Online Appendix to "Bubbles, Crashes, and Economic Growth: Theory and Evidence"

Pablo A. Guerron-Quintana, Tomohiro Hirano, and Ryo Jinnai*

January 10, 2023

A Household's Problem

A.1 With Loose Financial Constraints

This section solves the household's problem when financial constraints are loose. The financial constraint does not bind. We first show that the price of capital is equal to one in equilibrium. Suppose instead that it is greater than one. Then, the household can increase its utility by increasing i_t by Δ, and increasing both x_t^i and c_t^i by $(q_t - 1)\Delta$ for sufficiently small $\Delta > 0$. This is a contradiction to an equilibrium condition requiring that the household maximize its utility subject to the constraints.

Next, we show that the price of bubbly assets is equal to zero in the equilibrium. Suppose otherwise. Then, the Euler equation for bubbly assets,

$$\tilde{p}_t = E_t \left[\frac{\beta}{e^{d_{t+1}-d_t}} \left(\frac{c_t^i}{c_{t+1}^i} \right) \tilde{p}_{t+1} \mathbf{1}_{\{z_{t+1}=b\}} \right],$$

holds with a positive $\tilde{p}_t$ in some t in the bubbly regime. To simplify the argument, we assume without loss of generality that $a_t = d_t = 0$ holds for all $t \geq 0$. Multiplying both sides by M and dividing them by K_t, we obtain

$$1 = (1 - \sigma_b)\, \beta \left(\frac{\hat{c}_{b,t}^i}{\hat{c}_{b,t+1}^i} \right) \left(\frac{m_{b,t+1}}{m_{b,t}} \right),$$

*Guerron-Quintana: Department of Economics, Boston College, 140 Commonwealth Avenue, Chesnut Hill, Mass 02467 (pguerron@gmail.com). Tomohiro Hirano: Department of Economics, Royal Holloway, University of London, Egham Hill, Egham, TW20 0EX, United Kingdom, and Center for Macroeconomics, London School of Economics and Canon Institute for Global Studies (tomohih@gmail.com). Ryo Jinnai: Institute of Economic Research, Hitotsubashi University, 2-1 Naka, Kunitachi, Tokyo, 186-8603 Japan (rjinnai@ier.hit-u.ac.jp).

where $\hat{c}^i_{b,t}$ is the investor's consumption relative to the capital stock ($\hat{c}^i_t \equiv c^i_t/K_t$) in the bubbly regime, and $m_{b,t}$ is the market value of the bubbly assets relative to the capital stock $\left(m_t \equiv \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M/K_t\right)$ in the bubbly regime. To satisfy this equation, $\left(\frac{m_{b,t}}{\hat{c}^i_{b,t}}\right)$ has to grow exponentially at the rate $(1-\sigma_b)^{-1}\beta^{-1}$. But then, the transversality condition regarding bubbly assets is violated because

$$E_t\left[\beta^j\left(\frac{1}{c^i_{t+j}}\right)\tilde{p}_{t+j}\mathbf{1}_{\{z_{t+j}=b\}}M\right] = (1-\sigma_b)^j\beta^j\left(\frac{m_{b,t+j}}{\hat{c}^i_{b,t+j}}\right)$$

does not converge to zero. This is a contradiction to an equilibrium condition requiring that the household maximize its utility subject to the constraints.

Because $q_t = 1$ and $\tilde{p}_t = 0$ hold if ϕ is large, the household's problem becomes standard. It chooses a sequence of u_t, c^i_t, c^s_t, l_t, and n_{t+1} to maximize utility

$$E_0\left[\sum_{t=0}^{\infty}\frac{\beta^t}{e^{d_t}}\left(\pi\log\left(c^i_t\right)+(1-\pi)\left[\log\left(c^s_t\right)+\eta\left(1-l_t\right)\right]\right)\right]$$

subject to

$$\pi c^i_t + (1-\pi)\,c^s_t + n_{t+1} = \left[u_t r_t + (1-\delta(u_t))\right]n_t + w_t(1-\pi)\,l_t.$$

The first-order conditions are

$$c^i_t = c^s_t,$$

$$\eta\frac{c^s_t}{1-l_t} = w_t,$$

$$r_t - \delta'(u_t) = 0,$$

and

$$1 = E_t\left[\frac{\beta}{e^{d_{t+1}-d_t}}\left(\frac{c^i_t}{c^i_{t+1}}\right)\left(u_{t+1}r_{t+1}+1-\delta(u_{t+1})\right)\right].$$

A.2 With Tight Financial Constraints

This section considers the household's problem when financial constraints are tight. We derive the feasibility constraint for investment,

$$(1-\phi q_t)\,i_t = u_t r_t n_t + \phi q_t\left(1-\delta(u_t)\right)n_t + \mathbf{1}_{\{z_t=b\}}\tilde{p}_t\tilde{m}_t.$$

We first show that $x^i_t = 0$ if $q_t > 1$. Suppose that $x^i_t > 0$ holds even though $q_t > 1$. Then, the household can increase its utility by increasing i_t by Δ, increasing n^i_{t+1} by $(1-\phi)\Delta$, decreasing x^i_t by $(1-\phi q_t)\Delta$, decreasing n^s_{t+1} by $\frac{\pi}{1-\pi}(1-\phi)\Delta$, increasing x^s_t by $\frac{\pi}{1-\pi}(1-\phi)\,q_t\Delta$, and increasing both c^i_t and c^s_t by $\pi(q_t-1)\Delta$ for sufficiently small $\Delta > 0$. This is a contradiction to an equilibrium condition requiring that the household maximize its utility subject to the constraints.

Similarly, suppose that $\mathbf{1}_{\{z_t=b\}}\tilde{p}_t\tilde{m}^i_{t+1} > 0$ holds in some t even though $q_t > 1$. Then, the household can increase its utility by increasing i_t by Δ, increasing n^i_{t+1} by $(1-\phi)\Delta$, decreasing $\tilde{m}^i_{t+1}$ by $\frac{1}{\tilde{p}_t}(1-\phi q_t)\Delta$, decreasing n^s_{t+1} by $\frac{\pi}{1-\pi}(1-\phi)\Delta$, increasing $\tilde{m}^s_{t+1}$ by $\frac{\pi}{1-\pi}\frac{1}{\tilde{p}_t}(1-\phi q_t)\Delta$, and increasing both x^s_t and c^s_t by $\frac{\pi}{1-\pi}(q_t-1)\Delta$ for sufficiently small $\Delta > 0$. This is a contradiction to an equilibrium condition requiring that the household maximize its utility subject to the constraints.

With these observations, we can rewrite the investor's budget constraint

$$x^i_t + i_t + q_t\left(n^i_{t+1} - i_t - (1-\delta(u_t))n_t\right) + \mathbf{1}_{\{z_t=b\}}\tilde{p}_t\left(\tilde{m}^i_{t+1} - \tilde{m}_t\right) = u_t r_t n_t$$

and obtain the feasibility constraint for investment.

B Full Model

This section presents the full model.

B.1 Micro-Foundations of Financial Frictions

We describe the micro-foundations of financial frictions. Because they are related to the household, we describe its problem in detail.

The economy is populated by a continuum of households, with measure one. All households behave identically. Each household has a unit measure of members who are identical at the beginning of each period. During the period, members are separated from each other, and each member receives a shock that determines her role in the period. A member will be an investor with probability $\pi \in [0,1]$ and will be a saver/worker with probability $1-\pi$. These shocks are i.i.d. among members and across time.

A period is divided into three stages. In the first stage, all members of a household are together and pool their assets, which are holdings of capital and, if it is the bubbly regime, holdings of bubbly assets. Aggregate shocks to exogenous state variables are realized. The household decides how intensively to use the capital it owns (i.e., the capacity utilization rate). Because all the members of the household are identical in this stage, the household head evenly divides the assets among the members. The household head also gives contingency plans to each member, describing the actions a member should take if she becomes an investor or a saver/worker.

At the beginning of the second stage, each member receives the shock determining her role in the period. Markets open and competitive firms produce final goods. Compensation for productive factors is paid to their owners. A fraction of capital depreciates. Investors seek financing to undertake investment projects. They have the technology to transform any amount of final goods into the same amount of new capital.

Following Kiyotaki and Moore (2012), we assume that investors face a borrowing constraint due to their lack of commitment power. Namely, an investor who produces new capital cannot fully precommit to work with it even though her specific skill will be needed for capital to provide services. As a result, an investor can only issue new equity up to a fraction θ of her investment. Specifically, the inequality constraint

$$issue_t \leq \theta i_t \tag{1}$$

must be satisfied, where i_t denotes the amount of new capital produced by an investor and $issue_t$ denotes the amount of equity issued by the same investor. The rest of the new capital cannot be sold due to the lack of the commitment power. It must be held privately.

Investors however can still use privately held capital as collateral to borrow short-term funds. Specifically, investors can choose the amount of borrowing $loan_t^i$, but it has to satisfy the inequality constraint

$$loan_t^i \leq \tilde{\phi}_t \left(1 - \delta\left(u_t\right)\right) n_{p,t} \tag{2}$$

where $n_{p,t}$ is the amount of privately held capital the investor has and $\tilde{\phi}_t$ is a time-varying parameter. If $loan_t^i$ is negative, the investor is a lender. As we explain momentarily, loans are repaid from the household's budget in the consumption stage.

Investors have equity issued by other households in their portfolio. They can sell it in the stock market, but there is a limit to this activity. Specifically, following Kiyotaki and Moore (2012), we assume that an investor can sell a fraction $\phi < 1$ of her holdings of other households' equity before the investment opportunity disappears. This is equivalent to introducing transaction costs that are zero for the first fraction ϕ of equity sold, and then infinite. Let $n_{e,t}$ and $n_{e,t+1}^i$ denote the investor's holding of other households' equity at the beginning and at the end of the investment stage, respectively. The resalability constraint is given by

$$n_{e,t+1}^i \geq (1 - \phi)\left(1 - \delta\left(u_t\right)\right) n_{e,t}. \tag{3}$$

Finally, investors have bubbly assets in their portfolio if the economy is in the bubbly regime. They can sell them freely in the bubbly regime. In the fundamental regime, there are neither spot nor future markets for bubbly assets. Without markets, no one can purchase bubbly assets, which is formally stated as follows:

$$\mathbf{1}_{\{z_t=f\}} \tilde{m}_{t+1}^i = \mathbf{1}_{\{z_t=f\}} \tilde{m}_{t+1}^s = 0. \tag{4}$$

Our assumptions about asset tradings lead to the following flow budget constraint of investors:

$$\underbrace{x_t^i + i_t + \underbrace{q_t\left(n_{e,t+1}^i - \left(1 - \delta\left(u_t\right)\right) n_{e,t}\right)}_{\text{net equity purchase}} + \underbrace{\mathbf{1}_{\{z_t=b\}} \tilde{p}_t\left(\tilde{m}_{t+1}^i - \tilde{m}_t\right)}_{\text{net bubble purchase}}}_{\text{spending}} = \underbrace{\underbrace{u_t r_t\left(n_{e,t} + n_{p,t}\right)}_{\text{dividend}} + \underbrace{q_t\left(issue_t\right)}_{\text{equity finance}} + loan_t^i}_{\text{income + borrowing}}, \tag{5}$$

The saver's flow budget constraint is similar to the investor's:

$$\underbrace{x_t^s + q_t\left(n_{e,t+1}^s - (1-\delta(u_t))\, n_{e,t}\right) + \mathbf{1}_{\{z_t=b\}}\tilde{p}_t\left(\tilde{m}_{t+1}^s - \tilde{m}_t\right)}_{\text{spending}} = \underbrace{u_t r_t\left(n_{e,t}+n_{p,t}\right) + w_t l_t + loan_t^s}_{\text{income + lending}}. \quad (6)$$

Here, x_t^s, $n_{e,t+1}^s$, $\tilde{m}_{t+1}^s$, and $loan_t^s$ are saver's counterparts of x_t^i, $n_{e,t+1}^i$, $\tilde{m}_{t+1}^i$, and $loan_t^i$ in equation (5). Savers also face the same constraints regarding asset tradings as investors. But we omit them because they do not bind in equilibrium.

The members of the household get together in the consumption stage. The short-term loans are paid back from the household's budget. In a symmetric equilibrium,

$$\pi loan_t^i + (1-\pi)\, loan_t^s = 0$$

holds. Then, consumption takes place. The household's resource constraint at this point is

$$\pi x_t^i + (1-\pi)\, x_t^s = \pi c_t^i + (1-\pi)\, c_t^s. \quad (7)$$

After consumption, members' identities are lost. They start a new period as identical members. The household's portfolio at the beginning of period $t+1$ consists of holdings of other households' equity given by

$$n_{e,t+1} = \pi n_{e,t+1}^i + (1-\pi)\, n_{e,t+1}^s, \quad (8)$$

privately held capital given by

$$n_{p,t+1} = (1-\delta(u_t))\, n_{p,t} + \pi\left(i_t - issue_t\right), \quad (9)$$

and bubbly assets given by

$$\tilde{m}_{t+1} = \pi\tilde{m}_{t+1}^i + (1-\pi)\,\tilde{m}_{t+1}^s + \mathbf{1}_{\{z_t=f, z_{t+1}=b\}} M. \quad (10)$$

The household's problem is summarized as follows. It chooses a sequence of u_t, x_t^i, c_t^i, i_t, $n_{e,t+1}^i$, $\tilde{m}_{t+1}^i$, $loan_t^i$, $issue_t$, x_t^s, c_t^s, l_t, $n_{e,t+1}^s$, $\tilde{m}_{t+1}^s$, and $loan_t^s$ to maximize the utility function

$$E_0\left[\sum_{t=0}^{\infty}\frac{\beta^t}{e^{d_t}}\left(\pi\log\left(c_t^i\right) + (1-\pi)\left[\log\left(c_t^s\right) + \eta\left(1-l_t\right)\right]\right)\right]$$

subject to the constraints (1), (2), (3), (4), (5), (6), (7), (8), (9), and (10). The initial portfolio $\{n_{e,0}, n_{p,0}, \tilde{m}_0\}$ is given. Except for $loan_t^i$ and $loan_t^s$, the control variables must be non-negative.

B.2 Market Clearing Conditions

We have introduced new assets, i.e., equity and privately held capital. There is no market for the privately held capital because no one can sell it due to the lack of commitment power. The equity market clearing condition is given by

$$n_{e,t+1} = (1 - \delta(u_t)) n_{e,t} + \pi(issue_t).$$

Market clearing conditions for labor services, capital services, and final goods are the same as in the baseline model, and so is the market clearing condition for the bubbly assets in the bubbly regime. In addition, the consistency condition

$$n_{e,t} + n_{p,t} = K_t$$

is satisfied for all t.

B.3 Simplifying Assumptions

Because the aforementioned problem is hard to analyze in the general form, we make a simplifying assumption following Kiyotaki and Moore (2012) and Del Negro et al. (2017). Specifically, we assume that $\tilde{\phi}_t = \phi q_t$ always holds. It can be justified in several ways. For example, if lenders can convert a unit of uncommitted capital into ϕ units of general capital that can be easily used by anyone and hence sold in the equity market, $\tilde{\phi}_t = \phi q_t$ holds.

With this assumption, the household no longer has to keep track of these two assets separately, but the total capital it owns, $n_t \equiv n_{e,t} + n_{p,t}$, becomes the relevant state variable for the household. This is because the other households' equity and the household's privately held capital become perfect substitutes for the household, paying the same return per unit and providing the same amount of liquidity per period. q_t is not only the equity price but also the household's subjective valuation of privately held capital. Finally, following Kiyotaki and Moore (2012), we assume that $\theta = \phi$ holds to simplify the analysis. The full model is now effectively the same as the original model. But the distinction between $n_{e,t}$ and $n_{p,t}$ is still important for the measurement of the stock market value.

B.4 Stock Market Value

We assume that $n_{e,0} = \phi K_0$ holds in period 0, which implies that $n_{e,t+1} = \phi K_{t+1}$ holds for $t \geq 0$ too.[1] The stock market value is then given by

$$stock_t = \phi \left[q_t K_{t+1} \right] + \tilde{p}_t \mathbf{1}_{\{z_t = b\}} M.$$

This is identical to the stock market value we gave in the main text.

We close this section by discussing a caveat. Our assumption about uncommitted capital is slightly different from Kiyotaki and Moore's (2012). That is, while we assume that investors use it as collateral to borrow funds, they assume that investors gain additional commitment power to the uncommitted old capital every period and sell it in the equity market up to a certain limit. Our model behaves identically under their assumption except for the stock market value. Specifically, the equity-to-capital ratio has history dependence under their assumption, and we have to keep track of this ratio as an endogenous state variable in the estimation. This is technically demanding for our study, because our model has regime switches. Our assumption that investors borrow short-term funds avoids this issue because it makes the equity-to-capital ratio constant, simplifying the analysis.

C Model Summary

C.1 Fundamental Equilibrium With Loose Financial Constraints

When financial constraints are sufficiently loose, the equilibrium conditions are summarized as follows:

$$Y_t = \bar{A} e^{a_t} u_t^{\alpha} K_t \left((1 - \pi) l_t \right)^{1-\alpha},$$

$$\eta \frac{c_t}{1 - l_t} = w_t,$$

$$\delta' (u_t) = r_t,$$

$$1 = E_t \left[\frac{\beta}{e^{d_{t+1} - d_t}} \left(\frac{c_t}{c_{t+1}} \right) \left(u_{t+1} r_{t+1} + 1 - \delta (u_{t+1}) \right) \right],$$

$$r_t = \alpha \frac{Y_t}{u_t K_t},$$

$$w_t = (1 - \alpha) \frac{Y_t}{(1 - \pi) l_t},$$

and

$$c_t + K_{t+1} - (1 - \delta (u_t)) K_t = Y_t.$$

[1] $issue_t = \phi i_t$ is optimal if $q_t > 1$ holds. If $q_t = 1$ holds, any level of equity issuance between 0 and ϕi_t is optimal, and we assume that they choose $issue_t = \phi i_t$.

Detrending variables by K_t, we obtain

$$\hat{Y}_t = \bar{A} e^{a_t} u_t^{\alpha} \left((1-\pi) l_t \right)^{1-\alpha},$$

$$\eta \frac{\hat{c}_t}{1-l_t} = \hat{w}_t,$$

$$\delta'(u_t) = r_t,$$

$$1 = E_t \left[\frac{\beta}{e^{d_{t+1}-d_t}} \left(\frac{\hat{c}_t}{\hat{c}_{t+1}} \frac{1}{g_t} \right) \left(u_{t+1} r_{t+1} + 1 - \delta(u_{t+1}) \right) \right],$$

$$r_t = \alpha \frac{\hat{Y}_t}{u_t},$$

$$\hat{w}_t = (1-\alpha) \frac{\hat{Y}_t}{(1-\pi) l_t},$$

and

$$\hat{c}_t + g_t - (1 - \delta(u_t)) = \hat{Y}_t$$

where variables with a hat denote the original variables divided by K_t, for example, $\hat{Y}_t \equiv Y_t / K_t$.

C.2 Fundamental Equilibrium With Tight Financial Constraints

Suppose that the financial constraints are sufficiently tight that they are always binding. In addition, suppose that the economy is in the fundamental equilibrium. The equilibrium conditions are summarized as follows:

$$Y_t = \bar{A} e^{a_t} u_t^{\alpha} K_t \left((1-\pi) l_t \right)^{1-\alpha},$$

$$\eta \frac{c_t}{1-l_t} = w_t,$$

$$r_t - \delta'(u_t) q_t + \pi \lambda_t \left(r_t - \phi q_t \delta'(u_t) \right) = 0,$$

$$q_t = E_t \left[\frac{\beta}{e^{d_{t+1}-d_t}} \left(\frac{c_t}{c_{t+1}} \right) \left(u_{t+1} r_{t+1} + (1 - \delta(u_{t+1})) q_{t+1} + \pi \lambda_{t+1} \left(u_{t+1} r_{t+1} + \phi q_{t+1} (1 - \delta(u_{t+1})) \right) \right) \right],$$

$$r_t = \alpha \frac{Y_t}{u_t K_t},$$

$$w_t = (1-\alpha) \frac{Y_t}{(1-\pi) l_t},$$

$$Y_t = c_t + \pi \frac{\left[u_t r_t + \phi q_t (1 - \delta(u_t)) \right] K_t}{1 - \phi q_t},$$

$$K_{t+1} = (1 - \delta(u_t)) K_t + \pi \frac{\left[u_t r_t + \phi q_t (1 - \delta(u_t)) \right] K_t}{1 - \phi q_t},$$

and

$$\lambda_t = \frac{q_t - 1}{1 - \phi q_t}.$$

Detrending variables by K_t, we obtain

$$\hat{Y}_t = \bar{A} e^{a_t} u_t^{\alpha} \left((1 - \pi) l_t \right)^{1-\alpha},$$

$$\eta \frac{\hat{c}_t}{1 - l_t} = \hat{w}_t,$$

$$r_t - \delta'(u_t) q_t + \pi \lambda_t \left(r_t - \phi q_t \delta'(u_t) \right) = 0,$$

$$q_t = E_t \left[\frac{\beta}{e^{d_{t+1} - d_t}} \left(\frac{\hat{c}_t}{\hat{c}_{t+1}} \frac{1}{g_t} \right) \left(u_{t+1} r_{t+1} + (1 - \delta(u_{t+1})) q_{t+1} + \pi \lambda_{t+1} \left(u_{t+1} r_{t+1} + \phi q_{t+1} (1 - \delta(u_{t+1})) \right) \right) \right],$$

$$r_t = \alpha \frac{\hat{Y}_t}{u_t},$$

$$\hat{w}_t = (1 - \alpha) \frac{\hat{Y}_t}{(1 - \pi) l_t},$$

$$\hat{Y}_t = \hat{c}_t + \pi \frac{u_t r_t + \phi q_t (1 - \delta(u_t))}{1 - \phi q_t},$$

$$g_t = 1 - \delta(u_t) + \pi \frac{u_t r_t + \phi q_t (1 - \delta(u_t))}{1 - \phi q_t},$$

and

$$\lambda_t = \frac{q_t - 1}{1 - \phi q_t}.$$

C.3 Recurrent-Bubble Equilibrium

Suppose that the economy is in the recurrent-bubble equilibrium. The equilibrium conditions are summarized as follows:

$$Y_t = \bar{A} e^{a_t} u_t^{\alpha} K_t \left((1 - \pi) l_t \right)^{1-\alpha},$$

$$\eta \frac{c_t}{1 - l_t} = w_t,$$

$$r_t - \delta'(u_t) q_t + \pi \lambda_t \left(r_t - \phi q_t \delta'(u_t) \right) = 0,$$

$$q_t = E_t \left[\frac{\beta}{e^{d_{t+1} - d_t}} \left(\frac{c_t}{c_{t+1}} \right) \left(u_{t+1} r_{t+1} + (1 - \delta(u_{t+1})) q_{t+1} + \pi \lambda_{t+1} \left(u_{t+1} r_{t+1} + \phi q_{t+1} (1 - \delta(u_{t+1})) \right) \right) \right],$$

$$\mathbf{1}_{\{z_t = b\}} \tilde{p}_t = \mathbf{1}_{\{z_t = b\}} E_t \left[\frac{\beta}{e^{d_{t+1} - d_t}} \left(\frac{c_t}{c_{t+1}} \right) (1 + \pi \lambda_{t+1}) \tilde{p}_{t+1} \mathbf{1}_{\{z_{t+1} = b\}} \right],$$

$$r_t = \alpha \frac{Y_t}{u_t K_t},$$

$$w_t = (1-\alpha)\frac{Y_t}{(1-\pi)\,l_t},$$

$$Y_t = c_t + \pi\frac{[u_t r_t + \phi q_t\,(1-\delta\,(u_t))]\,K_t + \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M}{1-\phi q_t},$$

$$K_{t+1} = (1-\delta\,(u_t))\,K_t + \pi\frac{[u_t r_t + \phi q_t\,(1-\delta\,(u_t))]\,K_t + \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M}{1-\phi q_t},$$

and

$$\lambda_t = \frac{q_t - 1}{1-\phi q_t}.$$

Detrending variables by K_t, we obtain

$$\hat{Y}_t = \bar{A} e^{a_t} u_t^{\alpha}\left((1-\pi)\,l_t\right)^{1-\alpha},$$

$$\eta\frac{\hat{c}_t}{1-l_t} = \hat{w}_t,$$

$$r_t - \delta'\,(u_t)\,q_t + \pi\lambda_t\,(r_t - \phi q_t \delta'\,(u_t)) = 0,$$

$$q_t = E_t\left[\frac{\beta}{e^{d_{t+1}-d_t}}\left(\frac{\hat{c}_t}{\hat{c}_{t+1}}\frac{1}{g_t}\right)(u_{t+1}r_{t+1} + (1-\delta\,(u_{t+1}))\,q_{t+1} + \pi\lambda_{t+1}\,(u_{t+1}r_{t+1} + \phi q_{t+1}\,(1-\delta\,(u_{t+1}))))\right],$$

$$m_t = \mathbf{1}_{\{z_t=b\}} E_t\left[\frac{\beta}{e^{d_{t+1}-d_t}}\left(\frac{\hat{c}_t}{\hat{c}_{t+1}}\frac{1}{g_t}\right)(1+\pi\lambda_{t+1})\,m_{t+1} g_t\right],$$

$$r_t = \alpha\frac{\hat{Y}_t}{u_t},$$

$$\hat{w}_t = (1-\alpha)\frac{\hat{Y}_t}{(1-\pi)\,l_t},$$

$$\hat{Y}_t = \hat{c}_t + \pi\frac{u_t r_t + \phi q_t\,(1-\delta\,(u_t)) + m_t}{1-\phi q_t},$$

$$g_t = 1 - \delta\,(u_t) + \pi\frac{u_t r_t + \phi q_t\,(1-\delta\,(u_t)) + m_t}{1-\phi q_t},$$

and

$$\lambda_t = \frac{q_t - 1}{1-\phi q_t}$$

where $m_t \equiv \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M/K_t$. It is important that the system of equations summarized above does not have endogenous state variables. The endogenous variables in the system are therefore determined by exogenous state variables $\{z_t, a_t, d_t\}$.

It is convenient to make the regime-dependence explicit:

$$\hat{Y}_{f,t} = \bar{A} e^{a_t}\left(u_{f,t}\right)^{\alpha}\left((1-\pi)\,l_{f,t}\right)^{1-\alpha}, \tag{11}$$

$$\hat{Y}_{b,t} = \bar{A} e^{a_t}\left(u_{b,t}\right)^{\alpha}\left((1-\pi)\,l_{b,t}\right)^{1-\alpha}, \tag{12}$$

$$\eta \frac{\hat{c}_{f,t}}{1-l_{f,t}} = \hat{w}_{f,t}, \tag{13}$$

$$\eta \frac{\hat{c}_{b,t}}{1-l_{b,t}} = \hat{w}_{b,t}, \tag{14}$$

$$r_{f,t} - \delta'\left(u_{f,t}\right) q_{f,t} + \pi \lambda_{f,t} \left(r_{f,t} - \phi q_{f,t} \delta'\left(u_{f,t}\right)\right) = 0, \tag{15}$$

$$r_{b,t} - \delta'\left(u_{b,t}\right) q_{b,t} + \pi \lambda_{b,t} \left(r_{b,t} - \phi q_{b,t} \delta'\left(u_{b,t}\right)\right) = 0, \tag{16}$$

$$\begin{aligned} q_{f,t} = E_t \Bigg[& \left(1-\sigma_f\right) \frac{\beta}{e^{d_{t+1}-d_t}} \left(\frac{\hat{c}_{f,t}}{\hat{c}_{f,t+1}} \frac{1}{g_{f,t}} \right) \\ & \left(u_{f,t+1} r_{f,t+1} + \left(1-\delta\left(u_{f,t+1}\right)\right) q_{f,t+1} + \pi \lambda_{f,t+1} \left(u_{f,t+1} r_{f,t+1} + \phi q_{f,t+1} \left(1-\delta\left(u_{f,t+1}\right)\right)\right) \right) \\ & + \sigma_f \frac{\beta}{e^{d_{t+1}-d_t}} \left(\frac{\hat{c}_{f,t}}{\hat{c}_{b,t+1}} \frac{1}{g_{f,t}} \right) \\ & \left(u_{b,t+1} r_{b,t+1} + \left(1-\delta\left(u_{b,t+1}\right)\right) q_{b,t+1} + \pi \lambda_{b,t+1} \left(u_{b,t+1} r_{b,t+1} + \phi q_{b,t+1} \left(1-\delta\left(u_{b,t+1}\right)\right)\right) \right) \Bigg], \end{aligned} \tag{17}$$

$$\begin{aligned} q_{b,t} = E_t \Bigg[& \left(1-\sigma_b\right) \frac{\beta}{e^{d_{t+1}-d_t}} \left(\frac{\hat{c}_{b,t}}{\hat{c}_{b,t+1}} \frac{1}{g_{b,t}} \right) \\ & \left(u_{b,t+1} r_{b,t+1} + \left(1-\delta\left(u_{b,t+1}\right)\right) q_{b,t+1} + \pi \lambda_{b,t+1} \left(u_{b,t+1} r_{b,t+1} + \phi q_{b,t+1} \left(1-\delta\left(u_{b,t+1}\right)\right)\right) \right) \\ & + \sigma_b \frac{\beta}{e^{d_{t+1}-d_t}} \left(\frac{\hat{c}_{b,t}}{\hat{c}_{f,t+1}} \frac{1}{g_{b,t}} \right) \\ & \left(u_{f,t+1} r_{f,t+1} + \left(1-\delta\left(u_{f,t+1}\right)\right) q_{f,t+1} + \pi \lambda_{f,t+1} \left(u_{f,t+1} r_{f,t+1} + \phi q_{f,t+1} \left(1-\delta\left(u_{f,t+1}\right)\right)\right) \right) \Bigg], \end{aligned} \tag{18}$$

$$m_{f,t} = 0, \tag{19}$$

$$\begin{aligned} m_{b,t} = E_t \Bigg[& \left(1-\sigma_b\right) \frac{\beta}{e^{d_{t+1}-d_t}} \left(\frac{\hat{c}_{b,t}}{\hat{c}_{b,t+1}} \frac{1}{g_{b,t}} \right) \left(1+\pi \lambda_{b,t+1}\right) m_{b,t+1} g_{b,t} \\ & + \sigma_b \frac{\beta}{e^{d_{t+1}-d_t}} \left(\frac{\hat{c}_{b,t}}{\hat{c}_{f,t+1}} \frac{1}{g_{b,t}} \right) \left(1+\pi \lambda_{f,t+1}\right) m_{f,t+1} g_{b,t} \Bigg], \end{aligned} \tag{20}$$

$$r_{f,t} = \alpha \frac{\hat{Y}_{f,t}}{u_{f,t}}, \tag{21}$$

$$r_{b,t} = \alpha \frac{\hat{Y}_{b,t}}{u_{b,t}}, \tag{22}$$

$$\hat{w}_{f,t} = (1-\alpha)\frac{\hat{Y}_{f,t}}{(1-\pi)\,l_{f,t}}, \tag{23}$$

$$\hat{w}_{b,t} = (1-\alpha)\frac{\hat{Y}_{b,t}}{(1-\pi)\,l_{b,t}}, \tag{24}$$

$$\hat{Y}_{f,t} = \hat{c}_{f,t} + \pi\frac{u_{f,t}r_{f,t} + \phi q_{f,t}\left(1-\delta\left(u_{f,t}\right)\right) + m_{f,t}}{1-\phi q_{f,t}}, \tag{25}$$

$$\hat{Y}_{b,t} = \hat{c}_{b,t} + \pi\frac{u_{b,t}r_{b,t} + \phi q_{b,t}\left(1-\delta\left(u_{b,t}\right)\right) + m_{b,t}}{1-\phi q_{b,t}}, \tag{26}$$

$$g_{f,t} = 1-\delta\left(u_{f,t}\right) + \pi\frac{u_{f,t}r_{f,t} + \phi q_{f,t}\left(1-\delta\left(u_{f,t}\right)\right) + m_{f,t}}{1-\phi q_{f,t}}, \tag{27}$$

$$g_{b,t} = 1-\delta\left(u_{b,t}\right) + \pi\frac{u_{b,t}r_{b,t} + \phi q_{b,t}\left(1-\delta\left(u_{b,t}\right)\right) + m_{b,t}}{1-\phi q_{b,t}}, \tag{28}$$

$$\lambda_{f,t} = \frac{q_{f,t}-1}{1-\phi q_{f,t}}, \tag{29}$$

and

$$\lambda_{b,t} = \frac{q_{b,t}-1}{1-\phi q_{b,t}} \tag{30}$$

where subscripts f and b denote realizations of the variables in the fundamental and bubbly regimes, respectively; for instance, $\hat{Y}_{f,t}$ is the realization of $\hat{Y}_t$ in the fundamental regime. The regime-dependent steady states are obtained as the solutions of the system of nonlinear equations (11) to (30) under the assumption that both a_t and d_t are constant at zero. To capture the effects of a_t and d_t, we linearize the equations (11) to (30) around the regime-dependent steady states and obtain the impulse response functions.

D Some Results in the Simple Model

D.1 Crowding-In Effect of Realized Bubble

We show that the size of the bubble relative to the capital stock, m_b, decreases with the level of the financial development, ϕ. Given the analytical solution

$$m_b = \frac{r\beta}{\beta\pi + 1 - \beta + \sigma_b\beta\left(\phi-\pi\right)}\left[2-\pi-\frac{1}{\beta}-\left(\phi+\sigma_b\left(1-\pi\right)\right)\right],$$

we can directly prove it by taking a derivative. Namely, we have

$$\frac{\partial m_b}{\partial \phi} = -\left(\frac{1}{\beta\pi + 1 - \beta + \sigma_b\beta\left(\phi-\pi\right)}\right)^2 r\beta\left(1-\sigma_b\right)\left[1-\beta\left(1-\pi\right)\left(1-\sigma_b\right)\right] < 0.$$

D.2 Crowding-Out Effect of Realized Bubble

We show that the price of capital during the bubbly episode, q_b, is smaller than the price of capital after the bubbly episode, q_f^*. It can be shown in two steps. First, notice that q_b converges to q_f^* as σ_b converges to $\bar{\sigma}_b$, namely, $\lim_{\sigma_b \nearrow \bar{\sigma}_b} q_b = q_f^*$. This is obvious from the analytical solutions

$$q_f^* = \frac{\beta\left(1-\pi\right)}{\pi\left(1-\beta\right)+\beta\phi}$$

and

$$q_b = \frac{\left(1-\pi\right)\beta + \left(\sigma_b - \bar{\sigma}_b\right)\frac{\beta(1-\pi)^2}{1-\phi}}{\pi\left(1-\beta\right)+\beta\phi+\left(\sigma_b-\bar{\sigma}_b\right)\frac{\beta(\phi-\pi)(1-\pi)}{1-\phi}}.$$

Second, taking a derivative of q_b with respect to σ_b, we obtain

$$\frac{\partial q_b}{\partial \sigma_b} = \left(\frac{1-\pi}{\pi\left(1-\beta\right)+\beta\phi+\left(\sigma_b-\bar{\sigma}_b\right)\frac{\beta(\phi-\pi)(1-\pi)}{1-\phi}}\right)^2 \frac{\beta\pi}{1-\phi} > 0.$$

Hence, q_b is increasing in σ_b and converges to q_f^* as σ_b converges to $\bar{\sigma}_b$. Because we assume that σ_b is less than $\bar{\sigma}_b$, q_b is smaller than q_f^*.

D.3 Comovement Problem

This section shows that the simple model suffers from a comovement problem. Investment growth is given by

$$\frac{i_t}{i_{t-1}} = \frac{K_{t+1}}{K_t} = g_t = \begin{cases} g_f, & \text{if } z_\tau = f \text{ and for all } \tau \leq t, \\ g_b, & \text{if } z_t = b, \\ g_f^*, & \text{otherwise.} \end{cases}$$

Similarly, consumption growth is given by

$$\frac{c_t}{c_{t-1}} = \frac{\hat{c}_t}{\hat{c}_{t-1}}\frac{K_t}{K_{t-1}} = \frac{r-g_t}{r-g_{t-1}}g_{t-1} = \begin{cases} g_f, & \text{if } z_\tau = f \text{ and for all } \tau \leq t, \\ \left(1+\frac{g_f-g_b}{r-g_f}\right)g_f, & \text{if } \{z_{t-1}, z_t\} = \{f, b\}, \\ g_b, & \text{if } \{z_{t-1}, z_t\} = \{b, b\}, \\ \left(1+\frac{g_b-g_f^*}{r-g_b}\right)g_b, & \text{if } \{z_{t-1}, z_t\} = \{b, f\}, \\ g_f^*, & \text{otherwise.} \end{cases}$$

If $g_f < g_b$ holds, investment growth rises from g_f to g_b when the bubble emerges (in period t with $\{z_{t-1}, z_t\} = \{f, b\}$) but consumption growth drops. Similarly, if $g_b > g_f^*$ holds, investment growth drops from g_b to g_f^* when the bubble bursts (in period t with $\{z_{t-1}, z_t\} = \{b, f\}$) but consumption

growth rises. Clearly, it is impossible to have comovement between investment and consumption.

E Transversality Conditions

We show that the transversality conditions are satisfied along the balanced growth path. Let's assume without loss of generality that $a_t = d_t = 0$ for all $t \geq 0$. Then, we have

$$\left(\frac{1}{c_t^i}\right) q_t K_{t+1} = \left(\frac{1}{\hat{c}_t^i}\right) q_t g_t = \begin{cases} \left(\frac{1}{\hat{c}_f^i}\right) q_f g_f, & \text{if } z_t = f, \\ \left(\frac{1}{\hat{c}_b^i}\right) q_b g_b, & \text{if } z_t = b. \end{cases}$$

Similarly, we have

$$\left(\frac{1}{c_t^i}\right) \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M = \left(\frac{1}{\hat{c}_t^i}\right) m_t = \begin{cases} 0, & \text{if } z_t = f, \\ \left(\frac{1}{\hat{c}_b^i}\right) m_b, & \text{if } z_t = b. \end{cases}$$

The transversality conditions are satisfied because

$$0 \leq \lim_{t\to\infty} E_0 \left[\beta^t \left(\frac{1}{c_t^i}\right) q_t K_{t+1}\right] \leq \lim_{t\to\infty} \beta^t \times \left[\max\left\{\left(\frac{1}{\hat{c}_f^i}\right) q_f g_f, \left(\frac{1}{\hat{c}_b^i}\right) q_b g_b\right\}\right] = 0$$

and

$$0 \leq \lim_{t\to\infty} \mathbf{1}_{\{z_0=b\}} E_0 \left[\beta^t \left(\frac{1}{c_t^i}\right) \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M\right] \leq \lim_{t\to\infty} \beta^t \left(\frac{1}{\hat{c}_b^i}\right) m_b = 0.$$

F Partial-Collapse Model

This section examines an alternative assumption replacing the fundamental regime with a low-bubble regime in which a small fraction of bubbly assets survive from the previous regime. This model has two bubbly regimes with different amounts of bubbly assets. We call them high-bubble (H) and low-bubble (L) regimes respectively, in each of which M and $(1-\delta_M)M$ units of bubbly assets exist respectively. A fraction $\delta_M \in (0,1)$ of randomly chosen bubbly assets physically disappears when the regime switches to the low-bubble one, and $\delta_M M$ units of a new vintage of bubbly assets are created when the regime switches to the other direction. We omit the productivity and preference shocks to simplify the analysis.

Green circles and crosses in Figure 1 show the regime-dependent capital growth in this model. We set the depreciation rate of the bubbly asset at $\delta_M = 0.999$. Therefore, nearly all the bubbly assets disappear when the regime switches to the low-bubble regime. Nonetheless, the regime-dependent capital growth in the partial-collapse model does not resemble its counterpart in the original model plotted in red circles and crosses in the same figure. Specifically, the distance

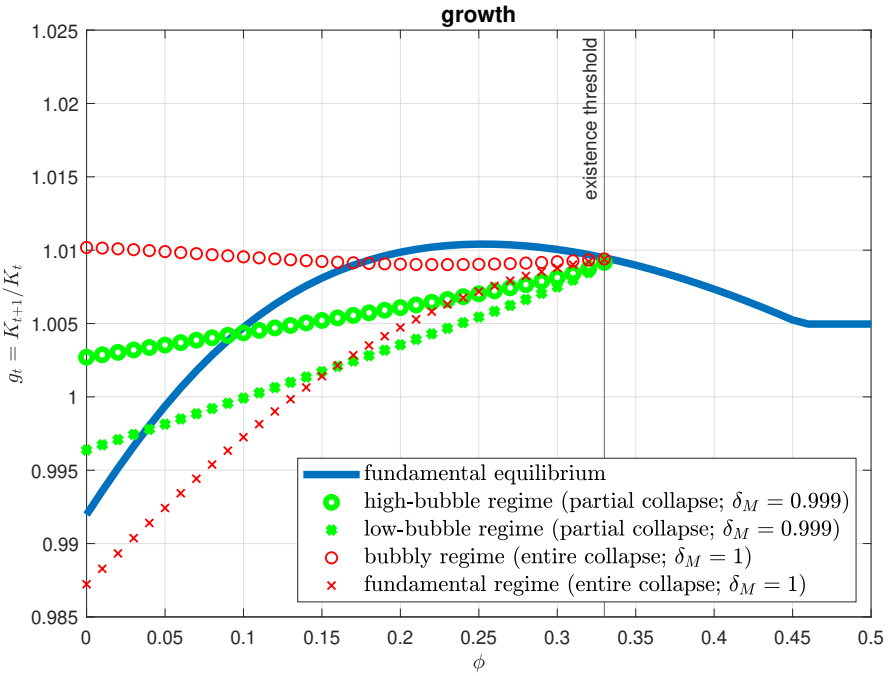

Figure 1: Partial Collapse vs. Entire Collapse (Capital Growth)

between green circles and crosses is a lot shorter than the distance between red circles and crosses.

Figure 2 explains why. It plots the regime-dependent bubble size relative to the capital stock. Importantly, a sizable bubble exists in the low-bubble regime. The mechanism is simple; even if most of the bubbly assets lose value (physically disappear in the model), the rest of the bubbly assets appreciate because liquid assets become scarce and demand for the remaining bubbly assets rises. This general equilibrium effect stabilizes the impact of the collapse. Our benchmark model is different in this respect; because we consider the entire collapse of bubbles as in Weil (1987), the supply of bubbly assets is zero in the fundamental regime, and therefore, the aforementioned general equilibrium effect is absent. As a consequence, the entire collapse of bubbles has a much stronger impact on growth.

Figure 3 plots the regime-dependent capital growth in this alternative model as a function of δ_M. We set $\phi = 0.15$, but the result is robust to other values of ϕ. At $\delta_M = 1$, we plot the regime-dependent capital growth in our benchmark model. We see no sign of "convergence" from the model with multiple partial collapses to the benchmark model as δ_M approaches 1. There is a discrete jump at $\delta_M = 1$. This is the same type of non-linearity that Brunnermeier and Sannikov (2014), He and Krishnamurthy (2013), and Gertler and Kiyotaki (2015) emphasize as an important factor to account for the financial crisis.

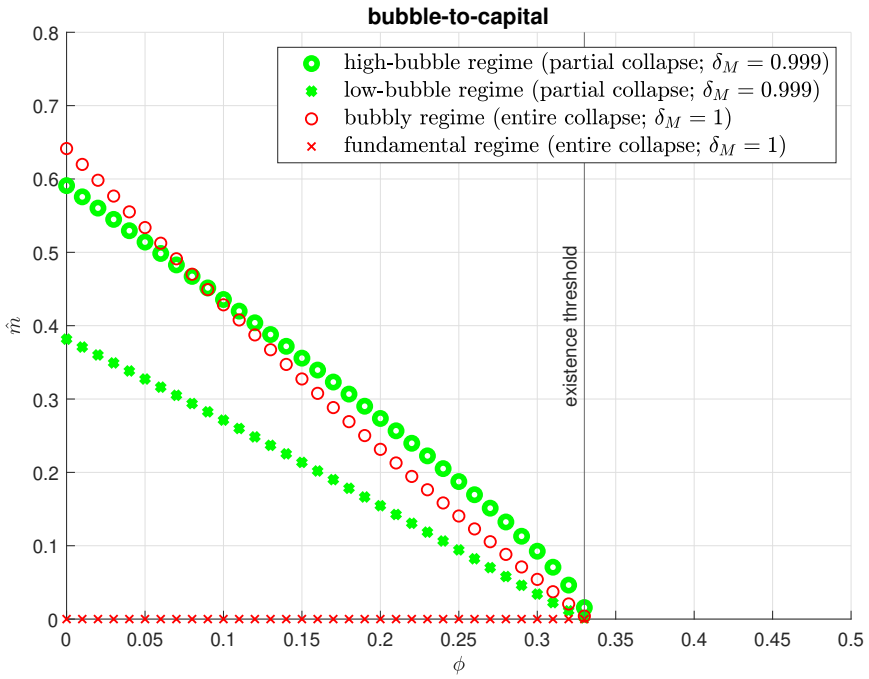

Figure 2: Partial Collapse vs. Entire Collapse (Bubble Size)

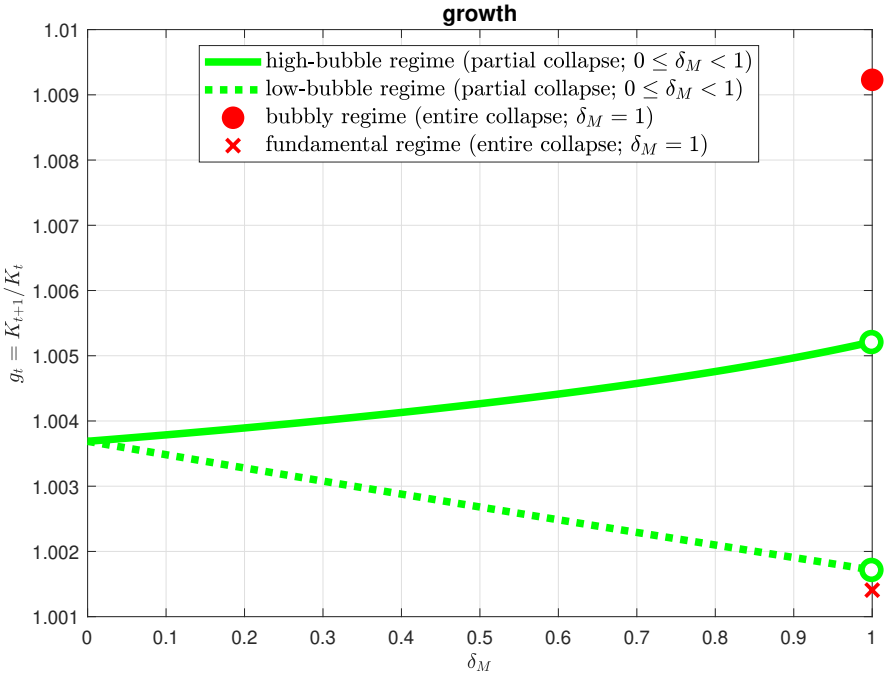

Figure 3: Partial Collapse vs. Entire Collapse (Depreciation and Growth)

G Data

In this section, we explain the observables used to estimate the model. The data consist of quarterly GDP growth and the stock-market-to-GDP ratio for the period 1984.Q1-2017.Q4.[2] The data come from the St. Louis Fed's FRED database. For the stock-market-to-GDP ratio, we use the quarterly not seasonally adjusted Wilshire 5000 Full Cap Price Index series. The raw unfiltered series was used to compute GDP growth. We pre-filtered the stock market-to-GDP ratio series with the HP filter to remove the trend in the data that is not present in our model; see the main text for a discussion of the properties of the filtered series. This approach is reasonable because we are interested in understanding how the fluctuations around this trend are influenced by the presence or the absence of bubbles. Furthermore, this de-trending approach is standard in policy institutions such as the Federal Reserve System when it analyzes the evolution of credit in the economy (Bassett et al. (2015)). The Bank of Japan takes a similar approach too. The bank constructs the "heat map" from several financial indicators, including stock market value, on which abnormal deviations of a variable from the trend are read as a sign of over-heating. Please see the Financial System Report, a biannual publication of the bank surveying the financial system.[3]

H Solution Method

The solution and estimation of the model requires a series of steps that we describe next.

1. We de-trend the model's equilibrium conditions by the stock of capital, resulting in a stationary model. It is easy to see that given the structural shocks and the regime today, the model is entirely forward looking (equations (11) to (30) in Section C.3).

2. Let X_t^f and Y_t^f denote the vectors containing the states and controls in the fundamental regime. Similarly, X_t^b and Y_t^b denote the vectors containing the states and controls in the bubbly regime. Then the de-trended model can be written as

$$E_t \Gamma_f(X_t^f, Y_t^f, X_{t+1}^f, Y_{t+1}^f, X_{t+1}^b, Y_{t+1}^b) = 0.$$
$$E_t \Gamma_b(X_t^b, Y_t^b, X_{t+1}^f, Y_{t+1}^f, X_{t+1}^b, Y_{t+1}^b) = 0.$$

   That is, we stack the model's equilibrium equations conditional on being in the fundamental and the bubbly regimes. Note that the notation makes clear that the economy may switch to a different regime tomorrow. The functional equations describing the equilibrium conditions are captured by $\Gamma_f(\cdot)$ and $\Gamma_b(\cdot)$.

[2] The data are from U.S. Bureau of Economic Analysis (2022) and Wilshire Associates (2022).
[3] https://www.boj.or.jp/en/research/brp/fsr/index.htm/.

3. We compute the steady state (w/o structural shocks) of each regime (X^f, Y^f, X^b, Y^b) by shutting down the structural shocks but preserving the regime switches. In other words, we look for X^f, Y^f, X^b, Y^b that solve the system:

$$\Gamma_f(X^f, Y^f, X^f, Y^f, X^b, Y^b) = 0.$$
$$\Gamma_b(X^b, Y^b, X^f, Y^f, X^b, Y^b) = 0.$$

   In doing so, our method respects the probability of switching from the fundamental steady state to the bubbly steady state and vice versa.

4. We perturb the model around the steady states and solve the resulting system to obtain the laws of motion for the endogenous states and controls. For simulations and estimation, we use a first-order perturbation approach (Schmitt-Grohe and Uribe, 2004).

5. It can be shown that the first-order approximation of the model can be written compactly as follows:

$$\mathbb{X}_t = \Lambda_x \mathbb{X}_{t-1} + \Omega_x \Xi_{x,t}.$$

   Here, $\mathbb{X}_t = [X^f, Y^f, X^b, Y^b]'$ and $\Xi_{x,t}$ contains the structural innovations at time t.

6. We supplement the transition equation in the previous point with a measurement equation of the form:

$$\mathbb{Y}_t = \Lambda_y \mathbb{X}_t + \Omega \Xi_{y,t}.$$

   The matrix Λ_y makes the necessary transformations to make the model's variables compatible with the observables in the data collected in vector $\mathbb{Y}_t$. We allow for classical measurement errors as captured by $\mathbb{Y}_t$.

7. To compute the likelihood of the model, we use the nonlinear filter discussed in chapter 5 in Kim and Nelson (1999).

8. The Bayesian estimation is implemented following Fernandez-Villaverde et al. (2016).

I Impulse Responses

This section discusses the impulse response functions of variables not discussed in the paper. Table 1 reports responses to a unit productivity shock and a unit preference shock. Their autocorrelations are 0.9 and 0.5, respectively. We report contemporaneous responses on impact of the shock alone, because they are sufficient to summarize the impulse responses for the variables reported in the table. This is because all the variables in the table are determined by the exogenous state variables $\{z_t, a_t, d_t\}$ alone.

	Bubbly Regime		Fundamental Regime	
	Productivity	Preference	Productivity	Preference
output-to-capital	1.18%	-0.00%	1.10%	-0.05%
consumption-to-capital	1.06%	-0.28%	1.03%	-0.26%
investment-to-capital	1.55%	0.84%	1.38%	0.87%
hours	0.09%	0.21%	0.05%	0.16%
utilization	0.36%	-0.42%	0.21%	-0.47%
capital price	0.77%	0.62%	0.95%	0.67%
bubble-to-capital	1.83%	0.79%	-	-
capital growth	0.05%	0.06%	0.04%	0.04%
productivity shock	1.00%	0%	1.00%	0%
preference shock	0%	1.00%	0%	1.00%

Table 1: Effects of Productivity and Preference Shocks

A positive productivity shock (a rise in a_t) increases output, consumption, investment, and hours worked simultaneously. In contrast, a positive preference shock (a rise in d_t) increases investment but decreases consumption. Remember that the preference shock decreases the level of the subjective discount factor on impact but it is mean reverting. Hence, after the shock, households end up assigning large weights to the utility flows in the distant future relative to those in the near future. Therefore, households become effectively more patient than before, hence increasing investment and decreasing consumption. Asset prices also increase because of the discount factor channel.

Comparing responses across regimes, we see larger responses in the bubbly regime than in the fundamental regime. Bubbles amplify the impact of the shocks because the bubble size positively responds to the shocks, supplying more liquidity to the economy. But the regime-dependence is relatively mild.

J Alternative Identification Strategies

In this section, we show the impact of alternative identification strategies on our empirical results. For our first check, we use quarterly U.S. data on GDP growth and the credit-to-GDP ratio.[4] Similar to the stock market value, the credit-to-GDP ratio in the model is higher during bubbly episodes than during the fundamental ones. Figure 4 presents the estimated probability of the economy being in the bubbly regime. It shows that the economy spent more time in the fundamental regime prior to the 2000s. This means that during the first 15 years of the sample, growth was driven by exogenous productivity shocks (not shown), not a surprise given the moderate credit-to-GDP ratio in the data.

The economy starts the 2000s in the fundamental regime, but as credit expands rapidly, the

[4]Credit data are from Bank for International Settlements (2022).

probability of being in the bubbly regime rises. By mid-2005, the bubble is becoming more likely, with a smoothed probability above 50%. Between 2007 and early 2009, our exercise reveals that the bubble was in full swing. Importantly, growth is bubble-driven in this period, which is an interesting contrast to the productivity-driven growth in the 1990s. At its peak, credit in the data is explained by a combination of bubbles and a favorable productivity shock. The bubble disappears in the early 2010s.

During the initial phase of the Great Recession, credit is in correction territory but still high compared to the 1990s. As a consequence, our approach identifies this stage of the crisis as the result of a sharp decline in investment demand due to an exogenous shock to preferences. But as the contraction in credit continued and the economy grew at lackluster rates, the fundamental regime becomes more likely, to the point where it is the prevalent regime since 2011. It is worth noting that our estimate of the bubbly episode lasts longer than other researchers have found (Jorda et al., 2015). This is due to the evolution of aggregate credit, peaking at the end of 2008 and slowly retrenching afterward, the latter of which Ivashina and Scharfstein (2010) attribute to the extensive use of existing lines of credit during 2009 and 2010. Ideally, we would use newly issued credit rather than total credit to better capture the narrative behind the crisis. However, to the best of our knowledge, such data are not available at the frequency and length required for our purpose.

It is worth noticing that a similar bubble regime would emerge if we used the Case-Shiller house price index. The main difference is that the bubble would collapse around the first quarter of 2008. The reason is that the credit-to-GDP ratio's dynamic tracks closely that of the Shiller-Case-to-GDP ratio except for the early collapse of the housing index.

For the financial constraints of $\phi = 0.19$ considered in the main text, the average growth rates and credit-to-GDP are off the values in the data seen during the bubbly episode in the 2000s. One possibility, used in the paper, is to introduce a constant and estimate it to offset the difference. Alternatively, one can change the financial constraints to match the average growth rate during the presumptive bubbly period, with the caveat that we impose the dates when the bubble exists a priori. Figure 5 shows the estimated path of the probability of the fundamental (upper panel) and bubbly (lower panel) regimes under this specification. Clearly, the paths are consistent with those reported in the paper.

In the main text, we estimate the regimes using the sample 1984.Q1-2017.Q4. One can extend the sample to include the pre-Great Moderation era 1960.Q1-1983.Q4 but this brings a complication. Growth was strong during that period and credit-to-GDP was above average. Through the lens of our benchmark model, this points to a bubble. However, most economic observers would agree that there was no bubble during those years. To cope with this issue, we add a third regime that allows for high growth and average credit. Figure 6 shows the probabilities of each regime from this alternative model. As one can see, the main message remains. The high growth/high credit of the 2000s was most likely associated with the occurrence of a bubble in the economy. We

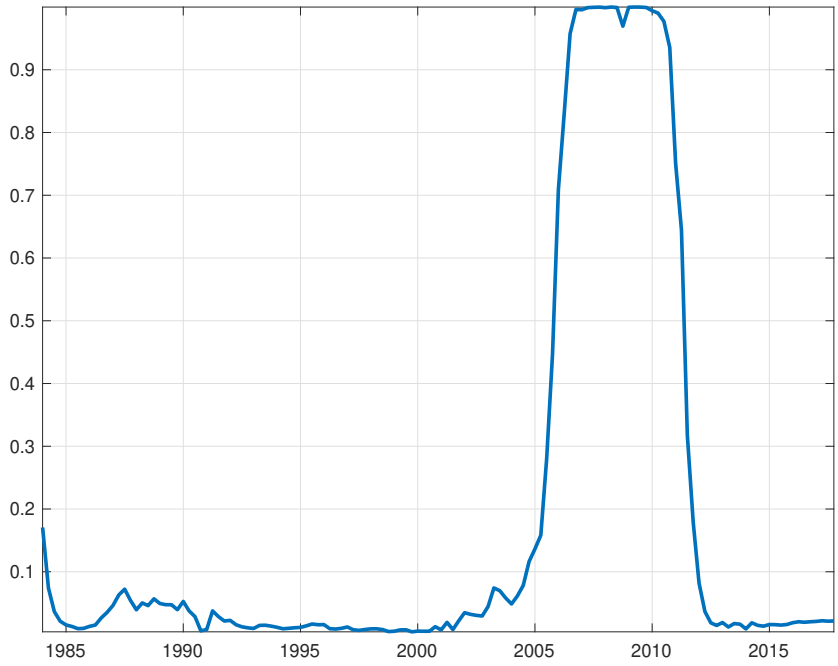

Figure 4: Probability of Bubbly Regime

also see that the economy spent most of the 1960s and 1970s in the third regime.

References

Bank for International Settlements (2022): "Total Credit to Private Non-Financial Sector, Adjusted for Breaks, for United States [CRDQUSAPABIS], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/CRDQUSAPABIS (accessed October 24, 2022)," Tech. rep.

Bassett, W. F., A. Daigle, R. M. Edge, and G. I. Kara (2015): "Credit-to-GDP Trends and Gaps by Lender- and Credit-Type," Tech. Rep. December 3, 2015, FEDS Notes.

Brunnermeier, M. K. and Y. Sannikov (2014): "A Macroeconomic Model with a Financial Sector," *American Economic Review*, 104, 379–421.

Del Negro, M., G. Eggertsson, A. Ferrero, and N. Kiyotaki (2017): "The Great Escape? A Quantitative Evaluation of the Fed's Liquidity Facilities," *American Economic Review*, 107, 824–857.

Fernandez-Villaverde, J., J. F. Rubio-Ramirez, and F. Schorfheide (2016): "Solution

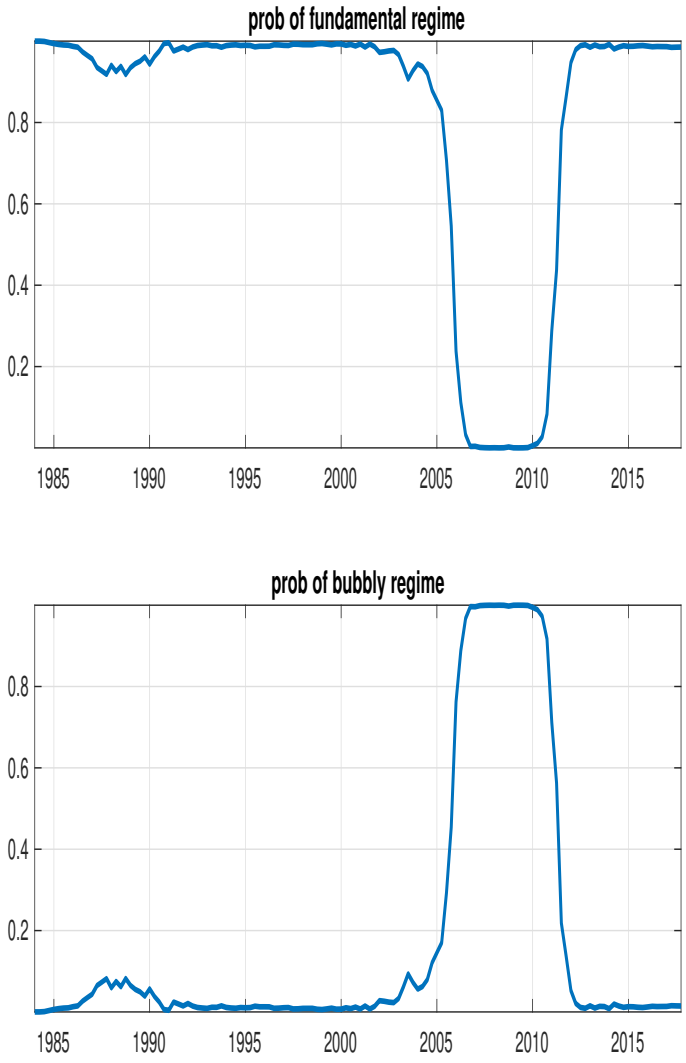

Figure 5: Regime Probabilities with Tighter Liquidity

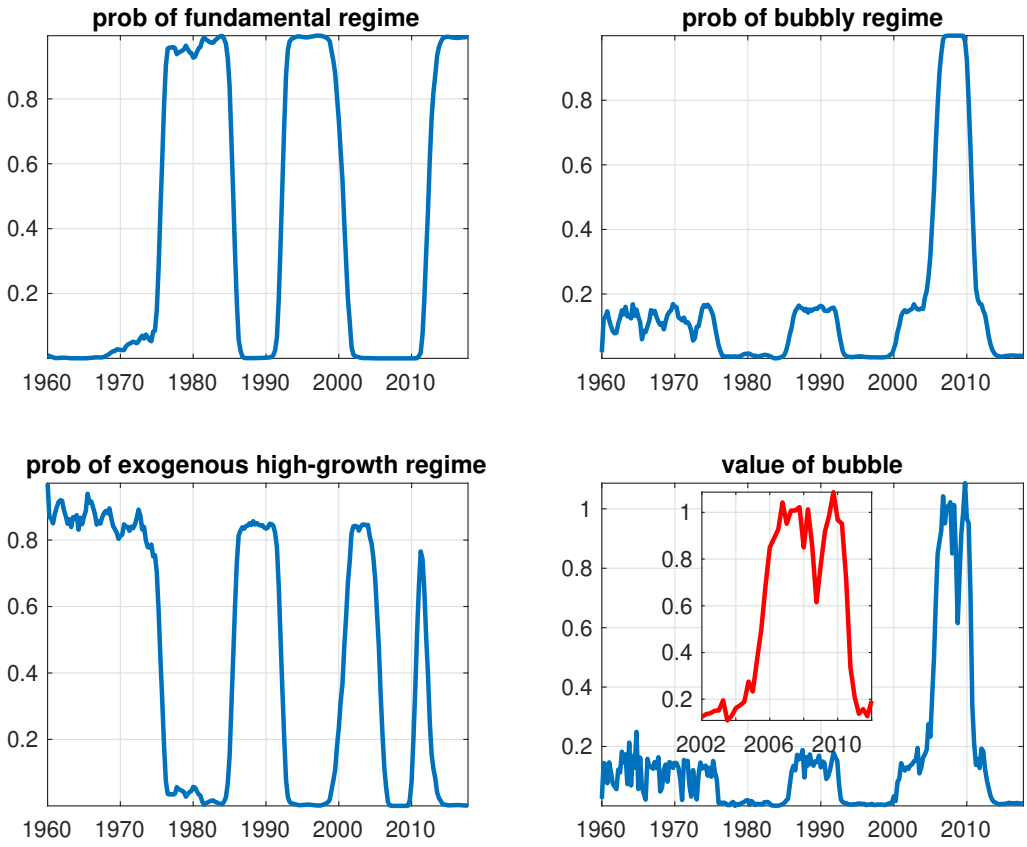

Figure 6: Regime Probabilities Extended Sample

and Estimation Methods for DSGE Models," in *Handbook of Macroeconomics*, ed. by J. B. Taylor and H. Uhlig, Elsevier, vol. 2, 527–724.

Gertler, M. and N. Kiyotaki (2015): "Banking, Liquidity, and Bank Runs in an Infinite Horizon Economy," *American Economic Review*, 105, 2011–2043.

He, Z. and A. Krishnamurthy (2013): "Intermediary Asset Pricing," *American Economic Review*, 103, 732–770.

Ivashina, V. and D. Scharfstein (2010): "Bank lending during the financial crisis of 2008," *Journal of Financial Economics*, 97, 319–338.

Jorda, O., M. Schularick, and A. M. Taylor (2015): "Leveraged bubbles," *Journal of Monetary Economics*, 76, S1–S20.

Kim, C.-J. and C. R. Nelson (1999): *State-Space Models with Regime Switching*, The MIT Press.

Kiyotaki, N. and J. Moore (2012): "Liquidity, Business Cycles, and Monetary Policy," Tech. Rep. 17934, NBER Working Papers.

Schmitt-Grohe, S. and M. Uribe (2004): "Solving dynamic general equilibrium models using a second-order approximation to the policy function," *Journal of Economic Dynamics and Control*, 28, 755–775.

U.S. Bureau of Economic Analysis (2022): "Real Gross Domestic Product [GDPC1], retrieved from FRED, Federal Reserve Bank of St. Louis," Tech. rep., https://fred.stlouisfed.org/series/GDPC1 (accessed October 24, 2022).

Weil, P. (1987): "Confidence and the Real Value of Money in an Overlapping Generations Economy," *The Quarterly Journal of Economics*, 102, 1.

Wilshire Associates (2022): "Wilshire 5000 Full Cap Price Index [WILL5000PRFC], retrieved from FRED, Federal Reserve Bank of St. Louis," Tech. rep., https://fred.stlouisfed.org/series/WILL5000PRFC (accessed October 24, 2022).