Markets with Multidimensional Private Information

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Online Appendix

Examples of Equilibria

For the following analysis, it is useful to introduce some additional notation. In parallel with $\underline{v} = \min V$, let $\bar{v} = \sup V$, $\underline{\beta} = \min B$, and $\bar{\beta} = \sup B$.

A1. Semi-Separating Equilibrium

Impose Assumption 1 and assume there exists a $\beta \in B$ with $\beta\Gamma(\underline{v}) > \underline{v}$. The semi-separating equilibrium is characterized by a discount factor for the marginal buyer, $\hat{\beta} \in B$, which is determined in equation (A1) below. For now, fix $\hat{\beta}$ and assume, as we verify below, that $\hat{\beta}\Gamma(\underline{v}) > \underline{v}$.

We next define two critical prices. The lowest price with trade is $\underline{p} \equiv \hat{\beta}\Gamma(\underline{v})$, the value that the marginal buyer places on an asset sold by the seller with the lowest continuation value. Note that, given our assumption, $\underline{p} > \underline{v}$, so a seller with the lowest continuation value strictly prefers selling his asset for $\underline{p}$ rather than retaining it. The second critical price is the highest one with trade. Let $\bar{p}$ be the smallest price satisfying $\bar{p} = \hat{\beta}\Gamma(\bar{p})$, or $\bar{p} = \infty$ if there is no such price. That is, $\hat{\beta}\Gamma(v) > v$ whenever $v < \bar{p}$.

In the semi-separating equilibrium, the equilibrium buyer-seller ratio satisfies

$$\Theta(p) = \begin{cases} \infty & p < \underline{p} \\ \exp\left(-\int_{\underline{p}}^{p} \frac{1}{p' - \Gamma^{-1}(p'/\hat{\beta})} dp'\right) \text{ if } & p \in [\underline{p}, \bar{p}] \\ 0 & p > \bar{p}. \end{cases}$$

Facing this market tightness, the first part of the definition of equilibrium implies that any seller (β, δ) with continuation value $\beta\delta < \bar{p}$ maximizes his profit by setting the sale price $p_s(\beta, \delta) = \hat{\beta}\Gamma(\beta\delta)$. A seller (β, δ) with a higher continuation value cannot sell his asset at any price satisfying $p \geq \beta\delta$ and $\Theta(p) > 0$. Although such a seller is indifferent between all sale prices at which he cannot sell his asset, i.e. with $\Theta(p) = 0$, his behavior still matters in equilibrium since it influences buyers' beliefs. We assume that such an investor sets price $p_s(\beta, \delta) = \max\{\beta\delta, \hat{\beta}\Gamma(\beta\delta)\}$.

Turn next to the buyers' belief about the quality of asset offered at each price. At prices $p < \underline{p}$, buyers are unable to find sellers, $\Theta(p) = \infty$, and so beliefs are undefined. Intermediate prices, $p \in [\underline{p}, \bar{p}]$, are offered only by investors with continuation value $v = \Gamma^{-1}(p/\hat{\beta})$. Since the average quality asset held by these sellers is $\Gamma(v) = p/\hat{\beta}$, part 3(a) of the definition of equilibrium imposes $\Delta(p) = p/\hat{\beta}$ when $p \in [\underline{p}, \bar{p}]$. Finally, at still higher prices, $\Delta(p) \leq p/\hat{\beta}$. Such beliefs are rational, since by construction an investor with continuation value $v > \bar{p}$ always sets a price at least equal to $\hat{\beta}\Gamma(\beta\delta)$.[15]

Given these beliefs, we now use part 2 of the definition of equilibrium. An investor with discount factor $\beta > \hat{\beta}$ maximizes his profit by buying at any price $p \in [\underline{p}, \bar{p}]$, weakly prefers buying at those prices rather than any higher price, and strictly prefers buying at these prices rather than a lower price where there are no sellers. An investor with discount factor $\beta < \hat{\beta}$ prefers to offer a price $p < \underline{p}$, which ensures that he fails to buy in equilibrium.

The last piece of equilibrium is the determination of the marginal discount factor. In order to ensure that the supply of assets is equal to the demand, we require

$$\text{(A1)} \qquad \int_{\beta > \hat{\beta}} g_b(\beta) d\beta = \iint_{\beta\delta < \bar{p}} p_s(\beta, \delta)\Theta(p_s(\beta, \delta)) g_s(\beta, \delta)\, d\beta d\delta$$

The left hand side is the total supply of the period 1 consumption good brought to the market by investors with discount factors greater than $\hat{\beta}$. The right hand side is the total cost of purchasing up the assets brought to the market by investors with continuation values $\beta\delta < \bar{p}$. We prove in the proof of Proposition 3 that there is a unique solution to this equation. Finally, we allocate buyers with $\beta > \hat{\beta}$ to markets so as to equate supply and demand at each price, in accordance with part 4 of the definition of equilibrium.

A2. One-Price Equilibrium

Assume that the support of the buyer's type distribution is an interval B and the support of the seller type distribution is a rectangle $S = B \times D$ for some interval D. Under these restrictions, we prove the existence of a one-price equilibrium characterized by two numbers, the trading price p^* and the identity of the marginal buyer $\hat{\beta} \in B$.

In a one-price equilibrium, an investor can purchase an asset at any price greater than or equal to p^* and can sell an asset at any price less than or equal to p^*:

$$\Theta(p) = \begin{cases} \infty \\ 1 \\ 0 \end{cases} \Leftrightarrow p \lesseqgtr p^*.$$

Part 1 of the definition of equilibrium implies that, taking $\Theta(p)$ as given, an investor (β, δ) with a continuation value $\beta\delta \leq p^*$ will choose to sell for $p_s(\beta, \delta) =$

[15]Part 3(a) of the definition of equilibrium, together with the assumption that $p_s(\beta, \delta) = \max\{\beta\delta, \hat{\beta}\Gamma(\beta\delta)\}$ imposes additional restrictions on $\Delta(p)$, but these are unimportant for our analysis.

p^*. Investors with a higher continuation value, $\beta\delta > p^*$, set a higher sale price. To support the equilibrium, we choose one such price, $p_s(\beta, \delta) = \bar{\beta}\delta$ if $\beta\delta > p^*$.

Turn next to buyers' beliefs. At prices $p < p^*$, buyers cannot find any seller so beliefs are undefined. At $p = p^*$, Part 3(a) of the definition of equilibrium implies that buyers expect

$$\Delta(p^*) = \frac{\iint_{\beta\delta \le p^*} \delta g_s(\beta, \delta)\, d\delta\, d\beta}{\iint_{\beta\delta \le p^*} g_s(\beta, \delta)\, d\delta\, d\beta},$$

the average quality asset held by investors with a continuation value below p^*. At $p > p^*$, beliefs are also pinned down by condition 3(a): $\Delta(p) = \max\{p/\bar{\beta}, \min D\}$ whenever $p \in (p^*, \bar{v}]$. This is the worst quality asset held by an investor with continuation value p. Finally, we assume $\Delta(p) = \max D$ when $p > \bar{v}$, consistent with condition 3(b).

Now turn to part 2 of the definition of equilibrium. Let $\hat{\beta} = p^*/\Delta(p^*)$. Given the beliefs we just constructed, buyers with discount factor $\beta > \hat{\beta}$ find it strictly optimal to buy at price p^*, while buyers with lower discount factors find it better to offer a price $p < p^*$ at which they cannot buy.

Finally, we close the model using the market clearing condition, part 4 of the definition of equilibrium:

$$\text{(A2)} \qquad \int_{\beta > \hat{\beta}} g_b(\beta) d\beta = p^* \iint_{\beta\delta < p^*} g_s(\beta, \delta)\, d\beta d\delta.$$

The left hand side is the amount of the period 1 consumption good held by investors with discount factors greater than $\hat{\beta}$ and the right hand side is the cost of buying the assets held by investors with continuation value less than p^*.

A one-price equilibrium is a pair $(\hat{\beta}, p^*)$ solving $\hat{\beta}\Delta(p^*) = p^*$ and equation (A2). Depending on functional forms, one or more one-price equilibrium may exist.

A3. Other Equilibria

We illustrate the full multiplicity of equilibria through a parametric example. Assume $G_s(\beta, \delta) = \beta\delta^2$ on $[0, 1]^2$, so $\underline{v} = 0$, $\bar{v} = 1$, $\Gamma(v) = \frac{1+v}{2}$, and $H(v) = v(2 - v)$.

Other Semi-Separating Equilibria. — We start by showing there is a continuum of semi-separating equilibria. These equilibria are indexed by the identity of the seller with the highest continuation value, $\bar{p} \in [0.456, 1]$. Given $\bar{p}$, let $\underline{p} = \bar{p}/(1+\bar{p})$, $\hat{\beta} = 2\underline{p}$, and

$$\text{(A3)} \qquad \hat{\theta} = \frac{(1 - \bar{p})(2 + \bar{p})(3 + \bar{p})}{4\bar{p}^2(6 - \bar{p})}.$$

The restriction on the range of $\bar{p}$ ensures that $\hat{\theta} \in [0,1]$. In such an equilibrium, the buyer-seller ratio is

$$\Theta(p) = \begin{cases} \infty & \text{if } p < \hat{\theta}\underline{p} \\ \hat{\theta}\underline{p}/p & \text{if } p \in [\hat{\theta}\underline{p}, \underline{p}) \\ \hat{\theta}\left(\frac{\bar{p}-p}{\underline{p}\bar{p}}\right)^{\bar{p}} & \text{if } p \in [\underline{p}, \bar{p}] \\ 0 & \text{if } p > \bar{p}, \end{cases}$$

while the expected quality of assets offered for sale at prices above $\hat{\theta}\underline{p}$ is $\Delta(p) \leq p/\hat{\beta}$, with equality if $p \in [\underline{p}, \bar{p}]$.

To prove this is an equilibrium, we need to discuss buying and selling behavior. Start with selling. For any investor (β, δ) with continuation value with $\beta\delta \in (0, \bar{p})$, the unique optimal selling price is $p_s(\beta, \delta) = \hat{\beta}\Gamma(\beta\delta)$. For investors with the lowest continuation value, $\beta\delta = 0$, any $p_s(\beta, \delta) \in [\hat{\theta}\underline{p}, \underline{p}]$ is optimal; we assume $p_s(\beta, \delta) = \underline{p}$. For investors with higher continuation values, $\beta\delta \geq \bar{p}$, any $p_s \geq \beta\delta$ is optimal; we assume $p_s(\beta\delta) = \beta\delta$.

Given these beliefs,

$$\Delta(p) = \begin{cases} 0 & \text{if } p \in [\hat{\theta}\underline{p}, \underline{p}) \\ p/\hat{\beta} & \text{if } p \in [\underline{p}, \bar{p}] \\ (1+p)/2 & \text{if } p > \bar{p}. \end{cases}$$

Note that we are free to assign any beliefs at prices $p \in [\hat{\theta}\underline{p}, \underline{p})$, since all investors with $\beta = 0$ find such prices optimal. We choose to assign beliefs that only those investors who have $\delta = 0$ set these prices. Given these beliefs, optimal buying behavior sets any price $p_b(\beta) \in [\underline{p}, \bar{p}]$ if $\beta \geq \hat{\beta}$ and any prices $p_b(\beta) < \hat{\theta}\underline{p}$ if $\beta < \hat{\beta}$.

Finally, we can verify that equation (A3) ensures that the goods market clears.

Building on this logic, we can construct a continuum of semi-separating equilibria whenever the lowest asset quality held by investors with the lowest continuation value is smaller than the average asset quality held by investors with the lowest continuation value. If the support of (β, δ) is a rectangle, this requires that the lowest continuation value is zero, but otherwise it may hold more generally.

Other One-Price Equilibria. — The same logic supports a continuum of one-price equilibria with rationing at the equilibrium trading price. Equilibria are now characterized by three numbers, the equilibrium trading price p^*, the probability of trade at that price $\theta_1 \in [0,1]$, and the discount factor of the marginal buyer $\hat{\beta}$, but only two equations. First, the marginal buyer must be indifferent about buying all the assets offered for sale at p^*:

$$p^* = \hat{\beta} \frac{3 - p^*2}{3(2 - p^*)},$$

where $(3 - p^*2)/3(2 - p^*)$ is the average quality of assets held by investors with continuation value $v < p^*$. Second, the goods market must clear:

$$1 - \hat{\beta} = \theta_1 p^* 2(2 - p^*),$$

where $p^*(2 - p^*)$ is the fraction of sellers at the price p^*. There is a solution to these equations with $\theta_1 \in [0, 1]$ if $p^* \in [0.426, 0.634]$, giving

$$\theta_1 = \frac{3 - 6p^* + 2p^*2}{p^*2(2 - p^*)(3 - p^*2)}$$

and

$$\hat{\beta} = \frac{3p^*(2 - p^*)}{3 - p^*2}.$$

In such an equilibrium, the buyer-seller ratio satisfies

$$\Theta(p) = \begin{cases} \infty & \text{if } p < \theta_1 p^* \\ \theta_1 p^*/p & \text{if } p \in [\theta_1 p^*, p^*] \\ 0 & \text{if } p > p^*, \end{cases}$$

while the expected quality of assets for sale relative to the price is maximized at p^*.

To construct an equilibrium of this sort, we again discuss buying and selling behavior. All investors (β, δ) with continuation value $\beta\delta < p^*$ set price p^* in equilibrium, while those with higher continuation values set price $p_s(\beta, \delta) = \delta$. This pins down buyers' beliefs at prices above p^*. At prices between $\theta_1 p^*$ and p^*, rational beliefs requires that investors anticipate meeting sellers with zero continuation value. To support the equilibrium, we assume that they anticipate meeting sellers with zero-quality assets:

$$\Delta(p) = \begin{cases} 0 & \text{if } p < p^* \\ \frac{3-p^*2}{3(2-p^*)} & \text{if } p = p^* \\ p & \text{if } p > p^* \end{cases}$$

One can verify that $\Delta(p)/p$ is maximized at p^* for all $p^* \leq 0.634$, so buyers with $\beta \geq \hat{\beta}$ in fact prefer to pay this single price: $p_b(\beta) = p^*$ if $\beta \geq \hat{\beta}$ and $p_b(\beta) = 0$ otherwise. Finally, one can verify that the goods market clears.

Again, this logic shows how to construct a continuum of one-price equilibria whenever the lowest asset quality held by investors with the lowest continuation value is smaller than the average asset quality held by investors with the lowest continuation value.

n-PRICE EQUILIBRIA. — Our model also admits an n-dimensional set of n-price equilibria. Denote the prices by $p^1 < \cdots < p_n$; in equilibrium all trade occurs at these prices. Also let $\theta_1 > \cdots > \theta_n$ denote the buyer-seller ratios at these prices, with $\theta_1 \in (0, 1]$. Let $v_1 < \cdots < v_n$ denote the n critical continuation values who are indifferent between neighboring prices (so v_i is indifferent between setting prices p_i and p_{i+1} and v_n is indifferent between setting price p_n and setting a higher price at which she cannot sell). Finally, let $\hat{\beta}$ denote the discount factor of the marginal buyer. This gives us a total of $3n + 1$ variables. These must satisfy $2n + 1$ equations. The first n equations come from the indifference conditions of the marginal sellers:

$$\theta_i(p_i - v_i) = \theta_{i+1}(p_{i+1} - v_i) \text{ for } i \in \{1, \ldots, n-1\}$$

and $p_n = v_n$. The next n equations come from the marginal buyer's indifference about buying at any price. With our functional forms, this gives

$$p_i = \hat{\beta}\frac{3 - v_{i-1}^2 - v_{i-1}v_i - v_i^2}{3(2 - v_{i-1} - v_i)},$$

where $v_0 = 0$. The fraction is the average value of the assets held by investors with continuation value $v \in [v_{i-1}, v_i]$. Finally, the goods market must clear:

$$1 - \hat{\beta} = \sum_{i=1}^{n} \theta_i p_i (v_i(2 - v_i) - v_{i-1}(2 - v_{i-1}),$$

where $v_i(2-v_i)-v_{i-1}(2-v_{i-1})$ is the measure of investors who set price p_i, those with continuation values $v \in [v_{i-1}, v_i]$.

In equilibrium, the buyer-seller ratio satisfies

$$\Theta(p) = \begin{cases} \infty & p < p_0 \\ \frac{\theta_i(p_i - v_i)}{p - v_i} \text{ if} & p \in [p_i, p_{i+1}],\ i \in \{0, \ldots, n-1\} \\ 0 & p > p_n, \end{cases}$$

where $p_0 = \theta_1 p_1$ and $\theta_0 = 1$. Given this structure, only sellers with continuation value v_i find prices $p \in (p_i, p_{i+1})$ optimal for $i \in \{0, \ldots, n-1\}$. To support the equilibrium, we assume buyers anticipate that investors (β, δ) with $\beta = 1$ and $\delta = v_i$ set these prices. Finally, investors with $\beta\delta > p_n$ and $\delta = p$ set price

$p > p_n$. This pins down buyers' beliefs. The remainder of the construction of equilibrium is now standard.

In our parametric example, first suppose $\theta_1 = 1$. We find that for any value of $\theta_2 \in [0, 0.832]$, it is possible to construct an equilibrium with trade at two prices. Higher values of θ_2 are associated with lower values of p_1 (falling from 0.426 to 0.371), lower values of $p_2 = v_2$ (falling from 0.527 to 0.446), lower values of v_1 (falling from 0.426 to 0), and higher values of $\hat{\beta}$ (rising from 0.714 to 0.743). It does not seem possible to construct equilibria with $\theta_2 > 0.832$, because the system of equations would imply $v_1 < 0$. For lower values of θ_1, there is a smaller interval of θ_2 corresponding to an equilibrium, but the interval always exists.

The possibility that $\theta_1 < 1$ again hinges on the assumption that the lowest asset quality held by investors with the lowest continuation value is smaller than the average asset quality held by these investors. However, the remaining construction does not rely on this restriction and so appears to be completely general. For example, there are many n-price equilibria in the independent Pareto example that we use throughout the text.

Qualitatively an n-price equilibrium looks very similar to the semi-separating equilibrium. Investors with higher continuation values set weakly higher sale prices and sell with a weakly lower probability. Indeed, we conjecture that in the limit as n converges to infinity, the functions $\Theta(p)$ and $\Delta(p)$ in any n price equilibrium will be close to their values in some semi-separating equilibrium in the sense of the sup-norm.

A4. Mixed Equilibria

Equilibria may also feature a mix of mass points and continuous distributions. We discuss how to construct such equilibria here; it will be useful later in our analysis, so we break it into a separate section.

Take a set of points $v_1 < \cdots < v_n$, and construct pools of positive radius $\varepsilon_1, \ldots, \varepsilon_n$ around those points with $v_1 - \varepsilon_1 > \underline{v}$ and $v_i - \varepsilon_i > v_{i-1} + \varepsilon_i$ for all $i \in 2, \ldots, n$. We look for an equilibrium where any two sellers with continuation values in the same pool set the same price. That is, for all (β, δ) and (β', δ'), $p_s(\beta, \delta) = p_s(\beta', \delta')$ if and only if there exists an $i \in \{1, \ldots, n\}$ with $\beta\delta \in (v_i - \varepsilon_i, v_i + \varepsilon_i)$ and $\beta'\delta' \in (v_i - \varepsilon_i, v_i + \varepsilon_i)$.

Within each pool, the equilibrium price reflects the quality of the pool:

$$p_i = \hat{\beta} \frac{\int_{v_i - \varepsilon_i}^{v_i + \varepsilon_i} \Gamma(v) dH(v)}{H(v_i + \varepsilon_i) - H(v_i - \varepsilon_i)} \tag{A4}$$

Using the known functional forms, it is possible to simplify this and the subsequent expressions. Outside of these pools, the price is the fair one, $p_s(\beta, \delta) = \hat{\beta}\Gamma(\beta\delta)$.

We turn next to the sale probability. For sellers (β, δ) with the lowest continuation value, $\beta\delta \in [\underline{v}, v_1 - \varepsilon_1]$, the sale price is $\hat{\beta}\Gamma(\beta\delta)$ and the sale probability is as in the semi-separating equilibrium with the same value of the marginal buyer $\hat{\beta}$,

$$\Theta(\hat{\beta}\Gamma(\beta\delta)) = \exp\left(-\int_{\underline{p}}^{\hat{\beta}\Gamma(\beta\delta)} \frac{1}{p' - \Gamma^{-1}(p'/\hat{\beta})} dp'\right)$$

We then proceed recursively. Assume for some $i \in \{1, \ldots, n\}$, we have already found $\Theta(\hat{\beta}\Gamma(v_i - \varepsilon_i))$, the trading probability at the bottom of the pool. A seller with this continuation value must be indifferent about charging the separating price $\hat{\beta}\Gamma(v_i - \varepsilon_i)$ or charging the pooling price p_i:

$$\text{(A5)} \qquad \Theta(\hat{\beta}\Gamma(v_i - \varepsilon_i))\big(\hat{\beta}\Gamma(v_i - \varepsilon_i) - (v_i - \varepsilon_i)\big) = \Theta(p_i)\big(p_i - (v_i - \varepsilon_i)\big).$$

This pins down the trading probability in the pool, $\Theta(p_i)$. Next, we turn to the seller with continuation value $v_i + \varepsilon_i$. He must be indifferent between separating and pooling as well,

$$\text{(A6)} \qquad \Theta(\hat{\beta}\Gamma(v_i + \varepsilon_i))\big(\hat{\beta}\Gamma(v_i + \varepsilon_i) - (v_i + \varepsilon_i)\big) = \Theta(p_i)\big(p_i - (v_i + \varepsilon_i)\big),$$

which we solve for $\Theta(\hat{\beta}\Gamma(v_i + \varepsilon_i))$. Finally, sellers (β, δ) with $\beta\delta \in [v_i + \varepsilon_i, v_{i+1} - \varepsilon_{i+1}]$ must find the price $\hat{\beta}\Gamma(\beta\delta)$ optimal. It is straightforward to prove that this price is locally optimal if and only if the sale probability is proportional to its level in the semi-separating equilibrium. The value of $\Theta(\hat{\beta}\Gamma(v_i + \varepsilon_i))$ pins down the constant of proportionality:

$$\text{(A7)} \qquad \Theta(\hat{\beta}\Gamma(\beta\delta)) = \Theta(\hat{\beta}\Gamma(v_i + \varepsilon_i)) \exp\left(-\int_{\hat{\beta}\Gamma(v_i+\varepsilon_i)}^{\hat{\beta}\Gamma(\beta\delta)} \frac{1}{p' - \Gamma^{-1}(p'/\hat{\beta})} dp'\right).$$

This completes the recursion.

All that remains is pinning down the beliefs of a buyer at prices without trade. We make those as pessimistic as possible. In our example, this means that a buyer believes that if a price p is weakly optimal for a seller with continuation value v but no such seller sets the price, $\Delta(p) = v$ (and so the seller's discount factor is $\beta = 1$). Such beliefs always support the mixed equilibrium.

Omitted Proofs

PROOF OF PROPOSITION 1:

Fix $(\beta_1, \delta_1) \in S$ and $(\beta_2, \delta_2) \in S$ with $\beta_1\delta_1 < \beta_2\delta_2$. Part 1 of the definition of equilibrium implies

(B1)
$\min\{\Theta(p_s(\beta_1,\delta_1)),1\}(p_s(\beta_1,\delta_1)-\beta_1\delta_1) \geq \min\{\Theta(p_s(\beta_2,\delta_2)),1\}(p_s(\beta_2,\delta_2)-\beta_1\delta_1)$
(B2)
$\min\{\Theta(p_s(\beta_2,\delta_2)),1\}(p_s(\beta_2,\delta_2)-\beta_2\delta_2) \geq \min\{\Theta(p_s(\beta_1,\delta_1)),1\}(p_s(\beta_1,\delta_1)-\beta_2\delta_2).$

Add the inequalities and simplify to get

$$(\min\{\Theta(p_s(\beta_1,\delta_1)),1\}-\min\{\Theta(p_s(\beta_2,\delta_2)),1\})(\beta_2\delta_2-\beta_1\delta_1)\geq 0$$

or $\min\{\Theta(p_s(\beta_1,\delta_1)),1\} \geq \min\{\Theta(p_s(\beta_2,\delta_2)),1\}$.

Now assume the seller with the higher continuation value sells with a positive probability, $\Theta(p_s(\beta_2,\delta_2)) > 0$. Divide the left hand side of inequality (B2) by $\min\{\Theta(p_s(\beta_2,\delta_2)),1\}$ and the right hand side by the larger quantity $\min\{\Theta(p_s(\beta_1,\delta_1)),1\}$ to prove $p_s(\beta_2,\delta_2) \geq p_s(\beta_1,\delta_1)$.

Finally, suppose to find a contradiction that $\Theta(p_s(\beta_1,\delta_1)) < \Theta(p_s(\beta_2,\delta_2))$. If $\Theta(p_s(\beta_1,\delta_1)) < 1$, this contradicts the first step. On the other hand, if $\Theta(p_s(\beta_1,\delta_1)) \geq 1$, $\min\{\Theta(p_s(\beta_2,\delta_2)),1\} = 1$ as well. But then inequality (B1) implies $p_s(\beta_1,\delta_1) \geq p_s(\beta_2,\delta_2)$, a contradiction.

PROOF OF PROPOSITION 2:

Let $\hat{\beta}$ be the infimal value of $p/\Delta(p)$ among p with $\Theta(p) < \infty$. This means that for all $\beta > \hat{\beta}$, there exists a p with $\Theta(p) < \infty$ such that $\beta > p/\Delta(p)$, or equivalent $\beta\Delta(p)/p > 1$. Part 2 of the definition of equilibrium implies that buyers with that discount factor buy at some such price. If $\beta < \hat{\beta}$, then for any p with $\Theta(p) < \infty$, $\beta\Delta(p)/p < 1$. Buyers with this discount factor are better off not buying, i.e. setting a price such that $\Theta(p_b(\beta)) = \infty$.

Now suppose there is a seller $(\beta,\delta) \in S$ with $0 < \Theta(p_s(\beta,\delta)) < \infty$. The definition of $\hat{\beta}$ implies $\hat{\beta} \leq p_s(\beta,\delta)/\Delta(p_s(\beta,\delta))$. If the inequality were strict, part 2 of the definition of equilibrium implies there is there is no buyer who finds the price $p_s(\beta,\delta)$ optimal, contradicting part 4 of the definition of equilibrium. Therefore $\hat{\beta} = p_s(\beta,\delta)/\Delta(p_s(\beta,\delta))$. It follows immediately that all buyers with $\beta \geq \hat{\beta}$ are indifferent about buying any of the assets sold in equilibrium.

We turn next to the proof of Proposition 3. To prove this, we first state and prove four preliminary Lemmas.

LEMMA 1: *Consider an equilibrium with multidimensional private information. Let* $\mathbb{V} : \mathbb{R}_+ \rightrightarrows V$ *denote the set of sellers' continuation values* v *for which the* *price p is weakly optimal:*

$$v \in \mathbb{V}(p) \Leftrightarrow p \in \arg\max_{p' \geq v} \quad \big(\min\{\Theta(p'), 1\}(p' - v)\big).$$

If $\Theta(p) < \infty$, the set $\mathbb{V}(p)$ is nonempty. In addition, for any $p_1 < p_2$ with $\mathbb{V}(p_1)$ nonempty, $\Theta(p_2) < 1$. For any $p_1 < p_2$ with $\Theta(p_1) = 0$, $\Theta(p_2) = 0$ as well.

PROOF OF LEMMA 1:

First observe that part 3(b) of the definition of equilibrium with multidimensional private information implies that the set $\mathbb{V}(p)$ is nonempty when $\Theta(p)$ is finite.

Now take any $p_1 < p_2$ with $v \in \mathbb{V}(p_1)$. For an invester with continuation value v, p_1 gives weakly higher profit than p_2:

$$\min\{\Theta(p_1), 1\}(p_1 - v) \geq \min\{\Theta(p_2), 1\}(p_2 - v). \tag{B3}$$

Notice $p_2 > p_1 \geq v$. If $p_1 = v$, (B3) implies $\Theta(p_2) = 0$. If $p_1 > v$, (B3) implies $\min\{\Theta(p_2), 1\} < \min\{\Theta(p_1), 1\} \leq 1$, so $\Theta(p_2) < 1$. Therefore, $\Theta(p_2) < 1$ as long as $\mathbb{V}(p_1)$ is not empty.

Moreover, if $\Theta(p_1) = 0$, (B3) implies $\Theta(p_2) = 0$.

LEMMA 2: *Consider an equilibrium with multidimensional private information. Take any $p_1 < p_2$ with $v_1 \in \mathbb{V}(p_1)$ and $v_2 \in \mathbb{V}(p_2)$. If $\Theta(p_1) > 0$ then $v_1 \leq v_2$. Moreover, $\mathbb{V}(p)$ is convex and closed.*

PROOF OF LEMMA 2:

A seller with continuation value v_2 either finds the price p_1 suboptimal because $p_1 < v_2$ or prefers p_2 to p_1. First suppose $p_1 < v_2$. Since $v_1 \in \mathbb{V}(p_1)$, $v_1 \leq p_1$, proving $v_1 < v_2$. Second suppose v_2 prefers p_2 to p_1 and $p_1 \geq v_2$:

$$\min\{\Theta(p_2), 1\}(p_2 - v_2) \geq \min\{\Theta(p_1), 1\}(p_1 - v_2). \tag{B4}$$

Similarly, a seller with continuation value v_1 weakly prefers p_1 to p_2, as in (B3). If $\Theta(p_2) = 0$, $\Theta(p_1) > 0$ implies $p_1 = v_2$ so again $v_1 \leq v_2$. If instead $\Theta(p_2) > 0$, multiply inequalities (B3) and (B4) and simplify to get $(p_2 - p_1)(v_2 - v_1) \geq 0$, which proves that $v_2 \geq v_1$.

To prove $\mathbb{V}(p)$ is convex, take any p and $v_1 < v_2$ with $v_1, v_2 \in \mathbb{V}(p)$. Fix any $\tilde{v} = \alpha v_1 + (1 - \alpha)v_2, \alpha \in (0, 1)$, so $p \geq v_2 > \tilde{v}$. We can find $\tilde{p}$ such that $\tilde{v} \in \mathbb{V}(\tilde{p})$

by setting $\tilde{p} = p_s(\beta, \delta)$ for some $\beta\delta = \tilde{v}$. Then

$$\begin{aligned}\min\{\Theta(\tilde{p}), 1\}(\tilde{p} - \tilde{v}) &= \alpha \min\{\Theta(\tilde{p}), 1\}(\tilde{p} - v_1) + (1-\alpha)\min\{\Theta(\tilde{p}), 1\}(\tilde{p} - v_2) \\ &\leq \alpha \min\{\Theta(p), 1\}(p - v_1) + (1-\alpha)\min\{\Theta(p), 1\}(p - v_2) \\ &= \min\{\Theta(p), 1\}(p - \tilde{v})\end{aligned}$$

where the inequality comes from the fact that sellers with continuation value v_1 or v_2 weakly prefer p to any $\tilde{p} \geq 0$. Therefore $\tilde{v} \in \mathbb{V}(p)$ for all $\tilde{v} \in (v_1, v_2)$.

To prove $\mathbb{V}(p)$ is closed, suppose there exists a sequence $\{v_n\} \to v$ with $v_n \in \mathbb{V}(p)$ for all n but $v \notin \mathbb{V}(p)$. Since $p \geq v_n$ for all n, $p \geq v$ as well. The definition of $\mathbb{V}$ then implies that there exists a $\tilde{p} \geq v$ with

$$\min\{\Theta(\tilde{p}), 1\}(\tilde{p} - v) - \min\{\Theta(p), 1\}(p - v) \equiv \varepsilon > 0$$

But since $\{v_n\} \to v$, there exists an N such that for all $n > N$,

$$\big(\min\{\Theta(\tilde{p}), 1\} - \min\{\Theta(p), 1\}\big)(v_n - v) < \varepsilon.$$

Using the definition of ε, this implies $\min\{\Theta(\tilde{p}), 1\}(\tilde{p} - v_n) > \min\{\Theta(p), 1\}(p - v_n) \geq 0$, and in particular $\tilde{p} > v_n$, which contradicts $v_n \in \mathbb{V}(p)$.

LEMMA 3: *Impose Assumption 1 and consider an equilibrium with unidimensional private information. Let* $\mathbb{T} = \{p : 0 < \Theta(p) < \infty\}$. *There exists a function* $\mathcal{V} : \mathbb{T} \to V$ *such that* $\mathbb{V}(p) = \{\mathcal{V}(p)\}$ *for all* $p \in \mathbb{T}$. *Moreover,* $\mathcal{V}$ *is continuous and non-decreasing.*

PROOF OF LEMMA 3:

First, we want to show that when $p \in \mathbb{T}$, $\mathbb{V}(p)$ is a singleton. Notice if $p = 0$, by definition, $\mathbb{V}(p)$ has at most one element, that is 0.

Now consider the case of $p > 0$. Suppose $\mathbb{V}(p)$ has more than one element. Then Lemma 2 implies there must be $v_1 < v_2$, such that $\mathbb{V}(p) = [v_1, v_2]$. Lemma 2 also implies that p is the only optimal sale price for $v \in (v_1, v_2)$, i.e., $P(v) = p$ if $v \in (v_1, v_2)$. Then part 3(a) of the definition of equilibrium with multidimensional private information implies the average quality of asset held by sellers with this continuation value is

$$\Delta(p) = \frac{\int_{v_1}^{v_2} \Gamma(v)h(v)dv}{\int_{v_1}^{v_2} h(v)dv}.$$

Monotonicity of Γ (Assumption 1) implies

$$\Gamma(v_1) < \Delta(p) < \Gamma(v_2) \leq \Delta(p')$$

for any price $p' > p$. Note that the last inequality uses the definition of the equilibrium with unidimensional private information. Thus $\lim_{x \to p^+} \frac{\Delta(x)}{x} > \frac{\Delta(p)}{p}$.

We can use this to prove that no buyer finds setting the price p optimal. If there were such a buyer, he must have $\beta\Delta(p) \geq p$ by part 2 of the definition of equilibrium with multidimensional private information. So take any $p' > p$ with $\frac{\Delta(p')}{p'} > \frac{\Delta(p)}{p}$; this is feasible because $\frac{\Delta(x)}{x}$ jumps up at p^+. Since $\Theta(p) < \infty$, Lemma 1 implies $\Theta(p') < 1$. Then

$$\min\{\Theta(p)^{-1}, 1\}\left(\frac{\beta\Delta(p)}{p} - 1\right) \leq \frac{\beta\Delta(p)}{p} - 1 < \frac{\beta\Delta(p')}{p'} - 1 = \min\{\Theta(p')^{-1}, 1\}\left(\frac{\beta\Delta(p')}{p'} - 1\right).$$

The first inequality uses $\min\{\Theta(p)^{-1}, 1\} \leq 1$ and $\beta\Delta(p) \geq p$. The second uses $\Delta(p)/p < \Delta(p')/p'$. The equality uses $\Theta(p') < 1$. This proves all buyers prefer p' to p.

We now have a contradiction. The measure of buyers setting price p is zero, $d\mu_b(p) = 0$, while the measure of sellers setting price p is positive, $d\mu_s(p) = \int_{v_1}^{v_2} h(v)dv$. This is inconsistent with part 4 of the definition of equilibrium with multidimensional private information, $d\mu_b(p) = \Theta(p)d\mu_s(p)$.

So far we show if $\Theta(p) > 0$, $\mathbb{V}(p)$ has at most 1 element. Then by Lemma 1, when $\Theta(p) \in \mathbb{T}$, there exists a $\mathcal{V}(p) \in V$ such that $\mathbb{V}(p) = \{\mathcal{V}(p)\}$. In addition, by Lemma 2, $\mathcal{V}$ is non-decreasing.

Now suppose $\mathcal{V}$ has a discontinuity at $p \in \mathbb{T}$. Since $\mathcal{V}$ is non-decreasing, either $\lim_{p' \to p^-} \mathcal{V}(p') < \mathcal{V}(p)$ or $\mathcal{V}(p) < \lim_{p' \to p^+}$. If $\lim_{p' \to p^-} \mathcal{V}(p') < \mathcal{V}(p)$, pick any $\tilde{v} \in (\lim_{p' \to p^-} \mathcal{V}(p'), \mathcal{V}(p))$, and let $\tilde{p}$ be a price that investors with continuation value $\tilde{v}$ find weakly optimal. By Lemma 2, $\tilde{p} = p$, so $\tilde{v} \in \mathbb{V}(p)$, contradicting with the first part of this lemma. Similarly, if $\mathcal{V}(p) < \lim_{p' \to p^+}$, by picking any $\tilde{v} \in (\mathcal{V}(p), \lim_{p' \to p^+}$, we can find another contradiction. Therefore, $\mathcal{V}$ is continuous on $\mathbb{T}$.

LEMMA 4: *Impose Assumption 1 and consider an equilibrium with unidimensional private information. If $p \in int(\mathbb{T})$, $\Theta'(p) = -\Theta(p)/(p - \mathcal{V}(p))$.*

PROOF OF LEMMA 4:

First want to show if $p \in \text{int}\mathbb{T}$, $p > \mathcal{V}(p)$. Suppose not, then $p = \mathcal{V}(p)$. Since $p \in \text{int}\mathbb{T}$, there exist $p' < p$ such that $\theta(p') > 0$. But then investors with continuation value $\mathcal{V}(p)$ will be better off posting p, contradicting the defination of $\mathcal{V}$.

Consider an arbitrary sequence of prices $\{p^k\}$ with $0 < \Theta(p^k) < \infty$ and converging to p. From $p > \mathcal{V}(p)$, we know $p^k > \mathcal{V}(p)$ for sufficiently large k. Since $\mathcal{V}$ is continuous at p by Lemma 3, $p^k > \mathcal{V}(p^k)$ for sufficiently large k. We have $\min\{\Theta(p), 1\}(p - \mathcal{V}(p)) \geq \min\{\Theta(p^k), 1\}(p^k - \mathcal{V}(p))$, from the optimization condition for seller with continuation value p; and $\min\{\Theta(p^k), 1\}(p^k - \mathcal{V}(p^k)) \geq$

$\min\{\Theta(p),1\}(p-\mathcal{V}(p^k))$ from the optimization condition for seller with continuation value p^k. In addition, by Lemma 1, if $p \in \mathrm{int}\mathbb{T}$, $\Theta(p)<1$ and $\Theta(p^k)<1$ when k is large enough. Combining all the conditions, we have

$$\Theta(p)\frac{p-\mathcal{V}(p)}{p^k-\mathcal{V}(p)} \geq \Theta(p^k) \geq \Theta(p)\frac{p-\mathcal{V}(p^k)}{p^k-\mathcal{V}(p^k)}.$$

The two bounds converge to $\Theta(p)$, proving that $\Theta(p^k)\to\Theta(p)$. Rearranging above inequalities, we also have

$$\frac{-\Theta(p^k)}{p-\mathcal{V}(p)} \geq \frac{\Theta(p^k)-\Theta(p)}{p^k-p} \geq \frac{-\Theta(p)}{p^k-\mathcal{V}(p^k)},$$

if $p^k>p$ and

$$\frac{-\Theta(p^k)}{p-\mathcal{V}(p)} \leq \frac{\Theta(p^k)-\Theta(p)}{p^k-p} \leq \frac{-\Theta(p)}{p^k-\mathcal{V}(p^k)},$$

if $p^k<p$. Again both bounds converge to $-\Theta(p)/(p-\mathcal{V}(p))$, establishing the result.

PROOF OF PROPOSITION 3:

First we assume that $\beta\Gamma(\underline{v}) \leq \underline{v}$ for all $\beta \in B$ and show that we can construct an equilibrium with no trade. Set $P(v)=\max\{\bar{\beta}\Gamma(v),v\}$ for all v, where $\bar{\beta}=\sup B$. Also set $\Theta(p)=0$ for all $p \geq P(\underline{v})$ and $\Theta(p)=\infty$ otherwise. Finally, assume $\Delta(p)=p/\bar{\beta}$ for all p and $p_b(\beta)=0$. It is easy to verify that this is an equilibrium with unidimensional private information.

Now to find a contradiction, suppose that $\beta\Gamma(\underline{v}) \leq \underline{v}$ for all $\beta \in B$ and there is an equilibrium where a positive measure of investors trades. This means $d\mu_b(p)>0$ for some $p>\underline{v}$, so we have $\Theta(p)>0$. Sellers' optimality implies $P(\underline{v})>\underline{v}$ with $\Theta(P(\underline{v}))>0$. Lemma 3 implies only sellers with the lowest continuation value set this price, and therefore $\Delta(P(\underline{v}))=\Gamma(\underline{v})$. Stringing together these inequalities gives $\beta\Delta(P(\underline{v}))<P(\underline{v})$ for all $\beta\in B$, and so part 2 of the definition of equilibrium with multidimensional private information implies no buyer sets this price, a contradiction.

For the remainder of the proof, assume $\bar{\beta}\Gamma(\underline{v})>\underline{v}$. Let $\underline{p}=\sup\{p:\Theta(p)=\infty\}$ and $\bar{p}=\inf\{p:\Theta(p)=0\}$. Using Lemma 1, we have $\underline{p}\leq\bar{p}$, $\Theta(p)=\infty$ if $p<\underline{p}$, $\Theta(p)=0$ if $p>\bar{p}$, and $\Theta(p)\in(0,1)$ if $p\in(\underline{p},\bar{p})$.

We first rule out the possibility of an equilibrium in which $\underline{p}=\bar{p}$. If $\bar{p}>\underline{v}$ and $\Theta(\bar{p})<1$, there is no $p_s(\beta,\delta)$ solves the maximization problem in part 1 of the definition of equilibrium with multidimensional private information for $\beta\delta\in[\underline{v},\bar{p})$, a contradiction. If $\bar{p}>\underline{v}$ and $\Theta(\bar{p})\geq 1$, $p_s(\beta,\delta)=\bar{p}$ for $\beta\delta\in[\underline{v},\bar{p})$, which contradicts Lemma 3. Therefore it must be the case that $\bar{p}\leq\underline{v}$ and hence $\bar{\beta}\Gamma(\underline{v})>\bar{p}$. Part 1 of the definition of equilibrium with multidimensional private information implies $p_s(\beta,\delta) \geq \beta\delta > \bar{p}$ for all $\beta\delta > \underline{v}$, hence sellers (β,δ) don't sell and $d\mu_s(\bar{p}) = 0$. If $\Theta(\bar{p}) < \infty$, the second part of the definition of equilibrium with unidimensional private information and Assumption 1 implies $\bar{\beta}\Delta(\bar{p}) \geq \beta\Gamma(\underline{v}) > \bar{p}$: all sufficiently patient buyers would post $\bar{p}$, so $\mu_b(\bar{p}) > 0$ contradicting the market clearing condition. If $\Theta(\bar{p}) = \infty$, the second part of the definition of equilibrium with unidimensional private information and Assumption 1 implies $\lim_{p\to\bar{p}^+} \bar{\beta}\Delta(p) \geq \beta\Gamma(\underline{v}) > \bar{p}$. Then there is no $p_b(\beta)$ solves the maximization problem in part 2 of the definition of equilibrium with multidimensional private information for sufficiently large β. This cannot be an equilibrium.

Now consider the case of $\underline{p} < \bar{p}$. From the definition of $\mathcal{V}$, we have for all $p \in (\underline{p}, \bar{p})$, $p > \mathcal{V}(p)$. The differential equation for Θ in Lemma 4 then applies

$$\Theta(p) = \lambda \exp\left(-\int_{\underline{p}}^{p} \frac{1}{\tilde{p} - \mathcal{V}(\tilde{p})} d\tilde{p}\right) \tag{B5}$$

for all $p \in (\underline{p}, \bar{p})$ and some constant of integration $\lambda > 0$. In addition, Lemma 1 ensures that $\lambda \leq 1$ so that $\Theta(p) < 1$ for all $p > \underline{p}$. Define $\mathcal{V}(\underline{p})$ and $\mathcal{V}(\bar{p})$ as the associated right limit / left limit of $\mathcal{V}$, respectively. Notice $\underline{p} > \mathcal{V}(\underline{p})$, otherwise sellers with continuation values $\mathcal{V}(p)$, with $p \to \underline{p}$ are better of posting $(\underline{p} + \bar{p})/2$ instead of p. If $\lambda < 1$, an investor with continuation value $v = \mathcal{V}(p)$ where p approaches $\underline{p}$ from above would rather earn a profit approaching $\underline{p} - \mathcal{V}(\underline{p})$ by selling with probability 1 at price $\underline{p} - \varepsilon$ for $\varepsilon \to 0^+$, than earn a profit approaching $\lambda(\underline{p} - \mathcal{V}(\underline{p}))$ by selling at price p, a contradiction. Therefore $\lambda = 1$.

Turn now to the buyers' problem. From Lemma 3, only a seller with continuation value $\mathcal{V}(p)$ sells at price p. Then from part 3 of the definition of equilibrium, $\Delta(p) = \Gamma(\mathcal{V}(p))$. For buyers to be willing to purchase at all prices $p \in (\underline{p}, \bar{p})$, it must be the case that $p/\Delta(p) = \hat{\beta}$ for some constant $\hat{\beta} \leq \bar{\beta}$, or equivalently $P(v) = \hat{\beta}\Gamma(v)$ for all $v \in (\mathcal{V}(\underline{p}), \mathcal{V}(\bar{p}))$. From assumption 1, P is a continuously differentiable function, so we can substitute this into equation (B5). Changing the variable of integration gives

$$\Theta(P(v)) = \exp\left(-\int_{\mathcal{V}(\underline{p})}^{v} \frac{\hat{\beta}\Gamma'(\tilde{v})}{\hat{\beta}\Gamma(\tilde{v}) - \tilde{v}} d\tilde{v}\right) \tag{B6}$$

for $v \in (\mathcal{V}(\underline{p}), \mathcal{V}(\bar{p}))$. First, Lemma 2 implies $\mathcal{V}(\underline{p}) = \underline{v}$. Second, want to show $\mathcal{V}(\bar{p}) = \bar{p}$. Suppose not, then $\mathcal{V}(\bar{p}) < \bar{p}$. Consider any $v \in (\mathcal{V}(\bar{p}), \bar{p})$. If $\Theta(P(v)) = 0$, sellers with continuation value v can do better off by posting price $\bar{p} - \varepsilon$ for sufficiently small positive ε, a contradiction. If $\Theta(P(v)) > 0$, from Lemma 2, $P(v) \geq \bar{p}$. Then it has to be $P(v) = \bar{p}$, which contradicts Lemma 3.

Now we have any seller with continuation value $v < \bar{p}$ sets price $P(v) = \hat{\beta}\Gamma(v)$, while any seller with continuation value higher than $\bar{p}$ is indifferent about all prices larger than or equal to his continuation value and in particular is willing to set a price such that $P(v) \geq \hat{\beta}\Gamma(v)$. This ensures that buyers with $\beta > \hat{\beta}$ are indifferent about buying at any price $p \in (\underline{p}, \bar{p})$ and prefer those prices to higher prices. Buyers with continuation values lower than $\hat{\beta}$ set lower prices and do not succeed in buying. To find an equilibrium, we simply allocate the buyers to the different prices in a way that ensures the appropriate buyer-seller ratio at each price. This is feasible if the total wealth of buyers with $\beta > \hat{\beta}$ is exactly enough to purchase the assets sold by sellers with $v \in [\underline{v}, \bar{p}]$:

$$\int_{\beta \geq \hat{\beta}} g_b(\beta) d(\beta) = \int_{v \leq \bar{p}} P(v)\Theta(P(v))h(v)dv. \tag{B7}$$

This is the same as equation (A1). The left hand side is decreasing in $\hat{\beta}$, equal to 0 when $\hat{\beta} = \bar{\beta}$. The right hand side is strictly positive when $\hat{\beta} = \bar{\beta}$ and increasing in $\hat{\beta}$. To prove monotonicity of right hand side, note first that $\bar{p}$, defined as the smallest solution to $x \geq \hat{\beta}\Gamma(x)$, is nondecreasing in $\hat{\beta}$. So are $P(v) = \hat{\beta}\Gamma(v)$ and $\Theta(P(v))$ defined in equation (B6). Therefore there is a unique $\hat{\beta}$ that solves equation (B7).

PROOF OF PROPOSITION 4:

Let $\hat{\beta}$ denote the marginal buyer in the semi-separating equilibrium and $\hat{\beta}_m$ denote the marginal buyer in the mixed equilibrium. We first prove that if a mixed equilibrium Pareto dominates the semi-separating equilibrium, it must have $\hat{\beta}_m = \hat{\beta}$. If $\hat{\beta}_m > \hat{\beta}$, all buyers with $\beta > \hat{\beta}$ are worse off in the mixed equilibrium, since they live in autarky rather than buying, and so it does not Pareto dominate the semi-separating equilibrium. If $\hat{\beta}_m < \hat{\beta}$, consider a seller with the lowest continuation value $\underline{v}$. He sells for sure in both equilibria, earning a price equal to $\Gamma(\underline{v})$ times the discount factor of the marginal buyer. A reduction in the discount factor of the marginal buyer therefore makes him worse off and so again the mixed equilibrium does not Pareto dominate the semi-separating equilibrium.

The bulk of the proof then compares a semi-separating equilibrium with a given value of $\hat{\beta}$ to a mixed equilibrium with the same value of $\hat{\beta}_m = \hat{\beta}$. Here we focus on the limiting case where the pool sizes are arbitrarily small. For the construction of the mixed equilibrium when the pool sizes are not necessarily small, see Appendix A.A4; we use those results here.

We start by approximating the price charged by each pool. Taking a Taylor expansion of equation (A4) in a neighborhood of v_i, we obtain

$$p_i \approx \hat{\beta}\Gamma(v_i) + \frac{\hat{\beta}}{6}\left(\Gamma''(v_i) + \frac{2\Gamma'(v_i)h'(v_i)}{h(v_i)}\right)\varepsilon_i^2 + O(\varepsilon_i^3). \tag{B8}$$

We next turn to the trading probabilities. For notational convenience, we let

$\Theta(P(v))$ and $\omega(v)$ denote the equilibrium trading probabilities for a seller with continuation value v in the semi-separating and mixed equilibrium. The latter probabilities are defined recursively in equations (A5), (A6) and (A7).

First, we claim that the mixed equilibrium makes all sellers better off than the the semi-separating equilibrium if and only if $\omega(v_i+\varepsilon_i) \geq \Theta(P(v_i+\varepsilon_i))$ for $i=1,\ldots,n$. For $v_i+\varepsilon_i$, pooling alters the probability of trade but not the price, and so this seller is better off if and only if $\omega(v_i+\varepsilon_i) \geq \Theta(P(v_i+\varepsilon_i))$. This proves the "only if" part of the statement. To prove the "if" part, we turn to the other sellers. In the mixed equilibrium, all individuals with $v \in (v_i-\varepsilon_i, v_i+\varepsilon_i)$ trade with the same probability and at the same price. Moreover, $v_i+\varepsilon_i$ is indifferent about trading at that price. If trading in the pool makes $v_i+\varepsilon_i$ better off, then it makes all the members of the pool, who have lower quality assets better off. Additionally, for individuals who are separating in the mixed equilibrium, $v \in (v_i+\varepsilon_i, v_{i+1}-\varepsilon_{i+1})$, pooling raises the probability of trade by the same proportion as it raises the probability of trade for $v_i+\varepsilon_i$ and again it has no effect on their trading price. This establishes the claim.

Next we compute $\omega(v_i+\varepsilon_i)/\Theta(P(v_i+\varepsilon_i))$. Use equations (A5) and (A6) to get

$$\frac{\omega(v_i+\varepsilon_i)}{\omega(v_i-\varepsilon_i)} = \frac{\left(\hat{\beta}\Gamma(v_i-\varepsilon_i)-(v_i-\varepsilon_i)\right)\left(p_i-(v_i+\varepsilon_i)\right)}{\left(\hat{\beta}\Gamma(v_i+\varepsilon_i)-(v_i+\varepsilon_i)\right)\left(p_i-(v_i-\varepsilon_i)\right)}.$$

Equation (A7) implies

$$\frac{\omega(v_i+\varepsilon_i)}{\omega(v_i-\varepsilon_i)} = \exp\left(-\int_{v_i-\varepsilon_i}^{v_i+\varepsilon_i} \frac{\hat{\beta}\Gamma'(\tilde{v})}{\hat{\beta}\Gamma(\tilde{v})-\tilde{v}} d\tilde{v}\right).$$

Finally, the relative trading probabilities between the mixed and semi-separating equilibria are constant within the separating regions:

$$\frac{\omega(v_i-\varepsilon_i)}{\Theta(P(v_i-\varepsilon_i))} = \frac{\omega(v_{i-1}+\varepsilon_{i-1})}{\Theta(P(v_{i-1}+\varepsilon_{i+1}))},$$

where $v_0+\varepsilon_0 \equiv \underline{v}$ and $\omega(\underline{v}) = \Theta(P(\underline{v})) = 1$. Combining these three equalities and solving the recursion gives

$$\frac{\omega(v_i+\varepsilon_i)}{\Theta(P(v_i+\varepsilon_i))} = \prod_{j=1}^{i} \frac{\left(\hat{\beta}\Gamma(v_j-\varepsilon_j)-(v_j-\varepsilon_j)\right)\left(p_j-(v_j+\varepsilon_j)\right)}{\left(\hat{\beta}\Gamma(v_j+\varepsilon_j)-(v_j+\varepsilon_j)\right)\left(p_j-(v_j-\varepsilon_j)\right)} \exp\left(\int_{v_j-\varepsilon_j}^{v_j+\varepsilon_j} \frac{\hat{\beta}\Gamma'(\tilde{v})}{\hat{\beta}\Gamma(\tilde{v})-\tilde{v}} d\tilde{v}\right)$$

for $i=1,\ldots,n$. Thus all sellers are better off in the mixed equilibrium than the semi-separating equilibrium if and only if

$$\sum_{j=1}^{i}\left(\log\left(\frac{\big(\hat\beta\Gamma(v_i-\varepsilon_i)-(v_i-\varepsilon_i)\big)\big(p_i-(v_i+\varepsilon_i)\big)}{\big(\hat\beta\Gamma(v_i+\varepsilon_i)-(v_i+\varepsilon_i)\big)\big(p_i-(v_i-\varepsilon_i)\big)}\right)+\int_{v_j-\varepsilon_j}^{v_j+\varepsilon_j}\frac{\hat\beta\Gamma'(\tilde v)}{\hat\beta\Gamma(\tilde v)-\tilde v}d\tilde v\right)\geq 0$$

for $i=1,\dots,n$. Finally, perform a Taylor expansion of this sum near $\varepsilon_j=0$, using the approximation for p_i given in equation (B8). This gives
(B9)

$$\log\left(\frac{\omega(v_i+\varepsilon_i)}{\Theta(P(v_i+\varepsilon_i))}\right)=\sum_{j=1}^{i}\left(\frac{2\hat\beta\Gamma'(v_j)}{3(\hat\beta\Gamma(v_j)-v_j)^2}\left(\frac{h'(v_j)}{h(v_j)}-\frac{2-\hat\beta\Gamma'(v_j)}{\hat\beta\Gamma(v_j)-v_j}\right)\varepsilon_j^3+O(\varepsilon_j^4)\right).$$

All sellers are better off in the mixed equilibrium if and only if this is non-negative.

Now we turn to the amount of consumption goods buyers use to purchase assets in the first period. We compute the difference in this cost in the mixed equilibrium compared to the semi-separating equilibrium:

$$\begin{aligned}\sum_{i=1}^{n}\left(p_i\omega(v_i)\big(H(v_i+\varepsilon_i)-H(v_i-\varepsilon_i)\big)+\frac{\omega(v_i+\varepsilon_i)}{\Theta(P(v_i+\varepsilon_i))}\int_{v_i+\varepsilon_i}^{v_{i+1}-\varepsilon_{i+1}}\hat\beta\Gamma(v)\omega(v)dH(v)\right)\\ -\int_{v_1-\varepsilon_1}^{v_{n+1}-\varepsilon_{n+1}}\hat\beta\Gamma(v)\Theta(P(v))dH(v).\end{aligned}$$

The first term on the first line is the cost within the pooling regions and the second term is the cost in the separating regions in the mixed equilibrium. The second line is the cost in the semi-separating equilibrium.

We substitute the previous expressions for $\omega(v_i)$ and $\omega(v_i+\varepsilon_i)/\Theta(P(v_i+\varepsilon_i))$ into this expressions, then perform a Taylor expansion of this sum near $\varepsilon_j=0$, using the approximation for p_i given in equation (B8). This gives that the increase in first period cost is

$$\text{(B10)}\quad\begin{aligned}\sum_{i=1}^{n}\frac{2\hat\beta\Gamma'(v_i)}{3(\hat\beta\Gamma(v_i)-v_i)^2}\Bigg(\left(\frac{h'(v_i)}{h(v_i)}-\frac{2-\hat\beta\Gamma'(v_i)}{\hat\beta\Gamma(v_i)-v_i}\right)\int_{v_i}^{\infty}\hat\beta\Gamma(v)\Theta(P(v))dH(v)\\ +\hat\beta\Theta(P(v_i))h(v_i)(2\Gamma(v_i)-v_i\Gamma'(v_i))\Bigg)\varepsilon_i^3+\sum_{i=1}^{n}O(\varepsilon_i^4).\end{aligned}$$

A quick comparison of this expression with equation (B9) shows that the first period cost of the mixed equilbrium exceeds the first period cost of the semi-separating equilibrium with the same value of $\hat\beta$ whenever the mixed equilibrium leaves all sellers better off and the elasticity of Γ is smaller than 2. Because $G_b(\beta)$ is continuous, this is inconsistent with Part 4 of the definition of equilibrium, market clearing.

Pareto Efficient Allocations

C1. Incentive Feasible Allocations

We start by using the revelation principle to define the set of incentive compatible and feasible allocations. Each seller and buyer reports his or her private information to a mechanism, which then recommends certain trades. Without loss of generality, we focus on incentive-compatible mechanisms and we verify that the resulting trades are feasible.

We start with buyers. Each buyer reports her discount factor β to the mechanism and receives consumption $c_1^B(\beta)$ in period 1 and $c_2^B(\beta)$ in period 2. The mechanism must be incentive compatible, so a buyer prefers to report her true type β rather than misreporting it as some other $\tilde{\beta}$:

$$u^B(\beta) = c_1^B(\beta) - 1 + \beta c_2^B(\beta) \geq c_1^B(\tilde{\beta}) - 1 + \beta c_2^B(\tilde{\beta}) \tag{C1}$$

for all β and $\tilde{\beta}$, where $u^B(\beta)$ is the buyer's gain from trade. In addition, the mechanism must satisfy the buyer's participation constraint, $u^B(\beta) \geq 0$ for all β.

Turning now to sellers, each seller reports his continuation value v to the mechanism, getting expected consumption $c^S(v)$ in period 1 and giving up his asset with probability $\omega(v)$.[16] Again, the mechanism must be incentive compatible, so a seller prefers to report his true type v rather than misreporting it as some other $\tilde{v}$:

$$u^S(v) = c^S(v) - v\omega(v) \geq c^S(\tilde{v}) - \omega(\tilde{v})v$$

for all v and $\tilde{v}$, where $u^S(v)$ is the seller's gain from trade. In addition, the mechanism must satisfy the seller's participation constraint, $u^S(v) \geq 0$ for all v.

Standard arguments imply that the seller's mechanism is incentive compatible if and only if $\omega(v) \in [0, 1]$ is non-increasing and

$$c^S(v) = \int_v^{\bar{v}} \omega(x)dx + v\omega(v) + k$$

for some constant k. Substituting this back into the expression for $u^S(v)$ in the previous paragraph gives

$$u^S(v) = \int_v^{\bar{v}} \omega(x)dx + k. \tag{C2}$$

[16] We assume that a seller only reports his continuation value, rather than both his discount factor and his asset quality. It is an open question whether a mechanism that allows a seller to separately report his asset quality and discount factor would do better still.

The seller's participation constraint imposes that $k \geq 0$, which in turn also guarantees that $c^S(v) \geq 0$ for all v.

We next turn to feasibility, i.e. the cost of this mechanism. We start with the buyers' cost. In period 1, a buyer with discount factor β consumes $c_1^B(\beta)$ units of the consumption good per unit of endowment. Allowing for free disposal, the cost is therefore

$$C_1^B \geq \int_{\underline{\beta}}^{\bar{\beta}} (c_1^B(\beta) - 1) dG_1(\beta). \tag{C3}$$

In period 2, the buyers have no endowment and receive $c_2^B(\beta)$ units of the consumption good per unit of endowment. Thus the cost is simply

$$C_2^B \geq \int_{\underline{\beta}}^{\bar{\beta}} c_2^B(\beta) dG_1(\beta). \tag{C4}$$

Now turn to the sellers' cost. In period 1, the sellers receive $c^S(v)$ units of the consumption good, so the cost is

$$C_1^S \geq \int_{\underline{v}}^{\bar{v}} c^S(v) h(v) dv = \int_{\underline{v}}^{\bar{v}} \omega(v)(H(v) + v h(v)) dv + k, \tag{C5}$$

where the equality uses incentive compatibility and integration by parts. The total cost of the mechanism in period 2 is negative, given by the amount of dividends collected from the sellers:

$$C_2^S \geq -\int_{\underline{v}}^{\bar{v}} \omega(v) \Gamma(v) h(v) dv. \tag{C6}$$

The buyers' and sellers' mechanisms are feasible if total costs are zero in each period, $C_1^B + C_1^S = C_2^B + C_2^S = 0$.

It is straightforward to verify that any equilibrium allocation is incentive compatible and feasible, but the converse is not true. The most important difference lies in the trading probability $\omega(v)$. Incentive compatibility and feasibility only restricts $\omega(v)$ to lie between 0 and 1 and be non-increasing. We argued in Section II.B that equilibrium imposes additional restrictions on ω; for example, a pair of discontinuities must surround any constant portion of ω. It follows that some incentive compatible and feasible allocations cannot be supported in any equilibrium.

C2. Buyer Efficiency

We say an allocation is *buyer efficient* if it is incentive-compatible, feasible, and Pareto optimal for buyers among all the incentive-compatible, feasible allocations with the same buyer cost (C_1^B, C_2^B). We prove that any buyer efficient allocation is characterized by a threshold $\hat{\beta}$. Buyers with discount factor $\beta < \hat{\beta}$ consume only in the first period, while buyers with $\beta > \hat{\beta}$ consume only in the second period. A buyer with discount factor $\hat{\beta}$ is indifferent between consuming in the two periods.

PROPOSITION 5: *Let b and $\hat{\beta}$ solve*

$$C_1^B = \big(bG_b(\hat{\beta}) - 1\big) \text{ and } C_2^B = \frac{(G_b(\bar{\beta}) - G_b(\hat{\beta}))b}{\hat{\beta}}. \tag{C7}$$

If this defines $b \geq 1$, then any buyer efficient allocation has

$$c_1^B(\beta) = \begin{cases} b & \text{if } \beta < \hat{\beta} \\ 0 & \text{if } \beta > \hat{\beta} \end{cases} \quad \text{and} \quad c_2^B(\beta) = \begin{cases} 0 & \text{if } \beta < \hat{\beta} \\ b/\hat{\beta} & \text{if } \beta > \hat{\beta} \end{cases}.$$

Otherwise there is no incentive-compatible, feasible allocation with cost (C_1^B, C_2^B).

PROOF OF PROPOSITION 5:

The proposed allocation is incentive compatible, has $c_1^B(\beta)$ and $c_2^B(\beta)$ nonnegative, and satisfies the feasibility constraints (C3) and (C4). Now consider a competitive equilibrium of an economy in which each individual with $\beta < \hat{\beta}$ has an endowment of b in period 1 and 0 in period 2, while each individual with $\beta \geq \hat{\beta}$ has an endowment of 0 in period 1 and $b/\hat{\beta}$ in period 2. It is easy to verify the equilibrium involves no trade. The first welfare theorem implies this allocation is Pareto optimal among all allocations satisfying the two feasibility constraints. It is therefore Pareto optimal among the smaller set of allocations that also satisfy the incentive constraint (C1).

A corollary of this result is that any equilibrium of our model is buyer efficient. This is not surprising, since there is no interesting information problem on the buyer's side of the market. A buyer is privately informed about her discount factor, but a seller does not care about the buyer's discount factor when they trade.[17] This contrasts with the seller's side of the market, since a buyer cares about a seller's expected valuation v, which is private information. We turn to the seller's problem next.

[17]Comparing Propositions 2 and 5 shows that a buyer efficient allocation with $b > 1$ can never be supported in equilibrium. Such an allocation would require an initial redistribution of $b - 1$ units of the period 1 consumption good to each buyer.

C3. Seller Efficiency

An allocation is *seller efficient* if it is incentive compatible, feasible, and Pareto optimal for sellers among all the incentive-compatible, feasible allocations with the same seller cost (C_1^S, C_2^S). This section provides necessary and sufficient conditions for a semi-separating equilibrium to be seller efficient:

PROPOSITION 6: *If Assumption 1 holds and $\Gamma(\underline{v}) > 0$, the semi-separating equilibrium is seller efficient if and only if there exist non-negative numbers ψ_1 and ψ_2 satisfying the following conditions:*

- $\psi_1 \geq 1$,
- $J(\underline{v}) = 0$,
- $J(v)$ *nondecreasing for* $v \in [\underline{v}, \bar{p}]$,
- $J(\bar{p}) = 1$, *and*
- $\int_{\bar{p}}^{v} J(x)dx/(v - \bar{p}) \geq 1$ *for* $v > \bar{p}$,

where $J(v) \equiv \psi_1(H(v) + vh(v)) - \psi_2\Gamma(v)h(v)$.

PROOF OF PROPOSITION 6:

To start, assume that the semi-separating equilibrium is seller efficient. This means that there are nondecreasing integrated Pareto weights $\Lambda(v)$ with $\Lambda(\underline{v}) \geq 0$ and $\Lambda(\bar{v}) = 1$,[18] such that the allocation maximizes the Pareto-weighted sum of seller utilities,

$$\int_{\underline{v}}^{\bar{v}} u^S(v)d\Lambda(v),$$

among all incentive compatible and feasible allocations. Eliminate $u^S(v)$ using equation (C2) and perform integration-by-parts to rewrite the Pareto-weighted sum of utilities as

$$\int_{\underline{v}}^{\bar{v}} \omega(v)(\Lambda(v) - \Lambda(\underline{v}))dv + k. \tag{C8}$$

Any seller-efficient allocation maximizes (C8) subject to $\omega(v) \in [0, 1]$ non-increasing, $k \geq 0$, and the two resource constraints (C5) and (C6) for some nondecreasing integrated Pareto weights $\Lambda(v)$.

Write the Lagrangian of the Pareto-weighted maximization problem, placing nonnegative multipliers ψ_1 and ψ_2 on the two constraints (C5) and (C6):

$$\mathcal{L} = \int_{\underline{v}}^{\bar{v}} \omega(v)\phi(v)dv + (1 - \psi_1)k + \psi_1 C_1^S + \psi_2 C_2^S \tag{C9}$$

[18]The integrated Pareto weight $\Lambda(v)$ is the sum of the Pareto weights on sellers with continuation value less than or equal to v, so the Pareto weight on v is $d\Lambda(v)$.

subject to $k \geq 0$, and $\omega(v) \in [0,1]$ non-increasing, where $\phi(v) \equiv \Lambda(v) - \Lambda(\underline{v}) - J(v)$ with

$$J(v) \equiv \psi_1(H(v) + vh(v)) - \psi_2\Gamma(v)h(v).$$

The Lagrangian is linear in k, which implies that $\psi_1 \geq 1$; otherwise raising k would increase the Lagrangian without bound. In addition, integration by parts implies

$$\int_{\underline{v}}^{\bar{v}} \omega(v)\phi(v)dv = \omega(\bar{v})\Phi(\bar{v}) - \int_{\underline{v}}^{\bar{v}} \Phi(v)d\omega(v),$$

where $\Phi(v) \equiv \int_{\underline{v}}^{v} \phi(x)dx$. Therefore the Lagrangian is also linear in $d\omega(v)$, which implies that $\omega(v)$ is constant at any v that does not maximize $\Phi(v)$. In the semi-separating equilibrium, $\Theta(P(v))$ is strictly decreasing for all $v \in [\underline{v}, \bar{p}]$. Therefore if the equilibrium is Pareto efficient, all values of v in this interval must maximize $\Phi(v)$. We use this to characterize the conditions for Pareto efficiency.

Now assume there is a pair (ψ_1, ψ_2) such that the five conditions in the statement of the proposition hold. Set $\Lambda(v) = J(v)$ for $v \in [\underline{v}, \bar{p}]$ and $\Lambda(v) = 1$ for $v > \bar{p}$. The first condition ensure that $k = 0$ is optimal with these Pareto weights and Lagrange multipliers. The next three conditions ensure that $\Lambda(\underline{v}) = 0$, $\Lambda(v)$ is nondecreasing, and $\Lambda(\bar{p}) = 1$, so $d\Lambda(v)$ are valid Pareto weights. By construction $\phi(v) = \Phi(v) = 0$ for all $v \in [\underline{v}, \bar{p}]$ and $\Phi(v) = \int_{\bar{p}}^{v}(1 - J(x))dx \leq 0$ for all $v > \bar{p}$ using the final condition. Therefore any function $\omega(v)$ that is strictly decreasing on $[\underline{v}, \bar{p}]$ and 0 at higher values of v maximizes the Lagrangian. In particular, the semi-separating equilibrium is Pareto optimal.

Conversely, suppose there is no pair (ψ_1, ψ_2) satisfying these five conditions. If the first condition failed, the Lagrangian would not have a maximum and so the semi-separating equilibrium allocation would not maximize it. If any of the next three conditions failed, any nondecreasing Pareto weight $\Lambda(v)$ would have $\phi(v) = \Lambda(v) - \Lambda(\underline{v}) - J(v) \neq 0$ for some $v \in [\underline{v}, \bar{p}]$; therefore not all $v \in [\underline{v}, \bar{p}]$ would maximize $\Phi(v)$ and any solution to the Lagrangian must have $d\omega(v)$ constant at such v, inconsistent with the semi-separating equilibrium allocation. And if the fifth condition failed, $\Phi(v) > 0$ at some $v > \bar{v}$, so again any solution to the Lagrangian must have $d\omega(v) = 0$ at all $v \leq \bar{v}$, inconsistent with the semi-separating equilibrium allocation.

We use this proposition to prove the results in Section IV.C. As in the text, assume

$$\Gamma(v) = \frac{1+v}{2} \text{ and } H(v) = 1 - (\alpha+1)v^{-\alpha} + \alpha v^{-\alpha-1}.$$

First assume $0 < \alpha \leq 2$.[19] If $\hat{\beta} < 2$, set $\psi_1 = \psi_2 = \big(H(\bar{p}) + \bar{p}h(\bar{p}) - \Gamma(\bar{p})h(\bar{p})\big)^{-1} > 1$. If $\hat{\beta} \geq 2$, set $\psi_1 = \psi_2 = 1$. It is easy to verify that $J(v)$ is increasing with

[19] With $\alpha \leq 1$, total dividends held by sellers are infinite, $\int_1^\infty \Gamma(v)H'(v)\,dv = \infty$, which might seem worrisome for constructing an equilibrium. Nevertheless, total dividends sold are bounded above by the dividends of the worst asset: $\int_1^\infty \Theta(P(v))\Gamma(v)H'(v)\,dv < 1$ for any value of $\hat{\beta}$, so the market clearing condition can hold.

$J(1) = 0$ and $J(\bar{p}) = 1$. Proposition 6 implies that the semi-separating equilibrium is seller efficient.

If instead $\alpha > 2$ and $\hat{\beta} \geq 2$ (so $\bar{p} = \infty$), then the semi-separating equilibrium is not seller efficient. If $\psi_1 < \psi_2$, $J'(1) < J(1) = 0$, which implies $J(v)$ is negative at values of v slightly above 1, inconsistent with a seller-efficient allocation. If $\psi_1 \geq \psi_2$, $J(v)$ is decreasing at sufficiently large values of v, again inconsistent with a seller efficient allocation when $\bar{p} = \infty$.

To construct a Pareto improvement in this example, it is not enough to pool a single group of sellers. That will always either reduce some sellers' utility or raise costs in one of the periods. Instead, we must pool investors within two separate intervals.

We illustrate this with a concrete example. As in the text, assume $\alpha = 3$ and $\hat{\beta} = 2$, so that $\Theta(P(v)) = e^{1-v}$. First consider a pool with radius ε in the neighborhood of some $v > 1+\varepsilon$, setting $\omega(v)$ equal to the average value of $\Theta(P(v))$ within this pool, $\omega(v) = e^{1-v}(e^{\varepsilon} - e^{-\varepsilon})/(2\varepsilon)$. By construction, this increases welfare relative to the semi-separating equilibrium for all $v' \in (v-\varepsilon, v+\varepsilon)$, while welfare is unchanged for other sellers; see equation (C2). Thus the pool is Pareto improving if it is cost feasible.

Taking a Taylor expansion of costs in a neighborhood of $\varepsilon = 0$, we find that the first period cost of the pool in excess of the cost of the semi-separating allocation, is

$$\int_{v-\varepsilon}^{v+\varepsilon} \big(\omega(v) - \Theta(P(x))\big)(H(x) + xH'(x))dx = \frac{8e^{1-v}(3-2v)}{v^5}\varepsilon^3 + O(\varepsilon^4)$$

which is negative if $v > 3/2$. The second period cost of this pool, again in excess of the cost of the semi-separating allocation, is

$$-\int_{v-\varepsilon}^{v+\varepsilon} \big(\omega(v) - \Theta(P(x))\big)\Gamma(x)H'(x)dx = \frac{4e^{1-v}(3v^2-5)}{v^6}\varepsilon^3 + O(\varepsilon^4),$$

which is negative if $v < \sqrt{5/3}$. Since these regions do not overlap, any single pool must raise costs in one of the two periods.

But now consider two such pools, one with radius ε_1 in a neighborhood of some $v_1 < \sqrt{5/3}$ and the other with radius ε_2 in a neighborhood of some $v_2 > 3/2$. Manipulating the above expressions, we find that for small values of ε_1 and ε_2, the costs are negative in both periods if

$$\left(\frac{e^{1-v_2}(2v_2-3)v_1^5}{e^{1-v_1}(3-2v_1)v_2^5}\right)^{1/3} > \frac{\varepsilon_1}{\varepsilon_2} > \left(\frac{e^{1-v_2}(3v_2^2-5)v_1^6}{e^{1-v_1}(5-3v_1^2)v_2^6}\right)^{1/3}$$

Simplifying these inequalities, we find that if $v_1 \in (1, 10/9)$ and $v_2 > \frac{5(3-2v_1)}{10-9v_1}$, the inequalities on $\varepsilon_1/\varepsilon_2$ define a non-empty open interval on the nonnegative real line. This means that in a neighborhood of such v_1 and v_2, we construct two small pools. Each pool alone would raise costs in one of the periods, but the two pools together reduce costs in both periods.

We can also compute the expected price within such a pool, the ratio of the buyer's cost in period 1 to the amount of dividends he gets in period 2. Again using a Taylor expansion, this is

$$\frac{\omega(v+\varepsilon)+(v+\varepsilon)\omega(v)}{\omega(v)\frac{\int_{v-\varepsilon}^{v+\varepsilon}\Gamma(x)H'(x)dx}{H(v+\varepsilon)-H(v-\varepsilon)}} = 2+\frac{2(v^2+3v-5)}{3v(v^2-1)}\varepsilon^2+O(\varepsilon^3),$$

which is bigger than 2 when v is bigger than $(\sqrt{29}-3)/2 \approx 1.19$ and smaller than 2 at lower values. In other words, buyers get a low price when they buy from the low pool, $v \in (v_1-\varepsilon_1, v_1+\varepsilon_1)$, but they pay a high price when they buy from the high pool, $v \in (v_2-\varepsilon_2, v_2+\varepsilon_2)$. Randomizing between both pools allows them to make money in expected value. The example in the text, with pools for $v \in [1, 1.01]$ and $v \in [8.3, 11.3]$, is based on this calculation but does not use limits as the radius of the pools vanish.

C4. Local Pareto Efficiency

In the previous two sections, we asked whether, starting from a semi-separating equilibrium, it is possible to improve the welfare first of buyers and then of sellers without affecting the other group of investors, i.e. taking the costs C_1^B, C_2^B, C_1^S, and C_2^S as given. This section examines the possibility of achieving a Pareto improvement by moving costs across periods in a manner consistent with the resource constraint.

To understand the scope for this, we need to understand how buyers' and sellers' utility is affected by changes in the costs. We focus here on marginal changes in the costs, again starting from a semi-separating equilibrium. We say an allocation is *locally Pareto efficient* if it is buyer- and seller-efficient and if no small resource-feasible change in the costs generates a Pareto improvement.

PROPOSITION 7: *Impose Assumption 1 and* $\Gamma(\underline{v}) > 0$ *and suppose that the semi-separating equilibrium is seller efficient. If there exists a pair* (ψ_1, ψ_2) *that not only satisfies all conditions in Proposition 6 but also*

$$\frac{G_b(\hat{\beta})+\hat{\beta}g_b(\hat{\beta})}{g(\hat{\beta})} > \frac{\psi_2}{\psi_1} > \frac{\hat{\beta}^2 g_b(\hat{\beta})}{1-G_b(\hat{\beta})+\hat{\beta}g_b(\hat{\beta})},$$

then the semi-separating equilibrium is locally Pareto efficient. Otherwise it is not locally efficient.

PROOF OF PROPOSITION 7:

Proposition 5 describes the buyer efficient allocation. Buyers' utility is

$$u^B(\beta) = \begin{cases} b - 1 & \text{if } \beta < \hat{\beta} \\ \beta b/\hat{\beta} - 1 & \text{if } \beta \geq \hat{\beta}, \end{cases}$$

where b and $\hat{\beta}$ depend on C_1^B and C_2^B through equation (C7). Implicitly differentiating this expression, we get that a change in (C_1^B, C_2^B) of magnitude (dC_1^B, dC_2^B) raises the utility of buyers with $\beta < \hat{\beta}$ if and only if

$$dC_1^B + \frac{\hat{\beta}^2 g_b(\hat{\beta})}{1 - G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})} dC_2^B > 0$$

The same change raises the utility of buyers with $\beta > \hat{\beta}$ if and only if

$$\frac{g_b(\hat{\beta})}{G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})} dC_1^B + dC_2^B > 0.$$

Note that

$$\frac{G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})}{g_b(\hat{\beta})} \geq \frac{\hat{\beta}^2 g_b(\hat{\beta})}{1 - G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})},$$

as can be confirmed algebraically. This means that if buyers $\beta < \hat{\beta}$ like the perturbation (dC_1^B, dC_2^B) with $dC_2^B \geq 0$, all buyers like the perturbation. And if buyers $\beta > \hat{\beta}$ like the perturbation (dC_1^B, dC_2^B) with $dC_2^B \leq 0$, all buyers like the perturbation.

Next, a feasible change in the costs satisfies $dC_1^S = -dC_1^B$ and $dC_2^S = -dC_2^B$ and so in particular $\psi_1 dC_1^S + \psi_2 dC_2^S = -\psi_1 dC_1^B - \psi_2 dC_2^B$. Proposition 6 then implies that if $\psi_1 dC_1^B + \psi_2 dC_2^B < 0$ for any (ψ_1, ψ_2) consistent with the conditions in the Proposition, the equilibrium is not locally Pareto efficient.

Putting these results together, the equilibrium is locally Pareto efficient if there exists a (ψ_1, ψ_2) consistent with the conditions in Proposition 6 such that

1) for any $dC_2^B > 0$, $dC_1^B + \frac{\hat{\beta}^2 g_b(\hat{\beta})}{1 - G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})} dC_2^B < 0$ or $\psi_1 dC_1^B + \psi_2 dC_2^B \geq 0$, and

2) for any $dC_2^B < 0$, $\frac{g_b(\hat{\beta})}{G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})} dC_1^B + dC_2^B < 0$ or $\psi_1 dC_1^B + \psi_2 dC_2^B \geq 0$.

Part 1 holds if and only if $\frac{\psi_2}{\psi_1} > \frac{\hat{\beta}^2 g_b(\hat{\beta})}{1 - G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})}$, while part 2 holds if and only if $\frac{G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})}{g(\hat{\beta})} > \frac{\psi_2}{\psi_1}$.

The Lagrange multipliers ψ_1 and ψ_2 give the marginal value of funds to the sellers in each period, thus their ratio is the marginal rate of substitution of funds across the two periods. The first ratio involving $G_b(\hat{\beta})$ is the marginal rate of substitution for active buyers, those with $\beta > \hat{\beta}$. The last ratio is the marginal rate of substitution for inactive buyers, those with $\beta < \hat{\beta}$. If the marginal rate of substitution for sellers lies in between these two marginal rates of substitution, there is no way to make all investors better off by reallocating resources across periods.

To see how to apply this Proposition, we build on our previous example with independent Pareto distributions. Assume $0 \leq \alpha_\delta \leq 1$. For any $\psi_2 > \psi_1$, $J'(1) < 0$, so there is no associated seller-efficient allocation, while any ratio $\psi_2/\psi_1 \geq 0$ gives us valid Pareto weights for the semi-separating equilibrium. Therefore the semi-separating equilibrium is locally Pareto efficient if and only if $\frac{\hat{\beta}^2 g_b(\hat{\beta})}{1-G_b(\hat{\beta})+\hat{\beta} g_b(\hat{\beta})} < 1$.[20] Since these conditions hinge on the value of $G_b(\hat{\beta})$ and $g_b(\hat{\beta})$, they may or may not hold in any particular economy.

[20]If $1 < \alpha_\delta \leq 2$, there is also a lower bound on the ratio ψ_2/ψ_1 for generating valid Pareto weights, say $\psi_2/\psi_1 \geq \bar{\psi}$, where $\bar{\psi} \in (0, 1]$. We therefore also require $\frac{G_b(\hat{\beta})+\hat{\beta} g_b(\hat{\beta})}{g(\hat{\beta})} > \bar{\psi}$ in order for the semi-separating equilibrium to be locally Pareto efficient.