

Online Appendix

Testing the effectiveness of unconventional monetary policy in Japan and the United States

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A Theoretical Model

Appendix A presents a simple New Keynesian model with an effective lower bound (ELB) and unconventional monetary policy (UMP). The model is a version of a preferred habitat model such as [Chen et al. \(2012\)](#), extended to incorporate two things: a policy rule for quantitative easing (QE) that is operated using a shadow rate as policy guidance and forward guidance (FG) in the spirit of [Reifschneider and Williams \(2000\)](#). To keep the analysis focused on the salient features of the transmission mechanisms of UMP, the model abstracts from capital accumulation and consumption habit formation. There are three shocks: a demand (preference) shock, a supply (productivity) shock, and a monetary policy shock.

A.1 Model building blocks

A.1.1 Long-term bonds

There is a long-term government bond (consol bond). The long-term bond issued at time t yields μ^{j-1} dollars at time $t + j$ over time. Let $R_{L,t+1}$ denote the gross nominal rate from time t to $t + 1$. The period- t price of the bond issued at time t , $P_{L,t}$, is defined as

$$\begin{aligned} P_{L,t} &= E_t \left(\frac{1}{R_{L,t+1}} + \frac{\mu}{R_{L,t+1}R_{L,t+2}} + \frac{\mu^2}{R_{L,t+1}R_{L,t+2}R_{L,t+3}} + \dots \right) \\ &= E_t \left(\frac{1}{R_{L,t+1}} + \frac{\mu}{R_{L,t+1}} P_{L,t+1} \right). \end{aligned} \tag{A.1}$$

The gross yield to maturity (or the long-term interest rate) at time t , $\bar{R}_{L,t}$ is defined as

$$E_t \left(\frac{1}{\bar{R}_{L,t}} + \frac{\mu}{(\bar{R}_{L,t})^2} + \frac{\mu^2}{(\bar{R}_{L,t})^3} + \dots \right) = P_{L,t},$$

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or

$$P_{L,t} = \frac{1}{\bar{R}_{L,t} - \mu}. \quad (\text{A.2})$$

Let $B_{L,t|t-s}$ denote period- t holdings of bonds that were issued at time $t - s$. Suppose that a household has $B_{L,t|t-s}$ for $s = 1, 2, \dots$ in the beginning of period t . The total amount of dividends the household receives in period t is

$$\sum_{s=1}^{\infty} \mu^{s-1} B_{L,t|t-s}.$$

Note that having one unit of $B_{L,t|t-s}$ is equivalent to having μ^{s-1} units of $B_{L,t|t-1}$ because they both yield μ^{s-1} dollars. The total amount of dividends then can be expressed in terms of $B_{L,t|t-1}$ as

$$\sum_{s=1}^{\infty} \mu^{s-1} B_{L,t|t-s} \equiv B_{L,t-1},$$

where $B_{L,t-1}$ denotes the amount of bonds in units of the bonds issued at time $t - 1$, held by the household in the beginning of period t . Let $P_{L,t|t-s}$ denote the time- t price of the bond issued at time $t - s$. Then, the value of all bonds at time t is

$$\sum_{s=1}^{\infty} P_{L,t|t-s} B_{L,t|t-s}.$$

The bond price satisfies

$$\begin{aligned} P_{L,t|t-s} &= E_t \left(\frac{\mu^s}{R_{L,t+1}} + \frac{\mu^{s+1}}{R_{L,t+1}R_{L,t+2}} + \frac{\mu^{s+2}}{R_{L,t+1}R_{L,t+2}R_{L,t+3}} + \dots \right) \\ &= \mu^s P_{L,t} \end{aligned}$$

Then the value of all bonds at time t is

$$\sum_{s=1}^{\infty} P_{L,t|t-s} B_{L,t|t-s} = P_{L,t} \mu \sum_{s=1}^{\infty} \mu^{s-1} B_{L,t|t-s} = \mu P_{L,t} B_{L,t-1}.$$

So the return of holding $B_{L,t-1}$ is given by the sum of dividends and the value of all bonds as:

$$B_{L,t-1} + \mu P_{L,t} B_{L,t-1} = (1 + \mu P_{L,t}) B_{L,t-1} = P_{L,t} \bar{R}_{L,t} B_{L,t-1} = \frac{\bar{R}_{L,t}}{\bar{R}_{L,t} - \mu} B_{L,t-1}.$$

A.1.2 Households

There are two types of households: unrestricted households (U-households) and restricted households (R-households). U-households, with population ω_u , can trade both short-term and long-term government bonds subject to a transaction cost ζ_t per unit of long-term bonds purchased. R-households, with population $\omega_r = 1 - \omega_u$, can trade only long-term government bonds. For $j = u, r$, each household chooses consumption c_t^j , hours worked h_t^j ,

the long-term government bond holdings $B_{L,t}^j$, and the short-term government bond holdings $B_{S,t}^j$ to maximize utility,

$$\sum_{t=0}^{\infty} \beta_j^t d_t \left[\frac{(c_t^j)^{1-\sigma}}{1-\sigma} - \psi \frac{(h_t^j)^{1+1/\nu}}{1+1/\nu} \right],$$

subject to: for a U-household,

$$P_t c_t^u + B_{S,t}^u + (1 + \zeta_t) P_{L,t} B_{L,t}^u = (1 + i_{t-1}) B_{t-1}^u + P_{L,t} \bar{R}_{L,t} B_{L,t-1}^u + W_t h_t^u - T_t^u + \Pi_t^u,$$

and for a R-household,

$$P_t c_t^r + P_{L,t} B_{L,t}^r = P_{L,t} \bar{R}_{L,t} B_{L,t-1}^r + W_t h_t^r - T_t^r + \Pi_t^r,$$

where P_t is the price level and i_t is the short-term interest rate. In addition $\bar{R}_{L,t}$ denotes the gross yield to maturity at time t on the long-term bond

$$\bar{R}_{L,t} = \frac{1}{P_{L,t}} + \mu, \quad 0 < \mu \leq 1.$$

The average duration of the bond is given by $\bar{R}_{L,t} / (\bar{R}_{L,t} - \mu)$. There is a shock d_t to the preference, and it is given by:

$$d_t = \begin{cases} e^{z_1^b} e^{z_2^b} \dots e^{z_t^b} & \text{for } t \geq 1 \\ 1 & \text{for } t = 0 \end{cases},$$

where z_t^b is a preference (demand) shock, which is assumed to follow an AR(1) process

$$z_t^b = \rho_b z_{t-1}^b + \epsilon_t^b,$$

with $\epsilon_t^b \sim \text{i.i.d. } N(0, \sigma_b^2)$.

We assume that the transaction cost of trading long-term bonds for the U-households is collected by financial firms and redistributed as a lump-sum profits to the U-households. Under the assumption, the transaction cost does not appear in the goods market clearing condition, which is given by:

$$y_t = \omega_u c_t^u + (1 - \omega_u) c_t^r. \quad (\text{A.3})$$

Arranging the first-order conditions of the U-household's problem yields the following optimality conditions:

$$w_t = \psi (c_t^u)^\sigma (h_t^u)^{1/\nu}, \quad (\text{A.4})$$

$$1 = E_t \beta_u e^{z_{t+1}^b} \left(\frac{c_{t+1}^u}{c_t^u} \right)^{-\sigma} \frac{1 + i_t}{\pi_{t+1}}, \quad (\text{A.5})$$

$$1 + \zeta_t = E_t \beta_u e^{z_{t+1}^b} \left(\frac{c_{t+1}^u}{c_t^u} \right)^{-\sigma} \frac{R_{L,t+1}}{\pi_{t+1}}, \quad (\text{A.6})$$

where $w_t \equiv W_t/P_t$ denotes the real wage, $\pi_t \equiv P_t/P_{t-1}$ denotes the inflation rate, and $R_{L,t+1}$ denotes the yield of the long-term bond between periods t and $t+1$, given by

$$R_{L,t+1} \equiv \frac{P_{L,t+1}}{P_{L,t}} \bar{R}_{L,t+1} = \frac{P_{L,t+1}}{P_{L,t}} \left(\frac{1}{P_{L,t+1}} + \mu \right) = \frac{1 + \mu P_{L,t+1}}{P_{L,t}}.$$

Similarly, arranging the first-order conditions of the R-household's problem yields

$$w_t = \psi (c_t^r)^\sigma (h_t^r)^{1/\nu}, \quad (\text{A.7})$$

$$1 = E_t \beta_r e^{z_{t+1}^b} \left(\frac{c_{t+1}^r}{c_t^r} \right)^{-\sigma} \frac{R_{L,t+1}}{\pi_{t+1}}. \quad (\text{A.8})$$

A.1.3 Firms

The firm sector consists of two types of firms: final-goods-producing firms and intermediate-goods-producing firms. The problems of these firms are standard except that the average discount rate between U-households and R-households is used in discounting the profits of these firms. The profits need to be derived explicitly because one of the two households' budget constraints constitutes an equilibrium condition as well as the goods market clearing condition.

Each final-goods-producing firm produces a unit of final goods y_t in a competitive market by combining intermediate goods $\{y_t(l)\}_{l=0}^1$ according to

$$y_t = \left[\int_0^1 y_t(l)^{\frac{1}{\lambda_p}} dl \right]^{\lambda_p}, \quad \lambda_p > 1.$$

The demand function for the l -th intermediate good is given by

$$y_t(l) = \left(\frac{P_t(l)}{P_t} \right)^{\frac{\lambda_p}{1-\lambda_p}} y_t.$$

Each intermediate-goods-producing firm uses labor and produce intermediate goods according to

$$y_t(l) = e^{z_t^a} h_t(l)^\theta, \quad 0 < \theta \leq 1.$$

where z_t^a is a productivity shock, which is assumed to follow

$$z_t^a = \rho_a z_{t-1}^a + \epsilon_t^a,$$

with $\epsilon_t^a \sim \text{i.i.d. } N(0, \sigma_a^2)$. Because there is no price dispersion in steady state, the aggregate output can be expressed up to the first-order approximation as:

$$\hat{y}_t = z_t^a + \theta \hat{h}_t, \quad (\text{A.9})$$

where \hat{y}_t and \hat{h}_t denote the aggregate output and hours worked in terms of deviation from the steady state. The total cost of producing $y_t(l)$ is equal to

$$W_t h_t(l) = W_t \left(\frac{y_t(l)}{e^{z_t^a}} \right)^{\frac{1}{\theta}}.$$

In each period, intermediate-goods-producing firms can change their price with probability ξ identically and independently across firms and over time. For each l , the l -th

intermediate-goods producing firm chooses the price, $\tilde{P}_t(l)$, to maximize the discounted sum of profits,

$$\max_{\tilde{P}_t(l)} E_t \sum_{s=0}^{\infty} (\xi\delta)^s \bar{\Lambda}_{t+s|t} \left[P_{t+s}(l) y_{t+s}(l) - W_{t+s} \left(\frac{y_{t+s}(l)}{e^{z_{t+s}^a}} \right)^{\frac{1}{\theta}} \right],$$

subject to the demand curve,

$$y_{t+s}(l) = \left(\frac{P_{t+s}(l)}{P_{t+s}} \right)^{\frac{\lambda_p}{1-\lambda_p}} y_{t+s},$$

where

$$\begin{aligned} \delta &= \omega_u \beta_u + (1 - \omega_u) \beta_r, \\ \bar{\Lambda}_{t+s|t} &\equiv d_{t+s|t} (\omega_u \Lambda_{t+s|t}^u + (1 - \omega_u) \Lambda_{t+s|t}^r), \\ \Lambda_{t+s|t}^j &= \left(\frac{c_{t+s}^j}{c_t^j} \right)^{-\sigma} \frac{1}{P_{t+s}}, \quad d_{t+s|t} = \begin{cases} 1 & \text{if } s = 0 \\ e^{z_{t+1}^b} e^{z_{t+2}^b} \dots e^{z_{t+s}^b} & \text{if } s = 1, 2, \dots \end{cases} \\ P_{t+s}(l) &= \tilde{P}_t(l) \Pi_{t,t+s}^p, \\ \Pi_{t+s|t}^p &= \begin{cases} 1 & \text{if } s = 0 \\ \prod_{k=1}^s (\pi_{t+k-1})^{\iota_p} (\pi)^{1-\iota_p} & \text{if } s = 1, 2, \dots \end{cases} \end{aligned}$$

The presence of $\Pi_{t+s|t}^p$ implies price indexation for firms that do not have a chance to change prices and $0 \leq \iota_p \leq 1$ governs the degree of indexation to the past inflation rates. Substituting the demand curve into the objective function yields

$$\max_{\tilde{P}_t(l)} E_t \sum_{s=0}^{\infty} (\xi\delta)^s \bar{\Lambda}_{t+s|t} \left[\tilde{P}_t(l) \Pi_{t,t+s}^p \left(\frac{\tilde{P}_t(l) \Pi_{t,t+s}^p}{P_{t+s}} \right)^{\frac{\lambda_p}{1-\lambda_p}} Y_{t+s} - W_{t+s} \left(\frac{\tilde{P}_t(l) \Pi_{t,t+s}^p}{P_{t+s}} \right)^{\frac{\lambda_p}{(1-\lambda_p)\theta}} \left(\frac{y_{t+s}}{e^{z_{t+s}^a}} \right)^{\frac{1}{\theta}} \right].$$

The first-order condition is

$$0 = E_t \sum_{s=0}^{\infty} (\xi\delta)^s \bar{\Lambda}_{t+s|t} \left[\frac{1}{1 - \lambda_p} \Pi_{t+s|t}^p y_{t+s}(l) - W_{t+s} \frac{\lambda_p}{(1 - \lambda_p)\theta} \left(\frac{y_{t+s}(l)}{e^{z_{t+s}^a}} \right)^{\frac{1}{\theta}} \frac{1}{\tilde{P}_t(l)} \right].$$

Since $\tilde{P}_t(l)$ does not depend on l , index l is omitted hereafter. Define $\tilde{p}_t \equiv \tilde{P}_t/P_t$ and

$$\tilde{\Pi}_{t+s|t}^p = \begin{cases} 1 & \text{if } s = 0 \\ \prod_{k=1}^s \frac{(\pi_{t+k-1})^{\iota_p} (\pi)^{1-\iota_p}}{\pi_{t+k}} & \text{if } s = 1, 2, \dots \end{cases}$$

The first-order condition can be transformed as

$$\begin{aligned} 0 &= E_t \sum_{s=0}^{\infty} (\xi\delta)^s \bar{\Lambda}_{t+s} P_{t+s} \left[\frac{1}{1 - \lambda_p} \frac{\Pi_{t+s|t}^p}{P_{t+s}} \left(\tilde{p}_t \tilde{\Pi}_{t+s|t}^p \right)^{\frac{\lambda_p}{1-\lambda_p}} y_{t+s} \right. \\ &\quad \left. - \frac{W_{t+s}}{P_{t+s}} \frac{\lambda_p}{(1 - \lambda_p)\theta} \left(\tilde{p}_t \tilde{\Pi}_{t+s|t}^p \right)^{\frac{\lambda_p}{(1-\lambda_p)\theta}} \left(\frac{Y_{t+s}}{e^{z_{t+s}^a}} \right)^{\frac{1}{\theta}} \frac{1}{\tilde{P}_t} \right], \end{aligned}$$

This equation can be written as:

$$\tilde{p}_t = \left(\frac{\lambda_p \omega_u K_{p,t}^u + (1 - \omega_u) K_{p,t}^r}{\theta \omega_u F_{p,t}^u + (1 - \omega_u) F_{p,t}^r} \right)^{\frac{(1-\lambda_p)\theta}{\theta-\lambda_p}}, \quad (\text{A.10})$$

where for $j \in \{r, u\}$

$$F_{p,t}^j = (c_t^j)^{-\sigma} y_t + \xi \delta E_t e^{z_{t+1}^b} (\tilde{\Pi}_{t+1|t}^p)^{\frac{1}{1-\lambda_p}} F_{p,t+1}^j, \quad (\text{A.11})$$

$$K_{p,t}^j = (c_t^j)^{-\sigma} \left(\frac{y_t}{e^{z_t^a}} \right)^{\frac{1}{\theta}} w_t + \xi \delta E_t e^{z_{t+1}^b} (\tilde{\Pi}_{t+1|t}^p)^{\frac{\lambda_p}{(1-\lambda_p)\theta}} K_{p,t+1}^j. \quad (\text{A.12})$$

The aggregate price level evolves following

$$P_t = \left[\xi [(\pi_{t-1})^{\lambda_p} (\pi)^{1-\lambda_p} P_{t-1}]^{\frac{1}{1-\lambda_p}} + (1 - \xi) \tilde{P}_t^{\frac{1}{1-\lambda_p}} \right]^{1-\lambda_p},$$

which can be written as

$$\tilde{p}_t = \left[\frac{1 - \xi (\tilde{\Pi}_{t|t-1}^p)^{\frac{1}{1-\lambda_p}}}{1 - \xi} \right]^{1-\lambda_p}. \quad (\text{A.13})$$

The conditions, (A.10)-(A.13), summarize the price setting behavior of intermediate-goods-producing firms.

The aggregate nominal profits earned by intermediate-goods-producing firms are given by:

$$\Pi_t^m = \int_0^1 \left(P_t(l) y_t(l) - W_t \left(\frac{y_t(l)}{e^{z_t^a}} \right)^{\frac{1}{\theta}} \right) dl = P_t y_t - W_t \left(\frac{y_t}{e^{z_t^a}} \right)^{\frac{1}{\theta}},$$

where the last equality holds up to the first-order approximation. Then, the aggregate real profits are given by $\pi_t^m = y_t - w_t (y_t/e^{z_t^a})^{1/\theta}$.

A.1.4 Government

The government flow budget constraint is

$$(1 + i_{t-1}) B_{S,t-1} + (1 + \mu P_{L,t}) B_{L,t-1} = B_{S,t} + P_{L,t} B_{L,t} + T_t,$$

where $T_t = \omega_u T_t^u + (1 - \omega_u) T_t^r$. We assume that the lump-sum tax is imposed on households equally so that $T_t^u = T_t^r = T_t$. To focus on the role of long-term government bonds, we assume that the amount of short-term bonds is constant at $b_{S,t} \equiv B_{S,t}/P_t = \bar{b}_S$.

A.1.5 Central bank

The nominal interest rate i_t set by the central bank is bounded below by the ELB as

$$i_t = \max \{ i_t^*, \underline{i} \}, \quad (\text{A.14})$$

where \underline{i} is the ELB and i_t^* is a shadow rate – the short-term rate the central bank would set if there were no ELB. The shadow rate i_t^* is given by¹

$$i_t^* = i_t^{\text{Taylor}} - \alpha \left(i_t - i_t^{\text{Taylor}} \right). \quad (\text{A.15})$$

The shadow rate i_t^* consists of two parts: i_t^{Taylor} and $\alpha(i_t - i_t^{\text{Taylor}})$. First, i_t^{Taylor} is the Taylor-rule-based rate that responds to inflation π_t , output y_t , and the lagged ‘effective’ interest rate $(1 - \lambda^*)i_{t-1} + \lambda^*i_{t-1}^*$:

$$i_t^{\text{Taylor}} - i = \rho_i \left((1 - \lambda^*)i_{t-1} + \lambda^*i_{t-1}^* - i \right) + (1 - \rho_i) \left[r_\pi \log(\pi_t/\pi) + r_y \log(y_t/y) \right] + \epsilon_t^i, \quad (\text{A.16})$$

where ϵ_t^i is a monetary policy shock and variables without subscripts denote those in steady state. The parameter λ^* will be derived later in this appendix. Second, $\alpha(i_t - i_t^{\text{Taylor}})$ in equation (A.15) encapsulates the strength of FG. A positive value for α will maintain the target rate i_t^* below the Taylor rate i_t^{Taylor} . Under the ELB of $i_t = \underline{i}$, the more the central bank has missed to set the interest rate at its Taylor rate, the lower the central bank sets its target rate i_t^* through equation (A.15) as long as $\rho_i \lambda^* > 0$ in equation (A.16).²

The central bank activates QE when the economy hits the ELB. The central bank continues using the shadow rate as policy guidance in an ELB regime as in a non-ELB regime. Specifically, the amount of long-term bond purchases depends on the shadow rate, and as a result the amount of long-term government bonds, $b_{L,t} \equiv B_{L,t}/P_t$, held by the private agents is given by:

$$\hat{b}_{L,t} = \begin{cases} 0 & \text{if } i_t^* \geq \underline{i} \\ \gamma \frac{i_t^* - \underline{i}}{1 + \underline{i}} & \text{if } i_t^* < \underline{i} \end{cases}, \quad (\text{A.17})$$

where the caret on a variable denotes a deviation from the steady state. This QE rule implies that asset purchases by the central bank is zero (relative to the steady state) when the ELB is not binding (i.e. $i_t = i_t^* \geq \underline{i}$) and, given $\gamma > 0$, the purchases are positive (i.e., $\hat{b}_{L,t} < 0$) when the shadow rate goes below the ELB (i.e., $i_t^* < \underline{i}$).

A.1.6 Market clearing and equilibrium

As well as the goods market clearing condition (A.3), there are market clearing conditions for labor, long-term government bonds, and short-term government bonds:

$$\omega_u h_t^u + (1 - \omega_u) h_t^r = h_t, \quad (\text{A.18})$$

$$\omega_u b_{L,t}^u + (1 - \omega_u) b_{L,t}^r = b_{L,t}, \quad (\text{A.19})$$

$$\omega_u b_{S,t}^u = b_{S,t} \quad (\text{A.20})$$

Also, either the U-household’s budget constraint or the R-household’s budget constraint should be added as an equilibrium condition. Here the latter budget constraint is added:

$$\bar{c}_t^r + P_{L,t} b_{L,t}^r = \left(\bar{R}_{L,t} / \pi_t \right) P_{L,t} b_{L,t-1}^r + w_t h_t^r - T_t^r / P_t + \Pi_t^r / P_t, \quad (\text{A.21})$$

¹Reifschneider and Williams (2000) use the following rule: $i_t^* = i_t^{\text{Taylor}} - \alpha Z_t$ and $Z_t = \rho_Z Z_{t-1} + (i_t - i_t^{\text{Taylor}})$ with $\rho_Z = 1$.

²Debortoli et al. (2019) consider the case of $\alpha = 0$ and $\lambda^* = 1$ in equation (A.16) and interpret ρ_i – the coefficient of interest rate smoothing – as FG when i_t^* is below the ELB.

where

$$\begin{aligned}\frac{T_t^r}{P_t} &= -(b_{S,t} + P_{L,t}b_{L,t}) + \frac{1 + i_{t-1}}{\pi_t} b_{S,t-1} + \frac{1 + \mu P_{L,t}}{\pi_t} b_{L,t-1}, \\ \frac{\Pi_t^r}{P_t} &= y_t - w_t h_t.\end{aligned}$$

The cost of trading long-term bonds, ζ_t , is specified as

$$\frac{\zeta_t}{\zeta} = \left(\frac{b_{L,t}}{b_L} \right)^{\rho_\zeta}, \quad \rho_\zeta > 0. \quad (\text{A.22})$$

The trading cost is increasing in the amount of long-term bonds relative to its steady state value. The trading cost is ζ in steady state.

The system of equations for the economy consists of 19 equations, (A.3)-(A.21), with the following endogenous variables:

$$c_t^u, c_t^r, h_t^u, h_t^r, h_t, b_{L,t}^u, b_{L,t}^r, b_{L,t}, b_t^u, y_t, w_t, i_t, i_t^*, i_t^{\text{Taylor}}, R_{L,t}, \pi_t, \tilde{p}_t, F_{p,t}^j, K_{p,t}^j.$$

A.2 Log-linearized equations

We log-linearize the equilibrium conditions of the theoretical model presented in Appendix A.1 around the steady state in which inflation is equal to the target rate of inflation set by the central bank. By doing so, we derive key equations in the system of equations (1)-(16) presented in Section I of the main text. We also derive a log-linearized equation for the long-term yield.

Euler equation. Log-linearizing equations (A.3), (A.5), (A.6), (A.8), and (A.22), we obtain³

$$\hat{y}_t = \frac{\omega_u c^u}{y} \hat{c}_t^u + \frac{(1 - \omega_u) c^r}{y} \hat{c}_t^r. \quad (\text{A.23})$$

$$0 = E_t \left[-\sigma (\hat{c}_{t+1}^u - \hat{c}_t^u) + \hat{i}_t - \hat{\pi}_{t+1} + z_{t+1}^b \right], \quad (\text{A.24})$$

$$\frac{\zeta}{1 + \zeta} \hat{\zeta}_t = E_t \left[-\sigma (\hat{c}_{t+1}^u - \hat{c}_t^u) + \hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b \right], \quad (\text{A.25})$$

$$0 = E_t \left[-\sigma (\hat{c}_{t+1}^r - \hat{c}_t^r) + \hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b \right], \quad (\text{A.26})$$

$$\hat{\zeta}_t = \rho_\zeta \hat{b}_{L,t}. \quad (\text{A.27})$$

Equation (A.23) can be written as:

$$\hat{c}_t^u = \frac{y}{\omega_u c^u} \left\{ \hat{y}_t - \frac{(1 - \omega_u) c^r}{y} \hat{c}_t^r \right\}.$$

³The variable \hat{i} represents the deviation of the gross interest rate from the steady state.

Subtracting \hat{c}_{t+1}^u from \hat{c}_t^u yields:

$$\begin{aligned}\hat{c}_{t+1}^u - \hat{c}_t^u &= \frac{y}{\omega_u c^u} \left\{ \hat{y}_{t+1} - \hat{y}_t - \frac{(1 - \omega_u) c^r}{y} (\hat{c}_{t+1}^r - \hat{c}_t^r) \right\}, \\ &= \frac{y}{\omega_u c^u} \left\{ \hat{y}_{t+1} - \hat{y}_t - \frac{(1 - \omega_u) c^r}{y} \frac{(\hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b)}{\sigma} \right\},\end{aligned}\quad (\text{A.28})$$

where equation (A.26) was used in the second equality. Substituting equation (A.28) into equation (A.24) yields:

$$\begin{aligned}0 &= E_t \left[-\sigma (\hat{c}_{t+1}^u - \hat{c}_t^u) + \hat{i}_t - \hat{\pi}_{t+1} + z_{t+1}^b \right], \\ &= E_t \left[-\frac{\sigma y}{\omega_u c^u} (\hat{y}_{t+1} - \hat{y}_t) + \frac{\sigma y}{\omega_u c^u} \frac{(1 - \omega_u) c^r}{y} \frac{(\hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b)}{\sigma} \right. \\ &\quad \left. + \hat{i}_t - \hat{\pi}_{t+1} + z_{t+1}^b \right],\end{aligned}$$

or, by using equation (A.3) in steady state,

$$0 = E_t \left[-\sigma (\hat{y}_{t+1} - \hat{y}_t) + \frac{(1 - \omega_u) c^r}{y} \hat{R}_{L,t+1} + \frac{\omega_u c^u}{y} \hat{i}_t - \hat{\pi}_{t+1} + z_{t+1}^b \right]. \quad (\text{A.29})$$

Equation (A.29) shows that the interest rate relevant to the aggregate variables such as output and inflation is the weighted sum of the return of holding the long-term bonds $\hat{R}_{L,t+1}$ and the short-term interest rate \hat{i}_t . Also, substituting equation (A.28) into equation (A.25) yields:

$$\begin{aligned}\frac{\zeta}{1 + \zeta} \hat{\zeta}_t &= E_t \left[-\sigma (\hat{c}_{t+1}^u - \hat{c}_t^u) + \hat{R}_{L,t+1} - \hat{\pi}_{t+1} \right] \\ &= E_t \left[-\sigma \frac{y}{\omega_u c^u} \left\{ \hat{y}_{t+1} - \hat{y}_t - \frac{(1 - \omega_u) c^r}{y} \frac{(\hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b)}{\sigma} \right\} \right. \\ &\quad \left. + \hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b \right], \\ &= E_t \left[-\frac{\sigma y}{\omega_u c^u} (\hat{y}_{t+1} - \hat{y}_t) + \frac{y}{\omega_u c^u} (\hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b) \right],\end{aligned}$$

or, substituting out $\hat{\zeta}_t$ by using equation (A.27) yields

$$E_t \left(\hat{R}_{L,t+1} - \hat{\pi}_{t+1} \right) = \sigma E_t (\hat{y}_{t+1} - \hat{y}_t) - E_t (z_{t+1}^b) + \frac{\omega_u c^u}{y} \frac{\zeta}{1 + \zeta} \rho_\zeta \hat{b}_{L,t}. \quad (\text{A.30})$$

Combining equations (A.29) and (A.30) yields:

$$\begin{aligned}
0 &= E_t \left[-\sigma (\hat{y}_{t+1} - \hat{y}_t) + \frac{(1 - \omega_u) c^r}{y} \hat{R}_{L,t+1} + \frac{\omega_u c^u}{y} \hat{i}_t - \hat{\pi}_{t+1} + z_{t+1}^b \right] \\
&= E_t \left[-\sigma (\hat{y}_{t+1} - \hat{y}_t) + \frac{(1 - \omega_u) c^r}{y} \left(\sigma (\hat{y}_{t+1} - \hat{y}_t) - z_{t+1}^b + \frac{\omega_u c^u}{y} \frac{\zeta}{1 + \zeta} \rho_\zeta \hat{b}_{L,t} + \hat{\pi}_{t+1} \right) \right. \\
&\quad \left. + \frac{\omega_u c^u}{y} \hat{i}_t - \hat{\pi}_{t+1} + z_{t+1}^b \right] \\
&= E_t \left[-\frac{\omega_u c^u \sigma}{y} (\hat{y}_{t+1} - \hat{y}_t) + \frac{\omega_u c^u}{y} \hat{i}_t - \frac{\omega_u c^u}{y} (\hat{\pi}_{t+1} - z_{t+1}^b) + \frac{(1 - \omega_u) c^r}{y} \frac{\omega_u c^u}{y} \frac{\zeta}{1 + \zeta} \rho_\zeta \hat{b}_{L,t} \right],
\end{aligned}$$

or

$$0 = E_t \left[-\sigma (\hat{y}_{t+1} - \hat{y}_t) + \hat{i}_t - \hat{\pi}_{t+1} + z_{t+1}^b + \frac{(1 - \omega_u) c^r}{y} \frac{\zeta}{1 + \zeta} \rho_\zeta \hat{b}_{L,t} \right],$$

or

$$\begin{aligned}
\hat{y}_t &= E_t \hat{y}_{t+1} - \frac{1}{\sigma} (\hat{i}_t - E_t \hat{\pi}_{t+1} + E_t z_{t+1}^b) - \frac{1}{\sigma} \frac{(1 - \omega_u) c^r}{y} \frac{\zeta}{1 + \zeta} \rho_\zeta \hat{b}_{L,t} \\
&= E_t \hat{y}_{t+1} - \frac{1}{\sigma} (\hat{i}_t - E_t \hat{\pi}_{t+1}) - \frac{1}{\sigma} \frac{(1 - \omega_u) c^r}{y} \frac{\zeta}{1 + \zeta} \rho_\zeta \hat{b}_{L,t} - \frac{\rho_b}{\sigma} z_t^b.
\end{aligned}$$

This equation shows that the central bank's government bond purchase – a decrease in $\hat{b}_{L,t}$ – stimulates output, given $E_t \hat{y}_{t+1}$ and the real rate $\hat{i}_t - E_t \hat{\pi}_{t+1}$. Since $\hat{b}_{L,t}$ follows the simple rule (A.17), the equation can be written as equation (12) in the main text, which is reproduced here for convenience:

$$\hat{y}_t = E_t \hat{y}_{t+1} - \frac{1}{\sigma} \left((1 - \lambda^*) \hat{i}_t + \lambda^* \hat{i}_t^* - E_t \hat{\pi}_{t+1} \right) - \chi_b z_t^b \quad (\text{A.31})$$

where

$$\begin{aligned}
\lambda^* &= \frac{(1 - \omega_u) c^r}{y} \frac{\zeta}{1 + \zeta} \rho_\zeta \gamma \\
\chi_b &= \frac{\rho_b}{\sigma}
\end{aligned} \quad (\text{A.32})$$

The case of $\lambda^* = 1$ (and $\alpha = 0$) corresponds to the fully effective UMP, which makes the ELB irrelevant. Such a case can be achieved, e.g., when the central bank responds to the shadow rate aggressively enough to satisfy

$$\gamma = \left[\frac{(1 - \omega_u) c^r}{y} \frac{\zeta}{1 + \zeta} \rho_\zeta \right]^{-1}.$$

Phillips curve. The Phillips curve can be derived from equations (A.10)-(A.13). Log-linearizing equation (A.13) yields:

$$\hat{p}_t = -\frac{\xi}{1 - \xi} \hat{\Pi}_{t|t-1}^p, \quad (\text{A.33})$$

where

$$\widehat{\Pi}_{t|t-1}^p = (1 - \nu_p) \widehat{\pi}_{t-1} - \widehat{\pi}_t.$$

Log-linearizing equation (A.10) yields:

$$\begin{aligned} \frac{\theta - \lambda_p}{(1 - \lambda_p)\theta} \widehat{p}_t &= \frac{\omega_u K_p^u}{\omega_u K_p^u + (1 - \omega_u) K_p^r} \widehat{K}_{p,t}^u + \frac{(1 - \omega_u) K_p^r}{\omega_u K_p^u + (1 - \omega_u) K_p^r} \widehat{K}_{p,t}^r \\ &\quad - \frac{\omega_u F_p^u}{\omega_u F_p^u + (1 - \omega_u) F_p^r} \widehat{F}_{p,t}^u - \frac{(1 - \omega_u) F_p^r}{\omega_u F_p^u + (1 - \omega_u) F_p^r} \widehat{F}_{p,t}^r. \end{aligned} \quad (\text{A.34})$$

Combining equations (A.33) and (A.34) leads to:

$$\begin{aligned} -\frac{\xi}{1 - \xi} \frac{\theta - \lambda_p}{(1 - \lambda_p)\theta} [(1 - \nu_p) \widehat{\pi}_{t-1} - \widehat{\pi}_t] &= \frac{\omega_u K_p^u}{\omega_u K_p^u + (1 - \omega_u) K_p^r} \widehat{K}_{p,t}^u + \frac{(1 - \omega_u) K_p^r}{\omega_u K_p^u + (1 - \omega_u) K_p^r} \widehat{K}_{p,t}^r \\ &\quad - \frac{\omega_u F_p^u}{\omega_u F_p^u + (1 - \omega_u) F_p^r} \widehat{F}_{p,t}^u - \frac{(1 - \omega_u) F_p^r}{\omega_u F_p^u + (1 - \omega_u) F_p^r} \widehat{F}_{p,t}^r. \end{aligned} \quad (\text{A.35})$$

Log-linearizing equation (A.11) and (A.12) yields:

$$\begin{aligned} \widehat{F}_{p,t}^j &= (1 - \xi\delta) (-\sigma \widehat{c}_t^j + \widehat{y}_t) + \xi\delta E_t \left(z_{t+1}^b + \frac{1}{1 - \lambda_p} \widehat{\Pi}_{t+1|t}^p + \widehat{F}_{p,t+1}^j \right), \\ \widehat{K}_{p,t}^j &= (1 - \xi\delta) \left(-\sigma \widehat{c}_t^j + \frac{1}{\theta} \widehat{y}_t - \frac{1}{\theta} z_t^a + \widehat{w}_t \right) + \xi\delta E_t \left(z_{t+1}^b + \frac{\lambda_p}{(1 - \lambda_p)\theta} \widehat{\Pi}_{t+1|t}^p + \widehat{K}_{p,t+1}^j \right), \end{aligned}$$

for $j \in \{r, u\}$. The term involving $\widehat{F}_{p,t}^u$ and $\widehat{F}_{p,t}^r$ in equation (A.35) is calculated as follows.

$$\begin{aligned} &\frac{\omega_u F_p^u}{\omega_u F_p^u + (1 - \omega_u) F_p^r} \widehat{F}_{p,t}^u + \frac{(1 - \omega_u) F_p^r}{\omega_u F_p^u + (1 - \omega_u) F_p^r} \widehat{F}_{p,t}^r \\ &= (1 - \xi\delta) \left(-\sigma \frac{\omega_u F_p^u \widehat{c}_t^u + (1 - \omega_u) F_p^r \widehat{c}_t^r}{\omega_u F_p^u + (1 - \omega_u) F_p^r} + \widehat{y}_t \right) \\ &\quad + \xi\delta E_t \left(z_{t+1}^b + \frac{1}{1 - \lambda_p} \widehat{\Pi}_{t+1|t}^p + \frac{\omega_u F_p^u}{\omega_u F_p^u + (1 - \omega_u) F_p^r} \widehat{F}_{p,t+1}^u + \frac{(1 - \omega_u) F_p^r}{\omega_u F_p^u + (1 - \omega_u) F_p^r} \widehat{F}_{p,t+1}^r \right). \end{aligned}$$

Similarly, the term involving $\widehat{K}_{p,t}^u$ and $\widehat{K}_{p,t}^r$ in equation (A.35) is calculated as:

$$\begin{aligned} &\frac{\omega_u K_p^u}{\omega_u K_p^u + (1 - \omega_u) K_p^r} \widehat{K}_{p,t}^u + \frac{(1 - \omega_u) K_p^r}{\omega_u K_p^u + (1 - \omega_u) K_p^r} \widehat{K}_{p,t}^r \\ &= (1 - \xi\delta) \left(-\sigma \frac{\omega_u K_p^u \widehat{c}_t^u + (1 - \omega_u) K_p^r \widehat{c}_t^r}{\omega_u K_p^u + (1 - \omega_u) K_p^r} + \frac{1}{\theta} \widehat{y}_t - \frac{1}{\theta} z_t^a + \widehat{w}_t \right) \\ &\quad + \xi\delta E_t \left(z_{t+1}^b + \frac{\lambda_p}{(1 - \lambda_p)\theta} \widehat{\Pi}_{t+1|t}^p + \frac{\omega_u K_p^u}{\omega_u K_p^u + (1 - \omega_u) K_p^r} \widehat{K}_{p,t+1}^u + \frac{(1 - \omega_u) K_p^r}{\omega_u K_p^u + (1 - \omega_u) K_p^r} \widehat{K}_{p,t+1}^r \right). \end{aligned}$$

Let the right-hand-side of equation (A.35) be denoted as \hat{X}_t . Then, using the above relationships just derived, \hat{X}_t can be written as:

$$\hat{X}_t = (1 - \xi\delta) \left[\left(\frac{1}{\theta} - 1 \right) \hat{y}_t - \frac{1}{\theta} z_t^a + \hat{w}_t \right] + \xi\delta E_t \left(-\frac{\lambda_p - \theta}{(\lambda_p - 1)\theta} \hat{\Pi}_{t+1|t}^p + \hat{X}_{t+1} \right)$$

Because \hat{X}_t is the right-hand-side of equation (A.35), equation (A.35) can be written as:

$$\begin{aligned} -\frac{\xi}{1 - \xi} \frac{\lambda_p - \theta}{(\lambda_p - 1)\theta} [(1 - \nu_p) \hat{\pi}_{t-1} - \hat{\pi}_t] &= (1 - \xi\delta) \left[\left(\frac{1}{\theta} - 1 \right) \hat{y}_t - \frac{1}{\theta} z_t^a + \hat{w}_t \right] \\ &+ \xi\delta E_t \left(-\frac{\lambda_p - \theta}{(\lambda_p - 1)\theta} \hat{\Pi}_{t+1|t}^p - \frac{\xi}{1 - \xi} \frac{\lambda_p - \theta}{(\lambda_p - 1)\theta} [(1 - \nu_p) \hat{\pi}_t - \hat{\pi}_{t+1}] \right), \end{aligned}$$

or

$$\hat{\pi}_t = \frac{\xi(1 - \nu_p)}{(\xi + 1 - \nu_p)} \hat{\pi}_{t-1} + \frac{(1 - \xi\delta)(1 - \xi)(\lambda_p - 1)\theta}{(\lambda_p - \theta)(\xi + 1 - \nu_p)} \left[\left(\frac{1}{\theta} - 1 \right) \hat{y}_t - \frac{1}{\theta} z_t^a + \hat{w}_t \right] + \frac{\xi\delta}{(\xi + 1 - \nu_p)} E_t \hat{\pi}_{t+1}.$$

From equations (A.4) and (A.7), the wage \hat{w}_t can be written as:

$$\begin{aligned} \hat{w}_t &= \omega_u \left(\sigma \hat{c}_t^u + \frac{1}{\nu} \hat{h}_t^u \right) + (1 - \omega_u) \left(\sigma \hat{c}_t^r + \frac{1}{\nu} \hat{h}_t^r \right), \\ &= \sigma \hat{y}_t + \frac{1}{\nu} \hat{h}_t = \left(\sigma + \frac{1}{\nu\theta} \right) \hat{y}_t - \frac{1}{\nu\theta} z_t^a, \end{aligned}$$

where the market clearing conditions (A.3) and (A.18) were used in the second equality and the production function (A.9) was used in the third equality. Since we assume $c^u = c^r$, the second equality holds. By using the expression for \hat{w}_t , the Phillips curve can be written as

$$\begin{aligned} \hat{\pi}_t &= \frac{\xi(1 - \nu_p)}{(\xi + 1 - \nu_p)} \hat{\pi}_{t-1} \\ &+ \frac{(1 - \xi\delta)(1 - \xi)(\lambda_p - 1)\theta}{(\lambda_p - \theta)(\xi + 1 - \nu_p)} \left[\frac{\nu + \nu\theta(\sigma - 1) + 1}{\nu\theta} \hat{y}_t - \frac{1 + \nu}{\nu\theta} z_t^a \right] + \frac{\xi\delta}{(\xi + 1 - \nu_p)} E_t \hat{\pi}_{t+1}. \end{aligned}$$

In the case of no price indexation to the past inflation rate and a linear production function, that is, in the case of $\nu_p = 1$ and $\theta = 1$, the Phillips curve collapses to the standard form:

$$\hat{\pi}_t = \frac{(1 - \xi\delta)(1 - \xi)}{\xi} \left(\sigma + \frac{1}{\nu} \right) \hat{y}_t + \delta E_t \hat{\pi}_{t+1} - \frac{(1 - \xi\delta)(1 - \xi)}{\xi} \frac{1 + \nu}{\nu} z_t^a.$$

This completes the derivation of equation (16) in the main text, where

$$\kappa = \frac{(1 - \xi\delta)(1 - \xi)}{\xi} \left(\sigma + \frac{1}{\nu} \right), \quad (\text{A.36})$$

$$\chi_a = \frac{(1 - \xi\delta)(1 - \xi)}{\xi} \frac{1 + \nu}{\nu}. \quad (\text{A.37})$$

Long-term yield. From equations (A.29) and (A.31), the interest rate relevant to the aggregate variables has the following equality:

$$\frac{(1 - \omega_u)c^r}{y} E_t \hat{R}_{L,t+1} + \frac{\omega_u c^u}{y} \hat{i}_t = (1 - \lambda^*) \hat{i}_t + \lambda^* \hat{i}_t^*. \quad (\text{A.38})$$

This equation can be written as

$$E_t \hat{R}_{L,t+1} = \begin{cases} \hat{i}_t^* & \hat{i}_t^* \geq \hat{i} \\ \lambda^* y \hat{i}_t^* + \frac{(1-\lambda^*)y - \omega_u c^u}{(1-\omega_u)c^r} \hat{i} & \hat{i}_t^* < \hat{i} \end{cases} \quad (\text{A.39})$$

By using $R_{L,t+1} = \bar{R}_{L,t+1}(\bar{R}_{L,t} - \mu)/(\bar{R}_{L,t+1} - \mu)$, which relates the return of holding long-term bonds $\hat{R}_{L,t+1}$ to the long-term yield $\hat{R}_{L,t}$, the long-term yield can be written as

$$\hat{R}_{L,t} = \frac{\bar{R}_L - \mu}{\bar{R}_L} E_t \hat{R}_{L,t+1} + \frac{\mu}{\bar{R}_L} E_t \hat{R}_{L,t+1}, \quad (\text{A.40})$$

where $\bar{R}_L > \mu$ in steady state. Substitution equation (A.39) into equation (A.40) yields

$$\hat{R}_{L,t} = \begin{cases} \frac{\bar{R}_L - \mu}{\bar{R}_L} \hat{i}_t^* + \frac{\mu}{\bar{R}_L} E_t \hat{R}_{L,t+1} & \hat{i}_t^* \geq \hat{i} \\ \frac{\bar{R}_L - \mu}{\bar{R}_L} \left[\frac{y\lambda^*}{(1-\omega_u)c^r} \hat{i}_t^* + \frac{(1-\lambda^*)y - \omega_u c^u}{(1-\omega_u)c^r} \hat{i} \right] + \frac{\mu}{\bar{R}_L} E_t \hat{R}_{L,t+1} & \hat{i}_t^* < \hat{i} \end{cases} \quad (\text{A.41})$$

Equation (A.41) shows that the long-term yield is the discounted sum of the current and future short-term returns, where the short-term return is given by the shadow rate in the non-ELB regime and in the ELB regime it is given by the first two terms in the square brackets in (A.41).

A.3 Parameterization of the model

Instead of parameterizing the model presented in Appendix A.1, we parameterize the system of log-linearized equations (1)-(16) in the main text. It is worth emphasizing that we use the parameterized model to illustrate the implications of the theoretical model, and not to study the quantitative implications, which would require a more complex system.

The relative risk aversion parameter is set at $\sigma = 2$. The discount factor is set close to unity at $\delta = 0.997$. The slope of the Phillips curve κ is set at $\kappa = 0.336$ using equation (A.36) with the Calvo parameter of $\xi = 0.75$ and the Frisch labor elasticity of $\nu = 0.5$. In the monetary policy rule, the persistence parameter is set at $\rho_i = 0.7$; the inflation coefficient is set at $r_\pi = 1.5$; the output coefficient is set at $r_y = 0.5$. The AR(1) coefficients for the supply and demand shocks are set at $\rho_a = \rho_b = 0.9$, and the coefficients χ_b and χ_a are set according to equations (A.32) and (A.37), respectively. The term premium in steady state is set at $\zeta = 0.01/4$. We consider different values for the parameters λ^* and α (reported in the main text) to study the effects of UMP.

A.4 Proof of Proposition 1

Part (i) Because of the equivalence established in Lemma 1, without loss of generality, consider the case of $\lambda^* = 1$ and $\alpha = 0$ in the theoretical model. In this case, the variables

\hat{y}_t , $\hat{\pi}_t$, and \hat{i}_t^* have a closed system of equations, consisting of equation (12) with $\lambda^* = 1$, equation (16), and $\hat{i}_t^* = \hat{i}_t^{\text{Taylor}}$, where $\hat{i}_t^{\text{Taylor}}$ is given by equation (3).

In this case, the state of the economy in period t can be summarized by \hat{i}_{t-1}^* , ϵ_t^i , z_t^a , and z_t^b . Then decision rules for \hat{y}_t and $\hat{\pi}_t$ have the following form:

$$\begin{aligned}\hat{y}_t &= d_{yi^*}\hat{i}_{t-1}^* + d_{yi}\epsilon_t^i + d_{ya}z_t^a + d_{yb}z_t^b, \\ \hat{\pi}_t &= d_{\pi i^*}\hat{i}_{t-1}^* + d_{\pi i}\epsilon_t^i + d_{\pi a}z_t^a + d_{\pi b}z_t^b,\end{aligned}$$

with coefficients $\{d_{yi^*}, d_{yi}, d_{ya}, d_{yb}, d_{\pi i^*}, d_{\pi i}, d_{\pi a}, d_{\pi b}\}$ uniquely determined under standard assumptions of the model (such as the Taylor principle). With these decision rules, the equation for \hat{i}_t^* can be written as

$$\begin{aligned}\hat{i}_t^* &= [\rho_i + (1 - \rho_i)(r_\pi d_{\pi i^*} + r_y d_{yi^*})]\hat{i}_{t-1}^* + [(1 - \rho_i)(r_\pi d_{\pi i} + r_y d_{yi}) + 1]\epsilon_t^i \\ &\quad + (1 - \rho_i)(r_\pi d_{\pi a} + r_y d_{ya})z_t^a + (1 - \rho_i)(r_\pi d_{\pi b} + r_y d_{yb})z_t^b \\ &= d_{i^*i^*}\hat{i}_{t-1}^* + d_{i^*i}\epsilon_t^i + d_{i^*a}z_t^a + d_{i^*b}z_t^b.\end{aligned}\tag{A.42}$$

Let $\mathbf{y}_t \equiv [\hat{y}_t, \hat{\pi}_t, \hat{i}_t^*]'$ denote the vector of endogenous variables. The decision rule implies

$$\begin{aligned}\mathbf{y}_t &= \begin{bmatrix} d_{yi^*} & d_{yi} & d_{ya} & d_{yb} \\ d_{\pi i^*} & d_{\pi i} & d_{\pi a} & d_{\pi b} \\ d_{i^*i^*} & d_{i^*i} & d_{i^*a} & d_{i^*b} \end{bmatrix} \begin{bmatrix} \hat{i}_{t-1}^* \\ \epsilon_t^i \\ \rho_a z_{t-1}^a + \epsilon_t^a \\ \rho_b z_{t-1}^b + \epsilon_t^b \end{bmatrix} \\ &= \begin{bmatrix} d_{yi^*} & \rho_a d_{ya} & \rho_b d_{yb} \\ d_{\pi i^*} & \rho_a d_{\pi a} & \rho_b d_{\pi b} \\ d_{i^*i^*} & \rho_a d_{i^*a} & \rho_b d_{i^*b} \end{bmatrix} \begin{bmatrix} \hat{i}_{t-1}^* \\ z_{t-1}^a \\ z_{t-1}^b \end{bmatrix} + \begin{bmatrix} d_{yi} & d_{ya} & d_{yb} \\ d_{\pi i} & d_{\pi a} & d_{\pi b} \\ d_{i^*i} & d_{i^*a} & d_{i^*b} \end{bmatrix} \begin{bmatrix} \epsilon_t^i \\ \epsilon_t^a \\ \epsilon_t^b \end{bmatrix} \\ &= \mathbf{C}\mathbf{x}_{t-1} + \mathbf{D}\epsilon_t.\end{aligned}\tag{A.43}$$

The law of motion for $\mathbf{x}_t \equiv [\hat{i}_t^*, z_t^a, z_t^b]'$ is:

$$\begin{aligned}\mathbf{x}_t &= \begin{bmatrix} d_{i^*i^*} & \rho_a d_{i^*a} & \rho_b d_{i^*b} \\ 0 & \rho_a & 0 \\ 0 & 0 & \rho_b \end{bmatrix} \mathbf{x}_{t-1} + \begin{bmatrix} d_{i^*i} & d_{i^*a} & d_{i^*b} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \epsilon_t \\ &= \mathbf{A}\mathbf{x}_{t-1} + \mathbf{B}\epsilon_t.\end{aligned}\tag{A.44}$$

Solving equation (A.43) for ϵ_t , and substituting the outcome in equation (A.44) yields:

$$\mathbf{x}_t = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})\mathbf{x}_{t-1} + \mathbf{B}\mathbf{D}^{-1}\mathbf{y}_t.$$

If $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} = \mathbf{0}$, the vector of endogenous variables, \mathbf{y}_t , has a VAR(1) representation:

$$\mathbf{y}_t = \mathbf{C}\mathbf{B}\mathbf{D}^{-1}\mathbf{y}_{t-1} + \mathbf{D}\epsilon_t.$$

The rest of the proof shows $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} = \mathbf{0}$. Substituting the matrices \mathbf{A} and \mathbf{B} in equation (A.44) into this condition yields:

$$\mathbf{D}^{-1}\mathbf{C} = \begin{bmatrix} d_{i^*i^*}/d_{i^*i} & 0 & 0 \\ 0 & \rho_a & 0 \\ 0 & 0 & \rho_b \end{bmatrix}.$$

Further substituting the matrices \mathbf{C} and \mathbf{D} in equation (A.43) into this condition leads to: $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} = \mathbf{0}$ if and only if $d_{yi^*} = d_{yi} (d_{i^*i^*}/d_{i^*i})$ and $d_{\pi i^*} = d_{\pi i} (d_{i^*i^*}/d_{i^*i})$. Substituting the decision rules into equation (12) yields:

$$\hat{y}_t = \left(d_{yi^*} - \frac{1}{\sigma} + \frac{d_{\pi i}}{\sigma} \right) d_{i^*i^*} \hat{i}_{t-1}^* + \left(d_{yi^*} - \frac{1}{\sigma} + \frac{d_{\pi i}}{\sigma} \right) d_{i^*i} \epsilon_t^i + \dots,$$

where terms related to z_t^a and z_t^b are omitted. Matching coefficients on \hat{i}_{t-1}^* and ϵ_t^i of both sides of the equation yields:

$$\begin{aligned} d_{yi^*} &= \left(d_{yi^*} - \frac{1}{\sigma} + \frac{d_{\pi i}}{\sigma} \right) d_{i^*i^*}, \\ d_{yi} &= \left(d_{yi^*} - \frac{1}{\sigma} + \frac{d_{\pi i}}{\sigma} \right) d_{i^*i}. \end{aligned}$$

These two equations imply $d_{yi^*} = d_{yi} (d_{i^*i^*}/d_{i^*i})$. Next, substituting the decision rules into equation (16) yields:

$$\hat{\pi}_t = (\delta d_{\pi i^*} + \kappa d_{yi^*}) \hat{i}_{t-1}^* + (\delta d_{\pi i} d_{i^*i} + \kappa d_{yi}) \epsilon_t^i + \dots,$$

where terms related to z_t^a and z_t^b are omitted. Matching coefficients on \hat{i}_{t-1}^* and ϵ_t^i of both sides of the equation yields:

$$\begin{aligned} d_{\pi i^*} &= \delta d_{\pi i^*} + \kappa d_{yi} \left(\frac{d_{i^*i^*}}{d_{i^*i}} \right), \\ d_{\pi i} &= \delta d_{\pi i} d_{i^*i} + \kappa d_{yi}, \end{aligned}$$

where $d_{yi^*} = d_{yi} (d_{i^*i^*}/d_{i^*i})$ is used in the first equation. Solving these two equations for $d_{\pi i^*}$ yields $d_{\pi i^*} = d_{\pi i} (d_{i^*i^*}/d_{i^*i})$.

Part (ii) Again, without loss of generality, consider the case of $\lambda^* = 1$ and $\alpha = 0$. Under Assumption 1 and the irrelevance hypothesis, the long-term yield can be written as (15) with $\lambda^* = 1$ as

$$\hat{R}_L = \frac{\bar{R}_L - \mu}{\bar{R}_L} \hat{i}_t^* + \frac{\mu}{\bar{R}_L} E_t \hat{R}_{L,t+1}.$$

Solving this equation forward yields

$$\hat{R}_{L,t} = \left(\frac{\bar{R}_L - \mu}{\bar{R}_L} \right) E_t \left[\hat{i}_t^* + \frac{\mu}{\bar{R}_L} \hat{i}_{t+1}^* + \left(\frac{\mu}{\bar{R}_L} \right)^2 \hat{i}_{t+2}^* + \dots \right].$$

Because the right-hand-side of the equation depends on information in period t , which consist of \hat{i}_t^* , z_t^a , and z_t^b , the long-term interest rate can be written as:

$$\hat{R}_{L,t} = f_{i^*} \hat{i}_t^* + f_a z_t^a + f_b z_t^b,$$

where f_{i^*} , f_a , and f_b are coefficients derived by using equation (A.42) as

$$\begin{aligned} f_{i^*} &= \frac{\bar{R}_L - \mu}{\bar{R}_L - d_{i^*i^*}\mu}, \\ f_a &= \frac{(\bar{R}_L - \mu)d_{i^*a}\rho_a\mu}{(\bar{R}_L - \rho_a\mu)(\bar{R}_L - d_{i^*i^*}\mu)}, \\ f_b &= \frac{(\bar{R}_L - \mu)d_{i^*b}\rho_b\mu}{(\bar{R}_L - \rho_b\mu)(\bar{R}_L - d_{i^*i^*}\mu)}. \end{aligned}$$

Again by using equation (A.42) the equation for the long-term yield can be written as:

$$\hat{R}_{L,t} = f_{i^*}d_{i^*i^*}\hat{i}_{t-1}^* + f_{i^*}d_{i^*i^*}\hat{\epsilon}_t^i + (f_{i^*}d_{i^*a} + f_a)z_t^a + (f_{i^*}d_{i^*b} + f_b)z_t^b.$$

Define $\mathbf{y}_t \equiv [\hat{y}_t, \hat{\pi}_t, \hat{R}_{L,t}]'$, $\mathbf{x}_t \equiv [\hat{i}_{t-1}^*, z_t^a, z_t^b]'$, and $\epsilon_t = [\epsilon_t^i, \epsilon_t^a, \epsilon_t^b]'$. Then, the state space representation for \mathbf{y} is

$$\begin{aligned} \mathbf{y}_t &= \begin{bmatrix} d_{yi^*} & \rho_a d_{ya} & \rho_b d_{yb} \\ d_{\pi i^*} & \rho_a d_{\pi a} & \rho_b d_{\pi b} \\ f_{i^*} d_{i^*i^*} & \rho_a (f_{i^*} d_{i^*a} + f_a) & \rho_b (f_{i^*} d_{i^*b} + f_b) \end{bmatrix} \mathbf{x}_t + \begin{bmatrix} d_{yi} & d_{ya} & d_{yb} \\ d_{\pi i} & d_{\pi a} & d_{\pi b} \\ f_{i^*} d_{i^*i} & f_{i^*} d_{i^*a} + f_a & f_{i^*} d_{i^*b} + f_b \end{bmatrix} \epsilon_t \\ &= \mathbf{C}\mathbf{x}_{t-1} + \mathbf{D}\epsilon_t. \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}_t &= \begin{bmatrix} d_{i^*i^*} & \rho_a d_{i^*a} & \rho_b d_{i^*b} \\ 0 & \rho_a & 0 \\ 0 & 0 & \rho_b \end{bmatrix} \mathbf{x}_{t-1} + \begin{bmatrix} d_{i^*i} & d_{i^*a} & d_{i^*b} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \epsilon_t \\ &= \mathbf{A}\mathbf{x}_{t-1} + \mathbf{B}\epsilon_t. \end{aligned}$$

Similar to the part (i) in Proposition 1, a solution for \mathbf{y}_t has a VAR(1) representation if and only if $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} = \mathbf{0}$. This condition holds if and only if $d_{yi^*} = d_{yi}(d_{i^*i^*}/d_{i^*i})$ and $d_{\pi i^*} = d_{\pi i}(d_{i^*i^*}/d_{i^*i})$. The latter two conditions hold as shown in Part (i).

A.5 Proof of Proposition 2

We show that equations (1), (2), (3), (12), and (16) can be written in the empirical structural form of equations (17a), (17b), and (17c). This will prove the proposition since the structural form has a piecewise linear representation, as explained in the main text. It is straightforward to see that equations (1), (2), and (3) in the theoretical model can be written in the form of equations (17a) and (17b) in the empirical model. Below, we are going to show that equations (12) and (16) can be represented by the structural form equation (17c).

Without loss of generality, consider a case in which agents forming expectations assuming: $\lambda^* = 1$ and $\alpha = 0$. When forming expectations about variables in period $t + 1$, the initial condition is given by $\tilde{\mathbf{x}}_t \equiv [(1 - \lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^*, z_t^a, z_t^b]'$. Under Assumption 2, the decision rule used for forming expectations about period $t + 1$ variables is $\mathbf{y}_{t+1} = \mathbf{C}\tilde{\mathbf{x}}_t + \mathbf{D}\epsilon_{t+1}$, where \mathbf{C} and \mathbf{D} are those defined in equation (A.43). From period $t + s$ onward, for $s = 2, 3, \dots$, time $t + s$ variables are expected *in period t* to follow $\mathbf{y}_{t+s} = \mathbf{C}\mathbf{x}_{t+s-1} + \mathbf{D}\epsilon_{t+s}$,

where $\mathbf{x}_t \equiv [\hat{i}_t^*, z_t^a, z_t^b]'$. But, once the time proceeds and becomes period $t + 1$, the initial condition is updated to $\tilde{\mathbf{x}}_{t+1}$ and this is used for forming expectations about $t + 2$ variables as $E_{t+1}\mathbf{y}_{t+2} = \mathbf{C}\tilde{\mathbf{x}}_{t+1}$. Hence, under the assumption about expectations, the decision rule is given by $\mathbf{y}_{t+s} = \mathbf{C}\tilde{\mathbf{x}}_{t+s-1} + \mathbf{D}\epsilon_{t+s}$ for $s = 1, 2, \dots$. In this system, in every period information is updated and $\tilde{\mathbf{x}}_{t+s-1}$ is used as an initial condition. The interest rate \hat{i}_{t+s-1} in the initial condition is treated as if it were an exogenous variable.

By substituting the decision rule into the expected variables, equations (12) and (16) can be written as:

$$\hat{y}_t = \left(-\frac{1}{\sigma} + d_{yi^*} + \frac{d_{\pi i^*}}{\sigma} \right) \left((1 - \lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^* \right) + \left(\rho_a d_{ya} + \frac{\rho_a d_{\pi a}}{\sigma} \right) z_t^a + \left(\rho_b d_{yb} + \frac{\rho_b d_{\pi b}}{\sigma} - \chi_z \right) z_t^b, \quad (\text{A.45})$$

$$-\kappa y_t + \hat{\pi}_t = \delta d_{\pi i^*} \left((1 - \lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^* \right) + (\delta \rho_a d_{\pi a} - \chi_a) z_t^a + \delta \rho_b d_{\pi b} z_t^b. \quad (\text{A.46})$$

Since z_t^a and z_t^b follow AR(1) processes, equations (A.45) and (A.46) can be written in a matrix form as:

$$\mathbf{H}_1 \begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} = \mathbf{H}_2 \left((1 - \lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^* \right) + \mathbf{H}_3 \begin{bmatrix} z_{t-1}^a \\ z_{t-1}^b \end{bmatrix} + \mathbf{H}_4 \begin{bmatrix} \epsilon_t^a \\ \epsilon_t^b \end{bmatrix}.$$

or

$$\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} = \mathbf{H}_1^{-1} \mathbf{H}_2 \left((1 - \lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^* \right) + \mathbf{H}_1^{-1} \mathbf{H}_3 \begin{bmatrix} z_{t-1}^a \\ z_{t-1}^b \end{bmatrix} + \mathbf{H}_1^{-1} \mathbf{H}_4 \begin{bmatrix} \epsilon_t^a \\ \epsilon_t^b \end{bmatrix}. \quad (\text{A.47})$$

Also, under Assumption 2, the expected values can be written as: $E_t \tilde{\mathbf{y}}_{t+1} = \mathbf{G} \tilde{\mathbf{y}}_t$, where $\tilde{\mathbf{y}}_t \equiv [\hat{y}_t, \hat{\pi}_t, (1 - \lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^*]'$ and $\mathbf{G} \equiv \mathbf{CBD}^{-1}$, as derived in the proof of Proposition 1. By using this equation, equations (12) and (16) can be written as:

$$\begin{aligned} \chi_z z_t^b &= \left(g_{yy} + \frac{g_{\pi y}}{\sigma} - 1 \right) \hat{y}_t + \left(g_{y\pi} + \frac{g_{\pi\pi}}{\sigma} \right) \hat{\pi}_t + \left(g_{yi^*} + \frac{g_{\pi i^*}}{\sigma} - \frac{1}{\sigma} \right) \left((1 - \lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^* \right), \\ \chi_a z_t^a &= (\delta g_{\pi y} + \kappa) \hat{y}_t + (\delta g_{\pi\pi} - 1) \hat{\pi}_t + \delta g_{\pi i^*} \left((1 - \lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^* \right), \end{aligned}$$

where g_{ij} 's correspond to elements in the matrix \mathbf{G} . Then, the lagged shocks z_{t-1}^b and z_{t-1}^a in equation (A.47) can be represented by a function of $\tilde{\mathbf{y}}_{t-1} \equiv [\hat{y}_{t-1}, \hat{\pi}_{t-1}, (1 - \lambda^*)\hat{i}_{t-1} + \lambda^*\hat{i}_{t-1}^*]'$. From this result, equation (A.47) is in the same form of equation (17c) in the structural form.

A.6 Impulse responses to demand and supply shocks

We study impulse responses to a demand shock and a supply shock, respectively, in an ELB regime, using the theoretical model presented in Section I of the main text. We show that the responses differ significantly depending on the effectiveness of UMP.

Figure 1 plots impulse responses of output and inflation in the theoretical model to the contractionary demand shock of $\epsilon_t^b = 0.25/400$ under the ELB. The responses are calculated exactly in the same way as those to a monetary policy shock, shown in Figure 2 in the main text. In the case of no UMP ($\xi = 0$), a negative demand shock causes the largest declines in output and inflation. As the effectiveness of UMP increases, i.e., as ξ increases, the negative responses of output and inflation become smaller.

Figure 1: Impulse responses to a demand shock at the ELB

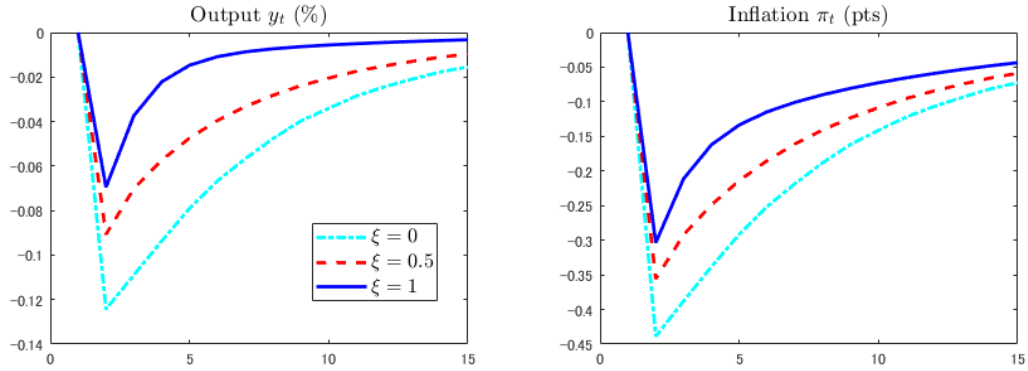


Figure 2: Impulse responses to a supply shock at the ELB

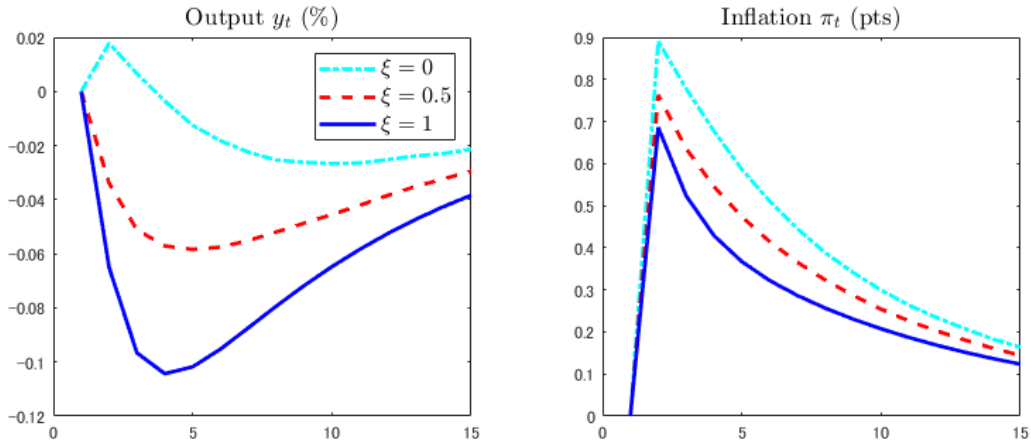


Figure 2 plots impulse responses of output and inflation to the negative supply shock of $\epsilon_t^a = -0.25/100$ under the ELB. In the case of fully effective UMP ($\xi = 1$), output decreases and inflation increases in response to the negative supply shock, as in the responses in a non-ELB regime. However, as the effectiveness of UMP decreases, the degree of a decrease in output shrinks, and output even increases on impact in response to the negative supply shock in the case of no UMP ($\xi = 0$). This is driven by a stronger increase in inflation under the ELB. Such an increase in inflation mitigates the negative impact of the ELB on output. This effect dominates the direct effect of the negative supply shock, resulting in an increase in output on impact.

While we exclusively focus on monetary policy shocks in this paper, the same co-movement of the variables in response to supply and demand shocks when the economy approaches the ELB would pose a challenge for identifying responses to demand and supply shocks.

B Derivation of the attenuation effect (24)

Start from the definition of the IRF to the monetary policy shock $\bar{\varepsilon}_{2t} := A_{22}^{*-1}\varepsilon_{2t}$.⁴ This is a function of the shock magnitude ς and horizon h :

$$IRF_{h,t}(\varsigma) = E(Y_{1t+h}|\bar{\varepsilon}_{2t} = \varsigma, x_t) - E(Y_{1t+h}|\bar{\varepsilon}_{2t} = 0, x_t), \quad (\text{B.1})$$

where $x_t = (x'_{1t}, x_{2t})'$, $x_{it} := C_i X_t + C_i^* X_t^*$ embodies all the relevant history of Y_t up to period $t-1$.⁵ We only need to discuss the impact effects, so we set $h = 0$ in (B.1) and write $IRF_{0,t}(\varsigma) = g(\varsigma; x_t) - g(0; x_t)$, where

$$g(\varsigma; x_t) := E(Y_{1t}|\bar{\varepsilon}_{2t} = \varsigma, x_t). \quad (\text{B.2})$$

Note that, despite the kink in the model, the fact that we are taking expectations with respect to the remaining shocks $\bar{\varepsilon}_{1t} := A_{11}^{-1}\varepsilon_{1t}$ implies that the function g is smooth in both arguments. So, we can consider infinitesimal interventions by computing $\lim_{\varsigma \rightarrow 0} \frac{g(\varsigma; x_t) - g(0; x_t)}{\varsigma} =: \frac{\partial g(\varsigma; x_t)}{\partial \varsigma}$.

Let $D_t = 1_{\{i_t^* < \underline{i}_t\}}$ denote the indicator that the interest rate is at the ELB. Then, using equation (19) in the main text, we obtain:

$$\begin{aligned} g(\varsigma, x_t) &= C_1 X_t + C_{12}^* X_{2t}^* + E(u_{1t}|\bar{\varepsilon}_{2t} = \varsigma, x_t) \\ &\quad - \tilde{\beta} E(D_t(C_2 X_t + C_{22}^* X_{2t}^* - \underline{i}_t + u_{2t})|\bar{\varepsilon}_{2t} = \varsigma, x_t) \\ &= x_{1t} + (I_{k-1} - \beta\gamma)^{-1} \beta\varsigma - \tilde{\beta} E(D_t|\bar{\varepsilon}_{2t} = \varsigma, x_t) \left(x_{2t} - \underline{i}_t + \frac{\varsigma}{1 - \gamma\beta} \right) \\ &\quad - \tilde{\beta} E\left(\frac{\gamma\bar{\varepsilon}_{1t}}{1 - \gamma\beta} D_t \middle| \bar{\varepsilon}_{2t} = \varsigma, x_t \right) \\ &= x_{1t} + (I_{k-1} - \beta\gamma)^{-1} \beta\varsigma - \tilde{\beta}\Phi\left(\frac{\underline{i}_t - x_{2t} - \frac{\varsigma}{1 - \gamma\beta}}{\varpi} \right) \left(x_{2t} - \underline{i}_t + \frac{\varsigma}{1 - \gamma\beta} \right) \\ &\quad + \tilde{\beta}\varpi\phi\left(\frac{\underline{i}_t - x_{2t} - \frac{\varsigma}{1 - \gamma\beta}}{\varpi} \right). \end{aligned} \quad (\text{B.3})$$

The second equality in equation (B.3) follows from the definitions of x_{it} and the fact that:

$$u_{1t} = (I_{k-1} - \beta\gamma)^{-1} (\bar{\varepsilon}_{1t} + \beta\bar{\varepsilon}_{2t}), \quad \text{and} \quad u_{2t} := \frac{\gamma\bar{\varepsilon}_{1t} + \bar{\varepsilon}_{2t}}{1 - \gamma\beta},$$

see (Mavroeidis, 2021, equations 32 and 33), and the third equality in equation (B.3) follows from:

$$\begin{aligned} E(D_t|\bar{\varepsilon}_{2t} = \varsigma, x_t) &= \Pr(u_{2t} < \underline{i}_t - x_{2t}|\bar{\varepsilon}_{2t} = \varsigma, x_t) \\ &= \Pr\left(\frac{\gamma\bar{\varepsilon}_{1t}}{1 - \gamma\beta} < \underline{i}_t - x_{2t} - \frac{\varsigma}{1 - \gamma\beta} \right) \\ &= \Phi\left(\frac{\underline{i}_t - x_{2t} - \frac{\varsigma}{1 - \gamma\beta}}{\varpi} \right), \quad \varpi^2 := \text{var}\left(\frac{\gamma\bar{\varepsilon}_{1t}}{1 - \gamma\beta} \right) \end{aligned}$$

⁴Note that the derivation of (24) remains the same if we worked with perturbations to ε_{2t} instead of $\bar{\varepsilon}_{2t}$, but we choose the latter to avoid carrying A_{22}^* around in the derivation.

⁵ x_t is the sufficient statistic for the entire history of Y_t in the conditional expectations, i.e., $E(Y_{t+h}|\bar{\varepsilon}_{2t}, Y_{t-1}, Y_{t-2}, \dots) = E(Y_{t+h}|\bar{\varepsilon}_{2t}, x_t)$.

and

$$\begin{aligned}
E\left(\frac{\gamma\bar{\varepsilon}_{1t}}{1-\gamma\beta}D_t\middle|\bar{\varepsilon}_{2t}=\varsigma,x_t\right) &= E\left(\frac{\gamma\bar{\varepsilon}_{1t}}{1-\gamma\beta}\middle|\frac{\gamma\bar{\varepsilon}_{1t}}{1-\gamma\beta}<\dot{i}_t-x_{2t}-\frac{\varsigma}{1-\gamma\beta},\bar{\varepsilon}_{2t}=\varsigma,x_t\right) \\
&\times \Pr\left(\frac{\gamma\bar{\varepsilon}_{1t}}{1-\gamma\beta}<\dot{i}_t-x_{2t}-\frac{\varsigma}{1-\gamma\beta}\middle|\bar{\varepsilon}_{2t}=\varsigma,x_t\right) \\
&= -\varpi\phi\left(\frac{\dot{i}_t-x_{2t}-\frac{\varsigma}{1-\gamma\beta}}{\varpi}\right),
\end{aligned}$$

where the second equality in the last expression follows from the independence of $\bar{\varepsilon}_{1t}$ from $\bar{\varepsilon}_{2t}$ and x_t , and the properties of the truncated standard normal distribution, i.e., $E(z|z < a) = -\phi(a)/\Phi(a)$.

Differentiating equation (B.3) with respect to ς yields:

$$\begin{aligned}
\frac{\partial g(\varsigma,x_t)}{\partial \varsigma} &= (I_{k-1}-\beta\gamma)^{-1}\beta - \frac{1}{1-\gamma\beta}\tilde{\beta}\Phi\left(\frac{\dot{i}_t-x_{2t}-\frac{\varsigma}{1-\gamma\beta}}{\varpi}\right) \\
&+ \tilde{\beta}\frac{1}{1-\gamma\beta}\phi\left(\frac{\dot{i}_t-x_{2t}-\frac{\varsigma}{1-\gamma\beta}}{\varpi}\right)\left(\frac{x_{2t}-\dot{i}_t+\frac{\varsigma}{1-\gamma\beta}}{\varpi}\right) \\
&+ \tilde{\beta}\frac{1}{1-\gamma\beta}\left(\frac{\dot{i}_t-x_{2t}-\frac{\varsigma}{1-\gamma\beta}}{\varpi}\right)\phi\left(\frac{\dot{i}_t-x_{2t}-\frac{\varsigma}{1-\gamma\beta}}{\varpi}\right) \\
&= (I_{k-1}-\beta\gamma)^{-1}\beta - \frac{1}{1-\gamma\beta}\tilde{\beta}\Phi\left(\frac{\dot{i}_t-x_{2t}-\frac{\varsigma}{1-\gamma\beta}}{\varpi}\right),
\end{aligned}$$

where the first equality follows from the fact that $\partial\phi(z)/\partial z = -z\phi(z)$. Evaluating the above expression at $\varsigma = 0$ yields the impact effect of a small monetary policy shock on Y_{1t} in period t , which is a $k-1$ vector, namely,

$$IR_t := \left.\frac{\partial g(\varsigma,x_t)}{\partial \varsigma}\right|_{\varsigma=0} = (I_{k-1}-\beta\gamma)^{-1}\beta - \frac{1}{1-\gamma\beta}\tilde{\beta}\Phi\left(\frac{\dot{i}_t-x_{2t}}{\varpi}\right).$$

If there is no attenuation effect, the impact effect of the monetary policy shock $\bar{\varepsilon}_{2t}$ on Y_{1t} is common across regimes and is given by:

$$IR_{NA} = (I_{k-1}-\beta\gamma)^{-1}\beta = \frac{\beta}{1-\gamma\beta},$$

where the second equality follows from the fact that $(I_{k-1}-\beta\gamma)\beta = \beta(1-\gamma\beta)$. Therefore,

$$IR_t = IR_{NA} - \frac{1}{1-\gamma\beta}\tilde{\beta}\Phi\left(\frac{\dot{i}_t-x_{2t}}{\varpi}\right).$$

The j th element of the $k-1$ vector IR_t above can be written as

$$\begin{aligned}
IR_{j,t} &= IR_{j,NA} - \frac{1}{1-\gamma\beta}\tilde{\beta}_j\Phi\left(\frac{\dot{i}_t-x_{2t}}{\varpi}\right) \\
&= \frac{\beta_j}{1-\gamma\beta} - \frac{\beta_j}{1-\gamma\beta}\frac{\tilde{\beta}_j}{\beta_j}\Phi\left(\frac{\dot{i}_t-x_{2t}}{\varpi}\right) \\
&= \left(1 - \frac{\tilde{\beta}_j}{\beta_j}\Phi\left(\frac{\dot{i}_t-x_{2t}}{\varpi}\right)\right)IR_{j,NA}.
\end{aligned}$$

Renaming $x_{2t} = C_2 X_t + C_{22}^* X_{2t}^*$, the one-step ahead forecast of the reduced-form shadow rate, as $i_{t|t-1}^*$ yields (24) as required.

C Data description

We construct our quarterly data by taking averages of monthly series. For the U.S., the inflation rate is computed from the implicit price deflator (GDPDEF) as $\pi_t = 400 \times \log(P_t/P_{t-1})$, where P_t is the GDP deflator. The output gap is calculated as $100\% \times (GDPC1 - GDPPOT)/GDPPOT$, where GDPC1 is the series for the U.S. real GDP and GDPPOT is the U.S. real potential GDP. The long-term interest rate is from the 10-year Treasury constant maturity rate (GS10). All these series are from the FRED database.⁶ Money growth data for the U.S. are computed from 12 alternative indicators as listed in Table 1 as $m_t = 400 \times \log(M_t/M_{t-1})$, where M_t is the particular money supply considered. All M_t values are quarterly and computed by taking averages of their corresponding monthly values. The traditional monetary aggregates (MB, M1, M2, M2M, MZM), and securities held outright are from the FRED database. The Divisia monetary aggregates (DIVM1, DIVM2, DIVM2M, DIVMZM, DIVM4) are from the Center for Financial Stability Divisia database.

For Japan, the quarterly call rate, bond yields, and the core CPI are computed as the averages of their monthly counterparts. The quarterly inflation rate is computed from the core CPI (consumption tax changes adjusted) as $\pi_t = 400 \times (CPI_t - CPI_{t-1})/CPI_{t-1}$. The GDP gap is that published by the Bank of Japan. The trend growth is defined by the annualised growth rate of potential GDP from the previous quarter, which comes from the estimates of the Cabinet Office. The interest on reserves (IOR) is constructed from the interest rate that the Bank of Japan applies to the Complementary Deposit Facility (Bank of Japan, 2008, 2016).⁷

⁶The data can be retrieved from the following websites: GDP deflator (U.S. Bureau of Economic Analysis, 2019a) <https://fred.stlouisfed.org/series/GDPDEF>; and series to construct the output gap (U.S. Bureau of Economic Analysis, 2019b; U.S. Congressional Budget Office, 2019): <https://fred.stlouisfed.org/series/GDPC1> and <https://fred.stlouisfed.org/series/GDPPOT>; the Federal Funds Rate (Board of Governors of the Federal Reserve System (US), 2019b) <https://fred.stlouisfed.org/series/FEDFUNDS>; and the long yield (Board of Governors of the Federal Reserve System (US), 2019f,g,h,i,j,k) <https://fred.stlouisfed.org/series/GS10>. The data for the different monetary aggregates (Board of Governors of the Federal Reserve System (US), 2019a,c,d,e,l,m; The Center for Financial Stability, 2019) is available at: <https://fred.stlouisfed.org/categories/24> and http://www.centerforfinancialstability.org/amfm_data.php.

⁷The data can be retrieved from the following websites: call rate (Bank of Japan, 2019a): http://www.stat-search.boj.or.jp/index_en.html; 9-year and 10-year government bond yields (Ministry of Finance of Japan, 2019): https://www.mof.go.jp/jgbs/reference/interest_rate/data/jgbcm_all.csv; GDP gap Bank of Japan (2019b): https://www.boj.or.jp/en/research/research_data/index.htm/; core CPI inflation (Statistics Bureau of Japan, 2019): <https://www.e-stat.go.jp/stat-search/file-download?statInfId=000031431696&fileKind=1>; trend growth rate (Cabinet Office of Japan, 2019): <https://www5.cao.go.jp/keizai3/getsurei-e/index-e.html>.

Table 1: Monetary Aggregates Data used in the Model

Monetary Aggregate (M_t)	Mnemonics in the Corresponding Database	Available Sample Periods
Monetary Base (MB)	BOGMBASE	1948Q1-2019Q1
M1	M1SL	1959Q2-2019Q1
M2	M2SL	1959Q2-2019Q1
M2M	M2MSL	1959Q2-2019Q1
MZM	MZMSL	1959Q2-2019Q1
Securities Held Outright	WSECOUT	1989Q3-2019Q1
Divisia M1 (DIVM1)	Divisia M1	1967Q2-2019Q1
Divisia M2 (DIVM2)	Divisia M2	1967Q2-2019Q1
Divisia M2M (DIVM2M)	Divisia M2M	1967Q2-2019Q1
Divisia MZM (DIVMZM)	Divisia MZM	1967Q2-2019Q1
Divisia M4 (DIVM4)	DM4	1967Q2-2019Q1

D Additional empirical results

Weaker version of IH_1 . Table 2 shows the results of the weaker version of IH_1 , i.e., $C_{12} = C_{12}^* = \tilde{\beta} = 0$ for inflation and output equations only. As in the baseline CKSVAR specification reported in Table 1 in the main text, 4 lags are selected for the U.S. and 2 lags are selected for Japan. The p -values reported in Table 2 show that the weaker version of IH_1 is firmly rejected for both countries.

Adding alternative measures of monetary policy. Table 3 shows the results of tests for exclusion of the Federal Funds Rate from a SVAR that includes inflation, the output gap, the 10-year bond yield, and various alternative measures of the growth of monetary aggregates outlined in column (1). Column (3) shows the order of the VAR selected by the AIC, which varies between 3 and 4 lags, consistent with the benchmark model in Table 1 in the main text. Columns (4) and (6) report the likelihood ratio test statistics for the joint exclusion hypothesis and the corresponding asymptotic p -values, respectively. These results show that the data strongly and consistently reject the joint exclusion restrictions on the Federal Funds Rate across all the alternative specifications for all measures of money supply, which corroborates the findings in the baseline 4-equation model in Table 1 in the main text.

Table 2: Test for excluding short rates from VAR that includes long rates

Panel A: KSVAR												
p	United States						Japan					
	loglik	pv-p	AIC	LR	df	<i>p</i> -val	loglik	pv-p	AIC	LR	df	<i>p</i> -val
5	-213.4	-	2.62	36.08	12	0.000	248.1	-	-2.18	12.74	12	0.389
4	-221.5	0.446	2.55	33.42	10	0.000	239.9	0.425	-2.30	14.13	10	0.167
3	-234.4	0.112	2.53	27.12	8	0.001	232.2	0.471	-2.42	14.89	8	0.061
2	-266.0	0.000	2.66	28.29	6	0.000	223.8	0.445	-2.53	15.70	6	0.015
1	-296.7	0.000	2.78	24.62	4	0.000	184.8	0.000	-2.19	25.15	4	0.000

Panel B: CKSVAR												
p	loglik	pv-p	AIC	LR	df	<i>p</i> -val	loglik	pv-p	AIC	LR	df	<i>p</i> -val
5	-191.3	-	2.60	52.08	22	0.000	284.7	-	-2.42	48.79	22	0.001
4	-202.7	0.290	2.53	44.48	18	0.000	277.1	0.766	-2.61	52.42	18	0.000
3	-223.0	0.011	2.53	33.32	14	0.003	258.1	0.081	-2.62	42.61	14	0.000
2	-256.3	0.000	2.64	31.10	10	0.001	242.1	0.018	-2.68	28.23	10	0.002
1	-290.2	0.000	2.76	27.19	6	0.000	204.8	0.000	-2.43	43.99	6	0.000

Note: Panel A reports results for a KSVAR(p) with inflation, output gap, long rate, and policy rate. Panel B reports corresponding results for a CKSVAR(p) that includes shadow rates. The sample period is 1960q1-2019q1 for the U.S. and 1985q3-2019q1 for Japan. Long rates are 10-year government bond yields for the U.S. and 9-year yields for Japan. Under the null hypothesis, the short rate is excluded from the equations for inflation and output only. loglik is the value of the log-likelihood. pv-p is the *p*-value of the test for lag reduction. AIC is the Akaike information criterion. LR is the value of the LR test statistic for excluding short rates from equations for inflation and output gap. df is the number of restrictions. *p*-val is the asymptotic χ^2_{df} *p*-value of the test.

Table 3: Test for excluding short rates from VARs that include long rates and money

Mon. Aggr.	sample	p	LR	df	p -val
MB	1960q1–2019q1	3	55.05	16	0.0000
M1	1960q3–2019q1	3	55.50	16	0.0000
M2	1960q3–2019q1	3	54.77	16	0.0000
M2M	1960q3–2019q1	4	73.78	20	0.0000
MZM	1960q3–2019q1	4	79.65	20	0.0000
DIVM1	1968q3–2019q1	4	80.68	20	0.0000
DIVM2	1968q3–2019q1	4	111.50	20	0.0000
DIVM2M	1968q3–2019q1	4	110.88	20	0.0000
DIVMZM	1968q3–2019q1	4	107.10	20	0.0000
DIVM4	1968q3–2019q1	4	135.38	20	0.0000
SHO	1990q4–2019q1	3	94.38	16	0.0000

Note: The estimated model is a KSVAR(p) for the U.S. with inflation, output gap, the Federal Funds Rate, the 10-year government bond yield, and a different measure of money growth in each row. Sample availability varies for each monetary aggregate used. LR is the value of the LR test statistic for the testing that lags of the Federal Funds Rate can be excluded from all other equations in the model, df is the number of exclusion restrictions, and p -val is the asymptotic χ^2_{df} p -value of the test.

Robustness of test results for the U.S. to the Great Moderation. The test results of the IH over the full sample are subject to a possible misspecification arising from the ‘Great Moderation’, a drop in U.S. macroeconomic volatility in the mid-1980s. Therefore, we assess the robustness of our results by estimating the model and performing the above tests of the IH over the sub-sample which starts in 1984q1. Tables 4 and 5 report the results over this subsample, which correspond to the results reported in Tables 1 and 3 in the main text for the full sample, respectively. The results of the tests of the IH remain the same: the hypothesis is firmly rejected.

Table 4: Test for excluding short rates form VAR that includes long rates post-1984

p	KSVAR(p)						CKSVAR(p)					
	loglik	pv-p	AIC	LR	df	p -val	loglik	pv-p	AIC	LR	df	p -val
5	97.92	-	-0.01	25.63	18	0.11	122.37	-	-0.07	61.65	33	0.002
4	92.83	0.857	-0.17	28.01	15	0.022	119.99	1.000	-0.33	70.78	27	0.000
3	85.07	0.776	-0.28	22.31	12	0.034	103.32	0.556	-0.37	48.44	21	0.001
2	66.06	0.064	-0.24	22.99	9	0.006	77.86	0.009	-0.30	40.06	15	0.000
1	14.30	0.000	0.27	5.33	6	0.502	19.29	0.000	0.25	13.59	9	0.138

Note: The estimated model is a (C)KSVAR(p) for the U.S. with inflation, output gap, Federal Funds Rate, and the 10-year government bond yield. Estimation sample is 1984q1–2019q1. loglik is the value of the log-likelihood. pv-p is the p -value of the test for lag reduction. AIC is the Akaike information criterion. LR is the test statistic for excluding short rates from equations for inflation, output gap and long rates. df is the number of restrictions. p -val is the asymptotic χ^2_{df} p -value of the test.

Table 5: Testing CSVAR against CKSVAR post-1984

Country	p	LR	df	p -val
U.S.	3	31.17	15	0.008

Note: The unrestricted model is a CKSVAR(3) for the U.S. with inflation, output gap, 10-year government bond yields, and the Federal Funds Rate. Sample: 1984q1-2019q1. LR is the test statistics of the restrictions that the model reduces to CSVAR(3). Lag order is chosen by AIC. df is the number of restrictions. p -val is the asymptotic χ^2_{df} p -value of the test.

Robustness of results to the inclusion of credit spreads in the VAR. For the U.S., we use Moody’s seasoned BAA corporate bond yield relative to 10-year treasury yield (Federal Reserve Bank of St. Louis, 2019) as the credit spreads, and the excess bond premium in Gilchrist and Zakrajsek (2012) and Favara et al. (2019) for Japan. The test results show that our baseline results for IH₁ (Tables 6-7) and IH₂ (Table 8) are robust to the inclusion of credit spreads in the VAR.

Table 6: Test for excluding short rates from VAR that includes long rates and credit spreads

Panel A: KSVAR												
	United States						Japan					
p	loglik	pv-p	AIC	LR	df	p -val	loglik	pv-p	AIC	LR	df	p -val
5	298.6	-	-2.34	40.59	24	0.018	391.8	-	-3.52	45.64	24	0.005
4	286.3	0.486	-2.54	42.69	20	0.002	373.9	0.073	-3.63	38.10	20	0.009
3	266.5	0.086	-2.62	37.49	16	0.002	358.7	0.061	-3.77	34.18	16	0.005
2	232.1	0.000	-2.47	18.24	12	0.109	346.3	0.100	-3.96	31.84	12	0.001
1	178.4	0.000	-2.02	10.79	8	0.214	303.4	0.000	-3.69	44.30	8	0.000

Panel B: CKSVAR												
p	loglik	pv-p	AIC	LR	df	p -val	loglik	pv-p	AIC	LR	df	p -val
5	328.6	-	-2.42	87.26	44	0.000	441.6	-	-3.89	131.19	44	0.000
4	309.9	0.167	-2.59	83.99	36	0.000	412.4	0.001	-3.90	100.99	36	0.000
3	288.0	0.035	-2.72	73.60	28	0.000	376.7	0.000	-3.82	62.78	28	0.000
2	243.0	0.000	-2.48	37.62	20	0.010	356.6	0.000	-3.97	47.15	20	0.001
1	183.5	0.000	-2.10	20.41	12	0.060	316.2	0.000	-3.81	62.95	12	0.000

Note: Panel A reports results for a KSVAR(p) with inflation, output gap, long rate, credit spread, and policy rate. Panel B reports corresponding results for a CKSVAR(p) that includes shadow rates. Estimation sample is 1987q2-2019q1 for the U.S. and 1985q3-2019q1 for Japan. Long rates are 10-year government bond yields for the U.S. and 9-year yields for Japan. The credit spreads are Moody’s seasoned BAA corporate bond yield relative to 10-year treasury yield for the U.S., and the excess bond premium introduced by Gilchrist and Zakrajsek (2012) for Japan. loglik is the value of the log-likelihood. pv-p is the p -value of the test for lag reduction. AIC is the Akaike information criterion. LR is the test statistic for excluding short rates from equations for inflation, output gap, credit spread, and long rates. df is the number of restrictions. p -val is the asymptotic χ^2_{df} p -value of the test.

Table 7: Test for excluding short rates from VAR that includes long rates and credit spreads

United States, with Excess Bond Premium												
	Panel A: KSVAR						Panel B: CKSVAR					
p	loglik	pv-p	AIC	LR	df	<i>p</i> -val	loglik	pv-p	AIC	LR	df	<i>p</i> -val
5	60.9	-	0.979	73.83	24	0.000	86.7	-	0.970	113.92	44	0.000
4	47.4	0.359	0.851	70.02	20	0.000	70.4	0.332	0.818	100.20	36	0.000
3	22.6	0.009	0.849	54.65	16	0.000	36.7	0.001	0.859	69.35	28	0.000
2	-13.8	0.000	0.975	33.35	12	0.001	0.4	0.000	0.929	49.91	20	0.000
1	-49.3	0.000	1.092	20.91	8	0.007	-36.6	0.000	1.006	37.12	12	0.000

Note: Panel A reports results for a KSVAR(*p*) with inflation, output gap, long rate, credit spread, and policy rate. Panel B reports corresponding results for a CKSVAR(*p*) that includes shadow rates. Estimation sample is 1974q2-2019q1. Credit spreads are the excess bond premium in [Gilchrist and Zakrajsek \(2012\)](#). loglik is the value of the log-likelihood. pv-p is the *p*-value of the test for lag reduction. AIC is the Akaike information criterion. LR is the test statistic for excluding short rates from equations for inflation, output gap, credit spread, and long rates. df is the number of restrictions. *p*-val is the asymptotic χ^2_{df} *p*-value of the test.

Table 8: Testing CSVAR against CKSVAR with credit spreads

Country	p	LR	df	<i>p</i> -val
U.S.(BAA)	3	49.58	19	0.000
U.S.(EBP)	4	53.56	24	0.000
Japan	2	40.62	14	0.000

Note: The unrestricted model is a CKSVAR(*p*) in inflation, output gap, long rate, credit spread, and policy rate. Long rate: 10-year government bond yield (U.S.), 9-year government bond yield (Japan). Policy rate: Federal Funds Rate (U.S.), call rate (Japan). Credit spread: Moody's seasoned BAA corporate bond yield relative to 10-year treasury yield (U.S.), the excess bond premium (U.S. and Japan). Sample: 1987q2-2019q1 (U.S. with BAA spread), 1974q2-2019q1 (U.S. with EBP), 1985q3-2019q1 (Japan). *p* chosen by AIC. LR is the test statistics of the restrictions that the model reduces to CSVAR(*p*). df is the number of restrictions. *p*-val is the asymptotic χ^2_{df} *p*-value of the test.

Robustness of Japanese results to 10-year rates. Similarly, we test the robustness of our results for the Japanese data by using the 10-year yields instead. This shortens the available sample for estimation to 1987q4 to 2019q1. Tables 9 and 10 report test statistics for the two types of tests for the IH. From Tables 9 and 10, the IH is rejected across all lags. For the CKSVAR alternative, 2 lags are selected based on the AIC. Table 10 also suggests the rejection of the IH.

Table 9: Test for excluding short rates from VAR for Japan using 10-year bond yields

p	KSVAR(p)						CKSVAR(p)					
	loglik	pv-p	AIC	LR	df	p-val	loglik	pv-p	AIC	LR	df	p-val
5	285.1	-	-2.92	37.43	18	0.005	320.5	-	-3.17	99.85	33	0.000
4	275.1	0.217	-3.02	31.62	15	0.007	307.4	0.159	-3.28	86.95	27	0.000
3	270.5	0.605	-3.20	32.05	12	0.001	290.6	0.023	-3.33	62.07	21	0.000
2	256.2	0.155	-3.23	24.60	9	0.003	274.3	0.004	-3.39	50.90	15	0.000
1	196.4	0.000	-2.53	22.84	6	0.001	212.8	0.000	-2.73	38.81	9	0.000

Note: The estimated model is a (C)KSVAR(p) for Japan with inflation, output gap, 10-year government bond yields, and the call rate. Estimation sample is 1987q4-2019q1. loglik is the value of the log-likelihood. pv-p is the p -value of the test for lag reduction. AIC is the Akaike information criterion. LR is the test statistic for excluding short rates from equations for inflation, output gap and long rates. df is the number of restrictions. p -val is the asymptotic χ_{df}^2 p -value of the test.

Table 10: Testing CSVAR against CKSVAR for Japan using 10-year bond yields

Country	p	LR	df	p-val
Japan	2	47.54	11	0.000

Note: The unrestricted model is a CKSVAR(2) for Japan with inflation, output gap, 10-year government bond yields, and the call rate. Estimation sample: 1987q4-2019q1. LR is the test statistics of the restrictions that the model reduces to CSVAR(2). Lag order is chosen by AIC. df is yje number of restrictions. p -val is the asymptotic χ_{df}^2 p -value of the test.

Power of the irrelevance tests IH_1 and IH_2 . We use the theoretical model to generate 100 artificial time series under values for the parameter ξ in the range $[0.7, 0.99]$. Table 11 and 12 report the number of rejections for the tests of our irrelevance hypotheses IH_1 and IH_2 , respectively. If our tests are powerful, we would expect the number of rejections to decline with ξ approaching the value of 1 for which the irrelevance hypothesis holds true in the simulated data.

The tables show that the irrelevance tests are powerful. For instance, in the case of the KSVAR as the unrestricted model, the test rejects IH_1 at a 1 percent significance level with the rejection rate (frequency) of 99 percent when $\xi = 0.7$, while the rejection rate is 1 percent when $\xi = 0.99$ at the same significance level. Similar results hold for alternative significance levels (columns 2, 3), the CKSVAR as the unrestricted model (Table 11, Panel B), and the test for IH_2 (Table 12).

Table 11: Test for excluding short rates from VAR that includes long rates, with simulated data

Panel A: KSVAR			
ξ	(1)	(2)	(3)
	$p \leq 0.01$	$p \leq 0.05$	$p \leq 0.1$
0.7	99	100	100
0.75	90	98	99
0.8	76	92	94
0.85	58	73	81
0.9	25	48	60
0.95	7	19	31
0.99	1	11	18

Panel B: CKSVAR			
ξ	(1)	(2)	(3)
	$p \leq 0.01$	$p \leq 0.05$	$p \leq 0.1$
0.7	99	100	100
0.75	93	99	100
0.8	85	92	94
0.85	69	84	88
0.9	42	62	71
0.95	20	33	45
0.99	13	25	32

Note: Panel A reports results for a KSVAR(1) with inflation, output gap, long rate, and policy rate. Panel B reports corresponding results for a CKSVAR(1) that includes shadow rates. Estimation sample is data simulated by the calibrated DSGE model for 237 quarters, which equals the length of the U.S. sample in section III.B. For each value of ξ , we run 100 simulations. Columns 2-4 report how many times the irrelevant hypothesis is rejected with 1 percent, 5 percent, and 10 percent significance levels, respectively.

Table 12: Test CSVAR against CKSVAR, with simulated data

ξ	(1)	(2)	(3)
	$p \leq 0.01$	$p \leq 0.05$	$p \leq 0.1$
0.7	100	100	100
0.75	100	100	100
0.8	99	100	100
0.85	91	97	97
0.9	58	81	85
0.95	13	20	34
0.99	3	6	14

Note: This table reports results for a CKSVAR(1) with inflation, output gap, long rate, and policy rate. Estimation sample is data simulated by the calibrated DSGE model for 237 quarters, which equals the length of the U.S. sample in section III.B. For each value of ξ , we run 100 simulations. Columns 2-4 report how many times the irrelevant hypothesis is rejected with 1 percent, 5 percent, and 10 percent significance levels, respectively.

Testing no attenuation effect. We repeat our test of no attenuation in the response of long rates to monetary policy shocks for different sample periods for the U.S. Table 13 shows that no attenuation hypothesis is not rejected if the same sample period of 1990q1–2012q4 is adopted as in Swanson and Williams (2014). If the sample period is extended backwards (starting from 1960q1), the null is rejected at a 5 percent significance level. These results suggest that the responses of the long rate to a monetary policy shock may differ between non-ELB and ELB regimes, depending on the sample period.

Table 13: Test for no attenuation, various sample periods

sample	p	LR	p -val
1990q1–2012q4	3	0.03	0.872
1960q1–2012q4	3	4.08	0.043

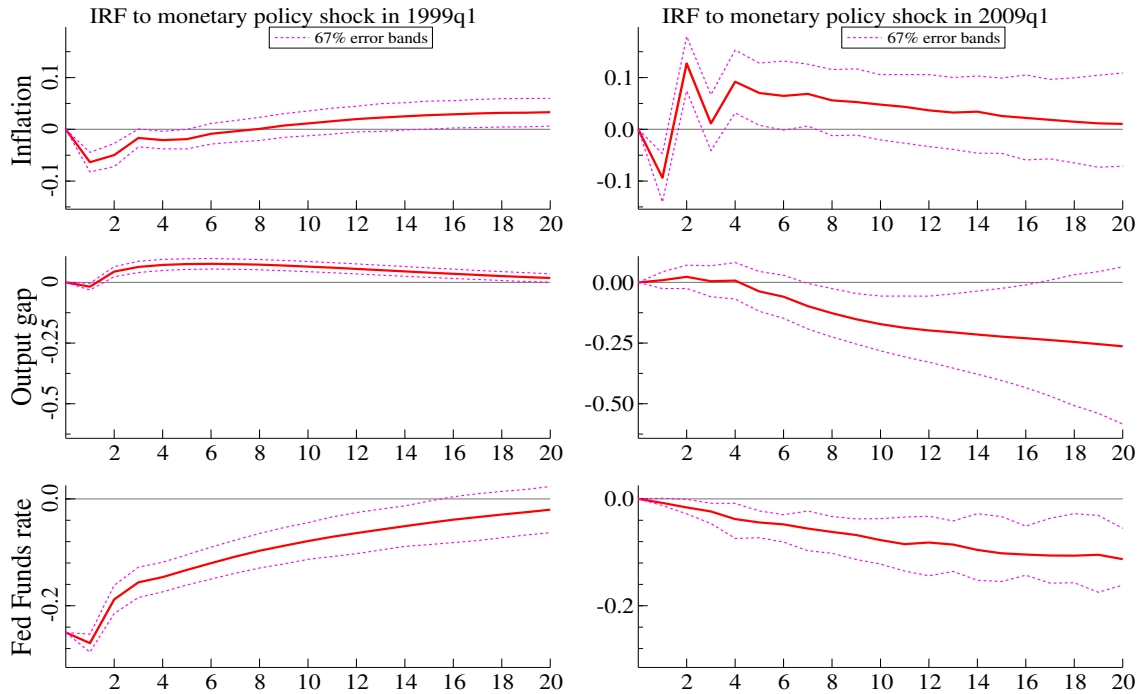
Note: The estimated model is a CKSVAR(p) for the U.S. with inflation, output gap, long-term rate, and policy rate. The long rate is the 10-year government bond yields. The hypothesis is tested with different sample periods, with the first one being consistent with Swanson and Williams. LR is the value of the likelihood ratio test statistic and asymptotic p -values are reported.

E Choleski identification

In our benchmark analysis we use the combination of the ELB identification developed by Mavroidis (2021) and the sign restrictions similar to those employed by Debortoli et al. (2019) to estimate the UMP parameter ξ . This appendix shows the results from using the standard Choleski identification. Figures 3 and 4 reports results for the U.S. and Japan, respectively. They show that the Choleski identification generates several puzzling responses such as the instantaneous decreases in output and inflation in reaction to a negative monetary policy shock. These responses are consistent with the findings in Gertler and Karadi

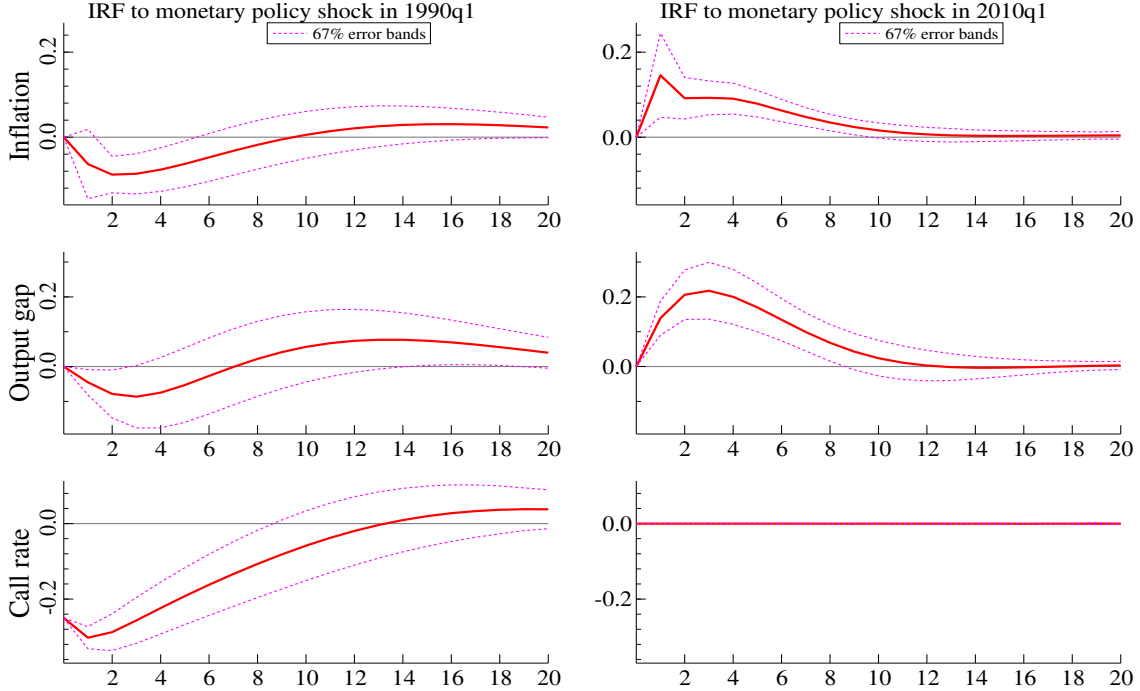
(2015) for the U.S. and Kubota and Shintani (2022) for Japan, who also show similar responses when using the Choleski identification. Thus, our analysis corroborates the results on the empirically-incongruous responses from the Choleski identification, while showing that the identification based on the combination of the ELB identification and sign restrictions provides plausible responses to monetary policy shocks for the U.S. and Japan when the economy is at the ELB. See Gortz et al. (2023) for a discussion of the issue and some additional corroborative evidence on U.K data.

Figure 3: Choleski identification: Impulse responses to a monetary policy shock in the U.S.



Note: Identified sets of IRFs in 1999q1 and 2009q1 to a -25bps monetary policy shock estimated from CKSVAR(3) model in inflation, output gap, and the Federal Funds Rate for the U.S. over the period 1960q1-2019q1, identified by the Choleski restrictions that the monetary policy shock has no contemporaneous effects on inflation and output. Dotted lines show the 67 percent asymptotic error bands.

Figure 4: Choleski identification: Impulse responses to a monetary policy shock in Japan



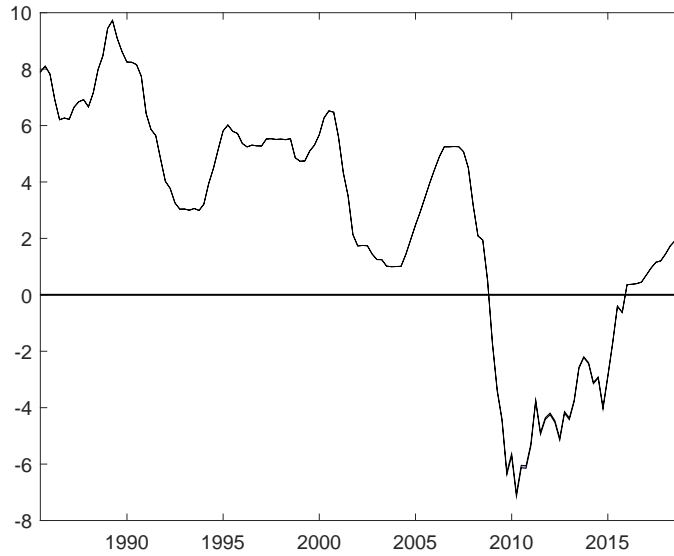
Note: Identified sets of IRFs in 1990q1 and 2010q1 to a -25bps point monetary policy shock estimated using a CKSVAR(2) model in inflation, output gap, and the call rate for Japan over the period 1985q3-2019q1, identified by the Choleski restrictions that the monetary policy shock has no contemporaneous effects on inflation and output. Dotted lines show the 67 percent asymptotic error bands.

F Shadow rates

Our analysis defines the shadow rate as the short-term interest rate that the central bank would set if there were no ELB. Thus defined, the shadow rate can be interpreted as an indicator of the desired monetary policy stance and we provide estimates of it for Japan and the U.S. Our estimates of the shadow rate do not impose the assumption that the model used to obtain them is constant across regimes, and therefore they explicitly account for the empirical relevance of the ELB over the estimation periods.

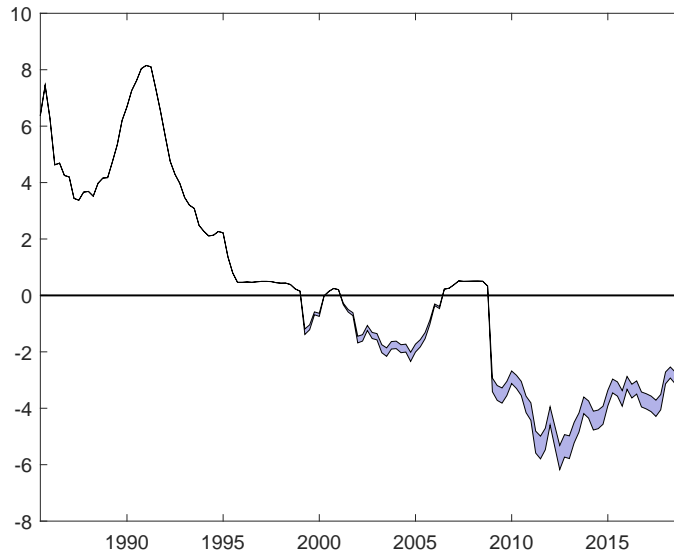
The important caveat is that the shadow rates are not identified under our present assumptions. As explained in Mavroeidis (2021), identifying the shadow rate i_t^* in the empirical model (17a)-(17c) in the main text requires knowledge of the parameter α , which scales the reaction function coefficients and policy shocks during the ELB regimes and is not identified without additional information. This parameter is needed *in addition* to the parameter ξ that measures the overall impact effect of UMP. In other words, to properly identify the shadow rate and interpret it as a measure of desired policy stance, we need to be able to isolate the effect of FG encapsulated by α . This exercise is beyond the scope of the present paper.

Figure 5: Shadow policy rate for the U.S.



Note: Estimated using a CKSVAR(3) model in inflation, output gap, and the Federal Funds Rate for the U.S. over the period 1960q1-2019q1 (plotted over the sub-sample 1985q3-2019q1), identified by the sign restrictions that a -25bp monetary policy shock has nonnegative effects on inflation and output and nonpositive effects on the short rate up to four quarters.

Figure 6: Shadow policy rate for Japan



Note: Estimated using a CKSVAR(2) model in inflation, output gap, and the call rate for Japan over the period 1985q3-2019q1, identified by the sign restrictions that a -25bp monetary policy shock has nonnegative effects on inflation and output and nonpositive effects on the short rate up to four quarters.

With the above caveat in mind, we report identified shadow rates under the assumption of $\alpha = 0$. The shadow rates are given in Figures 5 and 6 for the U.S. and Japan, respectively. Different values of α would scale those estimates by a factor $1 + \alpha$.⁸ Note that, even with $\alpha = 0$, the shadow rate is only partially identified because it also depends on the parameter ξ that is partially identified. This uncertainty due to ξ is reflected in the shaded areas below the ELB in the figures.⁹ In the case of the U.S., the shadow rate dropped sharply soon after the onset of the global financial crisis of 2007-2008. It reached its smallest value at the beginning of 2010 and gradually recovered until the exit from the ELB in 2016. In Japan, the behaviour of the shadow rate is different during the three ELB episodes. During the first episode, the shadow rate fell modestly. In the second episode, it exhibited a persistent decline until the beginning of 2005, followed by a quick reversal. In the third episode, which coincided with the ELB in the U.S., the decline was sharp, and followed by a second wave of declines that lasted until mid-2012. From that point on, the shadow rate exhibited a steady rise, but stayed far from zero even at the end of the sample, and remained near its trough in the second episode.

⁸Results are available on request.

⁹The shadow rate is equal to the observed policy rate above the ELB, see equation (17a) in the main text. Below the ELB, it is given by the equation $Y_{2t}^* = \kappa \bar{Y}_{2t} + (1 - \kappa)b_t$, where $\kappa = (1 + \alpha)(1 - \gamma\beta)/(1 - \xi\gamma\beta)$ and \bar{Y}_{2t} is a “reduced-form” shadow rate that can be filtered from the data using the likelihood, see Mavroeidis (2021).

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