Online Appendix to Financial Frictions, Capital Misallocation, and Input-Output Linkages

Hsuan-Li Su*

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*Department of Economics, National Taiwan University (email: shanesu@ntu.edu.tw).

A Technical Details on Financial Frictions

The Large Household Assumption

There is a representative household with a continuum of members of measure unity. The household consumes, saves, and supplies labor. Within the household there are two types of members: workers and entrepreneurs. Workers supply labor and return the wages to the household, and they consists of every type of differentiated labor supplied to every intermediate goods producing sector. Entrepreneurs manage intermediate goods producing firms and return part of the firms' profits as dividends to the household. Type j workers and entrepreneurs work and manage firms in sector j, respectively.

A type j entrepreneur in this period remains an entrepreneur of type j in the next period with probability κ_j, which is exogenously given. The average survival rate for a type j entrepreneur is $1/(1-\kappa_j)$. This finite horizon for entrepreneurs is used to ensure that over time they do not accumulate enough net worth to fund their capital fully. When entrepreneurs exit, they give their retained earnings, their net worth, back to the household as dividends and become workers. A similar number of workers randomly become entrepreneurs, keeping the relative proportion of each type fixed. The household then gives some startup funds to its new entrepreneurs for operation. Within the family there is perfect consumption insurance. The object of entrepreneurs is to maximize dividends.

Costly State Verification

An entrepreneur i in sector j has net worth $n^i_{j,t+1}$. The optimal contract is a standard debt contract that specifies the amount the entrepreneur can borrow, $B^i_{j,t+1} = Q_t k^i_{j,t+1} - n^i_{j,t+1}$, and the gross loan rate, $z^i_{j,t+1}$, that the entrepreneur needs to repay. So the specified total loan payment is $z^i_{j,t+1}B^i_{j,t+1}$. The optimal contract, contingent on $\sigma_{j,t+1}$, specifies $(L^i_{j,t+1}, \bar{\omega}_{j,t+1})$. Note that the threshold $\bar{\omega}_{j,t+1}$ satisfies

$$\bar{\omega}_{j,t+1} R^k_{j,t+1} Q_t k^i_{j,t+1} = z^i_{j,t+1} B^i_{j,t+1},$$

for every entrepreneur. Since $E_t[\omega^i_{j,t+1}] = 1$, $R^k_{j,t+1}Q_t k^i_{j,t+1}$ is the expected payoff by investing capital $k^i_{j,t+1}$, at period t, and $\omega^i_{j,t+1} R^k_{j,t+1} Q_t k^i_{j,t+1}$ is the realized payoff at period $t+1$.

The optimal contract that specifies the loan amount, $B^i_{j,t+1}$, and the loan payment, $z^i_{j,t+1}$, is equivalent to specifying the leverage, $L^i_{j,t+1} = Q_t k^i_{j,t+1}/n^i_{j,t+1}$, and the threshold, $\bar{\omega}_{j,t+1}$, through the above equation given $n^i_{j,t+1}$.

Denote $F_{jt}(\omega) = Pr[\omega^i_{jt} < \omega]$ a continuous probability distribution and $f_{jt}(\omega)$ the pdf of ω^i_{jt}. The zero-profit condition for the financial intermediary is

$$\Big\{ \underbrace{(1-\mu_j)\int_0^{\bar{\omega}_{j,t+1}} \omega f_{j,t+1}(\omega)d\omega}_{\text{from defaulted entrepreneurs}} + \underbrace{\bar{\omega}_{j,t+1}[1-F_{j,t+1}(\bar{\omega}_{j,t+1})]}_{\text{loan repayment}} \Big\} R^k_{j,t+1} Q_t k_{j,t+1} = R_{t+1} D_{j,t+1}, \forall j.$$

The first component of the left part is the amount of asset seized by the financial intermediary from defaulted entrepreneurs after paying the cost. The second component of the left part of is the amount of repayment the financial intermediary receives from those non-defaulted entrepreneurs.

In equilibrium, a competitive financial intermediary earns zero profit in every sector. Note that $\sum_j D_{j,t+1} = D_{t+1}$. Sum the above constraint over each sector j, we get

$$\sum_{j=1}^{N} \Big\{ (1-\mu_j)\int_0^{\bar{\omega}_{j,t+1}} \omega f_{j,t+1}(\omega)d\omega + \bar{\omega}_{j,t+1}[1-F_{j,t+1}(\bar{\omega}_{j,t+1})] \Big\} R^k_{j,t+1} Q_t k_{j,t+1} = R_{t+1} D_{t+1}.$$

To simplify the presentation, define $\Omega_{j,t+1}$ as the share of sector j's profits going to the financial intermediary at period $t+1$:

$$\Omega_{j,t+1}(\bar{\omega}_{j,t+1}, \sigma_{j,t+1}) = \int_0^{\bar{\omega}_{j,t+1}} \omega f_{j,t+1}(\omega)d\omega + \bar{\omega}_{j,t+1}[1-F_{j,t+1}(\bar{\omega}_{j,t+1})],$$

and $\mu_j G_{j,t+1}$ the share of sector j's profits as default cost at period $t+1$:

$$\mu_j G_{j,t+1}(\bar{\omega}_{j,t+1}, \sigma_{j,t+1}) = \mu_j \int_0^{\bar{\omega}_{j,t+1}} \omega f_{j,t+1}(\omega)d\omega.$$

For a given $\sigma_{j,t+1}$, $\Omega_{j,t+1}$ is strictly concave in $\bar{\omega}_{j,t+1}$, and $G_{j,t+1}$ is strictly increasing in $\bar{\omega}_{j,t+1}$. That means there exists an ω^* such that the net payoff to the lender, $\Omega_{j,t+1} - \mu_j G_{j,t+1}$, reaches the maximum. In equilibrium, the lender always chooses $\bar{\omega}_{j,t+1} \in (0, \omega^*)$.

For the profits in sector j at period $t+1$, entrepreneurs get $(1-\Omega_{j,t+1})$ share, and the financial intermediary gets $(\Omega_{j,t+1} - \mu_j G_{j,t+1})$ share. Thus the zero-profit condition at sector j can be written as

$$(\Omega_{j,t+1} - \mu_j G_{j,t+1}) R^k_{j,t+1} Q_t k_{j,t+1} = R_{t+1} D_{j,t+1}.$$

$\Omega_j(\bar{\omega}_{j,t+1}, \sigma_{j,t+1})$ and $G_j(\bar{\omega}_{j,t+1}, \sigma_{j,t+1})$ are functions of the endogenous threshold $\bar{\omega}_{j,t+1}$

and exogenous sectoral uncertainty $\sigma_{j,t+1}$.

$$\frac{\partial \Omega_j(\bar{\omega}_{j,t+1}, \sigma_{j,t+1})}{\partial \bar{\omega}_{j,t+1}} = 1 - F(\bar{\omega}_{j,t+1}) > 0, \frac{\partial^2 \Omega_j(\bar{\omega}_{j,t+1}, \sigma_{j,t+1})}{\partial \bar{\omega}^2_{j,t+1}} = -f(\bar{\omega}_{j,t+1}) < 0.$$

So $\Omega_j(\bar{\omega}_{j,t+1}, \sigma_{j,t+1})$ is strictly concave in $\bar{\omega}_{j,t+1}$. And

$$\frac{\partial G_j(\bar{\omega}_{j,t+1}, \sigma_{j,t+1})}{\partial \bar{\omega}_{j,t+1}} = \bar{\omega}_{j,t+1} f(\bar{\omega}_{j,t+1}) > 0.$$

Note that $D_{j,t+1} = Q_t k_{j,t+1} - N_{j,t+1}$, so $\frac{D_{j,t+1}}{N_{j,t+1}} = L_{j,t+1} - 1$. Dividing the zero-profit condition at sector j by $N_{j,t+1}$, we can rewrite the financial intermediary's expected zero-profit condition as the following:

$$\frac{1}{L_{j,t+1}} = \ 1 - E_t \frac{R^k_{j,t+1}}{R_{t+1}} (\Omega_{j,t+1} - \mu_j G_{j,t+1}). \tag{A.1}$$

The optimal contract then maximizes each entrepreneur's expected return,

$$\max_{L^i_{j,t+1}, \bar{\omega}_{j,t+1}} E_t \frac{[1 - \Omega_{j,t+1}(\bar{\omega}_{j,t+1})] R^k_{j,t+1} Q_t k^i_{j,t+1}}{R_{t+1} n^i_{j,t+1}} = E_t [1 - \Omega_{j,t+1}(\bar{\omega}_{j,t+1})] L^i_{j,t+1} \frac{R^k_{j,t+1}}{R_{t+1}},$$

by choosing the leverage and the threshold, $(L^i_{j,t+1}, \bar{\omega}_{j,t+1})$, subject to the zero-profit condition (A.1).

After solving the optimal contract for each individual, we get a linear relationship between individual capital and net worth:

$$Q_t k^i_{j,t+1} = \psi(E_t R^k_{j,t+1} / R_{t+1}) n^i_{j,t+1}.$$

Two remarks are in order here. First, after aggregating the above equation over entrepreneurs in sector j, it turns out that sectoral capital only depends on sectoral aggregate net worth, and there is no need to keep track of an individual's net worth. Second, every entrepreneur ends up with the same leverage. So sectoral leverage can be written as the ratio of the value of sectoral capital and sectoral net worth, $L_{j,t+1} = \frac{Q_t k_{j,t+1}}{N_{j,t+1}}$.

Aggregating the equilibrium condition of individual capital and net worth gives the relationship between sectoral spread and leverage as the following:

$$L_{j,t+1} = \psi(E_t R^k_{j,t+1} / R_{t+1}).$$

Rewrite the above equation gives the Eq. (2.15):

$$E_t \frac{R^k_{j,t+1}}{R_{t+1}} = \psi^{-1}(L_{j,t+1}) = \varphi(\bar{\omega}_{j,t+1}, \sigma_{j,t+1}). \tag{A.2}$$

Since $L_{j,t+1}$ is the result from the optimal contract and depends on the endogenous threshold $\bar{\omega}_{j,t+1}$ and exogenous sectoral uncertainty $\sigma_{j,t+1}$.

The exact formula of $\varphi(\bar{\omega}_{j,t+1}, \sigma_{j,t+1})$ is that

$$\varphi(\bar{\omega}_{j,t+1}, \sigma_{j,t+1}) = \frac{\lambda_{j,t+1}}{1 - \Omega_{j,t+1} + \lambda_{j,t+1}(\Omega_{j,t+1} - \mu_j G_{j,t+1})},$$

where

$$\lambda_{j,t+1} = \frac{\Omega'(\bar{\omega}_{j,t+1})}{\Omega'(\bar{\omega}_{j,t+1}) - \mu_j G'(\bar{\omega}_{j,t+1})} = \frac{1 - F(\bar{\omega}_{j,t+1})}{1 - F(\bar{\omega}_{j,t+1}) - \mu_j \bar{\omega}_{j,t+1} f(\bar{\omega}_{j,t+1}).}$$

At the end of period t, after the realization of ω^i_{jt}, production, and factor payments, an entrepreneur transfers all his or her asset into net worth, n^i_{t+1}, by selling capital to the capital producer. Before borrowing, $(1-\kappa_j)$ fraction of type j entrepreneurs exit and become workers, and upon doing so, they give their net worth back to the household as dividends. This dividend has two purposes. First, it makes entrepreneurs' net worth part of households' wealth. It is thus in the interest of the representative household to instruct its entrepreneurs to maximize expected net worth. Second, this setup ensures that entrepreneurs will not accumulate too much net worth and end up without the need to borrow. At the same time, a similar number of workers become type j entrepreneurs such that the relative proportion of each type is fixed. The household then gives these new entrepreneurs and defaulted entrepreneurs some startup funds as their net worth which they can use to borrow to buy new capital for the next period. For simplicity, since there is no need to keep track of individual net worth, let these startup funds for each type be $w^e_j N_{j,t}$, where w^e_j is set at 0.01 for every sector j. This small amount of wealth is used to ensure that every entrepreneur has positive net worth, for those who default and those new entrepreneurs. Therefore, the law of motion for sectoral net worth is:

$$N_{j,t+1} = \kappa_j(1 - \Omega_{jt}(\bar{\omega}_{jt}, \sigma_{jt}))R^k_{jt}Q_{t-1}k_{jt} + w^e_j N_{j,t}. \tag{A.3}$$

For each sector, the relation between the interest rate spread (the difference between $z_{j,t+1}$ and R_{t+1}) and the capital wedge (the spread between the return to capital and risk-free rate) is:

$$\frac{z_{j,t+1}}{R_{t+1}} = \bar{\omega}_{j,t+1} \frac{R^k_{j,t+1}}{R_{t+1}} \frac{L_{j,t+1}}{L_{j,t+1} - 1}. \tag{A.4}$$

B Proofs

Proof of *Proposition 1*: From firms' first-order condition on M_{ij}, Eq. (2.3),

$$\gamma_{ij} p_j Y_j = p_i M_{ij} = p_i \eta_{ij} Y_i, \Rightarrow \frac{p_i Y_i}{p_j Y_j} = \frac{\gamma_{ij}}{\eta_{ij}}.$$

Since $\frac{p_i Y_i}{p_j Y_j} = \frac{v_i}{v_j}$, we have $\eta_{ij} = \frac{\gamma_{ij} v_j}{v_i}$. It can be written in the vector form as: $\boldsymbol{\eta} = [\frac{1}{\boldsymbol{v}} * \boldsymbol{v}'] \circ \boldsymbol{\Gamma}$.

$\boldsymbol{x^l}$ can be calculated from the relation that $l_i = \frac{\lambda_i^l}{\lambda_j^l} l_j$.

$$\sum_i l_i = H = \sum_i \frac{\lambda_i^l}{\lambda_j^l} l_j = \frac{l_j}{\lambda_j^l} \sum_i \lambda_i^l \Rightarrow \frac{l_j}{H} = \frac{\lambda_j^l}{\sum_i \lambda_i^l}$$

.

For capital allocation, $\frac{k_i}{k_j} = \frac{\lambda_i^k \mu_{ki}^{-1}}{\lambda_j^k \mu_{kj}^{-1}}$ is used to solve for x_j^k in the same way.

Proof of *Proposition 2*: From Eq. (2.1), using $p_j Y_j = v_j Y$ and $k_j = x_j^k \bar{K}$, we have

$$MRPK_j = \frac{\lambda_j^k Y}{x_j^k \bar{K}} = (1 + \tau_j) r.$$

Denote aggregate MPK as $MPK_{agg} = \tilde{\alpha} \frac{Y}{\bar{K}}$. Thus the relation between $MRPK_j$ and MPK_{agg} is that

$$MRPK_j = \frac{\lambda_j^k}{x_j^k} \frac{MPK_{agg}}{\tilde{\alpha}} = (1 + \tau_j) r. \tag{B.1}$$

Since $\tilde{\alpha} = \sum_i \lambda_i^k$, and $\frac{x_j^k}{\sum_i \lambda_i^k}$ is the first-best capital allocation for producer j, rewrite $\frac{\lambda_j^k}{\tilde{\alpha}} = x_j^{k,FB}$, and we have .

$$MRPK_j = \frac{x_j^{k,FB}}{x_j^k} MPK_{agg}.$$

From Eq.(B.1),

$$MPK_{agg} = \frac{\tilde{\alpha} x_j^k}{\lambda_j^k \mu_{kj}^{-1}} r.$$

From Proposition 1, replacing $x_j^k = \frac{\lambda_j^k \mu_{kj}^{-1}}{\sum_i \lambda_i^k \mu_{ki}^{-1}}$ into the above equation, we get

$$MPK_{agg} = \frac{\tilde{\alpha}}{(\sum_i \lambda_i^k \mu_{ki}^{-1})} \times r.$$

Proof of *Proposition 3*: The marginal revenue product of capital is defined as

$$\text{MRPK}_{jt} = \frac{(1-m_j)\alpha_j p_{jt} Y_{jt}}{k_{jt}}.$$

From the relation that $k_{jt} = x^k_{jt} K_t$, substitute $p_{jt}Y_{jt} = v_j Y_t$ in the above equation, we get

$$\frac{k_{jt}}{k_{it}} = \frac{x^k_{jt}}{x^k_{it}} = \frac{MRPK_{it}(1-m_j)\alpha_j v_j}{MRPK_{jt}(1-m_i)\alpha_i v_i} = \frac{\lambda^k_j MRPK^{-1}_{jt}}{\lambda^k_i MRPK^{-1}_{it}}.$$

The rest of the proof follows the same logic as in Proposition 1.

Proof of *Theorem 1* & *2*: Take logarithm on sectoral production function and denote $\ln Y_j = y_j, \ln A_j = a_j$. Substitute k_j with $x^k_j K$, l_j with $x^l_j H$, and M_{ij} with $\eta_{ij} Y_i$:

$$\begin{aligned}
y_j =& a_j + (1-m_j)\alpha_j \ln(x^k_j K) + (1-m_j)(1-\alpha_j)\ln(x^l_j H) + \sum_{i=1}^{N} \gamma_{ij} \ln(\eta_{ij} Y_i), \\
\Rightarrow y_j =& a_j + \underbrace{(1-m_j)\alpha_j \ln x^k_j + (1-m_j)(1-\alpha_j)\ln x^l_j + \sum_{i=1}^{N}\gamma_{ij}\ln\eta_{ij}}_{\text{denote this as } c_{yj}} + \underbrace{(1-m_j)\alpha_j}_{\text{denote this as } \delta_{kj}} \ln K \\
& + \underbrace{(1-m_j)(1-\alpha_j)}_{\text{denote this as } \delta_{lj}} \ln H + \sum_{i=1}^{N} \gamma_{ij} \ln Y_i,
\end{aligned}$$

$$\Rightarrow y_j = a_j + c_{yj} + \delta_{kj}\ln K + \delta_{lj}\ln H + \sum_{i=1}^{N}\gamma_{ij} y_i.$$

The vector form of the above equation is

$$\begin{aligned}
\boldsymbol{y} =& \boldsymbol{a} + \boldsymbol{c_y} + \boldsymbol{\delta_k}\ln K + \boldsymbol{\delta_l}\ln H + \boldsymbol{\Gamma}'\boldsymbol{y}, \\
\Rightarrow \boldsymbol{y} =& [\boldsymbol{I_N} - \boldsymbol{\Gamma}']^{-1}(\boldsymbol{a} + \boldsymbol{c_y} + \boldsymbol{\delta_k}\ln K + \boldsymbol{\delta_l}\ln H).
\end{aligned}$$

Replace X_j with $(1-\eta_j)Y_j$, so $Y = \prod_{j=1}^{N}((1-\eta_j)Y_j)^{\beta_j}$. The logarithm of Y becomes $\ln Y = \boldsymbol{\beta}'\ln(1-\boldsymbol{\eta}) + \boldsymbol{\beta}'\boldsymbol{y}$. Note that $\boldsymbol{\beta}'[\boldsymbol{I_N} - \boldsymbol{\Gamma}']^{-1} = \boldsymbol{v}'$. Then

$$\ln Y = \underbrace{\boldsymbol{\beta}'\ln(1-\boldsymbol{\eta}) + \boldsymbol{v}'(\boldsymbol{a_t} + \boldsymbol{c_{yt}})}_{\text{denote these terms as } \ln\tilde{A}_t} + \underbrace{\boldsymbol{v}'\boldsymbol{\delta_k}}_{\tilde{\alpha}}\ln K + \underbrace{\boldsymbol{v}'\boldsymbol{\delta_h}}_{1-\tilde{\alpha}}\ln H_t.$$

Note that $\boldsymbol{v}'\boldsymbol{\delta_k} = \sum_j v_j(1-m_j)\alpha_j = \sum_j \lambda^k_j = \tilde{\alpha}$, and $\boldsymbol{v}'\boldsymbol{\delta_h} = \sum_j v_j(1-m_j)(1-\alpha_j) = \sum_j \lambda^l_j = 1-\tilde{\alpha}$. Thus log aggregate output is

$$\ln Y = \ln \tilde{A} + \tilde{\alpha} \ln \bar{K} + (1-\tilde{\alpha}) \ln H,$$

$$\ln(\tilde{A}) = \boldsymbol{v'a} + \boldsymbol{\lambda^{k'}} \ln \boldsymbol{x^k} + \boldsymbol{\lambda^{l'}} \ln \boldsymbol{x^l} + \sum_i \beta_i \sum_j (1-\eta_{ij}) + \sum_j v_j \sum_i \gamma_{ij} \ln \eta_{ij}.$$

By direct differentiation of the above equation, we have that

$$\frac{\mathrm{d} \ln \tilde{A}}{\mathrm{d} \ln A_j} = v_j + \sum_i \lambda_i^k \frac{\mathrm{d} \ln x_i^k}{\mathrm{d} \ln A_j},$$

and

$$\frac{\mathrm{d} \ln \tilde{A}}{\mathrm{d} \ln \mu_{kj}} = \sum_i \lambda_i^k \frac{\mathrm{d} \ln x_i^k}{\mathrm{d} \ln \mu_{kj}}.$$

For Theorem 2, the proof is the same except that the time subscripts are added. And $\boldsymbol{x_t^k}$ is subject to shocks. So aggregate output and aggregate TFP are

$$\ln Y_t = \ln \tilde{A}_t + \tilde{\alpha} \ln K_t + (1-\tilde{\alpha}) \ln H_t,$$

$$\ln(\tilde{A}_t) = \boldsymbol{v'a_t} + \boldsymbol{\lambda^{k'}} \ln \boldsymbol{x_t^k} + \sum_i \beta_i \sum_j (1-\eta_{ij}) + \boldsymbol{\lambda^{l'}} \ln \boldsymbol{x^l} + \sum_j v_j \sum_i \gamma_{ij} \ln \eta_{ij}.$$

C Strategy to Compute the Steady State

There are $9 + 8N + N^2$ equations for $9 + 8N + N^2$ unknowns in the main dynamic model (there are also 2 N shocks), where N stands for the number of sectors. Here is the step to compute the steady state of the model. All the following variables are evaluated at the steady state.

1. From 3N equations (A.1), (A.2), and (A.3), given parameters $\mu_j, \sigma_j, \kappa_j$, solve for $R_j^k, \omega_j, L_j, \rho_j(\omega_j)$ at the steady state. $Q = \frac{1}{\theta}\delta^{\frac{1-\theta}{\theta}}$ can be solved from Eq. (2.11).

   From $R_j^k = (1-m_j)\alpha_j p_j Y_j / Q k_j + (1-\delta)$. Denote $MRPK_j = Q(R_j^k - (1-\delta)) = (1-m_j)\alpha_j p_j Y_j / k_j$. At this stage, we know values of $MRPK_j$ from R_j^k and Q. From $p_j Y_j = v_j Y$, $k_j = (1-m_j)\alpha_j p_j Y_j / MRPK_j = \frac{\lambda_j^k Y}{MRPK_j}$.

2. Solve for $\boldsymbol{v}$ and $\boldsymbol{\eta}$ using the following formulas:

$$\boldsymbol{v} = [I_N - \Gamma]^{-1} \beta,$$
$$\boldsymbol{\eta} = [\frac{1}{\boldsymbol{v}} * \boldsymbol{v'}] \circ \Gamma.$$

Thus, λ_j^k and λ_j^l are known.

3. Solve for x_k, x_l using the following formulas:

$$
\begin{aligned}
x_{kj} =& \frac{\lambda_j^k (MRPK_j)^{-1}}{\sum_i \lambda_i^k (MRPK_i)^{-1}} \\
x_{lj} =& \frac{\lambda_j^l}{\sum_i \lambda_i^l}
\end{aligned}
$$

4. From step 1, $k_j = \frac{\lambda_j^k Y}{MRPK_j}$, so

$$\sum_j k_j = K = \sum_j \frac{\lambda_j^k}{MRPK_j} Y = \varphi_k Y.$$

and similarly, from sectoral MPL equal to the wage,

$$H = \frac{\sum_j \lambda_j^l}{w} Y = \varphi_h Y.$$

so $\ln K = \ln \varphi_k + \ln Y$, and $\ln H = \ln \varphi_h + \ln Y$. At this stage, φ_k is known, while φ_h depends on wage w.

5. Solve for wage. Define $\delta_{kj} = (1 - m_j)\alpha_j$, and $\delta_{hj} = (1 - m_j)(1 - \alpha_j)$, take log on sectoral production technology, and write in vector form, we get

$$
\begin{aligned}
\boldsymbol{y} =& [I_N - \Gamma']^{-1}(\boldsymbol{a} + \boldsymbol{c}_y + \boldsymbol{\delta_k} \ln K + \boldsymbol{\delta_h} \ln H) \\
=& [I_N - \Gamma']^{-1}(\boldsymbol{a} + \boldsymbol{c}_y + \boldsymbol{\delta_k} \ln \varphi_k + \boldsymbol{\delta}_h \ln \varphi_h) + 1_{N\times 1} \ln Y
\end{aligned}
$$

,where $c_{yj} = (1-m_j)\alpha_j \ln x_{kj} + (1-m_j)(1-\alpha_j)\ln x_{lj} + \sum_i \gamma_{ij} \ln \eta_{ij}$, and $a_j = \log A_j = 0$, assuming $A_j = 1, \forall j$.
Take log on $Y = \prod (1 - \eta_j)^{\beta_j} Y_j^{\beta_j} \implies \ln Y = \boldsymbol{\beta}' \ln(1 - \boldsymbol{\eta}) + \boldsymbol{\beta}' \boldsymbol{y}$, plug in $\boldsymbol{y}$, and since $\boldsymbol{\beta}' 1_{N\times 1} = 1$, $\ln Y$ on both sides cancels out. We get

$$\boldsymbol{\beta}' \ln(1 - \boldsymbol{\eta}) + \boldsymbol{v}'(\boldsymbol{c}_y + \boldsymbol{\delta_k} \ln \varphi_k + \boldsymbol{\delta_h} \ln \varphi_h)] = 0.$$

In the above equation, wage is the only unknown. w is then solved numerically from this equation, and we get φ_h.

6. Solve for output Y. From the resource constraint and the Euler equation,

$$Y = C + I + \sum_{j=1}^{N} \mu_j G_j R_j^K Q k_j = (\varphi_h Y)^{-1/\epsilon\sigma} w^{1/\sigma} + \varphi_k Y \delta^{1/\theta} + (\sum_j \mu_j G_j R_j^K Q x_{kj}) \varphi_k Y$$
$$\Longrightarrow [1 - \varphi_k \delta^{1/\theta} - (\sum_j \mu_j G_j R_j^K x_{kj}) Q \varphi_k] Y = (\varphi_h)^{-1/\epsilon\sigma} w^{1/\sigma} Y^{-1/\epsilon\sigma}$$

rewrite as $\quad AY = BY^{-1/\epsilon\sigma}$,

$$Y = (\frac{B}{A})^{\frac{\epsilon\sigma}{\epsilon\sigma+1}}.$$

Thus, other variables can be solved easily. The dynamic system is then solved using the standard method of log-linearization as in Uhlig 1999.

D Discussion on σ and External Finance

The standard deviation of idiosyncratic capital returns, σ, reflects the riskiness of investment, and there is a caveat to how we should interpret σ when we map the model into data. In the model, σ mostly affects leverage, which is an endogenous outcome of the optimal contract. It is the balance between the credit demand (the dependence on external finance of a sector) and supply. Since entrepreneurs behave as if they are risk-neutral, the dependence on an external finance channel is missing in the model. Therefore, the credit is mainly constrained by the supply side. That is, lenders are less willing to lend if the investment return is risky—the endogenous leverage is decreasing in σ. The calibration then maps a low leverage in the data to high σ in the model, and vice versa. However, in the real world, a firm may have low leverage because it does not require much external finance. It is thus important to consider the dependence on external finance of each sector when we examine the appropriateness of calibrated σ.

To do this, I follow the procedure described in Rajan and Zingales (1998) to construct the Rajan-Zingales (RZ) measure of dependence on external finance for each sector.[1] Table 1 reports the equilibrium leverage in data, calibrated σ, and the RZ measure, sorted by the leverage in descending order. The RZ measure of 13 sectors ranges from 1.28 (Minging) to 0.40 (Professional and Business), and the median is 0.53. Here, using the median as the benchmark, sectors depend relatively more on external finance if their RZ measures are larger than 0.53. Similarly, a sector has high (low) leverage if its leverage is larger (smaller)

1. The Rajan-Zingales measure is defined as capital expenditure minus cash flow from operations divided by capital expenditures. To calculate this, I first took the median value of firms in each sector and year in *Compustat* and then took the average from 1985 to 2012 for each sector.

Table 1: Equilibrium leverage, σ, and dependence on external finance (RZ).

13-sector	Leverage	σ	RZ
Utilities	2.09	0.22	0.88
Transportation	1.97	0.27	0.86
Construction	1.88	0.29	0.48
Arts, Entertainment	1.84	0.30	0.89
Other services	1.76	0.32	0.48
Wholesale	1.73	0.33	0.38
Agriculture	1.70	0.33	0.58
Education and Health Care	1.70	0.35	0.53
Retail	1.58	0.38	0.68
Information	1.50	0.43	0.48
Mining	1.45	0.43	1.28
Manufacturing	1.43	0.44	0.47
Professional and Business	1.37	0.49	0.40

than the median value of 1.7.

Since the model indicates that a high leverage sector is one with safe investment returns (low σ), a low RZ measure (small demand) would confirm this interpretation, and a high RZ measure suggests the model may overstate its safety. Consider the high leverage sectors, such as the Construction sector, which has a high leverage and a low RZ measure. For these sectors, a low value of σ_j is considered reasonable. But the values of σ_j for the Utilities, Transportation, and Arts sectors might be underestimated.

For those low leverage sectors, the risky return interpretation (high σ) is confirmed by a large value of the RZ measure, such as in the Mining sector. The Mining sector has a low leverage but a high RZ measure, reflecting higher demand for external financing but relatively smaller supply, so the risky investment interpretation—a high value of σ_j—is reasonable.[2] But for the Information, Manufacturing, and Professional and Business sectors, one cannot rule out the possibility that they have low leverage simply because they do not need much external financing. The model might overestimate the values of σ_j for these sectors.

2. Why are investments in the Mining sector risky? An important feature in the mining and oil industries is that their investment returns are highly related to oil price fluctuations. A plunge in the price of oil hurts mining and energy firms' profits and returns, and the default rate and spreads of their bonds often rise during these periods. Kellogg 2014 points out that firms' failures to respond to oil price volatility can lead to a significant cost (25 percent of the value of a drilling well in Texas). Since the price of oil is volatile, uncertainty in Mining investment returns is high, and this confirms the model's risky return interpretation.

E Different Levels of Disaggregation

E.1 Dispersion of Corporate Bond Spreads

In the model, the capital wedge is positively related to the credit spread. This can be shown from Eq.(A.4) and (2.16) such that

$$\frac{z_{j,t+1}}{R_{t+1}} \propto \frac{R^k_{j,t+1}}{R_{t+1}} \propto (1+\tau^k_{j,t+1}).$$

So the dispersion of sectoral credit spreads may serve as an indirect proxy for the dispersion of sectoral capital wedges, which then reflect the degree of misallocation.

It is quite common that the dispersion of financial frictions at the firm level is larger than that at the sectoral level. This implies the degree of misallocation differs at different aggregation levels, and thus the level of aggregation matters for quantitative exercises. In TRACE (2020), trades of corporate bonds are reported at the firm level, and the total number of firms in the sample period is 806. Table 2 reports the cross-sectional standard deviation of the calculated time average of corporate bond spreads at different disaggregation levels (excluding industries in the FIRE and Government sectors). However, note that in Eq.(A.4), credit spread is also affected by the endogenous threshold $\bar{\omega}$ and leverages. It is important to note that although credit spread dispersion is related to misallocation, it does not necessarily mean that higher credit spread dispersion indicates more misallocation. After all, the level of misallocation depends on the interaction among all three financial parameters.

Table 2: Standard deviations of spreads at different disaggregation levels.

NAICS Digit (Number of Industries/Firms)	Standard Deviation of Spreads
NAICS 2 (13)	0.00392
NAICS 3 (48)	0.00613
NAICS 4 (97)	0.00719
NAICS 6 (221)	0.00740
806 Firms	0.00809

E.2 Impulse Responses of Aggregate TFP

This subsection shows how the calibrated model responds to the financial discrepancy at the NAICS two-digit (13), three-digit (48), and four-digit (97) disaggregation levels. The credit spreads are recalculated from TRACE (2020), and the leverages are recalculated from Compustat (2020). All parameters are then recalibrated at different levels of disaggregation.

From Theorem 2, the direct impact of capital misallocation is on TFP through the allocative inefficiency channel:

$$\mathrm{d}\ln\tilde{A}_t \propto \boldsymbol{\lambda}^{\boldsymbol{k}\prime}\mathrm{d}\ln\boldsymbol{x}^{\boldsymbol{k}}_{\boldsymbol{t}}.$$

Figure 1 depicts the impulse response functions of TFP at 13, 48, and 97 industries under an aggregate uncertainty shock at 10%. The shock hits the model economy at period 0. Since capital is pre-determined, the effect of capital reallocation shows up at a period later. We can see that the financial discrepancy at different disaggregation level manifests itself in the impact on TFP. The drop in TFP at 97 industries (0.27%) is 2.7 times larger than that at the 13-sector level (0.1%).

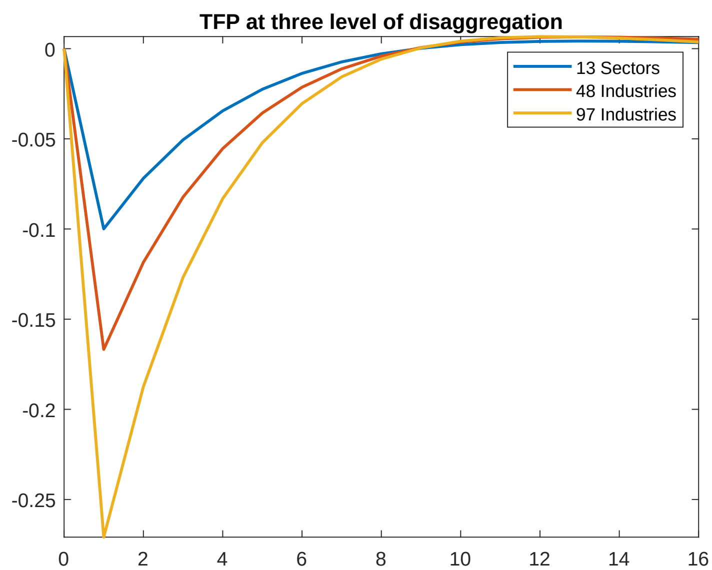

Figure 1: Impulse Response of TFP at three level of disaggregation

Figure 2 draws the impulse response functions of TFP and aggregate output between the IO and IM economy under a systematic uncertainty shock of 10%. As we look at the finer levels of disaggregation, the degree of capital misallocation increases, and both TFP and aggregate output demonstrate larger drops. The contribution of TFP to output drops in the IO economy are 14.7% at the 13-sector level, 22.4% at the 48-industry level, and 32.4% at the 97-industry level.

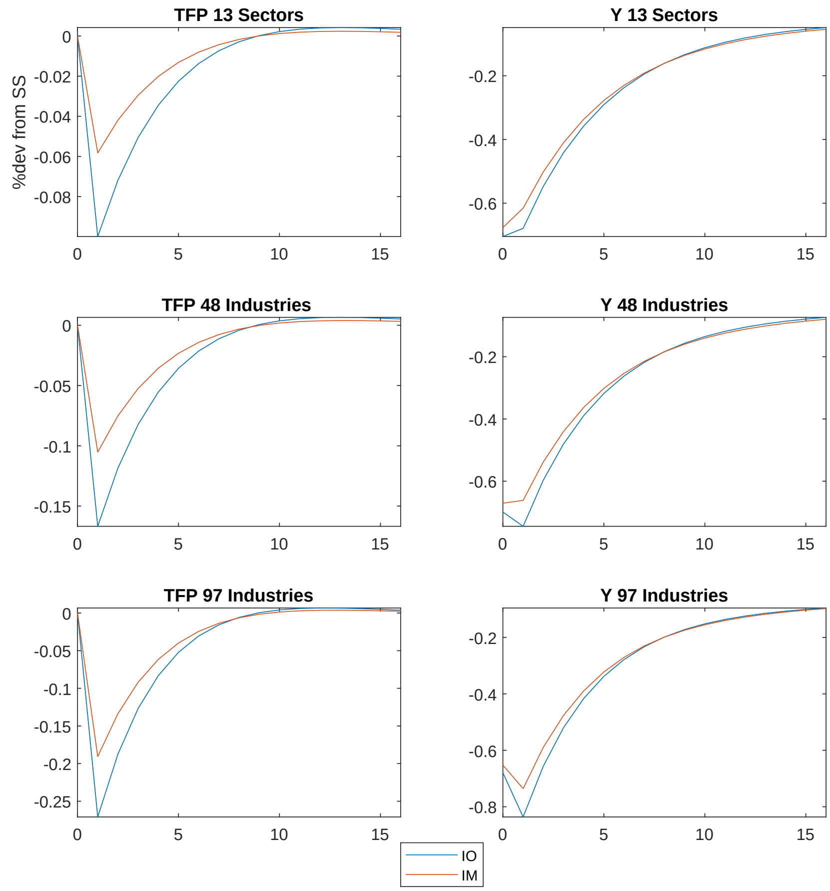

Figure 2: Impulse response functions of the IO and IM economy under an aggregate uncertainty shock

F Robustness

F.1 Varying Labor Supply Elasticity

Table 3 reports the aggregate TFP and output multiplier of uncertainty shocks, with different values of the Frisch elasticity of labor supply, ϵ. It shows that while the magnitude of output multiplier decreases with the Frisch elasticity, the magnitude of the TFP amplifier is robust the different values of the Frisch elasticity of labor supply.

F.2 Input-Output Multipliers at Different Disaggregation Levels

The Aggregate Input-Output Multiplier of Productivity Shocks

In Table 4, the four disaggregation levels correspond to the two-digit, three-digit, four-

Table 3: The network multiplier on TFP and Y with different values of the Frisch elasticity of labor supply.

48 Sectors	υ_σ=Δ lnTFP	Δ lnY
ϵ=0.5	1.5844	1.1546
ϵ=2	1.5855	1.1087
ϵ=3	1.5858	1.1025
ϵ=4	1.5859	1.0993

Table 4: Empirical input-output multiplier of productivity shocks at four disaggregation levels.

Number of Industries	υ_A
NAICS 2 (13)	1.68
NAICS 3 (48)	1.77
NAICS 4 (97)	1.58
NAICS 6 (221)	1.58

digit, and six-digit NAICS level. This multiplier is simply the sum of Domar weights, $\upsilon_A = \sum_j \upsilon_j$. The input-output multipliers of productivity shocks are similar across different disaggregation levels, slightly smaller at the NAICS 4 and 6 digit level.

The Aggregate Input-Output Multiplier of Uncertainty Shocks

Table 5: The network TFP multiplier of uncertainty shocks.

Number of Sectors	υ_σ=Δ lnTFP
13	1.73
48	1.58
97	1.42

Table 5 lists the input-output multipliers at three levels of disaggregation.

G Capacity Utilization

In this section, I add the capital capacity utilization into the model and analyze its effect as well as the impulse response functions. The functional form of capacity utilization is from

Christiano, Motto, and Rostagno (2014). By adding capacity utilization, the total return from capital in Eq (2.8) is adjusted as:

$$R_{jt}^K = \frac{[u_{jt} r_{j,t+1}^k - a_j(u_{jt})] + Q_{t+1}(1-\delta)}{Q_{t-1}}, \tag{G.1}$$

where $r_{j,t+1}^k$ is capital rent in Eq (2.3), u_{jt} is capacity utilization rate of sector j at time t, the convex function a_j represents the cost of capital utilization. In this set up, the choice of capacity utilization that maximized return from capital is independent of the entrepreneurs' net worth. The functional form of a_j is

$$a_j(u_{jt}) = r_{j,ss}^k[\exp(\sigma_a(u_{jt}-1)) - 1]\frac{1}{\sigma^a},$$

where the parameter σ^a captures the curvature of utilization cost, and $r_{j,ss}^k$ is the steady state value of capital rent of sector j.

The sectoral production function becomes

$$Y_{jt} = A_{jt}(u_{jt} x_{jt}^k K_t)^{\alpha_j m_j}(x_{lj} H_t)^{(1-\alpha_j)(1-m_j)} \Pi(\eta_{ij} Y_{it})^{\gamma_{ij}}.$$

The aggregate production function is $Y_t = \tilde{A}_t K_t^{\tilde{\alpha}} H_t^{1-\tilde{\alpha}}$. The capacity utilization is included in aggregate TFP, so

$$\ln \tilde{A}_t = \boldsymbol{v}' \mathrm{d} \ln \boldsymbol{A}_t + \boldsymbol{\lambda}^{\boldsymbol{k}\prime} \mathrm{d} \ln \boldsymbol{x}_{\boldsymbol{t}}^{\boldsymbol{k}} + \boldsymbol{\lambda}^{\boldsymbol{k}\prime} \mathrm{d} \ln \boldsymbol{u}_{\boldsymbol{t}}.$$

First, to show how the direction of capacity utilization and misallocation may differ, I consider the case of inelastic labor supply. Figure 3 depicts the impulse response functions of a systematic uncertainty shock hitting the IO economy with inelastic labor supply and capacity utilization. While misallocation induces negative TFP response, the effect of capacity utilization on TFP is positive. When labor is fixed, firms with capital outflow would increase their capacity utilization rate. This two opposite effects counteract with each other, and the response of aggregate output is small.

Figure 4 and 5 depict the impulse response functions with elastic labor supply in the IO and IM economy, respectively. When labor supply is elastic, the usage of capacity utilization declines with labor. The IO economy demonstrates a stronger decrease in aggregate TFP due to misallocation (-0.18% in IO and -0.11% in IM) and capacity utilization (-0.4% in IO and -0.18% in IM). In the IO economy, the aggregate output drops in the zero and first period are 1.64% and 1.17%. In the IM economy, the aggregate output drops in the zero

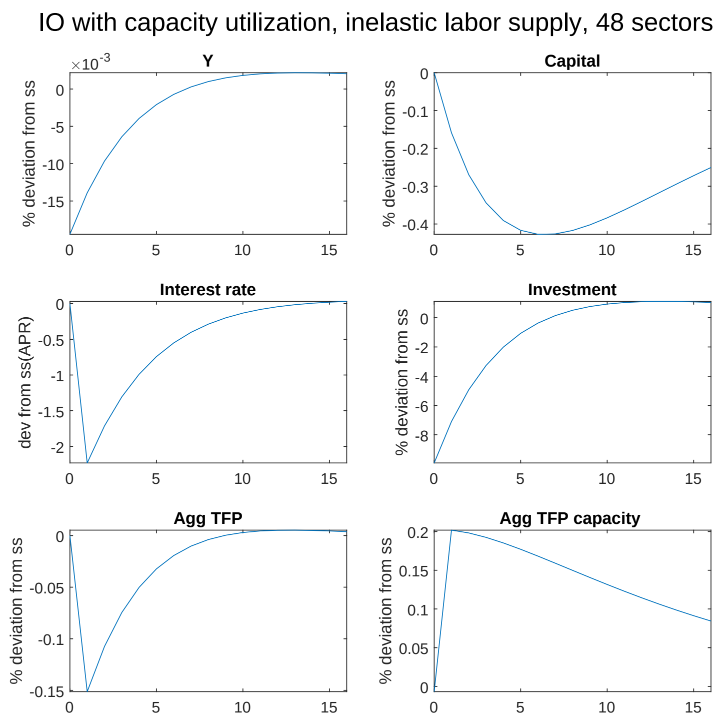

Figure 3: Impulse response functions of the IO economy with capacity utilization and inelastic labor supply. The bottom left panel "Agg TFP" represents the change of aggregate TFP due to misallocation. The bottom right panel "Agg TFP capacity" is the change of TFP due to changes in capacity utilization.

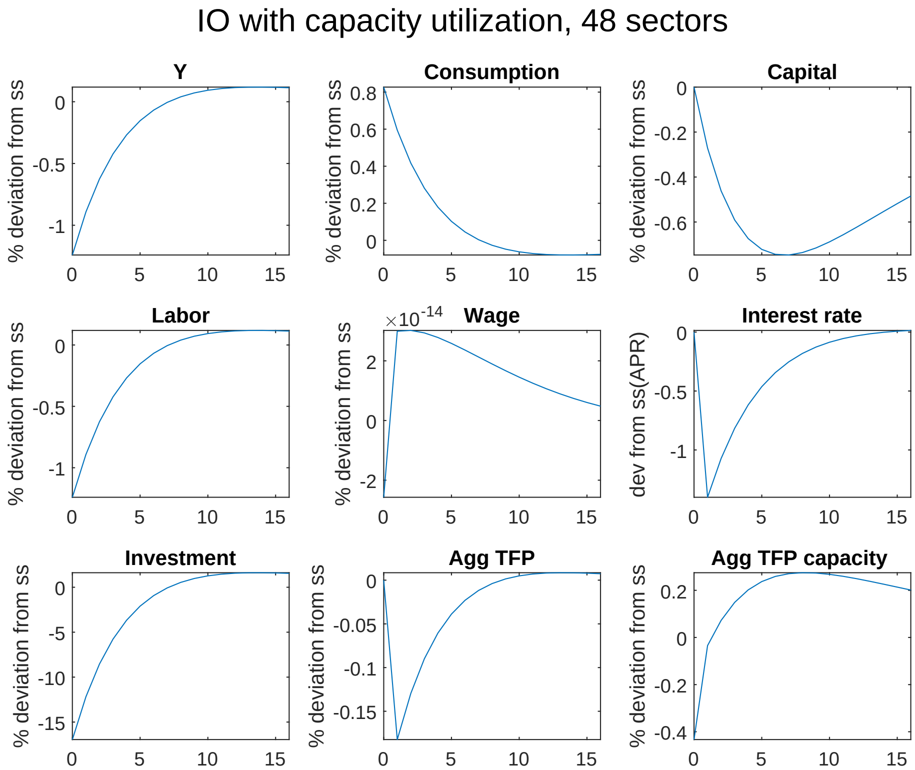

Figure 4: Impulse response functions of the IO economy with capacity utilization. The bottom middle panel "Agg TFP" represents the change of aggregate TFP due to misallocation. The bottom right panel "Agg TFP capacity" is the change of TFP due to changes in capacity utilization.

and first period are 0.9% and 0.8%.

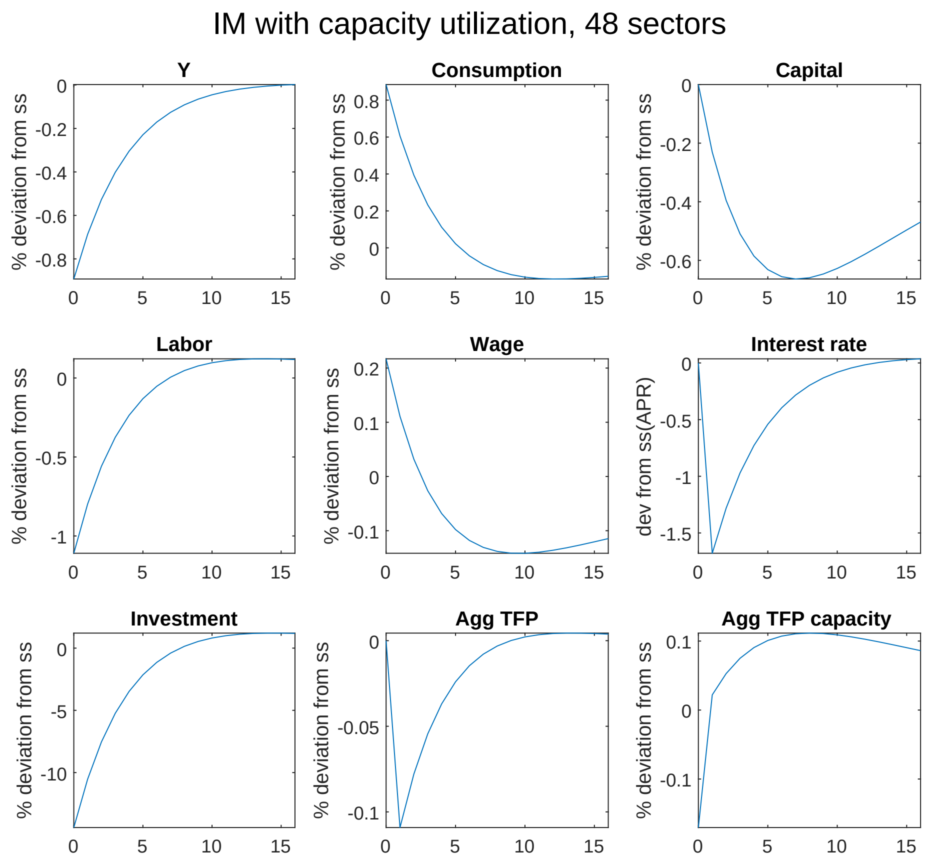

Figure 5: Impulse response functions of the IO economy with capacity utilization. The bottom middle panel "Agg TFP" represents the change of aggregate TFP due to misallocation. The bottom right panel "Agg TFP capacity" is the change of TFP due to changes in capacity utilization.

H A Static Model with General Distortions

Consider the same environment as in Section 2.1, but now with exogenous wedges on inputs and factor markets. So the intermediate goods producing firm of sector j's problem is adjusted as the following:

$$\max_{l_j, M_{ij}, k_j} p_j Y_j - (1+\tau_j^k) r k_j - (1+\tau_j^l) w l_j - \sum_{i=1}^{N} (1+\tau_{ij}^m) p_i M_{ij},$$

where τ_j^k, τ_j^l and τ_{ij}^m represent wedges on capital, labor and intermediate inputs for each sector j. r is the rental rate of capital, which includes both a real interest rate and a depreciation rate. w is wage. So far I suppose the rental rate and wage are the same across sectors. From the first order conditions, we can infer distortions as follows:

$$1+\tau_j^l = \frac{(1-\alpha_j)(1-m_j) p_j Y_j}{w l_j}, \tag{H.1}$$

$$1+\tau_j^k = \frac{\alpha_j (1-m_j) p_j Y_j}{r k_j}, \tag{H.2}$$

$$1+\tau_{ij}^m = \frac{\gamma_{ij} p_j Y_j}{p_i M_{ij}}, \tag{H.3}$$

From the market clearing condition, $Y_j = X_j + \sum_{i=1}^N M_{ji}$, and the Eq. (H.3), we can solve for $\boldsymbol{v}$:

$$v_j = \beta_j + \sum_{j=1}^{N} (1+\tau_{ji}^m)^{-1} \gamma_{ji} v_i.$$

Or, in the vector form,

$$\boxed{\boldsymbol{v} = [I_N - (1+\tau_m)^{-1} \circ \boldsymbol{\Gamma}]^{-1} \boldsymbol{\beta}}, \tag{H.4}$$

where $\boldsymbol{\tau}_m = \{\tau_{ij}^m\}$ is the matrix of distortions in intermediate inputs markets. Now Domar weights is affected by intermediate input wedges, τ_m.

The optimal allocation variables are the following:

$$\boldsymbol{\eta} = (1/\boldsymbol{v} * \boldsymbol{v}') \circ \boldsymbol{\Gamma} \circ (1+\boldsymbol{\tau}_m)^{-1}, \tag{H.5}$$

$$x_j^k = \frac{\lambda_j^k (1+\tau_j^k)^{-1}}{\sum_{i=1}^N \lambda_i^k (1+\tau_i^k)^{-1}}, \tag{H.6}$$

$$x_j^l = \frac{\lambda_j^l (1+\tau_j^l)^{-1}}{\sum_{i=1}^N \lambda_i^l (1+\tau_i^l)^{-1}}, \tag{H.7}$$

Finally, by substituting $k_j = x_j^k K, l_j = x_j^l H$, and $M_{ij} = \eta_{ij} Y_i$ into sectoral production functions and aggregate output, we can solve for an aggregate production function as in Theorem 1, and all distortions aggregate up into aggregate TFP.

Proposition 4. *The solution for aggregate output is*

$$Y = \tilde{A}(\tau_m, \tau_m, \tau_l) K^{\tilde{\alpha}} H^{1-\tilde{\alpha}},$$

where

$$\ln \tilde{A} = \boldsymbol{\beta}' \ln(1 - \boldsymbol{\eta}) + \boldsymbol{v}'\boldsymbol{a} + \boldsymbol{\lambda^k} \ln \boldsymbol{x^k} + \boldsymbol{\lambda^l} \ln \boldsymbol{x^l} + \boldsymbol{v}'(\boldsymbol{\Gamma}' \ln \boldsymbol{\eta}).$$

And aggregate TFP change is

$$d \ln \tilde{A} = \boldsymbol{v}' \ln \boldsymbol{a} + \boldsymbol{\lambda^k} d \ln \boldsymbol{x^k} + \boldsymbol{\lambda^l} d \ln \boldsymbol{x^l} + \boldsymbol{v}'(\boldsymbol{\Gamma}' d \ln \boldsymbol{\eta}).$$

The aggregate TFP, $\tilde{A}$, is a complex function of distortions, sectoral gross productivity, Domar weights, and other technological parameters. Domar weights is now a function of intermediate input wedges, $\boldsymbol{v}(\tau_m)$. Capital allocation depends on capital wedges, $\boldsymbol{\lambda^k}(\tau^k)$. And labor allocation depends on labor wedges, $\boldsymbol{\lambda^l}(\tau^l)$. Note that there is an identification issue on τ^m, since the observed intermediate input share is $\frac{p_i M_{ij}}{p_j Y_j} = \gamma_{ij}(1 + \tau_{ij}^m)^{-1}$. Without more information on τ_m, we cannot identify it from the observed $\frac{p_i M_{ij}}{p_j Y_j}$.

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