Dynamics of Markups, Concentration and Product Span
Online Appendix

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Comparative Dynamics

We first derive the slope of the $\dot{\lambda}_i$=0 curve. Differentiation of the right-hand side of (22) yields:

$$\hat{\Gamma}_i = -(\sigma-1)\,\hat{a}_i + \delta\hat{P} - \frac{\sigma\delta s_i}{(\sigma-\delta s_i-1)(\sigma-\delta s_i)}\hat{s}_i + \frac{\delta s_i(\sigma-2\delta s_i)}{(\sigma-\delta s_i-1)\,\sigma+\delta^2 s_i^2}\hat{s}_i.$$

This equation implies that the right-hand side of (22) is declining in r_i because Γ_i is declining in s_i and s_i is rising in r_i (see (11)). The former is seen from this equation by observing that $\sigma\delta s_i > \delta s_i(\sigma-2\delta s_i)$ and $(\sigma-\delta s_i-1)(\sigma-\delta s_i) < (\sigma-\delta s_i-1)\,\sigma + s_i^2\delta^2$. Collecting terms we can rewrite this equation as:
(A1)

$$\hat{\Gamma}_i = -(\sigma-1)\,\hat{a}_i + \delta\hat{P} - \delta^2 s_i^2 \frac{2(\sigma-\delta s_i-1)(\sigma-\delta s_i)+\sigma(\sigma-1)}{(\sigma-\delta s_i-1)(\sigma-\delta s_i)\left[(\sigma-\delta s_i-1)\,\sigma+\delta^2 s_i^2\right]}\hat{s}_i.$$

Next consider the total effect of a shift in the marginal cost a_i on Γ_i. From (11) we have:

$$\hat{s}_i = -\frac{\sigma-1}{1+(\sigma-1)\,\beta_i}\hat{a}_i = -\frac{(\sigma-1)(\sigma-\delta s_i-1)(\sigma-\delta s_i)}{(\sigma-\delta s_i-1)(\sigma-\delta s_i)+(\sigma-1)\,\delta s_i}\hat{a}_i.$$

Substituting this expression into (A1) we obtain the total impact of a_i on Γ_i:

$$\begin{aligned}\frac{\hat{\Gamma}_i}{(\sigma-1)\,\hat{a}_i} &= -1+\delta^2 s_i^2 \frac{2(\sigma-\delta s_i-1)(\sigma-\delta s_i)+\sigma(\sigma-1)}{\left[(\sigma-\delta s_i-1)\,\sigma+s_i^2\delta^2\right]^2}\\ &= \frac{(\sigma-1)\,s_i^2\delta^2-(\sigma-\delta s_i-1)^2\left(\sigma^2-\delta^2 s_i^2\right)}{\left[(\sigma-\delta s_i-1)\,\sigma+s_i^2\delta^2\right]^2}.\end{aligned}$$

It follows that a decline in the marginal cost a_i shifts upward the $\dot{\lambda}_i$=0 curve if

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and only if $(\sigma - 1)\, s_i^2 \delta^2 < (\sigma - \delta s_i - 1)^2 \left(\sigma^2 - \delta^2 s_i^2\right)$.

Empirical Analysis

Table B1—Average Number of Product Lines vs. Productivity Deciles

$Decile$	$Log(Prod)$	$MeanInd$	$MeanSegs$
1	10.05	1.89	2.93
2	11.54	2.14	3.65
3	12.04	2.27	4.00
4	12.31	2.48	4.47
5	12.54	2.64	4.84
6	12.77	2.67	4.98
7	13.06	2.63	4.83
8	13.42	2.53	4.79
9	13.91	2.29	4.57
10	15.31	1.92	3.99

Note: This table shows the deciles of average log labor productivity for firms in the Compustat database for the year 2018, available through WRDS. Labor productivity is defined as the ratio of total sales to employment. It also shows the mean number of industries and business segments that are reported in the Compustat Segments Data. The data was accessed on June 2, 2020.

We now provide additional information on the empirical analysis. Table B1 presents the data that has has been used to construct Figure 4 while Table B2 presents the regression results. As pointed out in the main text, the coefficient for log productivity is positive and significantly different from zero and the coefficient for the square of log productivity is negative and significantly different from zero in both specifications; i.e., when we use the number of industries or the number of segments to measure a firm's product span. While in the main text we reported in Figure 3 the curvature of this quadratic form for the number of segments as a proxy for the number of product lines, we now report a similar figure, Figure B1, for the case in which the number of industries is used as a proxy for the number of product lines. As is evident, the two figures are quite similar.

Optimal Allocation

In Section V we characterize the optimal allocation, showing that it differs from the market outcome. In this part of the appendix we propose policies that implement the optimal allocation in a market economy with taxes and subsidies. In particular, we show that there exist consumer subsidies for the purchase of varieties of the differentiated product and corporate taxes on operating profits that lead to a market allocation that coincides with the optimal allocation. These taxes and subsidies are firm specific and they vary over time. Moreover, implementation of the optimal allocation requires the policy maker to commit to the

Table B2—Quadratic Relationship of Productivity on Product Span

	Industries	Segments
log(Prod)	2.85**	5.50**
	(1.33)	(2.54)
$\log(\text{Prod})^2$	-0.11*	-0.21**
	(0.06)	(.11)
Primary Ind. FE	YES	YES
Obs	4126	4126
R^2	0.7334	0.4603

Robust standard errors clustered at the primary industry in parentheses.

$^{*}p < 0.10, ^{**}p < 0.05$.

Note: This table shows the results of an OLS quadratic regression of the number of industries or segments on the log of labor productivity. The data includes all firms with positive sales and employment in the Compustat database for the year 2018. Labor productivity is defined as the ratio of total sales to employment. Segments here refers to the total number of business segments listed in the Compustat Segments Data by firm. The number of industries is the number of primary and secondary SIC codes listed across all business segments. We also include fixed effects for 4 digit primary SIC code listed on Compustat. Data was accessed on June 2, 2020.

entire time path of these taxes and subsidies, which vary across firms and across time.

Let $\overline{\gamma}$ be the factor that converts a producer price $\overline{p}$ into a consumer price $\overline{\gamma}\,\overline{p}$ and by γ_i the factor that converts a producer price p_i into a consumer price $\gamma_i p_i$. We allow these conversion factors to vary over time, although we will find that the optimal value of $\overline{\gamma}$ is constant. Importantly, both consumers and producers treat these factors as exogenous variables. A γ smaller than one represents a subsidy to consumers while a γ larger than one represents a tax. Finally, we denote by τ_i the factor that converts gross operating profits of firm i, $r_i P^{\delta} (\gamma_i p_i)^{-\sigma} (p_i - a_i)$, into net operating profits $\tau_i r_i P^{\delta} (\gamma_i p_i)^{-\sigma} (p_i - a_i)$. The factors τ_i may also vary over time, but the firms treat them as exogenous variables. A τ_i smaller than one represents a corporate tax on operating profits while a τ_i larger than one represents a corporate subsidy to operating profits.

With these policies in place, the demand for varies of the differentiated product (3) can be expressed as:

$$\overline{x} = P^{*\delta} (\overline{\gamma}\,\overline{p})^{-\sigma},$$

$$x_i = P^{*\delta} (\gamma_i p_i)^{-\sigma},$$

where

$$P^* = \left[\overline{r} (\overline{\gamma}\,\overline{p})^{1-\sigma} + \sum_{j=1}^{m} r_j (\gamma_i p_j)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}.$$

In this exposition we use asterisks to denote equilibrium values of endogenous

Figure B1. Number of Industries vs. Labor Productivity

variables in the economy with taxes and subsidies. Large firms now maximize net operating profits $\tau_i r_i P^{*\delta} (\gamma_i p_i)^{-\sigma} (p_i - a_i)$ while small firms maximize operating profits $P^{*\delta} (\overline{\gamma}\,\overline{p})^{-\sigma} (\overline{p} - \bar{a})$. This yields the optimal pricing equations:

$$\overline{p}^* = \frac{\sigma}{\sigma - 1}\bar{a},$$

$$p_i^* = \frac{\sigma - \delta s_i^*}{\sigma - \delta s_i^* - 1} a_i, \tag{C1}$$

where s_i^* is the share of consumer spending on goods of firm i, equal to

$$s_i^* = \frac{r_i (\gamma_i p_i^*)^{1-\sigma}}{P^{*1-\sigma}}. \tag{C2}$$

We now propose the following numerical values of these policies:

$$\overline{\gamma} = \frac{\sigma - 1}{\sigma} \quad \text{and} \quad \gamma_i = \frac{\sigma - \delta s_i^* - 1}{\sigma - \delta s_i^*}, \tag{C3}$$

which yields $\overline{\gamma}\,\overline{p} = \overline{a}$ and $\gamma_i p_j = a_j$. In other words, these policies lead to consumer prices that equal marginal costs of production. Note that every γ is smaller than one. Therefore consumers enjoy subsidies on all varieties of the differentiated product and the subsidies are larger on products with larger market shares.

With these subsidies a small firm's operating profits are $P^{*\delta}\left(\overline{\gamma}\,\overline{p}^*\right)^{-\sigma}\left(\overline{p}^* - \bar{a}\right)$, and free entry ensures that these profits equal the entry cost f. Using the firm's optimal pricing equation (C1) and the subsidy policy (C3), this free entry condition yields

$$\frac{1}{\sigma - 1} P^{*\delta} \bar{a}^{1-\sigma} = f.$$

Comparing this to (33), we conclude that $P^* = C^*$, i.e., the price index equals the optimal resource cost of producing a unit of real consumption X. As a result, real consumption X is also at the optimal level, equal to $X^* = C^{*-\varepsilon}$, and the consumption levels of individual varies are at the optimal levels (see (34) and (35)):

$$\overline{x}^* = C^{*\delta}\overline{a}^{-\sigma} = (\sigma - 1)\overline{a}^{-1} f,$$

$$x_i^* = C^{*\delta} a_i^{-\sigma} = (\sigma - 1)\overline{a}^{\sigma-1} a_i^{-\sigma} f.$$

It remains to examine the investment policies of large firms.

Recognizing that $P^* = C^*$ is constant on the dynamic path, (C1) and (C2) implicitly define the optimal price of firm i as a function of its product span, $p_i^*\left(r_i\right)$, similarly to the analysis of the market economy without government intervention. The only difference is that now there are policy instruments that the firms treat as exogenous. As a result, profits of firm i net of taxes and investment costs are

$$\pi_i\left(\iota_i, r_i\right) = \tau_i r_i C^{*\delta}\left[\gamma_i p_i^*\left(r_i\right)\right]^{-\sigma}\left[p_i^*\left(r_i\right) - a_i\right] - \iota_i$$

and the current value Hamiltonian of the firm's optimal control problem is

$$\mathcal{H}(\iota_i, r_i, \lambda_i) = \tau_i r_i C^{*\delta}\left[\gamma_i p_i^*\left(r_i\right)\right]^{-\sigma}\left[p_i^*\left(r_i\right) - a_i\right] - \iota_i + \lambda_i\left[\phi\left(\iota_i\right) - \theta r_i\right].$$

The first-order conditions for the optimal control problem are therefore:

$$\frac{\partial \mathcal{H}}{\partial \iota_i} = -1 + \lambda_i \phi'\left(\iota_i\right) = 0,$$

$$-\frac{\partial \mathcal{H}}{\partial r_i} = -\tau_i \Gamma_i^*\left(r_i\right) + \theta\lambda_i = \dot{\lambda}_i - \rho\lambda_i,$$

where

$$\Gamma_i^*\left(r_i\right) \equiv \frac{\partial\left\{r_i C^{*\delta}\left[\gamma_i p_i^*\left(r_i\right)\right]^{-\sigma}\left[p_i^*\left(r_i\right) - a_i\right]\right\}}{\partial r_i},$$

and the transversality conditions are:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_i(t)\, r_i(t) = 0.$$

Now recall that the optimal investment in innovation is constant on the dynamic path and satisfies $\lambda_i^*\phi'(\iota_i^*) = 1$, where λ_i^* is is given in (39), i.e.,

$$\lambda_i^* = \frac{1}{\rho+\theta}\left(\frac{\overline{a}}{a_i}\right)^{\sigma-1} f.$$

The first-order conditions of the firm's optimal control problem imply that this investment pattern is attained if and only if:

$$\text{(C4)} \qquad \tau_i \Gamma_i^*(r_i^*) = \left(\frac{\overline{a}}{a_i}\right)^{\sigma-1} f$$

at every point in time, where r_i^* is the optimal product span. It follows from this result that operating profits of firm i are taxed ($\tau_i < 1$) if and only if

$$\Gamma_i^*(r_i^*) > \left(\frac{\overline{a}}{a_i}\right)^{\sigma-1} f.$$

We show next that $\tau_i \epsilon (0,1)$; that is, the optimal policy consists of taxing operating profits.

First note that

$$\frac{\Gamma_i^*(r_i)}{C^{*\delta}} = \frac{\partial\left\{r_i\left[\gamma_i p_i^*(r_i)\right]^{-\sigma}\left[p_i^*(r_i)-a_i\right]\right\}}{\partial r_i}$$

$$= \left[\gamma_i p_i^*(r_i)\right]^{-\sigma}\left[p_i^*(r_i)-a_i\right] - \left\{\sigma r_i \gamma_i^{-\sigma} p_i^*(r_i)^{-\sigma-1}\left[p_i^*(r_i)-a_i\right] - r_i\left[\gamma_i p_i^*(r_i)\right]^{-\sigma}\right\}\frac{\partial p_i^*(r_i)}{\partial r_i}$$

$$= \gamma_i^{-\sigma} p_i^*(r_i)^{-\sigma}\left(\left[p_i^*(r_i)-a_i\right] - \left\{\sigma r_i p_i^*(r_i)^{-1}\left[p_i^*(r_i)-a_i\right] - r_i\right\}\frac{\partial p_i^*(r_i)}{\partial r_i}\right)$$

$$= \gamma_i^{-\sigma} p_i^*(r_i)^{-\sigma}\left[\frac{1}{\sigma-\delta s_i^*(r_i)-1}a_i - r_i\left[\frac{\sigma}{\sigma-\delta s_i^*(r_i)}-1\right]\frac{\partial p_i^*(r_i)}{\partial r_i}\right]$$

$$= \gamma_i^{-\sigma} p_i^*(r_i)^{-\sigma}\left[\frac{1}{\sigma-\delta s_i^*(r_i)-1}a_i - r_i\frac{\delta s_i^*(r_i)}{\sigma-\delta s_i^*(r_i)}\frac{\partial p_i^*(r_i)}{\partial r_i}\right].$$

However,

$$\frac{\partial p_i^*(r_i)}{\partial r_i}\frac{r_i}{p_i^*} = \frac{\beta_i^*(r_i)}{1+(\sigma-1)\beta_i^*(r_i)}$$

where

$$\beta_i^*(r_i) = \frac{\delta s_i^*(r_i)}{[\sigma-\delta s_i^*(r_i)-1][\sigma-\delta s_i^*(r_i)]}.$$

Therefore

$$\begin{aligned}
\Gamma_i^*(r_i) &= C^{*\delta}\gamma_i^{-\sigma}p_i^*(r_i)^{-\sigma}\left[\frac{1}{\sigma-\delta s_i^*(r_i)-1}a_i - \left[p_i^*(r_i)\frac{\delta s_i^*(r_i)}{\sigma-\delta s_i^*(r_i)}\right]\frac{\beta_i^*(r_i)}{1+(\sigma-1)\beta_i^*(r_i)}\right] \\
&= \gamma_i^{-\sigma}a_i^{1-\sigma}C^{*\delta}\left[\frac{\sigma-\delta s_i^*(r_i)}{\sigma-\delta s_i^*(r_i)-1}\right]^{-\sigma}\frac{\sigma}{[\sigma-\delta s_i^*(r_i)-1]\sigma+\delta^2 s_i^*(r_i)^2} \\
&= \gamma_i^{-\sigma}\left(\frac{\overline{a}}{a_i}\right)^{\sigma-1} f\left[\frac{\sigma-\delta s_i^*(r_i)}{\sigma-\delta s_i^*(r_i)-1}\right]^{-\sigma}\frac{\sigma(\sigma-1)}{[\sigma-\delta s_i^*(r_i)-1]\sigma+\delta^2 s_i^*(r_i)^2},
\end{aligned}$$

where we used (33) in deriving the last line. Now compare this formula to (20). Since we showed that the expression on the right-hand side of (20) declines in r_i, it follows that— holding γ_i constant—$\Gamma_i^*(r_i)$ also declines in r_i. This ensures concavity in r_i of the firm's decision problem.

Finally, we show that $\tau_i \epsilon (0,1)$ in every time period, implying that the optimal policy consists of a tax on operating profits. To this end use the formula for the subsidy factor γ_i together with the optimal tax formula (C4) to obtain:

$$\begin{aligned}
\tau_i &= \Gamma_i^*(r_i)^{-1}\left(\frac{\overline{a}}{a_i}\right)^{\sigma-1} f = \frac{\sigma[\sigma-\delta s_i^*(r_i)-1]+\delta^2 s_i^*(r_i)^2}{\sigma(\sigma-1)}, \\
&= 1-\frac{[\sigma-\delta s_i^*(r_i)]\,\delta s_i^*(r_i)}{\sigma(\sigma-1)},
\end{aligned}$$

which shows that $\tau_i \epsilon (0,1)$ at every point in time.

For a firm with rising product span the share of consumer spending on its products rises over time, i.e., $s_i^*(r_i)$ is an increasing function. Therefore the corporate tax rate is rising over time (τ_i is decreasing) if and only $\sigma > 2\delta s_i^*(r_i)$. Since $\delta = \sigma - \varepsilon > 0$, it follows that for $\sigma > 2\varepsilon$ there exists a market share $s_c = \sigma/2(\sigma-\varepsilon)$ such that the tax rate is rising for market shares below s_c and declining for larger market shares. In the opposite case, when $\sigma > 2\varepsilon$, the corporate tax rate always rises for firms that expand their product span.

Comparative Statics: Given Number of Brands

In this section we examine the case in which the number of single-product firms, $\overline{r}$, as well the number of products available to each one of the large firms, r_i, are given. Equations (7) and (8) imply:

$$\hat{p}_i = \hat{a}_i + \frac{\delta s_i}{(\sigma - \delta s_i - 1)(\sigma - \delta s_i)}\hat{s}_i, \tag{D1}$$

$$\hat{s}_i = \hat{r}_i - \sum_{j=1}^{m} s_j \hat{r}_j - (\sigma - 1)(\hat{p}_i - \sum_{j=1}^{m} s_j \hat{p}_j).$$

Substituting the last equation into (D1) yields:

$$[1 + \beta_i(\sigma - 1)]\hat{p}_i - \beta_i(\sigma - 1)\sum_{j=1}^{m} s_j \hat{p}_j = \hat{a}_i + \beta_i(\hat{r}_i - \sum_{j=1}^{m} s_j \hat{r}_j), \text{ for all } i.$$

These equations can also be expressed as:

$$\mathbf{B}\hat{\mathbf{p}} = \mathbf{R}\hat{\mathbf{r}} + \hat{\mathbf{a}}, \tag{D2}$$

where $\mathbf{B}$ is an $m \times m$ matrix with elements:

$$b_{ii} = 1 + \beta_i(\sigma - 1)(1 - s_i),$$

$$b_{ij} = -\beta_i(\sigma - 1)s_j, \text{ for } j \neq i,$$

$\hat{\mathbf{p}}$ is an $m \times 1$ column vector with elements p_i, where a hat represents a proportional rate of change (i.e., $\hat{p}_i = dp_i/p_i$), $\mathbf{R}$ is an $m \times m$ matrix with elements:

$$r_{ii} = \beta_i(1 - s_i),$$

$$r_{ij} = -\beta_i s_j, \text{ for } j \neq i,$$

$\hat{\mathbf{r}}$ is an $m \times 1$ column vector with elements $\hat{r}_i$, where a hat represents a proportional rate of change, and $\hat{\mathbf{a}}$ is an $m \times 1$ column vector with elements $\hat{a}_i$, where a hat represents a proportional rate of change.

Since

$$|b_{ii}| - \sum_{j \neq i} |b_{ij}| = 1 + \beta_i(\sigma - 1)(1 - \sum_{j=1}^{m} s_j) > 1,$$

$\mathbf{B}$ is a diagonally dominant matrix with positive diagonal and negative off-diagonal elements. It therefore is an M-matrix and its inverse has all positive entries. This inverse, denoted by $\tilde{\mathbf{B}} = \mathbf{B}^{-1}$, is therefore an $m \times m$ matrix with elements $\tilde{b}_{ij} > 0$. Next note that $\mathbf{B}$ can be expressed as:

$$\mathbf{B} = \mathbf{I} + (\sigma - 1)\mathbf{R},$$

where $\mathbf{I}$ is the identity matrix. Therefore:

$$\mathbf{B}^{-1}\mathbf{B} = \tilde{\mathbf{B}} + (\sigma - 1)\tilde{\mathbf{B}}\mathbf{R} = \mathbf{I}. \tag{D3}$$

It follows from this equation that:

$$\tilde{b}_{ii} + (\sigma - 1)\sum_{j=1}^{m} \tilde{b}_{ij} r_{ji} = 1,$$

$$\tilde{b}_{ik} + (\sigma - 1)\sum_{j=1}^{m} \tilde{b}_{ij} r_{jk} = 0, \text{ for } k \neq i.$$

Summing these up yields:

$$\sum_{k=1}^{m} \tilde{b}_{ik} + (\sigma - 1)\sum_{j=1}^{m} \tilde{b}_{ij} \sum_{k=1}^{m} r_{jk} = 1, \text{ for all } i. \tag{D4}$$

Since:

$$\sum_{k=1}^{m} r_{jk} = \beta_j(1 - \sum_{k=1}^{m} s_k) > 0$$

and $\tilde{b}_{ik} > 0$ for all i and k, it follows from (D4) that:

$$0 < \tilde{b}_{ik} < 1 \text{ for all } i \text{ and } k.$$

Equation (D3) implies:

$$(\sigma - 1)\tilde{\mathbf{B}}\mathbf{R} = \mathbf{I} - \tilde{\mathbf{B}},$$

and therefore $\tilde{\mathbf{B}}\mathbf{R}$ has positive diagonal elements and negative off-diagonal elements.

Going back to the comparative statics equations (D2), we have:

$$\hat{\mathbf{p}} = \tilde{\mathbf{B}}\mathbf{R}\hat{\mathbf{r}} + \tilde{\mathbf{B}}\hat{\mathbf{a}}.$$

It follows from the properties of $\tilde{\mathbf{B}}$ that a decline in a_i reduces every price p_j, but less than proportionately. Equation (D1) than implies that all market share $s_j, j \neq i$, decline while the market share s_i rises. And it follows from the properties of $\tilde{\mathbf{B}}\mathbf{R}$ and (D1) that an increase in r_i raises the price and market share of firm i and reduces the price and market share of every other firm $j \neq i$. Noting that the markup of every firm i is larger the larger its market share, we therefore have:

Proposition 11. *Suppose that the number of firms and their product ranges are given. Then: (i) an increase in r_i raises the price, markup and market share of firm i, and reduces the price, markup and market share of every other large firm; (ii) a decline in a_i reduces the price of every large firm less than proportionately, raises the markup and market share of firm i, and reduces the markup and market share of every other large firms.*

D1. Aggregative Economy

In this section we show how to construct an aggregative economy with a continuum of industries, each one of the type analyzed in the main text of this paper.

We consider an economy with a continuum of individuals of mass 1, each one providing one unit of labor. The labor market is competitive and every individual earns the same wage rate.

There is a continuum of sectors of measure one, each one producing a differentiated product. Real consumption in sector k is:

$$X^k = \left[\int_0^{N^k} x^k(\omega)^{\frac{\sigma^k-1}{\sigma^k}} d\omega\right]^{\frac{\sigma^k}{\sigma^k-1}}, \ \sigma^k > 1,$$

where N^k is the number of varieties available in sector k, $x^k(\omega)$ is consumption of variety ω in sector k, and σ^k is the elasticity of substitution in sector k. Using this definition, the price index of X^k is:

$$P^k = \left[\int_0^{N^k} p^k(\omega)^{1-\sigma^k} d\omega\right]^{\frac{1}{\sigma^k-1}},$$

where $p^k(\omega)$ is the price of variety ω. The log utility of a representative individual is:

$$log\,(u) = \int_0^1 log\left(X^k\right) dk.$$

In these circumstances every individual spends an equal amount of money in

every sector. Therefore, if E denotes aggregate spending per capita, spending per capita in sector k also equals E. In this event, aggregate demand for variety ω in sector k is:

$$x^k(\omega) = A^k p(\omega)^{-\sigma}, \tag{D5}$$

$$A^k = E\left(P^k\right)^{\sigma^k - 1}. \tag{D6}$$

An individual's inter-temporal utility function is:

$$U = \int_0^\infty e^{-\rho t} log\left(u_t\right) dt,$$

where ρ is the subjective discount rate. As a result, the intertemporal allocation of spending satisfies:

$$\frac{\dot{E}_t}{E_t} = \zeta_t - \rho, \tag{D7}$$

where ζ_t is the interest rate at time t.

Two types of firms operate in sector k: atomless single-product firms and large multi-product firms, each one with a positive measure of product lines. Single-product firms produce $\overline{r}^k > 0$ varieties, each one specializing in a single brand. Large firm i in sector k has $r_i^k > 0$ product lines, $i = 1, 2, ..., m^k$, where m^k is the number of large firms in this sector. All the brands supplied to the market are distinct from each other.

All single-product firms share the same technology, which requires $\bar{a}^k$ unit of labor per unit output in sector k. Facing the demand function (D5), a single-product firm maximizes profits $A^k p(\omega)^{-\sigma}\left[p(\omega) - \bar{a}^k\right]$, taking as given the demand shifter A^k. Therefore, a single-product firm prices its brand ω according to $p(\omega) = \overline{p}^k$, where:

$$\overline{p}^k = \frac{\sigma^k}{\sigma^k - 1}\bar{a}^k. \tag{D8}$$

This yields the standard markup $\overline{\mu}^k = \sigma^k / \left(\sigma^k - 1\right)$ for a monopolistically competitive firm.

A large firm i has a technology that requires a_i^k units of labor per unit output, and it faces the demand function (D5) for each one of its brands. As a result, it prices every brand equally. We denote this price by p_i^k. The firm chooses p_i^k to maximize profits $r_i^k A^k p_i^{-\sigma}\left(p_i - a_i^k\right)$. However, unlike a single-product firm, a

large firm does not view A^k as given, because it recognizes that

$$P^k = \left(\bar{r}^k \left(\bar{p}^k\right)^{1-\sigma} + \sum_{j=1}^{m^k} r_j^k \left(p_j^k\right)^{1-\sigma^k} \right)^{\frac{1}{1-\sigma^k}}, \tag{D9}$$

and therefore that its pricing policy has a measurable impact on the price index of the differentiated product. It takes, however, the spending level E as given, because sector k is of measure zero. Accounting for this dependence of P^k on the firm's price, the profit maximizing price is:

$$p_i^k = \frac{\sigma^k - \left(\sigma^k - 1\right) s_i^k}{\left(\sigma^k - 1\right)\left(1 - s_i^k\right)} a_i^k, \tag{D10}$$

where s_i^k is the market share of firm i in sector k and:

$$s_i^k = \frac{r_i^k \left(p_i^k\right)^{1-\sigma^k}}{\left(P^k\right)^{1-\sigma^k}} = \frac{r_i^k \left(p_i^k\right)^{1-\sigma^k}}{\bar{r}^k \left(\bar{p}^k\right)^{1-\sigma} + \sum_{j=1}^{m^k} r_j^k \left(p_j^k\right)^{1-\sigma^k}}. \tag{D11}$$

Equations (D10) and (D11) jointly determine prices and market shares of large firms. The markup factor of firm i is $\mu_i^k = \left[\sigma^k - \left(\sigma^k - 1\right) s_i^k\right] / \left[\left(\sigma^k - 1\right)\left(1 - s_i^k\right)\right]$, which is increasing in its market share. When the market share equals zero the markup is $\sigma^k/(\sigma^k - 1)$, the same as the markup of a single product firm. The markup factor varies across firms as a result of differences in either the product span, r_i^k, or the marginal production cost, a_i^k. We analyze the dependence of prices, market shares and markups on marginal costs and product spans in the next section.

Entry of Single-Product Firms. — The number of large firms in every sector, m^k, is given. Unlike large firms, however, single-product firms enter the industry until their profits equal zero. In every sector the firms play a two-stage game: in the first stage single-product firms enter; in the second stage all firms play a Bertrand game as described above. Under these circumstances, (D8) and (D10) portray the equilibrium prices, except that the number of single product firms, $\bar{r}^k$, is endogenous. We seek to characterize a subgame perfect equilibrium of this game.

To determine the equilibrium number of single-product firms, assume that they face an entry cost f^k in sector k and they enter until profits equal zero. In a subgame perfect equilibrium every entrant correctly forecasts aggregate spending on the sector's products, the number of entrants, and the price that will be charged

for every variety in the second stage of the game. Therefore, every single-product firm correctly forecasts the price index and A^k. Using the optimal price (D8) and the profit function $A^k p(\omega)^{-\sigma}\left[p(\omega)-\bar{a}^k\right]$, this free entry condition can be expressed as:

$$\frac{1}{\sigma^k}A^k\left(\frac{\sigma^k}{\sigma^k-1}\bar{a}^k\right)^{1-\sigma^k}=f^k. \tag{D12}$$

The left-hand side of this equation describes the operating profits, which equal a fraction $1/\sigma^k$ of revenue, while the right-hand side represents the entry cost. In these circumstances the demand shifter A^k is determined by f^k and $\bar{a}^k$, and it is rising in both f^k and $\bar{a}^k$. Importantly, it does not depend on the number of large firms nor on their product spans. Moreover, given the spending level E, which is determined at the economy-wide level and is not influenced by product spans in sector k (because the sector is of measure zero), the price index P^k is also independent of product spans in sector k. In particular, changes over time in this price index are driven by changes in aggregate spending. For this reason (D6) and (D7) imply:

$$\frac{\dot{P}_t^k}{P_t^k}=\frac{1}{\sigma^k-1}\left(\rho-\zeta_t\right). \tag{D13}$$

Optimal Control. — We can now compute the response of p_i^k and s_i^k to changes in r_i^k as we did in the main text, and use the solution in the firm's optimal control problem. In the optimal control problem large firm i in sector k takes as given the path of the interest rate r_t and the path of spending E_t. After characterizing this solution we can use it to express the market clearing conditions. Spending E_t has to equal wage income and aggregate profits net of investment costs. This will give us the growth model. If we use the formulation from the main text, the steady state will have zero growth. But one could add a long-run growth mechanism, such as declining costs of innovation as a function of the cumulative experience in innovation, as is **?**. The steady state should be easy to analyze in either case.

As in the main text, investment is given by

$$\dot{r}_i^k=\phi(\iota_i^k)-\theta r_i^k,\ \text{for all}\, t\geq 0, \tag{D14}$$

At every point in time the firms play a two stage game. In the first stage single-product firms enter and large firms invest in innovation. Single-product firms live only one instant of time. For this reason they make profits only in this single instant. Under the circumstances the demand shifter A^k is determined by the

free entry condition, and it remains constant as long as the cost of entry and the cost of production of the single-product firms do no change. It follows that the profit flow of large firm i is:

$$\pi_i^k = r_i^k A^k \left(p_i^k\right)^{-\sigma} \left(p_i^k - a_i^k\right) - \iota_i^k, \text{ for all } t \geq 0,$$

where A^k is the same at every t while π_i^k, r_i^k, p_i^k and ι_i^k change over time, and p_i^k is given by $p_i^k = \frac{\sigma^k - (\sigma^k - 1) s_i^k}{(\sigma^k - 1)\left(1 - s_i^k\right)} a_i^k$. We can write the optimal control problem as:

$$\max_{\left\{\iota_i^k(t), r_i^k(t)\right\}_{t \geq 0}} \int_0^\infty e^{-\int_0^t \zeta_\tau d\tau} \pi_i^k \left[\iota_i^k(t), r_i^k(t)\right] dt$$

The main difference between this formulation and the formulation in the main text is that now we no longer have $\zeta_t = \rho$ at each point in time, but rather $\zeta_t = \frac{\dot{E}_t}{E_t} + \rho$. The current-value Hamiltonian of this problem is:

$$\mathcal{H}(\iota_i^k, r_i^k, \lambda_i^k) = \left\{r_i^k A^k p_i^k (r_i)^{-\sigma} \left[p_i^k (r_i) - a_i^k\right] - \iota_i^k\right\} + \lambda_i^k \left[\phi\left(\iota_i^k\right) - \theta r_i^k\right],$$

and the first-order conditions are:

$$\frac{\partial \mathcal{H}}{\partial \iota_i^k} = -1 + \lambda_i^k \phi'\left(\iota_i^k\right) = 0,$$

$$-\frac{\partial \mathcal{H}}{\partial r_i^k} = -\frac{\partial \left[r_i^k A^k \left(p_i^k\right)^{-\sigma} \left(p_i^k - a_i^k\right)\right]}{\partial r_i^k} + \theta \lambda_i^k = \dot{\lambda}_i^k - \zeta_t \lambda_i^k.$$

Note that the path of the price index P_t^k is determined by the growth rate of the aggregate economy that each firm takes as exogenous. Therefore, the resulting first-order conditions have a similar form to those we derived in the main text:

$$\lambda_i^k \phi'\left(\iota_i^k\right) = 1, \tag{D15}$$

(D16)

$$\dot{\lambda}_i^k = \left(\zeta_t + \theta\right) \lambda_i^k - A^k p_i^k \left(r_i^k\right)^{-\sigma^k} \left\{p_i\left(r_i^k\right) - a_i^k - r_i^k \left(\sigma^k p_i^k \left(r_i^k\right)^{-1} \left[p_i^k\left(r_i^k\right) - a_i^k\right] - 1\right) \frac{dp^k\left(r_i^k\right)}{dr_i^k}\right\}.$$

Substituting (D15) into (D14) yields:

$$\dot{r}_i = \phi\left[\iota_i\left(\lambda_i\right)\right] - \theta r_i. \tag{D17}$$

The second differential equation is obtained by substituting the pricing equation into (D16):

$$\dot{\lambda}_i^k = \left(\zeta_t + \theta\right)\lambda_i^k - \Gamma_i^k\left(r_i^k\right), \tag{D18}$$

where:
(D19)

$$\Gamma_i^k\left(r_i^k\right) \equiv a_i^{1-\sigma^k} A^k \sigma \left[\frac{\sigma^k - (\sigma^k - 1)s_i^k\left(r_i^k\right)}{\left(\sigma^k - 1\right)\left(1 - s_i^k\right)}\right]^{-\sigma^k} \frac{1}{\left(\sigma^k - 1\right)\left(1 - s_i^k\right)\sigma + s_i\left(r_i\right)^2\left(\sigma - 1\right)^2}.$$

Thus, our two differential equations are similar to the main text, with the caveat that the interest rate is evolving over time. Specifically, the dynamics are such that aggregate spending must satisfy $\zeta_t = \frac{\dot{E}_t}{E_t} + \rho$.

In steady state:

$$\phi\left[\iota_i^k\left(\lambda_i^k\right)\right] = \theta r_i^k, \tag{D20}$$

$$\left(\rho + \theta\right)\lambda_i^k = \Gamma_i^k\left(r_i^k\right), \tag{D21}$$

where we have used the fact that in steady state $\zeta_t = \rho$. The comparative statics of this system have the same form as in the main text. But note that while the key condition for having an inverted-U relationship between productivity and product span was $\left(\sigma - \delta - 1\right)^2\left(\sigma^2 - \delta^2\right) < \left(\sigma - 1\right)\delta^2$ in the main text, the formula is the same now with the exception that δ is replaced with $\sigma - 1$. This reduces the condition to $0 < \left(\sigma^k - 1\right)^3$, which is always satisfied. Thus, in this formulation we would expect every sector to have the inverted-U property. Another comparative static to note is the effect of an increase in the steady state expenditure level E. This shifts upward the curve associated with (D21) in the phase diagram, resulting in an instantaneous increase in λ_i^k and a trajectory of further expansion of r_i^k and rising profits. Thus, firms growing in other sectors reinforce the market dominance of large firms across industries through a pecuniary externality.

In order to close the model we need to solve for the steady state expenditure level. The market clearing condition is simply that revenue must equal net profits plus the total wage bill. With a unit mass of labor and the wage rate as the

numeraire, the resulting condition takes the form:

$$E_t = 1 + \int_{k \in K} \left[\sum_{i=1}^{m^k} r_i^k A^k \left(p_i^k \right)^{-\sigma} \left(p_i^k - a_i^k \right) - \iota_i^k \right] dk. \tag{D22}$$

We can further simplify this by recalling that $A^k = E_t \left(P^k \right)^{\sigma^k - 1}$. This means that we can use (D22) to obtain:

$$E_t = \left[1 - \int_{k \in K} \left[\sum_{i=1}^{m^k} r_i^k \left(P^k \right)^{\sigma^k - 1} \left(p_i^k \right) \left(p_i^k - a_i^k \right) - \iota_i^k \right] dk \right]^{-1}. \tag{D23}$$

Thus, the steady state expenditure level is increasing in the net profits of large firms across sectors. This equation also holds at every point in time, noting that the optimal investment levels depend on the path of aggregate expenditure through the interest rate. It follows that in order to solve the path of spending we need to ensure that the paths of profits of all firms aggregates to the path that rationalizes the optimal investments at each point in time.