Online Appendix for 'Investment Hangover and the Great Recession'

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APPENDIX A: CALIBRATION

This appendix describes the details of the calibration exercise in Section IV.

APPROXIMATION OF THE KEYNESIAN MULTIPLIER EMBEDDED IN GHH PREFERENCES. — In a model with GHH preferences, the labor wedge determines the local output multiplier with respect to demand shocks. This result was obtained by Auclert and Rognlie (2017), and in this appendix it is restated in the context of the current paper.

To see why the labor wedge is important, note that holding r_{t+1} (controlled by monetary policy) and c_{t+1} (tomorrow's net consumption) constant, c_t is uniquely pinned down by the Euler equation in (24). Now, equating production to demand, and writing $x_t \equiv k_{t+1} - (1-\delta^k)k_t + i_t^h$ for non-consumption demand, we have

$$y_t - v(l_t) = F(k_t, l_t) - v(l_t) = c_t + x_t$$

Assuming c_t fixed due to the observation above, and holding capital k_t fixed, totally differentiating gives

$$dl_t = \frac{1}{F_l(k_t, l_t) - v'(l_t)} dx_t$$

and noting that $dy_t = F_l(k_t, l_t) dl_t$, this implies

$$dy_t = \frac{1}{1 - \frac{v'(l_t)}{F_l(k_t, l_t)}} dx_t \equiv \frac{1}{\tau_t} dx_t \tag{A1}$$

where τ_t is the (conventionally defined) labor wedge at time t.

Hence the local response of total output to changes in non-consumption demand (either from residential or nonresidential investment), holding inherited capital, monetary policy, and tomorrow's net consumption fixed, is determined by the inverse of the labor wedge. If the labor wedge is zero, the implied multiplier is infinite; to obtain a more realistic multiplier in line with empirical evidence, the labor wedge must be sufficiently high.

Why is this? If there is no labor wedge ($\tau = 0$), then net output $F(k_t, l_t) - v(l_t)$ is at a local maximum in l_t, and the implied variation in l_t needed to increase net output is infinite. If there is more of a labor wedge ($\tau > 0$), then net output is still increasing in l_t, implying that less movement in l_t in response to a shock that changes net output is needed.

CHARACTERIZATION OF THE GENERALIZED MODEL FOR CALIBRATION PURPOSES. — As described in the main text, we generalize the baseline model (with housing adjustment costs) in two ways to make it more suitable for calibration. First, we make a distinction between the risk-free rate and the return to capital. This modification affects the threshold return at which the economy enters the liquidity trap region, as illustrated by Eq. (25), but it otherwise does not change the analysis. Second, we also allow for a labor tax that changes the firm's problem as in (26). With the labor tax, the constrained efficient levels of employment and output are determined by

$$\begin{aligned} L\left(k_t\right) &= \arg\max_{\tilde{l}} \left(1-\tau^l\right) F\left(k_t, \tilde{l}\right) - \upsilon\left(\tilde{l}\right), \\ \text{and } S\left(k_t\right) &= F\left(k_t, L\left(k_t\right)\right) - \upsilon\left(L\left(k_t\right)\right). \end{aligned}$$

In particular, the labor tax reduces the labor supply, which in turn lowers output. Note also that, when output is supply determined, the tax parameter corresponds to the labor wedge, $\tau^l = \tau_t$ (cf. Eq. (13)). The remaining equilibrium allocations are characterized by the following system,

$$\begin{aligned} & u'\left(c_t\right) = \beta\left(1+r_{t+1}\right) u'\left(c_{t+1}\right), \\ & i_t^h = i^{h*} + \frac{u'\left(c_t\right)}{\psi}\left(Q_t - 1\right) \text{ and } h_{t+1} = h_t\left(1-\delta^h\right) + i_t^h, \\ & Q_t \geq \frac{1-\delta^h}{1+r_{t+1}} Q_{t+1}, h_{t+1} \geq h^* \text{ and one of the inequalities hold as equality,} \\ & R\left(k, L\left(k\right)\right) - \delta^k = r_{t+1}, \\ & y_t = c_t + k_{t+1} - \left(1-\delta^k\right) k_t + i_t^h, \\ \text{and} \quad & y_t \leq S\left(k_t\right), r_{t+1} \geq \phi - \pi \text{ and one of the inequalities hold as equality.} \end{aligned}$$

These conditions are analogous to the equilibrium conditions in Section III, with the difference that the last condition incorporates the more general lower bound on the return to capital in Eq. (25).

CALIBRATING THE OVERBUILDING SHOCK. — We calibrate the magnitude of the overbuilding shock based on the analysis in Haughwout et al. (2013), who provide two measures of the excess supply of housing units during the Great Recession. Their first measure uses the Census data on housing vacancies. They calculate the stock of vacant housing units in excess of a baseline vacancy rate that we would expect to see in normal market conditions (which the authors estimate based on historical vacancy rates for each housing category). According to this measure, the excess vacant housing units peaked at around 3 million in mid-2010 and remained at around 2 million as of 2012 (see their Figure 2.7).

The second measure of oversupply in Haughwout et al. (2013) compares actual household production with an estimate of housing needs based on historical patterns of household formation and depreciation. As we describe in the main text, this measure implies around 3.4 million houses were overbuilt by mid-2007. We use this number to calibrate the initial excess supply of housing (see Eq. (27)). Their analysis also suggests that it would take the economy 6 years to work the excess supply (see their Figure 2.8). We use this observation to calibrate the housing adjustment costs in the model so that the adjustment in the model is also completed in 6 years (or 3 periods).

Haughwout et al. (2013) also analyze household formation rates, which speak to the demand for housing in recent years. After adjusting for demographics, they predict that the trend rate of growth of households since the mid-1990s should have been around 1.17 million per year. They then compare the cumulative household formation since 1995 relative to the predicted trend of 1.17 million per year. This analysis illustrates that household formation has been roughly in line with the predicted level until 2007, but it has been consistently below the predicted level in recent years. Using more recent Census data, we find that the household formation averaged 780 thousand per year between the first quarter of 2008 and the third quarter of 2016. This suggests that the low residential investment in recent years is at least in part driven by unusually low demand for housing in the aftermath of the bust (which could be due to, among other things, pessimism about the housing market or credit constraints in the mortgage market). In our calibration exercise, we abstract away from this additional demand shock for housing as it is difficult to quantify.

Appendix B: Omitted extensions

This appendix completes the analysis of the extensions of the baseline model discussed in the main text. Appendix C contains the proofs of omitted results in the main text as well as this appendix.

1. *Comparative statics with respect to durability*

A distinguishing feature of housing capital is its durability relative to other types of capital. A natural question is whether durability is conducive to triggering a demand-driven recession driven by overbuilding. In this section, we address this question in an extension of the baseline model with two types of housing capital, one more durable than the other. We show that overbuilding the more durable capital (relative to the less durable capital) is more likely to trigger a demand-driven recession.

Consider a slight variant of the model in Section II in which there are two types of housing capital that depreciate at different rates given by δ^{h^d} and δ^{h^n}, with $\delta^{h^d} < \delta^{h^n}$. Thus, type d (durable) housing capital has a lower depreciation rate than type n (non-durable) housing capital. Suppose the preferences in (2) are modified so that each type has a target level $h^*/2$. Suppose also that $\left(\delta^{h^d} + \delta^{h^n}\right)/2 = \delta^h$ so that the average depreciation rate is the same as before. Let $h_0^d = \left(1 + b_0^d\right)(h^*/2)$ and $h_0^n = \left(1 + b_0^n\right)(h^*/2)$, so that b_0^d and b_0^n capture the overbuilding in respectively durable and nondurable capital. The case with symmetric overbuilding, $b_0^d = b_0^n = b_0$, results in the same equilibrium as in Section II. Our next result investigates the effect of overbuilding one type of capital more than the other.

PROPOSITION 5 (Role of Durability): *Consider the model with two types of housing capital with different depreciation rates. Given the average overbuilding $b_0 = \left(b_0^d + b_0^n\right)/2$, the incidence of a demand-driven recession $\mathbf{1}\left[l_0 < L\left(k_0\right)\right]$ is increasing in overbuilding of the more durable housing capital b_0^d.*

To obtain an intuition, consider the maximum aggregate demand at date 0, which can be written as [cf. Eq. (19)],

$$\bar{y}_0 = \bar{k} - \left(1 - \delta^k\right) k_0 + \bar{c}_0 + \delta^h h^* - b_0^d \left(1 - \delta^{h^d}\right) \frac{h^*}{2} - b_0^n \left(1 - \delta^{h^n}\right) \frac{h^*}{2}. \tag{B.1}$$

Note that $1 - \delta^{h^d} > 1 - \delta^{h^n}$, and thus, overbuilding of the durable housing capital (relative to the nondurable capital) induces a greater reduction in aggregate demand at date 0. Intuitively, depreciation helps to “erase” the overbuilt capital naturally, thereby inducing a smaller reduction in investment and aggregate demand.

2. *Investment hangover with exogenous monetary policy*

A key ingredient of our analysis is constrained monetary policy. In the main text, we focus on the zero lower bound (ZLB) as the source of the constraint. In this section, we derive the analogue of our main result in Section II in an environment in which the money supply is determined by exogenous forces.

To introduce the money supply, we modify household preferences to introduce the demand for money explicitly. Specifically, the household's optimization problem can now be written as,

$$\text{(B.2)} \quad \max_{\{l_t, \hat{c}_t, a_{t+1}, M_t\}_t} \sum_{t=0}^{\infty} \beta^t u\left(\hat{c}_t - v\left(l_t\right) + \eta\left(\frac{M_t}{P_t}\right)\right) + u^h \mathbf{1}\left[h_t \geq h^*\right]$$

$$\text{s.t. } P_t\left(\hat{c}_t + a_{t+1} + i_t^h\right) + M_t = P_t\left(w_t l_t + a_t\left(1 + r_t\right) + \Pi_t\right) + M_{t-1},$$

$$\text{and } h_{t+1} = h_t\left(1 - \delta^h\right) + i_t^h.$$

Here, P_t denotes the aggregate price level. The household money balances are denoted by M_t, and the real money balances are given by M_t/P_t. The function, $\eta\left(\cdot\right)$ is strictly increasing, which captures the transaction services provided by additional real money balances. The household problem is the same as in Section I except for the presence of money balances in preferences as well as the budget constraint. The optimality condition for money balances, M_t, implies a money demand equation,

$$\text{(B.3)} \quad \eta'\left(\frac{M_t}{P_t}\right) = \frac{r_{t+1}^n}{1 + r_{t+1}^n}.$$

Here, $1 + r_{t+1}^n = \left(1 + r_{t+1}\right)\frac{P_{t+1}}{P_t}$ denotes the nominal interest rate, which captures the opportunity cost of holding money balances (as opposed to interest-bearing assets). The left hand side captures the marginal benefit of holding money balances.[27] The rest of the equilibrium is as described before.

We assume the money supply follows an exogenous path, $\left\{\overline{M}_t\right\}_{t=0}^{\infty}$. For analytical tractability, we focus on the case in which the money supply is fixed, $\overline{M}_t = \overline{M}$ for each t (the general case is similar). As before, the aggregate price level is also predetermined and constant, $P_t = P$ for each t. Combining these assumptions with Eq. $(B.3)$ implies that the nominal interest rate is also constant. There is one degree of freedom because different choices for the aggregate price level (which is a given of this model) lead to different levels for the interest rate. We assume the aggregate price level is such that the interest rate is equal to its steady-state level, that is,[28]

$$\text{(B.4)} \quad r_{t+1} = r_{t+1}^n = 1/\beta - 1 \text{ for each } t.$$

[27]With our specification, the marginal benefit does not depend the household's consumption or aggregate output. This is slightly different than conventional specifications of money demand but it does not play an important role beyond providing analytical tractability.

[28]This price level can be justified by assuming that the prices were set at a point in the past at which the economy was (and was expected to remain) at a steady state. In view of a New-Keynesian Phillips curve, the firms would not want to change their prices as long as they expected the discounted sum of the output gaps to be equal to zero. When the economy is at a steady state, this requirement implies a zero output gap for each period, which in turn implies the interest rate given by $(B.4)$.

The characterization of the remaining equilibrium allocations then parallels the baseline analysis. We conjecture an equilibrium in which, starting date 1 onwards, the employment and output are at their efficient levels. As before, this implies capital earns its marginal contribution to supply, $R_1 = S'(k_1)$ [cf. (9)]. Combining this with Eq. (6), and using $(B.4)$, we obtain $k_1 = k^*$. That is, the economy reaches the steady-state level of capital in a single period. This determines the investment at date 0 as

$$i_0^k = k_1 - \left(1 - \delta^k\right) k_0.$$

Next consider (net) consumption at date 0. Since the economy reaches the steady-state at date 1, we have $c_1 = c^*$. Combining this with the Euler equation and Eq. $(B.4)$, we also obtain $c_0 = c^*$. It follows that aggregate demand and output at date 0 is given by [cf. Eq. (19)],

$$y_0 = k^* - \left(1 - \delta^k\right) k_0 + c^* + \left(\delta^h - b_0 \left(1 - \delta^h\right)\right) h^*.$$

When $y_0 < S(k_0)$, the economy features a demand-driven recession at date 0. This is the case as long as the amount of overbuilding b_0 exceeds a threshold level [cf. (20)],

$$\overline{b}_0 \equiv \frac{k^* - \left(1 - \delta^k\right) k_0 + c^* + \delta^h h^* - S(k_0)}{\left(1 - \delta^h\right) h^*}.$$

It can also be checked that, it the initial capital stock is at its steady-state level $k_0 = k^*$, then the threshold is zero, $\overline{b}_0 = 0$: that is, any amount of overbuilding triggers a recession.

Hence, our main result generalizes to a setting with exogenous (and fixed) money supply. Intuitively, the key to the argument is that monetary policy is constrained and cannot lower the interest rate sufficiently to counter the aggregate demand reduction due to overbuilding. When monetary policy is exogenous—as in the case of an exogenous money supply, it is naturally constrained and cannot lower the interest rate in response to shocks. In fact, overbuilding in this case leads to a deeper recession because the nominal interest rate remains above zero during the recession, whereas monetary policy in the main text partially fights the recession by lowering the nominal interest rate to zero.

3. *Policy analysis with separable preferences*

We next complete the analysis of the model with separable preferences described and used in Section V. We first establish the analog of Proposition 1 for this setting. To this end, let $\overline{c}_0$ and $\overline{k}$ respectively denote the maximum level of consumption and investment characterized in Section II. The aggregate demand is then bounded from above, $y_0 \leq \overline{y}_0$, where

$$\overline{y}_0 \equiv \overline{k} - \left(1 - \delta^k\right) k_0 + \overline{c}_0 + \left(\delta^h - b_0 \left(1 - \delta^h\right)\right) h^*. \tag{B.5}$$

as in Eq. (19) in the main text.

Next consider the efficient level of employment at date 0. The efficiency implies the household's intratemporal condition holds, $w_0 u'(c_0) = v'(l_0)$, and the equilibrium wage level is determined by the labor's marginal product, $w_0 = F_l(k_0, l_0)$. Combining these conditions is equivalent to setting the labor wedge to zero, where the labor wedge is now given by,

$$\tau_0 = 1 - \frac{v_0'(l_0)}{u'(c_0) F_l(k_0, l_0)}. \tag{B.6}$$

Let $L_0(k_0)$ denote the efficient level of output at date 0 (when there is a liquidity trap) characterized by setting $\tau_0 = 0$ when $c_0 = \bar{c}_0$. This also implies an efficient level of output denoted by, $S_0(k_0) = F(k_0, L_0(k_0))$.

As in Section II, the equilibrium depends on a comparison of the maximum level of demand, $\bar{y}_0$, with the efficient supply, $S_0(k_0)$. Let $\bar{b}_0^{sep}$ denote the threshold level of overbuilding that ensures $\bar{y}_0 = S_0(k_0)$, that is,

$$\bar{b}_0^{sep} = \frac{\bar{k} - (1-\delta^k) k_0 + \bar{c}_0 + \delta^h h^* - S_0(k_0)}{(1-\delta^h) h^*}. \tag{B.7}$$

We then have the following analogue of Proposition 1.

LEMMA 2: *Consider the modified model with separable preferences at date* 0*. The competitive equilibrium decumulates the excess housing capital in a single period,* $h_1 = h^*$*. If the overbuilding is sufficiently large,* $b_0 > \bar{b}_0^{sep}(k_0)$*, then the date* 0 *equilibrium features a demand-driven recession with,*

$$r_1 = 0, \quad \tau_0 > 0, \quad y_0 = \bar{y}_0 < S_0(k_0), \text{ and } l_0 < L_0(k_0).$$

EX-POST WELFARE ANALYSIS. — Next suppose the overbuilding is sufficiently large so that the economy is in a recession. We next respectively define the household's and the planner's value functions and derive their optimality conditions. Note that choosing $h_1 < h^*$ is sub-optimal in view of the preferences (2). We thus consider the value functions over the region $h_1 \geq h^*$.

The household's problem can then be written as (cf. problem (5)),

$$\begin{aligned} W_0(h_1) &= \max_{\{c_t, a_{t+1}\}_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad & c_t + a_{t+1} + h_{t+1} = e_t + a_t(1+r_t) + \Pi_t + (1-\delta^h) h_t \\ \text{given } h_0 &\geq h^*, h_1 \geq h^* \text{ and } h_t = h^* \text{ for each } t \geq 2. \end{aligned}$$

Using the envelope theorem, we obtain,

$$\frac{dW_0(h_1)}{dh_1}|_{h_1=h^*} = \beta u'(c_1)\left(1-\delta^h\right) - u'(c_0).$$

Combining this with the Euler equation, $u'(c_0) = \beta(1+r_1)u'(c_1)$, establishes Eq. (28).

Next consider a constrained planner who can (only) control housing investment at date 0. When h_1 is in a neighborhood of h^*, the constrained planning problem can be written as,

$$\text{(B.8)} \qquad W_{0,pl}(h_1) = \max_{c_0,k_1,y_0,l_0} u(c_0) - v_0(l_0) + \beta V(k_1,h_1),$$

$$\text{s.t. } k_1 = \overline{k} \text{ and } u'(c_0) = \beta u'(C(h_1)),$$

$$\text{(B.9)} \qquad \text{and } y_0 = F(k_0,l_0) = k_1 - \left(1-\delta^k\right)k_0 + c_0 + h_1 - \left(1-\delta^h\right)(1+b_0)h^*.$$

Here, $V(k_1,h_1)$ denotes the efficient value function characterized as the solution to problem $(C.1)$, and $C(h_1)$ denotes the efficient level of consumption. The second line captures the zero lower bound constraint, which implies that nonhousing investment and consumption are determined by the zero interest rate. The third line captures that output and employment are determined by the aggregate demand at date 0. Importantly, output is increasing in h_1 because a greater level of housing investment increases aggregate demand.

To derive the optimality condition for problem $(B.8)$, note that the capital stocks k_0 and $k_1 = \overline{k}$ are constant, and that the remaining variables, $c_0(h_1), y_0(h_1), l_0(h_1)$, are determined as implicit functions of h_1. Implicitly differentiating the aggregate demand constraint $(B.9)$ with respect to h_1, we obtain,

$$\frac{dl_0}{dh_1} = \frac{1+\frac{dc_0}{dh_1}}{F_l(k_0,l_0)} = \left(1+\frac{dc_0}{dh_1}\right)\frac{(1-\tau_0)u'(c_0)}{v'(l_0)}.$$

Here, the second equality substitutes the labor wedge from Eq. $(B.6)$. Using problem $(C.1)$ along with the envelope theorem, we also obtain,

$$\frac{dV_1(k_1,h_1)}{dh_1} = \left(1-\delta^h\right)u'(c_1) = \left(1-\delta^h\right)\frac{u'(c_0)}{\beta}.$$

Here, the second equality uses the Euler equation. Differentiating the objective function of problem $(B.8)$ with respect to h_1, and using these expressions, we obtain,

$$\begin{aligned}\frac{dW_{0,pl}(h_1)}{dh_1} &= u'(c_0)\frac{dc_0}{dh_1} - v_0'(l_0)\frac{dl_0}{dh_1} + \beta\frac{dV_1(k_1,h_1)}{dh_1},\\ &= u'(c_0)\left(\frac{dc_0}{dh_1} - \left(1+\frac{dc_0}{dh_1}\right)(1-\tau_0) + 1 - \delta^h\right).\end{aligned}$$

Rearranging terms establishes Eq. (29). Using this expression, Appendix C proves Proposition 2 and completes the welfare analysis in Section V.A.

EX-ANTE WELFARE ANALYSIS. — Next consider the ex-ante welfare analysis in Section V.B. Recall that the representative household optimally chooses $h_0 = h^* \left(1 + \lambda^H\right)$, along with k_0 characterized as the solution to (30). The representative household recognizes that the rental rate of capital in state L, R_0^L, is below its efficient level (due to the demand shortage). This might induce her to choose a lower level of k_0 as a precaution. A sufficiently low level of k_0 can, in turn, raise the aggregate demand and prevent the demand-driven recession [cf. Eq. $(B.7)$]. Nonetheless, the following result establishes that the economy experiences a recession in state L, as long as the probability of the state is sufficiently low, and the demand for housing in the counterfactual state H is sufficiently high.

LEMMA 3: *Consider the modified model with the ex-ante date* -1*, with the initial conditions,* $h_{-1} = h^* \left(1 + \lambda^H\right)$ *and* $k_{-1} = k^*$*. Suppose* $\lambda^H > \overline{b}_0^{sep}(k^*)$*, where* $\overline{b}_0^{sep}(k^*)$ *denotes the overbuilding threshold in* $(B.7)$ *given* $k_0 = k^*$*. There exists* $\overline{\pi} < 1$ *such that, if* $\pi^H \in (\overline{\pi}, 1)$*, then the equilibrium features a demand-driven recession in state* L *of date* 0 *(but not in any other dates or states).*

The equilibrium path starting with the high-demand state H of date 0 is straightforward. It solves the neoclassical planning problem $(C.1)$ with a steady level of housing investment given by, $i_t^h = \delta \left(1 + \lambda^H\right) h^*$ for each $t \geq 0$. The zero lower bound does not bind and the rental rate of capital is given by $R_0^H = S'(k_0)$. The equilibrium path starting with the low-demand state L of date 0 is characterized as in Lemma 2 given the (endogenous) level of overbuilding, $b_0 = \lambda^H$.

Next consider a constrained planner who can (only) control households' date -1 allocations. As described in the main text, the planner optimally chooses $h_{0,pl} = h_0 = \left(1 + \lambda^H\right) h^*$. However, the planner's choice of nonhousing capital, $k_{0,pl}$, is potentially different. To characterize this choice, let $V_0^H(k_0, h_0)$ and $V_0^L(k_0, h_0)$ denote the welfare of the representative household in respectively states H and L of date 0. The ex-ante constrained planning problem can then be written as,

$$
\begin{aligned}
&\max_{c_{-1},k_0} u(c_{-1}) + \beta \left(\pi^H V_0^H(k_0, h_0) + \left(1 - \pi^H\right) V_0^L(k_0, h_0)\right), \\
&\text{s.t.} \quad c_{-1} + k_0 + h_{0,pl} = S(k_{-1}) + \left(1 - \delta^k\right) k_{-1} + \left(1 - \delta^h\right) h_{-1}.
\end{aligned}
\tag{B.10}
$$

In particular, the planner optimally trades off the ex-ante consumption, c_{-1}, with investment, k_0, evaluating the benefits of the latter in the competitive equilibrium that will obtain in each state. The optimality condition for the problem is then given by

$$
u'(c_{-1}) = \beta \left(\pi^H \frac{dV_0^H(k_0, h_0)}{dk_0} + \left(1 - \pi^H\right) \frac{dV_0^L(k_0, h_0)}{dk_0}\right). \tag{B.11}
$$

We next derive $\frac{dV_0^H(k_0,h_0)}{dk_0}$ and $\frac{dV_0^L(k_0,h_0)}{dk_0}$, and establish Eq. (31). If state H is realized, then the equilibrium solves the analog of problem $(C.1)$ (with appropriate modifications to capture the higher target level, $\left(1+\lambda^H\right)h^*$). Then, the envelope theorem implies,

$$\frac{dV_0^H\left(k_0,h_0\right)}{dk_0} = \left(S'\left(k_0\right)+1-\delta^k\right)u'\left(c_0^H\right).$$

Suppose instead state L is realized. The continuation allocation is characterized by Lemma 2, and it solves problem $(B.8)$ with $h_1 = h^*$ (since we rule out ex-post policies). This problem implies that the following variables are constant, $k_1 = \overline{k}$, $c_0 = \overline{c}_0$, $h_1 = h^*$ (and thus, the continuation value V_1 is also constant). In contrast, output and employment, $y_0\left(k_0\right), l_0\left(k_0\right)$, are determined as implicit functions of k_0. Implicitly differentiating the aggregate demand constraint $(B.9)$ with respect to k_0, we obtain,

$$\frac{dl_0}{dk_0} = -\frac{F_k\left(k_0,l_0\right)+\left(1-\delta^k\right)}{F_l\left(k_0,l_0\right)} = -\left(F_k\left(k_0,l_0\right)+\left(1-\delta^k\right)\right)\frac{\left(1-\tau_0\right)u'\left(\overline{c}_0\right)}{v'\left(l_0\right)}.$$

Here, the second equality substitutes the labor wedge from Eq. $(B.6)$. Differentiating the objective function with respect to k_0, and using this expression, we further obtain,

$$\frac{dV_0^L\left(k_0,h_0\right)}{dk_0} = -v_0'\left(l_0\right)\frac{dl_0}{dk_0} = \left(1-\tau_0\right)\left(F_k\left(k_0,l_0\right)+\left(1-\delta^k\right)\right)u'\left(\overline{c}_0\right).$$

Plugging in $R_0^L = \left(1-\tau_0\right)F_k\left(k_0,l_0\right)$ from Lemma 2 implies,

$$\frac{dV_0^L\left(k_0,h_0\right)}{dk_0} = \left(R_0^k+\left(1-\tau_0\right)\left(1-\delta^k\right)\right)u'\left(\overline{c}_0\right).$$

Plugging the expressions for $\frac{dV_0^H(k_0,h_0)}{dk_0}$ and $\frac{dV_0^L(k_0,h_0)}{dk_0}$ into $(B.11)$ implies the planner's optimality condition (31). Appendix C proves Propositions 3 and 4, and completes the welfare analysis in Section V.B.

APPENDIX C: OMITTED PROOFS

This appendix presents the omitted characterizations and proofs.

1. *Proofs for the baseline model analyzed in Sections I and II*

Characterization of the efficient benchmark. Consider a planner that maximizes households' welfare starting date t onwards, given the initial state h_t, k_t, and the feasibility constraints of the economy. The planner's problem can then be written as,

$$\max_{\left\{\hat{c}_{\tilde{t}}, l_{\tilde{t}}, k_{\tilde{t}+1}, \tilde{h}_{t+1}, \left[l_{\tilde{t}}(\nu), k_{\tilde{t}}(\nu)\right]_\nu\right\}_{\tilde{t}=t}^{\infty}} \sum_{\tilde{t}=t}^{\infty} \beta^{\tilde{t}} \left(u\left(\hat{c}_{\tilde{t}} - v\left(l_{\tilde{t}}\right)\right) + u^h \mathbf{1}\left[h_t \geq h^*\right]\right),$$
$$\text{s.t. } \hat{c}_{\tilde{t}} + k_{\tilde{t}+1} + h_{\tilde{t}+1} \leq \hat{y}_{\tilde{t}} + \left(1-\delta^k\right) k_{\tilde{t}} + \left(1-\delta^h\right) h_{\tilde{t}}, \text{ where}$$
$$\hat{y}_{\tilde{t}} = \left(\int_0^1 \left(F\left(k_{\tilde{t}}(\nu), l_{\tilde{t}}(\nu)\right)\right)^{\frac{\varepsilon-1}{\varepsilon}} d\nu\right)^{\varepsilon/(\varepsilon-1)}, k_{\tilde{t}} = \int k_{\tilde{t}}(\nu)\, d\nu, \text{ and } l_{\tilde{t}} = \int l_{\tilde{t}}(\nu)\, d\nu.$$

By concavity, the planner chooses $k_{\tilde{t}}(\nu) = k_{\tilde{t}}$ and $l_{\tilde{t}}(\nu) = l_{\tilde{t}}$ for each $\tilde{t}$. The optimality condition for labor then implies Eq. (9). Combining these observations, the planner's problem reduces to the neoclassical planning problem,

$$\text{(C.1)} \quad V(k_t, h_t) = \max_{\left\{c_{\tilde{t}}, k_{\tilde{t}+1}, h_{t+1}\right\}_{\tilde{t}=t}^{\infty}} \sum_{\tilde{t}=t}^{\infty} \beta^{\tilde{t}} \left(u\left(c_{\tilde{t}}\right) + u^h \mathbf{1}\left[h_t \geq h^*\right]\right),$$
$$\text{s.t.} \quad c_{\tilde{t}} + k_{\tilde{t}+1} - \left(1-\delta^k\right) k_{\tilde{t}} + h_{\tilde{t}+1} - \left(1-\delta^h\right) h_{\tilde{t}} = S\left(k_{\tilde{t}}\right).$$

Here, the function $S(\cdot)$ describes the supply-determined net output defined in (9).

Equilibrium in the aftermath of overbuilding. Suppose the economy reaches date 1 with $h_1 = h^*$ and $k_1 \leq \bar{k}$. We claim that the continuation equilibrium is the same the efficient benchmark. To this end, consider the solution to the planner's problem $(C.1)$ starting with $h_1 = h^*$ and $k_1 \leq \bar{k}$. We conjecture a solution in which $h_{t+1} = h^*$ for each $t \geq 1$, as in (3), and the remaining allocations are characterized as the solution to the neoclassical system,

$$\text{(C.2)} \qquad \begin{aligned} S\left(k_{\tilde{t}}\right) &= c_{\tilde{t}} + k_{\tilde{t}+1} - \left(1-\delta^k\right) k_{\tilde{t}} + \delta^h h^*, \\ u'\left(c_{\tilde{t}}\right) &= \beta\left(1 + S'\left(k_{\tilde{t}}\right) - \delta^k\right) u'\left(c_{\tilde{t}+1}\right), \end{aligned}$$

together with a standard transversality condition. The steady-state to this system is characterized by,

$$\beta\left(1 - \delta^k + S'\left(k^*\right)\right) = 1 \text{ and } S\left(k^*\right) = c^* + \delta^k k^* + \delta^h h^*.$$

We assume the parameters satisfy, $\min\left(S\left(k_0\right), S\left(k^*\right)\right) > \delta^k k^* + \delta h^*$, which ensures that the economy can afford the required investment at all periods. Then, using standard arguments, there is a unique interior path that solves the system in $(C.2)$ and converges to the steady state. Moreover, since capital converges monotonically to its steady-state level, and since we have $k_1 \leq \overline{k}$ and $k^* < \overline{k}$, we also have $k_{t+1} \leq \overline{k}$ for each $t \geq 1$. This in turn implies the interest rate satisfies, $r_{t+1} = S'(k_{t+1}) - 1 \geq S'\left(\overline{k}\right) - 1 = 0$ for each $t \geq 1$.

In particular, the implied real interest rate is nonnegative along the socially optimal path, which has two implications. First, the planner finds it optimal to choose $h_{t+1} = h^*$ as we have conjectured (since the gross return on investment, $1+r_{t+1}$, exceeds the return on empty houses, $1 - \delta^h$). Second, and more importantly, the lower bound constraint (7) does not bind along the socially optimal path. This implies that the monetary policy rule in (10) replicates the dynamically efficient outcomes. That is, the competitive equilibrium from date 1 onwards (starting $h_1 = h^*$ and $k_1 \leq \overline{k}$) coincides with the efficient benchmark. This completes the characterization of the equilibrium in the aftermath of overbuilding. ∎

Proof of Lemma 1. First consider the case $r_{t+1} > 0$. In this case, monetary policy implements the efficient allocation with $l_t = L(k_t)$ and $y_t = S(k_t)$**.** In addition, the first order conditions for problems (9) and (4) further imply, $F_l(k_t, L(k_t)) = \upsilon'(L(k_t)) = w_t$. Combining this with Eq. (12) implies that the labor wedge is zero, $\tau_t = 0$. Combining Eqs. (12) and (9) then imply the rental rate of capital is given by $F_k(k_t, L(k_t)) = S'(k_t)$, completing the proof for the first part.

Next consider the case $r_{t+1} = 0$. In this case, Eq. (12) implies $F_l(k_t, l_t) \geq \upsilon'(l_t)$. This in turn implies that $l_t \in [0, L(k_t)]$. By feasibility, net output satisfies

$$y_t = c_t + i_t^h + i_t^h = F(k_t, l_t) - \upsilon(l_t).$$

This right hand side is strictly increasing in l_t over the range $[0, L(k_t)]$. The minimum and the maximum are respectively given by 0 and $S(k_t)$, which implies $y_t \in [0, S(k_t)]$. Moreover, given y_t that satisfies these resource constraints, there is a unique l_t that solves (11). Combining this with Eq. (12), we further obtain the labor wedge as, $1 - \tau_t = \frac{\upsilon'(l_t)}{F_l(k_t,l_t)}$. Plugging this into Eq. (12) for capital, we obtain the rental rate of capital as, $R(k_t, l_t) = \frac{\upsilon'(l_t)}{F_l(k_t,l_t)} F_k(k_t, l_t)$. It can be checked that $R_k < 0$, $R_l > 0$ over $l \in [0, L(k_t)]$, and that $R(k_t, L(k_t)) = S'(k_t)$, completing the proof. ∎

Proof of Proposition 1. As we have shown above, the equilibrium at date 1 starting with $h_1 = h^*$ and $k_1 \leq \overline{k}$ coincides with the efficient benchmark. Note also that, by standard arguments, the neoclassical system in $(C.2)$ can be described by an increasing consumption function, $c_1 = C(k_1)$.

To characterize the equilibrium at date 0, we define $K_1(r_0)$ for each $r_0 \geq 0$ as the solution to

$$S'(K_1(r_0)) - \delta^k = r_0.$$

Note that $K_1(r_0)$ is decreasing in the interest rate, with $K_1(0) = \overline{k}$ and

$\lim_{r_0 \to \infty} K_1(r_0) = 0$. Similarly, define the function $C_0(r_0)$ as the solution to the Euler equation

$$u'(C_0(r_0)) = \beta(1+r_0)\,u'(C(K_1(r_0)))\,.$$

Note that $C_0(r_0)$ is decreasing in the interest rate, with $C_0(0) = \overline{c}_0$ and $\lim_{r_0 \to \infty} C_0(r_0) = 0$. Finally, define the aggregate demand function

$$Y_0(r_0) = C_0(r_0) + K_1(r_0) - \left(1-\delta^k\right)k_0 + i_0^h.$$

Note that $Y_0(r_0)$ is also decreasing in the interest rate, with

$$Y_0(0) = \overline{y}_0 \text{ and } \lim_{r_0 \to \infty} Y_0(r_0) = i_0^h - \left(1-\delta^k\right)k_0.$$

Next consider the date 0 equilibrium for the case $b_0 \le \overline{b}_0$. Note that this implies $S(k_0) \le \overline{y}_0 = Y_0(0)$, and that we also have $\lim_{r_0 \to \infty} Y_0(r_0) < S(k_0)$ (since we assume housing investment is feasible). By the intermediate value theorem, there is a unique equilibrium interest rate $r_0 \in [0,\infty)$ such that $Y_0(r_0) = S(k_0)$. The equilibrium features $c_0 = C_0(r_0)$ and $K_1(r_0) = k_1$, along with $y_0 = S(k_0)$ and $l_0 = L(k_0)$.

Next consider the date 0 equilibrium for the case $b_0 > \overline{b}_0$. In this case, $Y_0(0) < S(k_0)$. Thus, the unique equilibrium features $r_0 = 0$ and $y_0 = \overline{y}_0 < S(k_0)$. Consumption and investment are given by $c_0 = \overline{c}_0$ and $k_1 = \overline{k}_1$. Labor supply l_0 is determined as the unique solution to (11) over the range $l_0 \in (0, L(k_0))$. Finally, Eq. $(B.5)$ implies the equilibrium output, $y_0 = \overline{y}_0$, is declining in the initial overbuilding b_0.

In either case, it can also be checked that the economy reaches date 1 with $h_1 = h^*$ and $k_1 \ge \min(k_0, k^*)$. Thus, the continuation equilibrium is characterized as described above, completing the proof. ∎

Proof of Proposition 5. Note that the recession is triggered if $\overline{y}_0 < S(k_0)$, where $\overline{y}_0$ is given by Eq. $(B.1)$. Since $1-\delta^{h^d} > 1-\delta^{h^n}$, increasing b_0^d (while keeping $b_0 = \left(b_0^d + b_0^n\right)/2$ constant) reduces $\overline{y}_0$, proving the result. ∎

2. *Proofs for the policy analysis in Section V and Appendix B.3*

Proof of Lemma 2. Most of the proof is described in Appendix B.3. If $b_0 < \overline{b}_0^{sep}$, then the maximum aggregate demand is above the efficient level, $\overline{y}_0 > S_0(k_0)$. In this case, the zero lower bound constraint does not bind and outcomes are efficient. If instead $b_0 > \overline{b}_0^{sep}$, then output is below the efficient level and it is determined by aggregate demand, $y_0 = \overline{y}_0 < S_0(k_0)$. The employment is also below the efficient level, $l_0 < L_0(k_0)$, and it is characterized by solving, $y_0 = \overline{y}_0 = F(k_0, l_0)$. The labor wedge is characterized by solving, $1 - \tau_0 = \frac{v_0'(l_0)}{F_l(k_0,l_0)u'(\overline{c}_0)}$, and it satisfies $\tau_0 > 0$.

Proof of Proposition 2. We first show that the planner's marginal utility, $\frac{d_+ W_{0,pl}(h_1)}{dh_1}|_{h_1 = h^*}$, is increasing in the labor wedge, τ_0. Note that the Euler equation in problem $(B.8)$

implies,

$$\frac{dc_0}{dh_1}|_{h_1=h^*} = \frac{\beta u''\left(C\left(h^*\right)\right)}{u''\left(\overline{c}_0\right)} C'\left(h^*\right) > 0.$$

Here, the inequality follows because the solution to the neoclassical problem $(C.1)$ implies $C'\left(h^*\right) > 0$. Note also that the derivative $\frac{dc_0}{dh_1}|_{h_1=h^*}$ is independent of b_0 or τ_0. Combining this with Eq. (29) proves that $\frac{d_+ W_{0,pl}(h_1)}{dh_1}|_{h_1=h^*}$ is increasing τ_0.

Next note from the proof of **2** that the labor wedge, τ_0, is strictly decreasing in aggregate demand, $y_0 = \overline{y}_0$. Since the maximum demand, $\overline{y}_0$, in Eq. $(B.5)$ is strictly decreasing in overbuilding, b_0, this implies that the labor wedge is strictly increasing in overbuilding, b_0. This in turn implies that the planner's marginal utility, $\frac{d_+ W_{0,pl}(h_1)}{dh_1}|_{h_1=h^*}$, is strictly increasing in b_0. It can also be checked that $\frac{d_+ W_{0,pl}(h_1)}{dh_1}|_{h_1=h^*} > 0$ for sufficiently high levels of b_0. Let $\tilde{b}_0 > \overline{b}_0^{sep}$ denote the level of overbuilding such that $\frac{d_+ W_{0,pl}(h_1)}{dh_1}|_{h_1=h^*} = 0$. It follows that, $\frac{d_+ W_{0,pl}(h_1)}{dh_1}|_{h_1=h^*} > 0$ if and only if $b_0 > \tilde{b}_0$. This also implies $h_{1,pl} > h^*$ if and only if $b_0 > \tilde{b}_0$. ∎

Proof of Lemma 3. First consider the limiting case with $\pi^H = 1$. In this case, given the initial conditions, the economy is at an efficient steady-state with,

$$h_t = h^*\left(1+\lambda^H\right), k_t = k^* \text{ and } c^* = S\left(k^*\right) - \delta^h\left(1+\lambda^H\right)h^* - \delta^k k^*.$$

In particular, the competitive equilibrium features $k_0 = k^*$. In this equilibrium, the economy does not feature a demand shortage at date 0 or state H of date 1. In fact, we have $r_1 = r_2^H = 1/\beta > 0$. However, since $\lambda^H > \overline{b}_0^{sep}\left(k^*\right)$, the economy features a demand shortage in the (zero probability) state L.

Next note that the capital choice in competitive equilibrium is a continuous function of the probability of the high state, $k_0\left(\pi^H\right)$. By Eq. $(B.7)$, $\overline{b}_0^{sep}\left(k_0\right)$ is also a continuous function of k_0. It follows that there exists $\overline{\pi}^1$ (which could also be $\overline{\pi}^1 = 0$) such that $\lambda^H > \overline{b}_0^{sep}\left(k^*\right)$ if and only if $\pi^H > \overline{\pi}^1$. Similarly, note that the interest rates r_1 and r_2^H are also continuous functions of π^H. Using continuity once again, there exists $\overline{\pi}^2 < 1$ (which could also be $\overline{\pi}^2 = 0$) such that the economy does not feature a demand shortage at date 0 or at state H if and only if $\pi^H > \overline{\pi}^2$. Taking $\overline{\pi} = \max\left(\overline{\pi}^1, \overline{\pi}^2\right)$ proves the statement. ∎

Proof of Proposition 3. The planner's optimality condition (31) implies $k_{0,pl} < k_0$ since $\tau_0 > 0$, $\pi^H > 0$, and $1-\delta^k > 0$. ∎

Proof of Proposition 4. In this case, the difference is that the planner can also control the ex-ante employment and net output, l_{-1}, y_{-1}, by deviating from the monetary policy in (10). Thus, the analogue of the planner's problem in $(B.10)$ is given by,

$$\max_{l_{-1},y_{-1},c_{-1},k_0} u\left(c_{-1}\right)+\beta\left(\pi^H V_0^H\left(k_0,h_0\right)+\left(1-\pi^H\right)V_0^L\left(k_0,h_0\right)\right),$$
$$\text{s.t.} \quad c_{-1}+k_0+h_{0,pl}=S\left(k_{-1}\right)+\left(1-\delta^k\right)k_{-1}+\left(1-\delta^h\right)h_{-1},$$
$$\text{and } y_{-1}=F\left(k_{-1},l_{-1}\right)-\upsilon\left(l_{-1}\right)\leq S\left(k_{-1}\right).$$

It is easy to check that the first order conditions maximize the net output, $y_{-1}=S\left(k_{-1}\right)$ and $l_{-1}=L\left(k_{-1}\right)$. This in turn leads to the same problem $(B.10)$ as before, as well as the same first order conditions (31). In particular, the planner sets the interest rate, $r_0=r_0^*$, which (by definition) replicates the statically efficient allocations at date -1, completing the proof. ∎