Macroeconomic Frameworks:
Reconciling Evidence and Model Predictions from Demand Shocks

Alan J. Auerbach, Yuriy Gorodnichenko, and Daniel Murphy

Online Appendix

Here we derive expressions from the model's equilibrium as well as the response of macro metrics to increases in government spending.

Mass of Surviving Firms

Surviving firms are those for which

$$\theta_{j\ell t}^2 > \frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{\ell t})}$$

Given our distributional assumption on θ^2, this implies that the mass of surviving firms is

$$J_{\ell t} = \int_{\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{\ell t})}}^{\infty} \alpha(\theta^2)^{-\alpha-1}\, d\theta^2 = -\infty^{-\alpha} + \left[\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{\ell t})}\right]^{-\alpha} = \left[\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{\ell t})}\right]^{-\alpha}$$

$$J_{\ell} = \left[\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{\ell t})}\right]^{-\alpha}$$

Revenues

Total local revenues from the private sector are

$$R_{\ell t}^{P} = \frac{1}{4\gamma\lambda_t}\int_{\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{\ell t})}}^{\infty} \theta^2 f(\theta^2) d\theta^2 = \frac{1}{4\gamma\lambda_t}\int_{\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{\ell t})}}^{\infty} \alpha(\theta^2)^{-\alpha} d\theta^2 = \frac{1}{4\gamma\lambda_t}\frac{\alpha}{1-\alpha}(\theta^2)^{1-\alpha}\Big|_{\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{\ell t})}}^{\infty}$$

$$= \frac{1}{4\gamma\lambda_t}\frac{\alpha}{1-\alpha}\left(-\left[\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{\ell t})}\right]^{1-\alpha}\right) = \frac{1}{4\gamma\lambda_t}\frac{\alpha}{\alpha-1}\left[\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{\ell t})}\right]^{1-\alpha}$$

$$= (4\gamma\lambda_t)^{-\alpha}\frac{\alpha}{\alpha-1}\left[\frac{\underline{R}}{(1+\phi_{\ell t})}\right]^{1-\alpha}$$

Revenues from the government in a location are

$$G_{\ell t} = \phi_{\ell t} R_{lt}^{P}.$$

Total local revenues are the sum of private-sector revenues and revenues from government spending:

$$R_{\ell t} = R^P_{\ell t} + G_{\ell t} = (4\gamma\lambda_t)^{-\alpha}\frac{\alpha}{\alpha-1}\left[\frac{\underline{R}}{(1+\phi_{\ell t})}\right]^{1-\alpha}(1+\phi_{lt}).$$

The government share of GDP in a location is $\frac{G_{\ell t}}{R_{\ell t}} = \frac{\phi_{\ell t}}{1+\phi_{\ell t}}$.

GDP Multiplier

The multiplier is the change in total revenues for every dollar of spending from the government:

$$\frac{dR_{\ell t}}{dG_{\ell t}} = \frac{d\{(1+\phi_{\ell t})R^p_{\ell t}\}}{d\{\phi_{\ell t}R^P_{\ell t}\}} = \frac{\left((1+\phi)d(R^P_{\ell t}) + R^P_{\ell t}d\phi\right)}{\phi d(R^P_{\ell t}) + R^P_{\ell t}d\phi}$$

where $d(R^P_{\ell t}) = d(4\gamma\lambda_t)^{-\alpha}\frac{\alpha}{\alpha-1}\left[\frac{\underline{R}}{(1+\phi_{\ell t})}\right]^{1-\alpha} = (4\gamma\lambda_t)^{-\alpha}\underline{R}^{1-\alpha}\frac{\alpha}{\alpha-1}(\alpha-1)(1+\phi_{\ell t})^{\alpha-2}d\phi$

So

$$\frac{dR_{\ell t}}{dG_{\ell t}} = \frac{(1+\phi)(\alpha-1)(1+\phi_{\ell t})^{\alpha-2} + (1+\phi_{\ell t})^{\alpha-1}}{\phi(\alpha-1)(1+\phi_{\ell t})^{\alpha-2} + (1+\phi_{\ell t})^{\alpha-1}} = \frac{(\alpha-1)+1}{\phi(\alpha-1)(1+\phi_{\ell t})^{-1}+1}$$
$$= 1 + \frac{\alpha-1}{\alpha\phi+1}$$

(Inverse of) Labor Share

In the model, wage income w corresponds to earnings WH in the data, and revenues R correspond to PQ (GDP). Hence, the model analogue of the inverse of labor share is

$$\frac{PQ}{WH} = \frac{R_{\ell t}}{n\psi[R_{\ell t}+r]}.$$

We examine two measures of the response of the labor share to a demand shock: the elasticity of GDP with respect to earnings, and the change in GDP relative to the change in earnings.

We first derive the elasticity of GDP with respect to earnings, $\frac{d\log PQ}{d\log WH}$, driven by a change in local government spending $\phi_{\ell t}$.

$$\frac{d\log PQ}{d\phi_{\ell t}} = \frac{d\log R_{\ell t}}{d\phi_{\ell t}} = \frac{1}{R_{\ell t}}\frac{dR_{\ell t}}{d\phi_{\ell t}}$$

$$\frac{d\log WH}{d\phi_{\ell t}}=\frac{1}{n\psi[R_{\ell t}+r]}\frac{n\psi\, dR_{\ell t}}{d\phi_{\ell t}}$$

$$\frac{d\log PQ}{d\log WH}=\frac{R_{\ell t}+r}{R_{\ell t}}.$$

Next, we derive the change in revenues as a ratio of the change in earnings $\frac{dPQ}{dWH}$:

$$\frac{dPQ}{dWH}=\frac{dR_{\ell t}}{d\{n\psi[R_{\ell t}+r]\}}=\frac{dR_{\ell t}}{n\psi\times dR_{\ell t}}=\frac{1}{n\psi}$$

Elasticity of Nontradable Prices with Respect to GDP

The household's first order condition relates expenditure on local nontradables to the local household's budget multiplier $\lambda_{\ell t}$:[1]

$$\mathcal{L}_{\ell t}p_{\ell t}^{\mathcal{L}}=\frac{1}{\lambda_{\ell t}}.$$

To determine how this responds to an increase in government spending, we examine deviations around a steady state in which the local household's expenditure equals its income (e.g., there is balanced trade):

$$\mathcal{L}_{\ell t}p_{\ell t}^{\mathcal{L}}+\int_0^1\int_{\underline{\theta}_{\ell t}^2}^{\infty}p_{jmt}q_{jm\ell t}dj\,dm+T_{\ell t}=n\psi(R_{\ell t}+r)+\Pi_{\ell t}+I_{\ell t}.$$

Totally differentiating this budget constraint with respect to locally-determined variables and dividing through by R (and assuming $\mathcal{L}$ is fixed by locally endowed production factors, $\Pi_{\ell t}$ and I_{lt} are independent of local conditions due to diversification, $T_{\ell t}$ is independent of local DOD spending, and prices p_{jmt} are independent of local conditions due to price setting at the aggregate level), this becomes:

$$\frac{\mathcal{L}_{\ell t}dp_{\ell t}^{\mathcal{L}}}{R_{\ell t}}+\frac{1}{R_{\ell t}}\int_0^1\int_{\underline{\theta}_{\ell t}^2}^{\infty}p_{jmt}dq_{jm\ell t}dj\,dm=n\psi\frac{dR_{\ell t}}{R_{\ell t}}.^{2} \qquad \text{(A1)}$$

[1] $\lambda_{\ell t}$ denotes the budget multiplier for the local household while λ_t is the average multiplier across locations.

[2] Note that the comparative statics at the national level would include changes in taxes. This implies that national land prices do not change in response to national spending.

Note that demand for $q_{jm\ell t}$ is given by

$$q_{jm\ell t}^{d} = \frac{1}{\gamma}\left(\theta_{jm\ell t} - \lambda_{\ell t} p_{jmt}\right) = \frac{1}{\gamma}\left(\theta_{jm\ell} - \frac{1}{\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}}\frac{\theta_{jmt}}{2\lambda_t}\right) = \frac{\theta_{jmt}}{\gamma}\left(1 - \frac{1}{2\lambda_t \mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}}\right).$$

Totally differentiating this expression yields

$$dq_{jm\ell t} = \frac{\theta_{jmt}}{\gamma 2\lambda_t}\left(\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}\right)^{-2} d\left(\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}\right). \quad \text{(A2)}$$

$$= \frac{\theta_{jmt}}{\gamma 2\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}} d\left(\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}\right).$$

$$= \frac{\theta_{jmt}}{2\gamma} d\log\left(\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}\right).$$

Substituting in for $dq_{jm\ell t}$ in (A1) yields

$$\frac{\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}}{R_{\ell t}} d\log \mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}} + \frac{1}{R_{\ell t}}\int_0^1 \int_{\underline{\theta}_{\ell t}^2}^{\infty} \frac{\theta_{jmt}}{2\lambda_t}\frac{\theta_{jmt}}{2\gamma} d\log\left(\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}\right) dj\, dm = n\psi \frac{dR_{\ell t}}{R_{\ell t}}$$

$$\frac{\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}}{R_{\ell t}} d\log \mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}} + \frac{\lambda_t}{\gamma R_{\ell t}}\int_0^1 \int_{\underline{\theta}_{\ell t}^2}^{\infty} \left(\frac{\theta_{jmt}}{2\lambda_t}\right)^2 d\log\left(\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}\right) dj\, dm = n\psi \frac{dR_{\ell t}}{R_{\ell t}}$$

$$d\log \mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}} \left[\frac{\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}}{R_{\ell t}} + \frac{\lambda_t}{\gamma R_{\ell t}}\int_0^1 \int_{\underline{\theta}_{\ell t}^2}^{\infty} \left(\frac{\theta_{jmt}}{2\lambda_t}\right)^2 dj\, dm\right] = n\psi \frac{dR_{\ell t}}{R_{\ell t}}$$

Substitute in $\int_0^1 \int \left(\frac{\theta_{jmt}}{2\lambda_t}\right)^2 d\theta^2\, dm = R_{\ell t}^{P}\frac{4\gamma\lambda}{4\lambda^2} = R^P \frac{\gamma}{\lambda}$

$$d\log \mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}} \left[\frac{\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}}{R_{\ell t}} + \frac{\lambda_t}{\gamma R_{\ell t}} R_{\ell t}^{P} \frac{\gamma}{\lambda_t}\right] = n\psi \frac{dR_{\ell t}}{R_{\ell t}}$$

$$\frac{d\log \mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}}{d\log R_{\ell t}} = \frac{n\psi}{\frac{\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}}{R_{\ell t}} + \frac{R_{\ell t}^{P}}{R_{\ell t}}}$$

$$\frac{d\log \mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}}{d\log R_{\ell t}} = \frac{R_{\ell t} n\psi}{\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}} + R_{\ell t}^{P}}$$

Because in a symmetric equilibrium $\lambda_{\ell t} = \lambda_t$, it follows that

$$\frac{d\log \mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}}{d\log R_{\ell t}} = \frac{\lambda_t R_{\ell t}}{1 + \lambda_t R_{\ell t}^P} n\psi$$

Elasticity of Consumption with Respect to Output

Appendix equation (A2) gives the response of a variety of consumption to a change in spending on "land" $\mathcal{L}$. To turn this into an elasticity, note that $q_{jm\ell t} = \frac{\theta_{jmt}}{2\gamma}$. Then equation (A2) can be written as

$$d\log q_{jm\ell t} = d\log(\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}).$$

Note that the right-hand-side of this equation is the same for all j_m and therefore consumption bundle of tradable goods increases by $d\log(\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}})$. It follows that the response of consumption to $d\log R_{\ell t}$ is given by

$$\frac{d\log q_{jm\ell t}}{d\log R_{\ell t}} = \frac{d\log q_{jm\ell t}}{d\log \mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}} \frac{d\log \mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}}{d\log R_{\ell t}} = 1 \times \frac{\lambda_t R_{\ell t}}{1 + \lambda_t R_{\ell t}^P} n\psi = \frac{\lambda_t R_{\ell t}}{1 + \lambda_t R_{\ell t}^P} n\psi$$

Elasticity of Employment with Respect to Output

Employment in the model is proportional to the number of firms: $Emp_{\ell t} = nJ_{\ell t}$. Total output is the sum of firm-level output in a location, the private-sector component of which is given by equation (8) in the main text. Total private-sector output is

$$Q_{\ell t}^P = \frac{1}{2\gamma} \int_{\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{mt})}}^{\infty} \theta f(\theta^2) d\theta^2 = \frac{1}{2\gamma} \int_{\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{mt})}}^{\infty} \alpha\,(\theta^2)^{-\frac{1}{2}-\alpha} d\theta^2$$

$$= \frac{1}{2\gamma} \frac{\alpha}{.5-\alpha} (\theta^2)^{\frac{1}{2}-\alpha} |_{\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{mt})}}^{\infty}$$

$$= \frac{1}{2\gamma} \frac{\alpha}{\alpha - \frac{1}{2}} \left(\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{mt})}\right)^{-(\alpha-.5)}.$$

Total output is the sum of Q^P and Q^G:

$$Q = (1+\phi_{mt}) \frac{1}{2\gamma} \frac{\alpha}{\alpha - \frac{1}{2}} \left(\frac{4\gamma\lambda_t \underline{R}}{(1+\phi_{mt})}\right)^{-(\alpha-.5)} = \frac{1}{2\gamma} \frac{\alpha}{\alpha - \frac{1}{2}} \left(4\gamma\lambda_t \underline{R}\right)^{-(\alpha-.5)} (1+\phi_{mt})^{(1+\alpha-.5)}$$

Hence

$$d \log Q = \left(\frac{1}{2} + \alpha\right) \frac{\phi}{1+\phi} d \log \phi$$

Employment is $\frac{n}{N}\left[\frac{4\gamma\lambda_t R}{(1+\phi_{mt})}\right]^{-\alpha}$, so

$$d \log Emp = \alpha \frac{\phi}{1+\phi} d \log \phi$$

Hence,

$$\frac{d \log Emp}{d \log Q} = \frac{\alpha}{.5 + \alpha}$$

Since $d \log J = d \log Emp$, it follows that $\frac{d \log J}{d \log Q} = \frac{\alpha}{.5+\alpha}$.

We can also derive

$$d \log R = \alpha \frac{\phi}{1+\phi} d \log \phi$$

Hence

$$\frac{d \log Q}{d \log R} = \frac{0.5 + \alpha}{\alpha},$$

which we use when converting elasticities with respect to nominal GDP to elasticities with respect to real GDP.

Household Labor Wedge

We can write the labor wedge in growth rates as:

$$\tau^H = w - p_c - \frac{1}{\xi} h - c$$

In our model, the wage is the same as the wage bill $n\psi(R_{\ell t} + r)$ per employee. Therefore,

$$w = d \log n\psi(R_{\ell t} + r) - d \log Emp = d \log(R_{\ell t} + r) - d \log Emp = \frac{1}{R_{\ell t} + r} dR_{\ell t}$$

$$= \frac{R_{\ell t}}{R_{\ell t} + r} \, d \log R_{\ell t} - d \log Emp.$$

In our model, let the consumption price be

$$\mathrm{dlog}\, p_{\ell t}^{c} = s^{\mathcal{L}}\ \mathrm{dlog}\, p_{\ell t}^{\mathcal{L}} + (1 - s^{\mathcal{L}})\, \mathrm{dlog}\, p_{jmt},$$

where $s^{\mathcal{L}} = 0.4$ is the share of land expenditure in total household spending. In our model, p_{jmt} is invariant to local demand shocks. Recall also that

$$\frac{d \log \mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}}{d \log R_{\ell t}} = \frac{d \log p_{\ell t}^{\mathcal{L}}}{d \log R_{\ell t}} = \frac{\lambda_t R_{\ell t}}{1\ + \lambda_t R_{\ell t}^{P}} n\psi$$

Hence,

$$\mathrm{dlog}\, p_{\ell t}^{c} = s^{\mathcal{L}} \times \mathrm{dlog}\, p_{\ell t}^{\mathcal{L}} = s^{\mathcal{L}} \times \frac{\lambda_t R_{\ell t}}{1\ + \lambda_t R_{\ell t}^{P}} n\psi \times d \log R_{\ell t}$$

Finally, we have the response of consumption of tradable goods:

$$d \log q_{\ell t} = d \log\left(\mathcal{L}_{\ell t} p_{\ell t}^{\mathcal{L}}\right) = \frac{\lambda_t R_{\ell t}}{1\ + \lambda_t R_{\ell t}^{P}} n\psi \times d \log R_{\ell t}$$

It follows that the response of labor wedge is

$$\tau^{H} = (average\ wage) - p_c - \frac{1}{\xi} hours - consumption$$

$$= \left(\frac{R_{\ell t}}{R_{\ell t} + r}\ d \log R_{\ell t} - d \log Emp_{\ell t}\right) - \mathrm{dlog}\, p_{\ell t}^{c} - \frac{1}{\xi} d \log Emp_{\ell t} - d \log q_{\ell t}$$

$$= \frac{R_{\ell t}}{R_{\ell t} + r}\ d \log R_{\ell t} - \mathrm{dlog}\, p_{\ell t}^{c} - \left(1 + \frac{1}{\xi}\right) d \log Emp_{\ell t} - d \log q_{\ell t}$$

$$= \frac{R_{\ell t}}{R_{\ell t} + r}\ d \log R_{\ell t} - s^{\mathcal{L}} \times \frac{\lambda_t R_{\ell t}}{1\ + \lambda_t R_{\ell t}^{P}} n\psi \times d \log R_{\ell t} - \left(1 + \frac{1}{\xi}\right) d \log Emp_{\ell t}$$
$$- \frac{\lambda_t R_{\ell t}}{1\ + \lambda_t R_{\ell t}^{P}} n\psi \times d \log R_{\ell t}$$

$$= \frac{R_{\ell t}}{R_{\ell t} + r}\ d \log R_{\ell t} - (1 + s^{\mathcal{L}}) \times \frac{\lambda_t R_{\ell t}}{1\ + \lambda_t R_{\ell t}^{P}} n\psi \times d \log R_{\ell t} - \left(1 + \frac{1}{\xi}\right) d \log Emp_{\ell t}$$

Substituting in $d \log R_{\ell t}$ for $d \log Emp_{\ell t}$, we have

$$\tau^{H} = \left\{\frac{R_{\ell t}}{R_{\ell t} + r} - (1 + s^{\mathcal{L}}) \times \frac{\lambda_t R_{\ell t}}{1\ + \lambda_t R_{\ell t}^{P}} n\psi - \left(1 + \frac{1}{\xi}\right)\right\}\ d \log R_{\ell t}.$$