7 Social interactions with many types of networks

We consider a social network based on the work of Ballester, Calvó-Armengol, and Zenou (2006) (as summarized in the review article of Jackson and Zenou (2015)) where agents' payoffs depend both on their own actions as well as the actions of others in their social network. We extend that framework to incorporate (a) flexible impacts of others' actions on one's own payoffs; (b) many different types of actions; and (c) many different types of networks. Allowing agents' different types of actions through different networks—and for those choices to flexibly affect the payoffs of all other agents' actions—enables the study of a variety of empirically relevant social interactions, including e.g. the interdependent nature of different types actions on different social networks (friends, family, work, etc).

Setup There are $i \in \{1, ..., N\} \equiv \mathcal{N}$ agents, each of whom decides how much effort x_{ih} to exert on each activity $h \in \{1, ..., H\} \equiv \mathcal{H}$. Agent i's payoff from activity h is:

$$u_{ih}\left(\left\{x_{jh'}\right\}_{j\in\mathcal{N}}^{h'\in\mathcal{H}}\right) = c_{ih}x_{ih} - \frac{x_{ih}^2}{2} + x_{ih}\sum_{j\neq i} f_{ijh}\left(\left\{x_{jh'}\right\}_{h'\in\mathcal{H}}\right),$$

where $c_{ih} > 0$ is the (constant) marginal own benefit of effort, costs are quadratic in effort, and $f_{ijh}(\cdot) \geq 0$ is a function capturing the network of type h and how others' efforts in all activities affect agent i's payoff in activity h. Agent i's overall utility is given by:

$$u_i(x) = m\left[u_{i1}(x), ..., u_{iH}(x)\right],$$

where $m(\cdot)$ is a monotonic function increasing in each of its arguments.

Example For the purpose of illustration, consider a simple example with $H = 2$ where $f_{ij1}\left(\left\{x_{jh'}\right\}_{h'\in\mathcal{H}}\right) = K_{ij1}x_{j1}^{\alpha_{11}}x_{j2}^{\alpha_{12}}$ and $f_{ij2}\left(\left\{x_{jh'}\right\}_{h'\in\mathcal{H}}\right) = K_{ij2}x_{j1}^{\alpha_{21}}x_{j2}^{\alpha_{22}}$. Here, $\left\{x_{j1}\right\}_{j\in\mathcal{N}}$ and $\left\{x_{j2}\right\}_{j\in\mathcal{N}}$ can be agents' incomes and educations, respectively; correspondingly, $\left\{K_{ij1}\right\}_{i,j\in\mathcal{N}}$ and $\left\{K_{ij2}\right\}_{i,j\in\mathcal{N}}$ reflect the economic and education networks, through which incomes and educations are determined, with $\left\{\alpha_{ij}\right\}_{i,j\in\{1,2\}}$ capturing the interaction between incomes and educations.

Equilibrium Agent i choose her efforts $\left\{x_{ih}\right\}_{h\in\mathcal{H}}$ to maximize her utility $u_i(x)$. The first order conditions give the best response function of agent $i \in \mathcal{N}$ for action $h \in \mathcal{H}$ to all other agents actions:

$$x_{ih} = c_{ih} + \sum_{j\neq i} f_{ijh}\left(\left\{x_{jh'}\right\}_{h'\in\mathcal{H}}\right), \tag{43}$$

which is a special case of equation (1) (where $f_{iih}(x_{j1}, ..., x_{jH}) = c_{ih}$). We note that Ballester, Calvó-Armengol, and Zenou (2006) consider a single network ($H = 1$) and a linear spillover function ($f_{ij}(x_j) = g_{ij}x_j$).

Theorem 1, part (i): General spillovers Suppose that the elasticities of the spillover function can be bounded, i.e. for all $h, h' \in \mathcal{H}$ there exists an $\alpha_{hh'} \geq 0$ such that $\left|\frac{\partial \ln f_{ijh}}{\partial \ln x_{jh'}}\right| \leq \alpha_{hh'}$ for all $\left\{x_{jh'}\right\}_{h'\in\mathcal{H}}$. Let $\mathbf{A}$ be the $H \times H$ matrix whose (h, h') element is $\alpha_{hh'}$. From Theorem 1 part (i), there exists a unique strictly positive equilibrium if $\rho(\mathbf{A}) < 1$. Moreover, that equilibrium can be reached from any initial strictly positive starting point $\left\{x_{jh'}^0\right\}_{j\in\mathcal{N}}^{h'\in\mathcal{H}}$ by iteration of equation (43). Note that the iterative procedure here has the simple economic intuition as an application of best-response dynamics, i.e. from any initial starting point, the unique equilibrium can be reached as an iterative application of agents' best-responses (see e.g. section 6 of Parise and Ozdaglar (2019)). Finally, while there may also be weakly positive solutions, from Remark 2 any such solutions will be asymptotically unstable, in the sense of e.g. Weibull (1997).[14]

Theorem 1, part (ii): Constant elasticity spillovers Consider the above example of income and education networks. Then from Theorem 1 part (ii), there is at most one equilibrium if $\rho(\mathbf{A}) \leq 1$.

Theorem 1, part (iii): Multiplicity It is sufficient to consider a two agent single network ($N = 2$, $H = 1$) with constant elasticity social spillover $f_{ij} = g_{ij}x_j^{\alpha}$. For any $\alpha > 1$ it is straightforward to confirm that the payoff structure of $c_1 = c_2 = 1 - \frac{1}{2\alpha}$ and $g_{12} = g_{21} = \frac{1}{2\alpha}$ has at least two solutions: a low-effort symmetric equilibrium $(x_1, x_2) = (1, 1)$ and a high-effort symmetric equilibrium $(x_1, x_2) = (M, M)$, where M is the maximal root of the equation $x^{\alpha} - 2\alpha x + 2\alpha - 1 = 0$.

[14] In Online Appendix A.11, we extend the analysis here to consider the uniqueness of weakly positive solutions in the setting where the best response functions are linear (i.e. $\rho(\mathbf{A}) = 1$).

Comparison to previously known results We view our results are complementary to existing results in the social network literature.

In the baseline network model where $H = 1$ and $f_{ij} = g_{ij} x_j$, Ballester, Calvó-Armengol, and Zenou (2006) show that there exists a unique interior solution if $\rho(\mathbf{G}) < 1$, where $\mathbf{G}$ is the $N \times N$ matrix with (i, j) element g_{ij}, i.e. they provide a condition on the spectral radius of the *network structure*. In contrast, Theorem 1.ii.a shows there exists at most one interior equilibrium, as the *elasticities* of $f_{ij} = g_{ij} x_j$ and $f_{ii} = c_i$ are equal to and smaller than 1, respectively. Combining these results in the $\alpha = 1$ case illustrates that the condition $\rho(\mathbf{G}) < 1$ guarantees the existence of an interior equilibrium, but not uniqueness (indeed, if $\rho(\mathbf{G}) \geq 1$, there exists no interior solution).

Bramoullé, Kranton, and D'amours (2014) and Galeotti, Golub, and Goyal (2020) extend the Ballester, Calvó-Armengol, and Zenou (2006) framework to the case where actions can be substitutes by allowing possibly negative $\mathbf{G}$ and offer similar conditions for uniqueness as Ballester, Calvó-Armengol, and Zenou (2006) based on the network structure. While Theorem 1 does not allow negative f_{ijh}, it can incorporate situations where actions are substitutes through negative payoff elasticities.[15] In the example above, income and education can be substitutes if α_{12} and α_{21} are negative. Moreover, while there may exist non-interior weakly positive equilibria, Remark 2 guarantees that the only stable equilibria is the unique strictly positive solution when the spectral radius $\rho(\mathbf{A}) < 1$.

As in Allouch (2015); Acemoglu, García-Jimeno, and Robinson (2015) and Chen, Zenou, and Zhou (2018), the setup above also extends the Ballester, Calvó-Armengol, and Zenou (2006) framework to include non-linearity and multiple activities. However, the setup above also extends the framework to allow for multiple networks, something (to the best of our knowledge) for which positive properties have not been previously characterized, despite the empirical importance of simultaneous social interactions across multiple types networks (see e.g. Christakis and Fowler (2009); Banerjee, Chandrasekhar, Duflo, and Jackson (2013)). Our characterization emphasizes that the positive properties of the equilibrium multi-network system can be characterized in terms of a single statistic: the spectral radius of the matrix of the *elasticities* of the social interactions.

A.8 Discrete choice over many actions with social interactions

Here we consider a discrete choice framework with social interactions as in the seminal paper of Brock and Durlauf (2001), generalized to include agents simultaneously choosing over many types of actions with flexible social spillovers across agents and actions.

A.8.1 The model

Setup Suppose there are N agents. Each agent $i \in \{1, ..., N\} \equiv \mathcal{N}$ makes a discrete choice over $\mathcal{H} \equiv \{1, ..., H\}$, a set of H actions.

$$V_{ih}(\mu_i) = u_{ih} + S_{ih}(\mu_i) + \varepsilon_{ih},$$

where u_{ih} is the private utility associated with action h, $S_{ih}(\mu_i)$ is the social utility, μ_i is agent i's belief of other agents' actions, and ε_{ih} is a random utility term, independently and identically

[15] Note that Theorem 1's parallel result, Remark 3, does allow negative f_{ijh}. Remark 3 is also complementary with existing works on social networks by enabling the characterization of non-symmetric networks and settings with multiple actions in multiple networks (see Online Appendices A.10 and A.11).

distributed across agents and actions. Agent chooses action

$$\arg\max_{h\in\mathcal{H}} V_{ih}(\mu_i)$$

that maximizes her payoffs given her belief of the actions of others.Define μ_{ijh} to be the conditional probability measure agent i places on the probability that agent j chooses action h. We assume that $S_{ih}(\mu_i)$ takes the following form:

$$S_{ih}(\mu_i) = \sum_{h'\in\mathcal{H}} J_{hh'} \ln(\overline{\mu}_{ih'}),$$

where $J_{hh'}$ reflects the impact of action h' by others on agent i's utility when she chooses action h, $\overline{\mu}_{ih'} \equiv \sum_{j\neq i} w_{ijh'}\mu_{ijh'}$ is her (weighted) expected number of agents taking action h', $w_{ijh'}$ is the corresponding weight, and $\mu_{ijh'}$ is her belief of agent i taking action h'. We note that the log transform on the social utility function – not present in the primary case considered by Brock and Durlauf (2001) – ensures that the uniqueness of the equilibrium can be characterized without reference to an (endogenous) threshold value (c.f. Brock and Durlauf (2001) Proposition 2).

Equilibrium Retaining the assumption from Brock and Durlauf (2001) that the random utility term follows a type I extreme value distribution with shape parameter β_h and agent's conditional probabilities are rational (so that $\mu_{ijh} = \mu_{jh}$ for all $j \in \{1, ..., N\}$ and μ_{jh} is equal to the probability agent j actually chooses action h) results in the following equilibrium conditions for all $i \in \{1, ..., N\}$ and for all $h \in \{1, ..., H\}$:

$$\mu_{ih} = \frac{e^{\beta_h u_{ih}}\left(\Pi_{h'\in\mathcal{H}}\left(\sum_{j\neq i} w_{ijh'}\mu_{jh'}\right)^{\beta_h J_{hh'}}\right)}{\sum_{k\in\mathcal{H}} e^{\beta_k u_{ik}}\left(\Pi_{h'\in\mathcal{H}}\left(\sum_{j\neq i} w_{ijh'}\mu_{jh'}\right)^{\beta_k J_{kh'}}\right)} \tag{44}$$

Note this is a system of $N\times H$ equilibrium conditions in $N\times H$ unknown probabilities μ_{jh}. Equation (44) is a special case of (1). From Remark 5, define $x_{ih} \equiv \sum_{j\neq i} w_{ijh}\mu_{jh}$, which, when combined with equation (44), becomes:

$$x_{ih} = \sum_{j\neq i} w_{ijh} \frac{e^{\beta_h u_{jh}}\Pi_{h'\in\mathcal{H}}\left(x_{jh'}\right)^{\beta_h J_{hh'}}}{\left(\sum_{k\in\mathcal{H}} e^{\beta_k u_{jk}}\left(\Pi_{h'\in\mathcal{H}}\left(x_{jh'}\right)^{\beta_k J_{kh'}}\right)\right)} \tag{45}$$

Finally, defining $f_{ijh} \equiv w_{ijh}\frac{e^{\beta_h u_{jh}}\Pi_{h'\in\mathcal{H}}(x_{jh'})^{\beta_h J_{hh'}}}{\left(\sum_{k\in\mathcal{H}} e^{\beta_k u_{jk}}\left(\Pi_{h'\in\mathcal{H}}(x_{jh'})^{\beta_k J_{kh'}}\right)\right)}$ if $j\neq i$ and $f_{iih} = 0$ results in equation (45) be written as:

$$x_{ih} = \sum_{j\in\mathcal{N}} f_{ijh}(x_{j1}, ..., x_{jH}),$$

as in (1).

A.8.2 Applying Theorem 1

It is straightforward to calculate the elasticities of interactions as follows:

$$\frac{\partial \ln f_{ij,h}}{\partial \ln x_{j,h'}} = \beta_h J_{hh'} - \sum_k \frac{e^{\beta_k u_{jk}} \left(\Pi_{h' \in \mathcal{H}} \left(x_{jh'} \right)^{\beta_k J_{kh'}} \right)}{\sum_{k \in \mathcal{H}} e^{\beta_k u_{jk}} \left(\Pi_{h' \in \mathcal{H}} \left(x_{jh'} \right)^{\beta_k J_{kh'}} \right)} \beta_k J_{kh'},$$

which is between $\underline{J}_{hh'} \equiv \beta_h J_{hh'} - \max_{k \in \mathcal{H}} \beta_k J_{kh'}$ and $\overline{J}_{hh'} \equiv \beta_h J_{hh'} - \min_{k \in \mathcal{H}} \beta_k J_{kh'}$. Thus if we define:

$$(\mathbf{A})_{hh'} \equiv \max \left(-\underline{J}_{hh'}, \overline{J}_{hh'} \right)$$

then we have for all h, h':

$$\left| \frac{\partial \ln f_{ij,h}}{\partial \ln x_{j,h'}} \right| \leq (\mathbf{A})_{hh'} .$$

From Theorem 1, there is a unique solution if $\rho(\mathbf{A}) < 1$, i.e. as long as the social spillovers are not too heterogeneous across actions.

A.9 Choosing many (continuous) actions with social interactions

Here we consider a framework with non-market interactions as in Glaeser and Scheinkman (2002), generalized to include many actions and a general network structure.

A.9.1 The Model

Setup Suppose there are N agents where each agent $i \in \{1, ..., N\} \equiv \mathcal{N}$ who chooses actions $\{x_{ih}\}$, indexed by $h \in \{1, ..., H\} \equiv \mathcal{H}$. Let agent $i's$ payoffs depend on her own actions and the actions of others as follows:

$$U_i \left(\{x_{ih}\}_{h \in \mathcal{H}} ; \left\{ \sum_{j \neq i} g_{ijh'} x_{jh'} \right\}_{h' \in \mathcal{H}} \right), \tag{46}$$

where the utility function U_i is strictly concave in each x_{ih}, $g_{ijh'} \geq 0$, and $\sum_{j \neq i} g_{ijh'} x_{jh'}$ measures the aggregate behavior of agent i's peers. Note that this generalizes Glaeser and Scheinkman (2002) to include an unrestricted network structure $\{g_{ijh'}\}$ and arbitrary H.

Equilibrium Suppose there exists a unique solution to the utility maximization problem of equation (46) that can be written as:

$$x_{ih} = f_{ih} \left(\left\{ \sum_{j \neq i} g_{ijh'} x_{jh'} \right\}_{h' \in \mathcal{H}} \right), \tag{47}$$

where f_{ih} is the best response function. Following Remark 5, we define $y_{ih} \equiv \sum_{j \neq i} g_{ijh} x_{jh}$ and substitute the expression (47), yielding:

$$y_{ih} = \sum_{j \neq i} g_{ijh} f_{jh} \left(\{y_{jh'}\}_{h' \in \mathcal{H}} \right). \tag{48}$$

A.9.2 Applying Theorem 1

It is immediately evident that equation (48) is a special case of equation (1). Suppose that the elasticities of the spillover function can be bounded, i.e. for all $h, h' \in \mathcal{H}$ there exists an $\alpha_{hh'} \geq 0$ such that $\left|\frac{\partial \ln f_{jh}}{\partial \ln y_{jh'}}\right| \leq \alpha_{hh'}$ for all $\left\{y_{jh'}\right\}_{h' \in \mathcal{H}}$. Let $\mathbf{A}$ be the $H \times H$ matrix whose (h, h') element is $\alpha_{hh'}$. From Theorem 1 part (i), there exists a unique equilibrium if $\rho(\mathbf{A}) < 1$. Moreover, that equilibrium can be reached from any initial starting point $\left\{y^0_{jh'}\right\}_{j \in \mathcal{N}}^{h' \in \mathcal{H}}$ by iteration of equation (48).

Glaeser and Scheinkman (2002) prove uniqueness in the $H = 1$ case where $\sum_{j \neq i} g_{ij} = 1$ if a "Moderate Social Influence" condition holds, i.e. $\left|\frac{\partial f_j}{\partial y_j}\right| < 1$ for all y_j. Notice that their results are actually implied by Remark 3 and depend on the particular network structure. In the $H = 1$ case, our condition, obtained from Theorem 1, simplifies to $\left|\frac{\partial \ln f_j}{\partial \ln y_j}\right| < 1$ for all y_j, regardless of the structure of $\{g_{ij}\}$. More generally, ours is the first characterization (of which we are aware) for the $H > 1$ case with general $\{g_{ijh}\}$.

A.10 Public goods in social networks

Here we consider a framework where agents decide how much of their own resources to contribute to public goods whose payoff depends on the contributions of others. To do so, we extend the work of Allouch (2015) and Acemoglu, García-Jimeno, and Robinson (2015) to allow agents to contribute multiple types of public goods on multiple social networks.[16]

A.10.1 The model

Setup Consider a world of $i \in \{1, ..., N\} \equiv \mathcal{N}$ agents. Agent $i \in \mathcal{N}$ is endowed with wealth w_i and chooses how allocate that wealth to private consumption (c_i) or contributions to H public goods ($\{q_{ih}\}_{h \in \{1,2,3...H\} \equiv \mathcal{H}}$), where her payoff depends on the contributions of all other agents. In particular, agent $i \in \mathcal{N}$ solves:

$$\max_{c_i, q_i} U_i\left(c_i, \{Q_{ih}\}_h\right)$$

$$\text{s.t. } c_i + \sum_h q_{ih} = w_i + \sum_h Q_{-ih} \text{ and } q_i \geq 0,$$

where $U_i(\cdot)$ is the utility function $Q_{ih} = q_{ih} + Q_{-ih}$ is the public good bundle with $Q_{-ih} = \sum_{j \neq i} g_{ijh} q_{jh}$ the contributions of all other agents, and the $N \times N$ matrix $\mathbf{G}_h \equiv [g_{ijh}]$ governs the payoff of j's contribution of public good h to agent i (thereby defining the social network h).

Equilibrium Suppose that $U_i(\cdot)$ is continuous, strictly increasing in all its arguments, and strictly quasi-concave. Solving agent's utility maximization problem gives rise to agent i demand function of public good h,$\gamma_{ih}\left(w_i + \sum_{h'} Q_{-ih'}\right)$. Notice that if this demand function is less than other agents' contribution Q_{-ih}, agent i will contribute nothing in public goods. That is in equilibrium we have:

$$q_{ih} = \max\left(\gamma_{ih}\left(w_i + \sum_{h'} Q_{-ih'}\right) - Q_{-ih}, 0\right). \tag{49}$$

[16] Allouch (2015) and Acemoglu, García-Jimeno, and Robinson (2015) extend the work of Bramoullé, Kranton, and D'amours (2014), who applies the seminal work of the private provision of public goods in Bergstrom, Blume, and Varian (1986) to a network setting, but with non-linear best response functions.

We assume that $\gamma_{ih}(\cdot)$ is differentiable and the private and public goods are all normal goods i.e. $0 < \gamma'_{ih} < 1$.

A.10.2 Applying Theorem 1's Remark 3

Denote the right-side of equation (49) as $F_{ih}(\cdot)$. The right and left partial derivatives of $F_{ih}(\cdot)$ with respect to $q_{jh'}$ are either 0 or $\gamma'_{ih} g_{ijh'}$ if $h \neq h'$ and $(\gamma'_{ih} - 1) g_{ijh}$ if $h = h'$. Thus, according to Remark 3, equation (49) has a unique solution if there exists an N-by-N matrix $\mathbf{B}$ satisfying $\rho(\mathbf{B}) < 1$ such that for all $i, j \in \mathcal{N}$, $h \in \mathcal{H}$, $\sum_{h'} \max\left(\left|\frac{\partial_+ F_{ih}(x)}{\partial x_{jh'}}\right|, \left|\frac{\partial_- F_{ih}(x)}{\partial x_{jh'}}\right|\right) \leq \left|\gamma'_{ih} \sum_{h'} g_{ijh'} - g_{ijh}\right| \leq (\mathbf{B})_{ij}$. Intuitively, as long as the aggregate spillovers between agents' public goods contributions are not too strong, the incentives of any agent to shirk her contribution to public goods are not large enough to result in multiplicity. When $H = 1$, this condition can be reduced to $\rho(G) < \frac{1}{1-\gamma'_i}$ where G represents the only network. This condition is very similar to (and stronger than) Allouch (2015)'s $-\lambda_{min}(G) < \frac{1}{1-\gamma'_i}$, since $-\lambda_{min}(G) \leq \rho(G)$. However, the well-definedness of $\lambda_{min}(G)$ crucially relies on network G being symmetric (the symmetry guarantees all the eigenvalues of G are real and can be ranked); in contrast, the condition provided here is valid for asymmetric networks as well.

A.11 Multiple activities in social networks

We extend the work of Chen, Zenou, and Zhou (2018) where agent's payoffs depend on their own multiple actions as well as the actions of others in their social networks to more than two types of actions on multiple social networks. Unlike in Section A.7, here we focus on linear best response functions in order to extend the domain of solutions to include zero and negative values.

A.11.1 The model

Setup Consider a system of $h \in \{1, 2, 3...H\} \equiv \mathcal{H}$ social networks with N agents. Each agent $i \in \{1, 2, 3...N\} \equiv \mathcal{N}$ has preferences over actions $\{x_{ih}\}_{h\in\mathcal{H}}$ which take real numbers. We assume that agents' preferences are represented by the quadratic utility function:

$$U_i = \sum_{h=1}^{H}\left(c_{ih}x_{ih} - \frac{1}{2}x_{ih}^2\right) + \sum_{h=1}^{H}\sum_{h'=1}^{H} d_{ihh'}x_{ih}x_{ih'} + \sum_{h=1}^{H}\sum_{j=1}^{N} g_{ijh}x_{ih}x_{jh},$$

where c_{ih}, $d_{ihh'}$, and g_{ijh} are exogenously given constants and for all i, h, $d_{ihh} = 0$ and $g_{iih} = 0$. The first term in the above expression, reflects decreasing returns to scale of agent i's own actions; the second term reflects substitution or complementary effects between agent i's different actions; the last term reflects the network externality from other agents and $\{g_{ijh}\}_{i,j\in\mathcal{N}}$ represents the corresponding social network.

Equilibrium We assume the above utility function is concave. Thus its maximum can be characterized by the first order condition:

$$x_{ih} = c_{ih} + \sum_{h'\neq h}(d_{ihh'} + d_{ih'h})\,x_{ih'} + \sum_{j\neq i} g_{ijh}x_{jh}.$$

Define the H-by-H symmetric matrix D_i such that $(D_i)_{hh'} = d_{ihh'} + d_{ihh'}$. The concavity assumption implies that matrix $I - D_i$ is positive definite and thus invertible. Denote the element of the inverse of $I - D_i$ as $\delta_{ihh'}$. Then we can rewrite the above first order condition as:

$$x_{ih} = \sum_{h'} \delta_{ihh'} \left(c_{ih'} + \sum_{j \neq i} g_{ijh'} x_{jh'} \right). \tag{50}$$

This equation then represents the Nash equilibrium of this network game.

A.11.2 Applying Theorem 1's Remark 3

Denote the right-side of equation 50 as $F_{ih}(\cdot)$. Notice that $\frac{\partial F_{ih}(x)}{\partial x_{jh'}} = \delta_{ihh'} g_{ijh'}$. Thus, according to Remark 3, equation (50) has a unique solution if there exists an N-by-N matrix $\mathbf{B}$ satisfying $\rho(\mathbf{B}) < 1$ such that for all $i, j \in \mathcal{N}$, $h \in \mathcal{H}$, $\sum_{h'} \left| \delta_{ihh'} g_{ijh'} \right| \leq (\mathbf{B})_{ij}$. Intuitively, if the aggregate connections of different networks are low, agents' influences on each other are weak enough such that the multiplicity as in coordination games then is excluded.

We note that our condition simplifies to the one given by Chen, Zenou, and Zhou (2018) in the special case considered there of $H = 2$, $d_{ihh'} + d_{ihh'} = -\beta$ $(h = 1, h' = 2)$, and for all i, j, $g_{ij1} = g_{ij2} = g_{ij}$ i.e. there is a single network G. To see this, note that by calculating the inverse of $I - D_i$, we have $\sum_{h'} \left| \delta_{ihh'} g_{ijh'} \right| = \frac{1}{1-|\beta|} g_{ij}$. Then our condition can be written as $\rho(G) < 1 - |\beta|$, which is the one used in Chen, Zenou, and Zhou (2018).

A.12 Inverting a demand system with multiple types of goods

Here we consider the question of the invertibility of demand systems based on the seminal work of Berry, Levinsohn, and Pakes (1995). In Berry, Levinsohn, and Pakes (1995), agents makes a choice over a single type of goods, e.g. which cellphone to buy. Here, we extend the framework to consider a situation where consumers simultaneously make decisions across multiple types of goods, e.g. which cellphone and computer to buy. We suppose that the market shares for each type are observed and ask if that is enough information to recover the unobserved demand for each good.[17]

A.12.1 The model

Setup There are H types of goods for agents to buy (e.g. cellphones, computers, and automobiles). Within each type $h \in \{1, ..., H\} \equiv \mathcal{H}$ of good, there are N_h products over which to choose (e.g. in the case of cellphones, there are the Google Pixel 6, the iPhone 13, etc.). One of these N_h products may be the choice to purchase nothing.

Let J be a H-by-1 vector representing agent's choice over the bundle of products. Specifically, $J \equiv [j_h]_{h \in \mathcal{H}}$, where $j_h \in \{1, ..., N_h\} \equiv \mathcal{N}_h$ is agent's choice of product type h to purchase. Suppose that the latent utility of agent k's choice J is:

$$U_k(J) = \sum_{h \in \mathcal{H}} \delta_{j_h, h} + \mu(J, \nu_k) + \varepsilon_{kJ} \tag{51}$$

where $\delta_{j_h,h}$ represents the (unobservable) good characteristics of product j_h in type h, $\mu(J, \nu_k)$ is a function of (observable) good characteristics of the bundle J and consumer characteristics ν_k and ε_{kJ} is a random variable representing agents' idiosyncratic preference. Note that $\mu(J, \nu_k)$ flexibly

[17] While the choice of buying two products can be technically modeled as a single choice over pairs of products, applying the inversion results of Berry, Levinsohn, and Pakes (1995) would then require knowledge of the market shares of each *pair* of products, which is typically not observed.

allows for any sort of (observed) complementarity or substitutability across products of different types, which can potentially vary with consumer characteristics ν_k. We assume $\nu_k \sim P$ where P is a known distribution and ε_{kJ} has type I extreme value distributions independent of k and J.

Suppose for each $h \in \mathcal{H}$ we observe the fraction of agents that choose product $i \in \mathcal{N}_h$, i.e. the market share $s_{i,h}$. Our goal is to identify the set of unobservable good characteristics $\{\delta_{i,h}\}$.

Market share Given the extreme value distribution of ε_{ih}, the market share can be written as:

$$s_{i,h} = \int \frac{\exp\left(\delta_{i,h}\right) \sum_{j_1=1}^{N_1} \cdots \sum_{j_{h-1}=1}^{N_{h-1}} \sum_{j_{h+1}=1}^{N_{h+1}} \cdots \sum_{j_H=1}^{N_H} \exp\left(\sum_{h' \neq h} \delta_{j_{h'},h'} + \mu\left(\left[j_1, ..., j_{h-1}, i, j_{h+1}, ..., j_H\right], \nu\right)\right)}{\sum_{j_h=1}^{N_h} \exp\left(\delta_{j_h,h}\right) \sum_{j_1=1}^{N_1} \cdots \sum_{j_{h-1}=1}^{N_{h-1}} \sum_{j_{h+1}=1}^{N_{h+1}} \cdots \sum_{j_H=1}^{N_H} \exp\left(\sum_{h' \neq h} \delta_{j_{h'},h'} + \mu\left(\left[j_1, ..., j_{h-1}, j_h, j_{h+1}, ..., j_H\right], \nu\right)\right)} dP\left(\nu\right). \tag{52}$$

A.12.2 Applying Theorem 1

The case of $H = 1$ (Berry, Levinsohn, and Pakes (1995)) We first consider the case of $H = 1$, as in Berry, Levinsohn, and Pakes (1995). In this case, equation (52) becomes:

$$s_i = \int \frac{\exp\left(\delta_i + \mu\left(i, \nu\right)\right)}{\sum_{j=1}^{N} \exp\left(\delta_j + \mu\left(j, \nu\right)\right)} dP\left(\nu\right).$$

Define $x_i \equiv \exp\left(\delta_i\right)$. Then $x_i = s_i f_i\left(x\right)$,where $f_i\left(x\right) \equiv \left(\int \frac{\exp(\mu(i,\nu))}{\sum_{j=1}^{N} x_j \exp(\mu(j,\nu))} dP\left(\nu\right)\right)^{-1}$. We then have:

$$\frac{\partial \ln f_i}{\partial \ln x_j} = f_i \int \frac{\exp\left(\mu\left(i, \nu\right)\right) x_j \exp\left(\mu\left(j, \nu\right)\right)}{\left(\sum_{k=1}^{N} x_k \exp\left(\mu\left(k, \nu\right)\right)\right)^2} dP\nu$$

which in turn implies:

$$\begin{aligned}\sum_{j \in \mathcal{N}} \left|\frac{\partial \ln f_i}{\partial \ln x_j}\right| &= f_i \int \frac{\exp\left(\mu\left(i, \nu\right)\right) \sum_j x_j \exp\left(\mu\left(j, \nu\right)\right)}{\left(\sum_{k=1}^{N} x_k \exp\left(\mu\left(k, \nu\right)\right)\right)^2} dP\nu \\ &= f_i\left(x\right) / f_i\left(x\right) = 1.\end{aligned}$$

According to part (ii) of Theorem 1 and Remark 1, there is at most one set of $\{\delta_i\}$ (up to an unknown constant), as in Berry, Levinsohn, and Pakes (1995).

The case of $H = 2$ We now consider the case of $H = 2$, under the special case where $\mu\left(\left[i, j\right], \nu\right) \equiv \mu_p\left(\left[i, j\right]\right) + \mu_c\left(\nu\right)$, i.e. that there is separability between any complementarity or substitutability of product characteristics and any heterogeneity in consumer preferences. Also, we assume $N_1 = N_2 = N$.

Define $x_{i,h} \equiv \exp\left(\delta_{i,h}\right)$. Equation (52) can be written as:

$$x_{i,1} z_{i,2} = \sum_{j=1}^{N} s_{i,1} x_{j,1} z_{j,2}$$

$$x_{i,2} z_{i,1} = \sum_{j=1}^{N} s_{i,2} x_{j,2} z_{j,1},$$

where:

$$z_{i,1} \equiv \sum_{j=1}^{N} x_{j,1} \exp\left(\mu_p\left([j,i]\right)\right)$$

$$z_{i,2} \equiv \sum_{j=1}^{N} x_{j,2} \exp\left(\mu_p\left([i,j]\right)\right)$$

It is immediately evident that this system of $4N$ equations in $4N$ unknowns takes the form of equation (8), which is a special case of equation (1), so by applying Remark 5, it is sufficient to characterize the spectral radius of matrix $\mathbf{A} \equiv \left|\mathbf{B}\mathbf{\Gamma}^{-1}\right|$, where:

$$\mathbf{B} \equiv \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ \mathbf{\Gamma} \equiv \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so that:

$$\mathbf{A} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

which has a spectral radius equal to 1, so that from Theorem 1 part (ii) there exists at most one set of $\{\delta_{i,h}\}$ consistent with the observed market shares, up to an unknown constant for each $h \in \mathcal{H}$, thereby extending the results of Berry, Levinsohn, and Pakes (1995) to the case of $H = 2$ under the special case where $\mu\left([i,j], \nu\right) \equiv \mu_p\left([i,j]\right) + \mu_c\left(\nu\right)$.

B Additional Details

B.1 Details of Remarks

In this section, we provide further details for the remarks discussed in the paper.

B.1.1 Remark 1

Remark 1: Suppose there exists an H-by-H matrix $\mathbf{A}$ such that for all $i, j \in \mathcal{N}$, $h, h' \in \mathcal{H}$, and $x_j \in \mathbb{R}^H_{++}$, $\sum_{j\in\mathcal{N}} \left|\frac{\partial \ln \sum_{k\in\mathcal{N}} f_{ikh}(x)}{\partial \ln x_{jh'}}\right| \leq (\mathbf{A})_{hh'}$. Then:

(i). If $\rho(\mathbf{A}) < 1$, there exists a unique solution to equation (1) which can be computed by iteratively applying equation (1) with a rate of convergence $\rho(\mathbf{A})$;

(ii). If $\rho(\mathbf{A}) = 1$ and:

a. For all $i \in \mathcal{N}$ and $h, h' \in \mathcal{H}$ when $(\mathbf{A})_{hh'} \neq 0$ there exists some j such that for all $x_j \in \mathbb{R}^H_{++}$, $\sum_{j\in\mathcal{N}} \left|\frac{\partial \ln \sum_{k\in\mathcal{N}} f_{ikh}(x)}{\partial \ln x_{jh'}}\right| < (\mathbf{A})_{hh'}$, then equation (1) has at most one solution;

b. For all x_j, $\sum_{j\in\mathcal{N}} \frac{\partial \ln \sum_{k\in\mathcal{N}} f_{ikh}(x)}{\partial \ln x_{jh'}} = \alpha_{hh'} \in \mathbb{R}$ where $|\alpha_{hh'}| = (\mathbf{A})_{hh'}$ then there is at most one column-wise up-to-scale solution, i.e. for any $h \in \mathcal{H}$ and two solutions x and x' it must be $x'_{.h} = c_h x_{.h}$ for some scalar $c_h > 0$.

Proving this remark requires only a small change to the proof of Theorem 1. Equality (2) becomes $\frac{\partial g_{ih}}{\partial y_{jh'}} = \frac{\partial \ln \sum_k f_{ikh}(x)}{\partial \ln x_{jh'}}$.

Correspondingly, in part (i), inequality (4) becomes

$$\begin{aligned}\left|g_{ih}(y) - g_{ih}(y')\right| &= \left|\sum_{h' \in \mathcal{H}} \sum_{j \in \mathcal{N}} \frac{\partial g_{ih}(\hat{y})}{\partial y_{jh'}} \left(y_{jh'} - y'_{jh'}\right)\right| \\ &\leq \sum_{h' \in \mathcal{H}} \sum_{j \in \mathcal{N}} \left|\frac{\partial \ln \sum_k f_{ikh}(\hat{x})}{\partial \ln x_{jh'}}\right| \max_{j \in \mathcal{N}} \left|y_{jh'} - y'_{jh'}\right| \\ &\leq \sum_{h' \in \mathcal{H}} (\mathbf{A})_{hh'} \max_{j \in \mathcal{N}} \left|y_{jh'} - y'_{jh'}\right|.\end{aligned}$$

And in part (ii.b), equation (6) becomes

$$y_{ih} - y'_{ih} + \hat{s}_h = \sum_{h' \in \mathcal{H}} \sum_{j \in \mathcal{N}} \frac{\partial \ln \sum_k f_{ikh}(x)}{\partial \ln x_{jh'}} \left(y_{jh'} - y'_{jh'} + s_{h'}\right)$$

The rest of the proof of Theorem 1 remains unchanged.

B.1.2 Remark 2

Remark 2: Consider the special case of equation (1) where $f_{ijh} : \mathbb{R}_+^H \to \mathbb{R}_+ \equiv K_{ijh} g_{ijh}(x_{j1}, ..., x_{jH})$, where $K_{ijh} \geq 0$ and $g_{ijh}(x_j)$ is continuous, differentiable, and $g_{ijh}(x_j) > 0$ for all $x_j > 0$ so that equation (1) becomes $x_{ih} = \sum_{j=1}^N K_{ijh} g_{ijh}(x_{j1}, ..., x_{jH})$. Then if $\rho(\mathbf{A}) < 1$ and matrices $(K_{ijh})_{i,j \in \mathcal{N}}$ for all h are irreducible, there exists only one strictly positive solution. Weakly positive solutions, where for some i, h, $x^*_{i,h} = 0$, may exist but will be asymptotically unstable,

The condition that matrices $(K_{ijh})_{i,j \in \mathcal{N}}$ for all h are irreducible implies that for any strictly positive x, $\sum_{j=1}^N K_{ijh} g_{ijh}(x_{j1}, ..., x_{jH}) > 0$. Thus we can apply Remark 1 to obtain the existence, uniqueness, and convergence of the solution. Observe that the convergence simply implies the unstableness of other weakly positive solutions.

B.1.3 Remark 3

Remark 3: Suppose for all $i, j \in \mathcal{N}$, $h, h' \in \mathcal{H}$,$F_{ih}(x)$ has right and left partial derivatives with respect to $x_{jh'}$ and denote they as $\frac{\partial_+ F_{ih}(x)}{\partial x_{jh'}}$ and $\frac{\partial_- F_{ih}(x)}{\partial x_{jh'}}$. Equation$x_{ih} = F_{ih}(x)$ has a unique solution if (1) there exists an H-by-H matrix $\mathbf{A}$ satisfying $\rho(\mathbf{A}) < 1$ such that for all $i \in \mathcal{N}$, $h, h' \in \mathcal{H}$, $\sum_j \max\left(\left|\frac{\partial_+ F_{ih}(x)}{\partial x_{jh'}}\right|, \left|\frac{\partial_- F_{ih}(x)}{\partial x_{jh'}}\right|\right) \leq (\mathbf{A})_{hh'}$ or (2) there exists an N-by-N matrix $\mathbf{B}$ satisfying $\rho(\mathbf{B}) < 1$ such that for all $i, j \in \mathcal{N}$, $h \in \mathcal{H}$, $\sum_{h'} \max\left(\left|\frac{\partial_+ F_{ih}(x)}{\partial x_{jh'}}\right|, \left|\frac{\partial_- F_{ih}(x)}{\partial x_{jh'}}\right|\right) \leq (\mathbf{B})_{ij}$. Due to symmetry, we only need to prove the first condition.

Given any x and x', let $m_{ih}(t_{ih}) \equiv F_{ih}((1 - t_{ih})x + t_{ih}x')$ be a function of t_{ih} on interval $[0, 1]$. Since each $F_{ih}(x)$ is left and right differentiable, so is $m_{ih}(t_{ih})$. Suppose $m'_{ih+}(t_{ih})$ and $m'_{ih-}(t_{ih})$ are function $m_{ih}(\cdot)$'s right and left derivatives at t_{ih}. According to a generalized mean value theorem, there exists some $t_{ih} \in (0, 1)$ such that $\frac{m_{ih}(1) - m_{ih}(0)}{1-0}$ is between $m'_{ih+}(t_{ih})$ and $m'_{ih-}(t_{ih})$. Observe that $m'_{ih+}(t_{ih}) = \sum_{j,h'} \left[\frac{\partial_+ F_{ih}(\hat{x})}{\partial x_{jh'}} \mathbb{1}_{\Delta x_{jh'} > 0} + \frac{\partial_- F_{ih}(\hat{x})}{\partial x_{jh'}} \mathbb{1}_{\Delta x_{jh'} < 0}\right] \Delta x_{jh'}$ where $\hat{x} \equiv (1 - t_{ih})x + t_{ih}x'$ and $\Delta x_{jh'} \equiv x'_{jh'} - x_{jh'}$. Similarly, $m'_{ih-}(t_{ih}) = \sum_{j,h'} \left[\frac{\partial_+ F_{ih}(\hat{x})}{\partial x_{jh'}} \mathbb{1}_{\Delta x_{jh'} < 0} + \frac{\partial_- F_{ih}(\hat{x})}{\partial x_{jh'}} \mathbb{1}_{\Delta x_{jh'} > 0}\right] \Delta x_{jh'}$.

Thus we have $m'_{ih+}(t_{ih})$ and $m'_{ih-}(t_{ih})$ must be within interval $[-M_{ih}, M_{ih}]$ where

$$M_{ih} \equiv \sum_{j,h'} \max\left(\left|\frac{\partial_+ F_{ih}(\hat{x})}{\partial x_{jh'}}\right|, \left|\frac{\partial_- F_{ih}(\hat{x})}{\partial x_{jh'}}\right|\right) |\Delta x_{jh'}|.$$

Thus $F_{ih}(x') - F_{ih}(x) = m_{ih}(1) - m_{ih}(0)$ must be also within interval $[-M_{ih}, M_{ih}]$. That is

$$\begin{aligned} |F_{ih}(x') - F_{ih}(x)| &\leq \sum_{j,h'} \max\left(\left|\frac{\partial_+ F_{ih}(\hat{x})}{\partial x_{jh'}}\right|, \left|\frac{\partial_- F_{ih}(\hat{x})}{\partial x_{jh'}}\right|\right) |\Delta x_{jh'}| \\ &\leq \sum_{h'} (\mathbf{A})_{hh'} \max_j |\Delta x_{jh'}|. \end{aligned}$$

Since the above expression holds for any i, h, we have $\max_j |F_{ih}(x') - F_{ih}(x)| \leq \sum_{h'} (\mathbf{A})_{hh'} \max_j |\Delta x_{jh'}|$. Thus we establish a contraction mapping as in Theorem A.1, which gives us the existence and uniqueness of the solution in equation $x_{ih} = F_{ih}(x)$.

B.1.4 Remark 4

Consider first the equilibrium system (7) with constant elasticities, which can be written as follows:

$$\lambda_h x_{ih} = \sum_{j \in \mathcal{N}} K_{ijh} \prod_{h' \in \mathcal{H}} x_{jh'}^{\alpha_{hh'}}, \tag{53}$$

where $\lambda_h > 0$ is endogenous. If $\rho(\mathbf{A}) \leq 1$, we have the same conclusion as in part (ii)b: the $\{x_{ih}\}$ of any solution is column-wise up-to-scale unique. The proof of this result is exactly the same as part (ii)b of Theorem 1.

For $\rho(\mathbf{A}) < 1$, particularly it is possible to subsume the endogenous scalars into the equilibrium outcomes through a change in variables, expressing equation (53) as in equation (1). To do so, define $\tilde{x}_{ih} \equiv x_{ih} \prod_{h' \in \mathcal{H}} \lambda_{h'}^{d_{hh'}}$, where $d_{hh'}$is the hh'^{th} element of the $H \times H$ matrix $(\mathbf{I} - \boldsymbol{\alpha})^{-1}$ and $\boldsymbol{\alpha} \equiv (\alpha_{hh'})$ (i.e. $\boldsymbol{\alpha}$ is the matrix of elasticities without the absolute value taken) so the system becomes:

$$\tilde{x}_{ih} = \sum_{j \in \mathcal{N}} K_{ijh} \prod_{h' \in \mathcal{H}} \tilde{x}_{jh'}^{\alpha_{hh'}}.$$

Note that because $\rho(\mathbf{A}) < 1$, then so too is $\rho(\boldsymbol{\alpha}) < 1$, so that $(\mathbf{I} - \boldsymbol{\alpha})^{-1}$ exists. From Theorem 1 part (i), the $\{\tilde{x}_{ih}\}$ are unique and can be calculated using an iterative algorithm, which in turn implies that the $\{x_{ih}\}$ are column-wise up-to-scale unique. (Separating the $\{x_{ih}\}$ and $\{\lambda_h\}$ to determine the scale of $\{x_{ih}\}$ requires the imposition of further equilibrium conditions, e.g. aggregate labor market clearing conditions).

Consider now equilibrium system (7) with H additional aggregate constraints $\sum_{i=1}^N x_{ih} = c_h$ for known constants $c_h > 0$.

The second result concerns the general case with an endogenous scalar:

$$\lambda_h x_{ih} = \sum_{j=1}^N f_{ijh}(x_{j1}, ..., x_{jH})$$

with H additional aggregate constraints $\sum_{i=1}^N x_{ih} = c_h$ for known constants $c_h > 0$. Substituting in the aggregate constraints allows us to express the equilibrium system as:

$$x_{ih} = \sum_{j=1}^{N} \left(\frac{f_{ijh}\left(x_{j1},...,x_{jH}\right)}{\frac{1}{c_h}\sum_{k=1}^{N}\sum_{l=1}^{N} f_{klh}\left(x_{l1},...,x_{lH}\right)} \right),$$

where the denominator is equal to the endogenous scalar, i.e. $\lambda_h = \frac{1}{c_h}\sum_{k=1}^{N}\sum_{l=1}^{N} f_{klh}\left(x_{l1},...,x_{lH}\right)$. We can define the new function:

$$g_{ij,h}\left(x\right) \equiv \frac{f_{ijh}\left(x_{j1},...,x_{jH}\right)}{\frac{1}{c_h}\sum_{k=1}^{N}\sum_{l=1}^{N} f_{klh}\left(x_{l1},...,x_{lH}\right)}$$

so that the equilibrium system becomes:

$$x_{ih} = \sum_{j=1}^{N} g_{ijh}\left(x\right).$$

We can then bound the elasticities, following Remark 1. Note:

$$\frac{\partial \ln g_{ij,h}}{\partial \ln x_{m,l}} = \frac{\partial \ln f_{ij,h}}{\partial \ln x_{j,l}} \mathbf{1}_{m=j} - \sum_{o} \left(\frac{\partial \ln f_{om,h}}{\partial \ln x_{m,l}} \right) \frac{f_{om,h}\left(\{x_{m,l}\}\right)}{\sum_o \sum_p f_{op,h}\left(\{x_{p,l}\}\right)}$$

where $\mathbf{1}_{m=j} = \begin{cases} 1 & \text{if } m=j \\ 0 & \text{if } m \neq j \end{cases}$ is an indicator function. Thus,

$$\left| \frac{\partial \ln g_{ij,h}}{\partial \ln x_{m,l}} \right| \leq \left|A_{hl}\right| \mathbf{1}_{m=j} + \left|A_{hl}\right| \frac{\sum_o f_{om,h}\left(\{x_{m,l}\}\right)}{\sum_o \sum_p f_{op,h}\left(\{x_{p,l}\}\right)}.$$

Furthermore,

$$\sum_m \left| \frac{\partial \ln g_{ij,h}}{\partial \ln x_{m,l}} \right| \leq \sum_m \left|A_{hl}\right| \mathbf{1}_{m=j} + \left|A_{hl}\right| \frac{\sum_m \sum_o f_{om,h}\left(\{x_{m,l}\}\right)}{\sum_o \sum_p f_{op,h}\left(\{x_{p,l}\}\right)} = 2\left|A_{hl}\right|.$$

Hence, from Remark 1, we have uniqueness as long as $\rho\left(\mathbf{A}\right) < \frac{1}{2}$, as required.

B.2 Details of the Urban Spatial Model

B.2.1 Theorem 1, part (i): General spillovers

The following sufficient condition for uniqueness

$$\rho \begin{pmatrix} 2\boldsymbol{\theta\alpha} & 2\boldsymbol{\theta\beta} \\ 2\boldsymbol{\theta\alpha} & 2\boldsymbol{\theta\beta} \end{pmatrix} < 1.$$

is equivalent with $\rho\left(\boldsymbol{\theta}\left(\boldsymbol{\alpha}+\boldsymbol{\beta}\right)\right) < \frac{1}{2}$. To see this simplify spectral radius $\rho \begin{pmatrix} 2\boldsymbol{\theta\alpha} & 2\boldsymbol{\theta\beta} \\ 2\boldsymbol{\theta\alpha} & 2\boldsymbol{\theta\beta} \end{pmatrix}$ as r and denote its associated eigenvector as $\begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix}$ where both v_α and v_β are H-by-1 vectors. Thus we have

$$\begin{pmatrix} 2\boldsymbol{\theta\alpha} & 2\boldsymbol{\theta\beta} \\ 2\boldsymbol{\theta\alpha} & 2\boldsymbol{\theta\beta} \end{pmatrix} \begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix} = r \begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 2\boldsymbol{\theta\alpha} v_\alpha + 2\boldsymbol{\theta\beta} v_\beta \\ 2\boldsymbol{\theta\alpha} v_\alpha + 2\boldsymbol{\theta\beta} v_\beta \end{pmatrix} = r \begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix},$$

which implies $v_\alpha = v_\beta$. Thus, $2\boldsymbol{\theta}(\boldsymbol{\alpha}+\boldsymbol{\beta})\, v_\alpha = r v_\alpha$. According to the Perron–Frobenius theorem, there is a positive number $\frac{r}{2} = \rho(\boldsymbol{\theta}(\boldsymbol{\alpha}+\boldsymbol{\beta}))$. Therefore $\rho(\boldsymbol{\theta}(\boldsymbol{\alpha}+\boldsymbol{\beta})) < \frac{1}{2}$ implies $r < 1$, as desired.

B.2.2 Theorem 1, part (ii): Constant elasticity spillovers

Now we show that $c \equiv \rho\left(\begin{pmatrix} \mathbf{0} & \left|\boldsymbol{\theta\beta}(\mathbf{I}-\boldsymbol{\theta\beta})^{-1}\right| \\ \left|\boldsymbol{\theta\alpha}(\mathbf{I}-\boldsymbol{\theta\alpha})^{-1}\right| & \mathbf{0} \end{pmatrix}\right) \leq 1$, the uniqueness condition here, is weaker than $\rho(\boldsymbol{\theta}(|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|)) < \frac{1}{2}$, the condition required in the case of general spillovers given above. Suppose that for v_α and v_β H-by-1 vectors,

$$\begin{pmatrix} \mathbf{0} & \left|\boldsymbol{\theta\beta}(\mathbf{I}-\boldsymbol{\theta\beta})^{-1}\right| \\ \left|\boldsymbol{\theta\alpha}(\mathbf{I}-\boldsymbol{\theta\alpha})^{-1}\right| & \mathbf{0} \end{pmatrix} \begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix} = c \begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} \left|\boldsymbol{\theta\beta}(\mathbf{I}-\boldsymbol{\theta\beta})^{-1}\right| v_\beta \\ \left|\boldsymbol{\theta\alpha}(\mathbf{I}-\boldsymbol{\theta\alpha})^{-1}\right| v_\alpha \end{pmatrix} = c \begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix} \Rightarrow$$

$$\left|\boldsymbol{\theta\alpha}(\mathbf{I}-\boldsymbol{\theta\alpha})^{-1}\right| \left|\boldsymbol{\theta\beta}(\mathbf{I}-\boldsymbol{\theta\beta})^{-1}\right| v_\beta = c^2 v_\beta.$$

Thus it is equivalent to show that $\rho\left(\left|\boldsymbol{\theta\alpha}(\mathbf{I}-\boldsymbol{\theta\alpha})^{-1}\right| \left|\boldsymbol{\theta\beta}(\mathbf{I}-\boldsymbol{\theta\beta})^{-1}\right|\right) = c^2 \leq 1$. Define H-by-H matrix $\boldsymbol{\delta}$ where $(\boldsymbol{\delta})_{hh'} = \max\left((\boldsymbol{\theta}|\boldsymbol{\alpha}|)_{hh'}, (\boldsymbol{\theta}|\boldsymbol{\beta}|)_{hh'}\right)$. Clearly, $\left|\boldsymbol{\theta\beta}(\mathbf{I}-\boldsymbol{\theta\beta})^{-1}\right| \leq \sum_{n=1}^{\infty}(\boldsymbol{\theta}|\boldsymbol{\beta}|)^n \leq \sum_{n=1}^{\infty}\boldsymbol{\delta}^n$ where the inequality is element-wise; similarly, $\left|\boldsymbol{\theta\alpha}(\mathbf{I}-\boldsymbol{\theta\alpha})^{-1}\right| \leq \sum_{n=1}^{\infty}\boldsymbol{\delta}^n$. Thus,

$$\rho\left(\left|\boldsymbol{\theta\alpha}(\mathbf{I}-\boldsymbol{\theta\alpha})^{-1}\right| \left|\boldsymbol{\theta\beta}(\mathbf{I}-\boldsymbol{\theta\beta})^{-1}\right|\right) \leq \rho\left(\sum_{n=1}^{\infty}\boldsymbol{\delta}^n \sum_{n=1}^{\infty}\boldsymbol{\delta}^n\right) = \rho\left(\sum_{n=1}^{\infty}\boldsymbol{\delta}^n\right)^2.$$

Here, $\rho\left(\sum_{n=1}^{\infty}\boldsymbol{\delta}^n\right) = \frac{\rho(\boldsymbol{\delta})}{1-\rho(\boldsymbol{\delta})}$. Furthermore, $\rho(\boldsymbol{\delta}) \leq \rho(\boldsymbol{\theta}(|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|)) < \frac{1}{2}$. Thus, $\rho\left(\sum_{n=1}^{\infty}\boldsymbol{\delta}^n\right) < 1$, which is as desired.