Online Appendix:
Prediction: The long and the short of it.

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A A microeconomic interpretation of the model

Here we provide an interpretation of our model in a familiar microeconomic setting. Consider a competitive firm that produces a quantity q_t of product at time n, and faces linear marginal costs of production $C'(q) = c_0 + c_1 q$. The firm faces uncertain prices $(p_t)_{t\geq 0}$ in the future, and quadratic adjustment costs $k(q_t - q_{t-1})^2$. These costs are a reduced form representation of costs sustained due to rejigging operations at the intensive and extensive margins. Small changes in production are handled at the intensive margin, and are thus cheap – current employees work longer/shorter hours, installed capital is used more/less intensively, and orders from existing suppliers are tweaked to meet small fluctuations in demand. However, large changes in production require extensive margin changes – large numbers of new employees must be hired/fired, new machinery must be bought or rented, and new suppliers found and terms negotiated. The form of our adjustment costs amounts to assuming that extensive margin adjustments are cheaper if done in a planned sequence of steps, rather than in an abrupt transition. There are several reasons why this might be. With early warning the firm could find creative ways of adapting its existing resources to new market conditions. Early warning may also place the firm in a stronger negotiating position with respect to employment and supply contracts (because of the lack of urgency), and could reduce the opportunity costs associated with under/over capacity. Quadratic adjustment costs are an analytically convenient reduced form way of representing these inertial forces on adjustment.

Since the firm takes prices p_t as given, its instantaneous profit function can be written as:

$$\begin{aligned}\Pi_t &= p_t q_t - (c_0 q_t + \frac{1}{2} c_1 q_t^2) - k(q_t - q_{t-1})^2 \\ &= -\frac{c_1}{2}(q_t - \theta_t)^2 - k(q_t - q_{t-1})^2 + M_t\end{aligned}$$

where $\theta_t = (p_t - c_0)/c_1$, and M_t is a decision irrelevant constant, which may be neglected when computing the value of information about the sequence of values $(\theta_t)_{t\geq 0}$. Thus the firm's profit function is of the form (2), up to an irrelevant factor of c_1.

B Proof of Proposition 1

We use the Bellman equation (10) to solve for the optimal policy function $X^{n+1} = \pi(X^n, Y^n)$. When referring to functions and operations on functions, we will adopt a notation in which primed variables denote next period quantities, and unprimed variables denote current period quantities, i.e., $W = W(X', X, Y)$ and $V = V(X, Y)$. So $\frac{\partial W}{\partial X'}$, for example, refers to the function whose value is the partial derivative of W with respect to its first argument, i.e., the *next period* value of X. When we evaluate functions and their derivatives at specific times, we will still use e.g. X^n, X^{n+1} to denote function arguments. Thus $\frac{\partial W}{\partial X'}(X^{n+1}, X^n, Y^n)$ is the partial derivative of W with respect to its first argument, evaluated at (X^{n+1}, X^n, Y^n). With this notation, the first order condition for X^{n+1} is

$$\frac{\partial W}{\partial X'}(\pi(X^n, Y^n), X^n, Y^n) + \beta \int_{\mathbb{R}^\infty} \frac{\partial V}{\partial X}(\pi(X^n, Y^n), F(Y^n, S^n))Q(S^n; Y^n)dS^n = 0. \quad \text{(A.1)}$$

By the envelope theorem,

$$\frac{\partial V}{\partial X}(X^n, Y^n) = \frac{\partial W}{\partial X}(\pi(X^n, Y^n), X^n, Y^n). \quad \text{(A.2)}$$

From (2), and (A.2) evaluated at time $n + 1$, we have

$$\begin{aligned}
\frac{\partial W}{\partial X'}(\pi(X^n, Y^n), X^n, Y^n) &= \mu_0 + \alpha X^n - (1 + \alpha)\pi(X^n, Y^n) \\
\frac{\partial V}{\partial X}(\pi(X^n, Y^n), F(Y^n, S^n)) &= \frac{\partial W}{\partial X}(\pi(\pi(X^n, Y^n), F(Y^n, S^n)), \pi(X^n, Y^n), F(Y_n, S^n)) \\
&= \alpha(X' - X)|_{X'=\pi(\pi(X^n,Y^n),F(Y^n,S^n)),X=\pi(X^n,Y^n)} \\
&= \alpha(\pi(\pi(X^n, Y^n), F(Y^n, S^n)) - \pi(X^n, Y^n)).
\end{aligned}$$

Substituting into (A.1), we find that the policy rule must satisfy

$$\mu_0^n + \alpha X^n - (1+\alpha)\pi(X^n, Y^n) + \beta \int_{\mathbb{R}^\infty} [\alpha(\pi(\pi(X^n, Y^n), F(Y^n, S^n)) - \pi(X^n, Y^n))] Q(S^n; Y^n)dS^n = 0. \quad \text{(A.3)}$$

We solve this equation by the 'guess and verify' method. The certainty equivalence property of the quadratic control problem suggests that we should look for a control rule of the form

$$\pi(X, Y) = aX + \sum_{t=0}^{\infty} b_t \mu_t$$

where the coefficients $(a, (b_t)_{t\geq 0})$ are to be determined. Plugging this guess into (A.3), and now suppressing the index n, we find:

$$[\mu_0 + \alpha X - (1+\alpha)(aX + \sum_{t=0}^{\infty} b_t \mu_t)] +$$
$$\beta\alpha \left[\int_{\mathbb{R}^\infty} \left(a(aX + \sum_{t=0}^{\infty} b_t\mu_t) + \sum_{t=0}^{\infty} b_t \mu_t'(s_{t+1}) - (aX + \sum_{t=0}^{\infty} b_t\mu_t) \right) Q(S,Y)dS \right] = 0$$

where $\mu_t'(s_{t+1})$ is the next period value of μ_t conditional on receiving a signal s_{t+1}, given by (5). Since $E_{s_t}\mu_t'(s_{t+1}) = \mu_{t+1}$, we can simplify this to:

$$\mu_0 + \alpha X - (1+\alpha)(aX + \sum_{t=0}^{\infty} b_t\mu_t) +$$
$$\beta[\alpha a^2 X + a\alpha \sum_{t=0}^{\infty} b_t\mu_t + \alpha \sum_{t=0}^{\infty} b_t \mu_{t+1} - a\alpha X - \alpha \sum_t b_t \mu_t] = 0.$$

Since this equation must hold for all values of X, μ_t, we must equate the coefficients of each state variable to zero. The equation for the coefficient of X is:

$$\alpha\beta a^2 - (1 + \alpha(1+\beta))a + \alpha = 0 \tag{A.4}$$
$$\Rightarrow a = \frac{1 + \alpha(1+\beta) \pm \sqrt{(1+\alpha(1+\beta))^2 - 4\alpha^2\beta}}{2\alpha\beta} \tag{A.5}$$

To pick the correct root, note that if $\alpha \to 0$, the policy rule should reduce to

$$\pi(X, Y) = \mu_0.$$

This follows since when adjustment is costless, the optimal policy simply maximizes period payoffs. For the positive root we have

$$\lim_{\alpha\to 0} a(\alpha) \to \infty,$$

thus giving incorrect behaviour. By contrast, we show below that the correct behaviour is obtained if we select the negative root. Thus we conclude that

$$a = a(\alpha, \beta) = \frac{1 + \alpha(1+\beta) - \sqrt{(1+\alpha(1+\beta))^2 - 4\alpha^2\beta}}{2\alpha\beta} \tag{A.6}$$

The equation for b_0 is:

$$1-(1+\alpha)b_0+a\beta\alpha b_0-\alpha\beta b_0=0$$
$$\Rightarrow b_0=\frac{1}{1+\alpha+\alpha\beta(1-a)}. \tag{A.7}$$

For $t\geq 1$, the equation for b_t is:

$$-(1+\alpha)b_t+a\beta\alpha b_t+\alpha\beta b_{t-1}-\alpha\beta b_t=0$$
$$\Rightarrow b_t=\frac{\alpha\beta}{1+\alpha+\alpha\beta(1-a)}b_{t-1}.$$

Thus for all $t\geq 0$,

$$b_t=\frac{1}{1+\alpha+\alpha\beta(1-a)}\left[\frac{\alpha\beta}{1+\alpha+\alpha\beta(1-a)}\right]^t \tag{A.8}$$

We can simplify this further by using the equation for a in (A.4). Define

$$\Lambda\equiv 1+\alpha+\alpha\beta(1-a) \tag{A.9}$$

From (A.4) we have

$$(\alpha\beta)a^2-(1+\alpha(1+\beta))a+\alpha=0$$

Now

$$1+\alpha(1+\beta)=\Lambda+\alpha\beta a$$
$$\Rightarrow (\alpha\beta)a^2-(\Lambda+\alpha\beta a)a+\alpha=0$$
$$\Rightarrow \Lambda=\frac{\alpha}{a}.$$

Thus

$$b_t=\frac{a}{\alpha}\left(a\beta\right)^t. \tag{A.10}$$

We now prove the properties of the coefficients a, b_t, stated below the proposition:

1. $\lim_{\alpha\to 0} a(\alpha,\beta)=0$

   Use l'Hopital's rule: differentiate the numerator and denominator of a with respect to α, and evaluate the limit of each as $\alpha \to 0$:

$$\lim_{\alpha \to 0} a(\alpha, \beta) = \frac{1 + \beta - \frac{1}{2 \times 1}(2 \times 1 \times (1+\beta) - 0)}{2\beta}$$
$$= 0.$$

2. $\lim_{\alpha \to \infty} a(\alpha, \beta) = 1$:

$$\lim_{\alpha \to \infty} a(\alpha, \beta) = \frac{1+\beta}{2\beta} - \frac{1-\beta}{2\beta}$$
$$= 1.$$

3. $\frac{\partial a}{\partial \alpha} > 0$.

   From (A.6) we have

$$\frac{\partial a}{\partial \alpha} = -\frac{1}{2} \frac{-\alpha\beta + \sqrt{\alpha^2(1-\beta)^2 + 2\alpha(1+\beta) + 1} - \alpha - 1}{\alpha^2 \beta \sqrt{\alpha^2(1-\beta)^2 + 2\alpha(1+\beta) + 1}} \,. \tag{A.11}$$

   Hence, $\frac{\partial a}{\partial \alpha} > 0$ iff

$$\begin{aligned} & -\alpha\beta + \sqrt{\alpha^2(1-\beta)^2 + 2\alpha(1+\beta) + 1} - \alpha - 1 < 0 \\ \iff & \sqrt{\alpha^2(1-\beta)^2 + 2\alpha(1+\beta) + 1} < 1 + \alpha + \alpha\beta \\ \iff & \alpha^2(1-\beta)^2 + 2\alpha(1+\beta) + 1 < \alpha^2(1+\beta)^2 + 2\alpha(1+\beta) + 1 \end{aligned}$$

   which is obviously satisfied for all $\alpha > 0, \beta \in (0,1)$.

4. $a + \sum_{t=0}^{\infty} b_t = 1$.

   From the previous calculations we know that $a \in [0,1] \Rightarrow a\beta \in [0,1]$. It follows from (A.10) that

$$a + \sum_{t=0}^{\infty} b_t - 1 = a + \frac{a}{\alpha} \frac{1}{1 - a\beta} - 1$$
$$= \frac{-\alpha\beta a^2 + a(1 + \alpha(1+\beta)) - \alpha}{\alpha(1 - a\beta)}$$
$$= 0$$

where the last equality follows from the defining equation for a in (A.4).

5. $\frac{\partial}{\partial \alpha}(b_{t+1}/b_t) > 0$, $\frac{\partial b_0}{\partial \alpha} < 0$.

   Since $a + \sum_{t=0}^{\infty} b_t = 1$, and a is increasing in α, we know that $\sum_{t=0}^{\infty} b_t$ must be decreasing in α. From (A.10) we see that

$$\frac{b_{t+1}}{b_t} = a\beta$$

   and hence this ratio is increasing in α. Since b_t declines more slowly as α increases, it must be the case that $\frac{\partial b_0}{\partial \alpha} < 0$ in order to ensure that $\sum_{t=0}^{\infty} b_t$ is decreasing in α.

C Illustration of the dependence of optimal policies on adjustment costs α.

To illustrate how the adjustment cost parameter α affects decisions quantitatively, consider a deterministic version of the model in which the values $\tilde{\theta}^n$ are chosen be a fixed sequence of draws from an arbitrary univariate random variable with finite variance σ^2. When $\alpha = 0$, optimal decisions coincide with the current value of $\tilde{\theta}^n$, i.e., $X^n = \mu_0^n = \tilde{\theta}^n$ for all n. As α increases, adjustment becomes more costly, and the values of X^n fluctuate less than $\tilde{\theta}^n$ itself. Using the formula (11) and some simple ergodic arguments one can show that

$$\begin{aligned}\lim_{n\to\infty} \mathrm{Var}(X^n, X^{n+1}, \ldots) &= \frac{\sum_{t=0}^{\infty} b_t^2}{1-a^2}\sigma^2 \\ &= \left[\left(\frac{a}{\alpha}\right)^2 \frac{1}{(1-a^2)(1-a^2\beta^2)}\right]\sigma^2\end{aligned}$$

for arbitrary initial condition X^0. Figure F.1 plots the asymptotic variance of the sequence of decisions as a function of α for several β. The figure illustrates how α controls the magnitude of the adjustments the decision-maker makes to adapt to fluctuations in a stationary environment. For a wide range of β, $\alpha > 1.5$ implies that the decision maker adjusts to less than 20% of the variability in $\tilde{\theta}$, and $\alpha > 3$ implies adjustment to less than 10% of the variability. In addition, changes in α have a greater effect on behaviour when α is small (e.g. $\alpha < 1$) than when it is large.

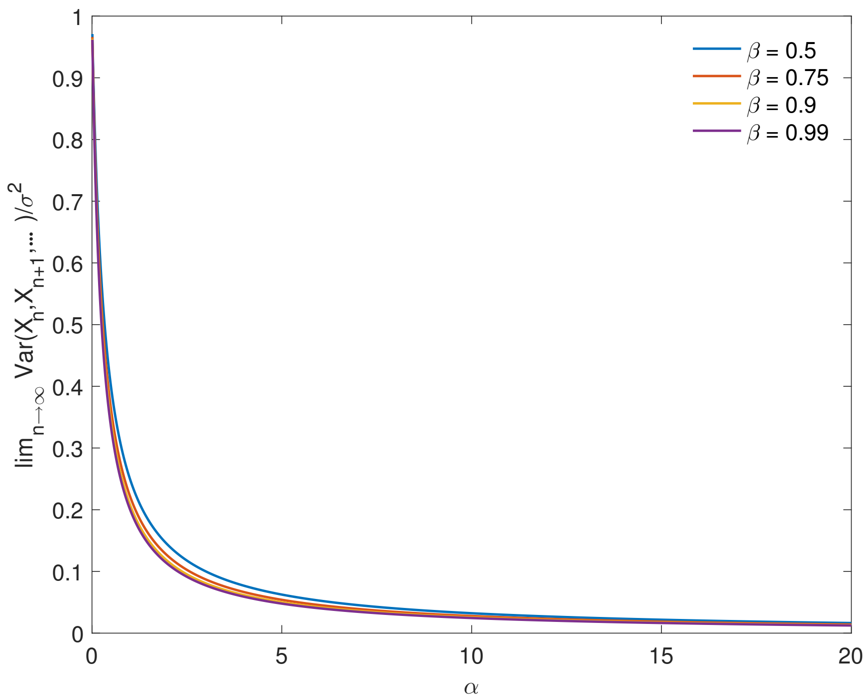


Figure F.1: Asymptotic variability of the decision variable X relative to the variability of the loss-minimizing decisions $\tilde{\theta}$, assuming that the values of $\tilde{\theta}^n$ are deterministic and given by a fixed sequence of draws from a random variable with variance σ^2.

D Proof of Proposition 2

As in the derivation of the optimal policy function, we use the 'guess and verify' method. Begin by guessing that the value function has the form

$$V(X,Y) = kX^2 + \sum_{t=0}^{\infty} c_t\mu_t X + \sum_{t=0}^{\infty}\sum_{p=t+1}^{\infty} D_{t,p}\mu_t\mu_p + \sum_{t=0}^{\infty} d_t\mu_t^2 + \sum_{t=0}^{\infty}\sum_{i=0}^{\infty}\frac{f_{i,t}}{\lambda_t + h_{i,t}}. \quad \text{(A.12)}$$

All except the last term of this expression are straightforward to guess simply by inspection of the formula for the period payoff in (2). The last term will however be the most important, as it will turn out that this is the only term that depends on the precision sequence $\vec{\tau} = (\tau_t)_{t\geq 1}$.

Consider the quadratic terms in this guess of the form $\mu_t\mu_p$. We are going to need to know how these will transform under the updating rule (5) and after the expectation over signal realizations has been applied. Letting a prime denote the next period value of a variable, we are interested in computing expectations of the form

$$\mathbf{E}_S \mu'_t \mu'_p = \mathbf{E}_{s_{t+1}, s_{p+1}} \mu'_t(s_{t+1}) \mu'_p(s_{p+1})$$

where signals are distributed according to the agents' current posterior predictive distribution, given by (7). Recall that

$$\mu'_t(s_{t+1}) = \frac{\tau_{t+1}}{\tau_{t+1} + \lambda_{t+1}} s_{t+1} + \frac{\lambda_{t+1}}{\tau_{t+1} + \lambda_{t+1}} \mu_{t+1}$$

When $t \neq p$, we can immediately write down the answer, as means are martingales, and signals are independent:

$$\mathbf{E}_{s_{t+1}, s_{p+1}} \mu'_t(s_{t+1}) \mu'_p(s_{p+1}) = \mu_{t+1} \mu_{p+1}$$

For $t = p$ however, things are different:

$$\mathbf{E}_{s_{t+1}} \mu'_t(s_{t+1}) \mu'_t(s_{t+1}) = \mathbf{E}_{s_{t+1}} \left[\frac{\tau_{t+1}}{\tau_{t+1} + \lambda_{t+1}} s_{t+1} + \frac{\lambda_{t+1}}{\tau_{t+1} + \lambda_{t+1}} \mu_{t+1} \right]^2$$

Consider the quadratic term in s_{t+1} in this expression:

$$\begin{aligned}
\mathbf{E}_{s_{t+1}} \left(\frac{\tau_{t+1}}{\tau_{t+1} + \lambda_{t+1}} \right)^2 s_{t+1}^2 &= \left(\frac{\tau_{t+1}}{\tau_{t+1} + \lambda_{t+1}} \right)^2 [\mathrm{Var}(s_{t+1}) + \mu_{t+1}^2] \\
&= \left(\frac{\tau_{t+1}}{\tau_{t+1} + \lambda_{t+1}} \right)^2 [\frac{\lambda_{t+1} + \tau_{t+1}}{\lambda_{t+1} \tau_{t+1}} + \mu_{t+1}^2] \\
&= \frac{\tau_{t+1}}{\lambda_{t+1}(\lambda_{t+1} + \tau_{t+1})} + \left(\frac{\tau_{t+1}}{\tau_{t+1} + \lambda_{t+1}} \right)^2 \mu_{t+1}^2
\end{aligned}$$

When we combine this expression with the other terms in the expression for $\mathbf{E}_{s_{t+1}} \mu'_t(s_{t+1}) \mu'_t(s_{t+1})$, the factor in front of μ_{t+1}^2 in the second term will cancel to 1 (as occurs in the case $t \neq p$), and we are left with

$$\mathbf{E}_{s_{t+1}} \mu'_t(s_{t+1}) \mu'_t(s_{t+1}) = \frac{\tau_{t+1}}{\lambda_{t+1}(\lambda_{t+1} + \tau_{t+1})} + \mu_{t+1}^2. \tag{A.13}$$

Hence, in summary:

$$\mathbf{E}_{s_{t+1}, s_{p+1}} \mu'_t(s_{t+1}) \mu'_p(s_{p+1}) = \begin{cases} \mu_{t+1} \mu_{p+1} & t \neq p \\ \frac{\tau_{t+1}}{\lambda_{t+1}(\lambda_{t+1} + \tau_{t+1})} + \mu_{t+1}^2 & t = p. \end{cases} \tag{A.14}$$

It will be more convenient in what follows to write the terms that depend on λ_{t+1} is this expression as

$$\frac{\tau_{t+1}}{\lambda_{t+1}(\lambda_{t+1}+\tau_{t+1})} = \frac{1}{\lambda_{t+1}} - \frac{1}{\lambda_{t+1}+\tau_{t+1}}. \tag{A.15}$$

We now want to write down the Bellman equation for our assumed functional form for the value function. The first step is to compute the period payoff:

$$\begin{aligned}
W(\pi(X,Y),X,Y) &= -\frac{1}{2}\left[(1+\alpha)[\pi(X,Y)]^2 + \alpha X^2 - 2\pi(X,Y)(\mu_0+\alpha X) + \frac{1}{\lambda_0} + (\mu_0)^2\right] \\
&= -\frac{1}{2}\left[(1+\alpha)(aX + \sum_{t=0}^{\infty} b_t\mu_t)^2 + \alpha X^2 - 2(aX + \sum_{t=0}^{\infty} b_t\mu_t)(\mu_0+\alpha X) + \frac{1}{\lambda_0} + (\mu_0)^2\right] \\
&= -\frac{1}{2}\left[(1+\alpha)(a^2X^2 + 2aX\sum_{t=0}^{\infty} b_t\mu_t + \sum_{t=0}^{\infty}\sum_{p=t+1}^{\infty} 2b_tb_p\mu_t\mu_p + \sum_{t=0}^{\infty} b_t^2\mu_t^2) + \alpha X^2\right. \\
&\quad \left. -2aX\mu_0 - 2a\alpha X^2 - 2\mu_0\sum_{t=0}^{\infty} b_t\mu_t - 2\alpha X\sum_{t=0}^{\infty} b_t\mu_t + (\mu_0)^2 + \frac{1}{\lambda_0}\right]
\end{aligned}$$

We also have

$$\begin{aligned}
\mathbf{E}_S V(\pi(X,Y),F(Y,S)) &= \mathbf{E}_S\left[k(\pi(X,Y))^2 + \sum_{t=0}^{\infty} c_t\mu_t'(s_{t+1})\pi(X,Y) + \sum_{t=0}^{\infty}\sum_{p=t+1}^{\infty} D_{t,p}\mu_t'(s_{t+1})\mu_p'(s_{p+1})\right. \\
&\quad \left. + \sum_{t=0}^{\infty} d_t(\mu_t'(s_{t+1}))^2 + \sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\frac{f_{i,t}}{\lambda_t'+h_{i,t}}\right] \\
&= k[aX + \sum_t b_t\mu_t]^2 + \sum_{t=0}^{\infty} c_t\mu_{t+1}[aX + \sum_{p=0}^{\infty} b_p\mu_p] + \sum_{t=0}^{\infty}\sum_{p=t+1}^{\infty} D_{t,p}\mu_{t+1}\mu_{p+1} \\
&\quad + \sum_{t=0}^{\infty} d_t(\mu_{t+1})^2 + \sum_{t=0}^{\infty} d_t\left[\frac{1}{\lambda_{t+1}} - \frac{1}{\lambda_{t+1}+\tau_{t+1}}\right] + \sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\frac{f_{i,t}}{\lambda_{t+1}+\tau_{t+1}+h_{i,t}}
\end{aligned}$$

We now have expressions for each of the three terms $V(X,Y)$, $W(\pi(X,Y),X,Y)$, $\mathbf{E}_S V(\pi(X,Y),F(Y,S))$, and must choose the free coefficients of the value function so that

$$V(X,Y) = W(\pi(X,Y),X,Y) + \beta\mathbf{E}_S V(\pi(X,Y),F(Y,S))$$

holds as an identity. We begin by focussing on the terms that depend on λ_t. If we focus just on these terms, the Bellman equation reads

$$\sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\frac{f_{i,t}}{\lambda_t+h_{i,t}}=-\frac{1}{2}\frac{1}{\lambda_0}+\beta\left(\sum_{t=0}^{\infty}d_t\left[\frac{1}{\lambda_{t+1}}-\frac{1}{\lambda_{t+1}+\tau_{t+1}}\right]+\sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\frac{f_{i,t}}{\lambda_{t+1}+\tau_{t+1}+h_{i,t}}\right) \tag{A.16}$$

We must determine values for the sequences $f_{i,t}, h_{i,t}$ such that this equation holds as an identity. Since the right hand side of this equation contains terms of the form $1/\lambda_t$ for all t, we must have terms of this form on the left hand side as well. We thus begin by choosing

$$h_{0,t}=0$$

for all $t\geq 0$. Then if (A.16) is to hold as an identity for all λ_t, τ_t we require

$$f_{0,0}=-\frac{1}{2} \tag{A.17}$$

$$f_{0,t}=\beta d_{t-1} \text{ for } t\geq 1. \tag{A.18}$$

Notice that setting $h_{0,t}=0$ creates an imbalance of terms of the form

$$\sum_{t=0}^{\infty}\frac{f_{0,t}}{\lambda_{t+1}+\tau_{t+1}}$$

on the right hand side of the Bellman equation through the last term in (A.16). To correct this imbalance through terms on the left hand side, we must choose

$$h_{1,t}=\tau_t$$

implying in turn that we must choose

$$f_{1,0}=0$$

$$f_{1,t}=\beta[-d_{t-1}+f_{0,t-1}] \text{ for } t\geq 1.$$

Again we create an imbalance of terms on the right hand side, which we correct by picking

$$h_{2,t}=\tau_t+h_{1,t-1}=\tau_t+\tau_{t-1}$$

and we find that

$$f_{2,0} = 0$$
$$f_{2,t} = \beta f_{1,t-1}.$$

We can complete this imbalance/rebalance procedure indefinitely to solve for all the coefficients $f_{i,t}, h_{i,t}$. We find:

$$h_{0,t} = 0; \quad h_{i,t} = \tau_t + h_{i-1,t-1} \qquad i \geq 1 \tag{A.19}$$
$$f_{0,0} = -\frac{1}{2}; \quad f_{0,t} = \beta d_{t-1} \qquad t \geq 1 \tag{A.20}$$
$$f_{1,0} = 0; \quad f_{1,t} = \beta[-d_{t-1} + f_{0,t-1}] \qquad t \geq 1 \tag{A.21}$$
$$f_{i,0} = 0; \quad f_{i,t} = \beta f_{i-1,t-1}. \qquad n \geq 2, t \geq 1. \tag{A.22}$$

It is straightforward to solve the set of recurrence relations for $f_{i,t}$. It is convenient to write the solution as an infinite dimensional matrix:

$$\mathbf{f} = \begin{pmatrix} -\frac{1}{2} & \beta d_0 & \beta d_1 & \beta d_2 & \beta d_3 & \dots \\ 0 & -\beta(d_0 + \frac{1}{2}) & \beta(\beta d_0 - d_1) & \beta(\beta d_1 - d_2) & \beta(\beta d_2 - d_3) & \dots \\ 0 & 0 & -\beta^2(d_0 + \frac{1}{2}) & \beta^2(\beta d_0 - d_1) & \beta^2(\beta d_1 - d_2) & \dots \\ 0 & 0 & 0 & -\beta^3(d_0 + \frac{1}{2}) & \beta^3(\beta d_0 - d_1) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix} \tag{A.23}$$

The i, t entry of this matrix corresponds to $f_{i-1,t-1}$, i.e., the rows correspond to fixed values of i, and the columns to fixed values of t, both starting at zero.[1]

Clearly $f_{i,t} = 0$ for any $i > t$. Thus the only parameters $h_{i,t}$ that are relevant have

[1]Notice that $\sum_{i=0}^{\infty} f_{i,t} = -\frac{1}{2}\beta^t$. To understand this suppose that $\tau_t = 0$ for all t, i.e., the agent receives no forecasts. Then her beliefs will not change over time, and the variance of her beliefs about $\tilde{\theta}^{n+t}$ will be the same once time $n+t$ rolls around as they are in the current period n. The contribution of the variance terms to the value function in this case is thus straightforward to compute, since variance terms only enter the period payoff through the term $-\frac{1}{2}\lambda_0$. Thus, when $\tau_t = 0$, we should expect the following term in the value function: $-\frac{1}{2}\sum_{t=0}^{\infty}\beta^t\frac{1}{\lambda_t}$. Now when $\tau_t = 0$ for all t, we have

$$\begin{aligned} \sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\frac{f_{i,t}}{\lambda_t + h_{i,t}} &= \sum_{t=0}^{\infty}\frac{\sum_{i=0}^{\infty} f_{i,t}}{\lambda_t} \\ &= -\frac{1}{2}\sum_{t=0}^{\infty}\beta^t\frac{1}{\lambda_t} \end{aligned}$$

as expected.

$0 \leq i \leq t$. It is straightforward to solve the recurrence relation (A.19) to find

$$h_{0,t} = 0$$
$$h_{i,t} = \sum_{k=t+1-i}^{t} \tau_k, \; 1 \leq i \leq t$$

The matrix $\mathbf{f}$ makes it clear that we will need to understand the parameters d_t if we are to solve for $f_{i,t}$. We can find these parameters by solving the μ_t^2 terms of the Bellman equation. Define

$$\delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{A.24}$$

Then the Bellman equation for the μ_t^2 terms yields

$$\begin{aligned} d_t &= -\frac{1}{2}[(\alpha+1)(b_t)^2 + (1-2b_0)\delta_{t,0}] + \beta(k(b_t)^2 + c_{t-1}b_t(1-\delta_{t,0}) + d_{t-1}(1-\delta_{t,0}) \\ &= \left(k\beta - \frac{1}{2}(\alpha+1)\right)(b_t)^2 - \frac{1}{2}(1-2b_0)\delta_{t,0} + \beta c_{t-1}b_t + \beta d_{t-1} \end{aligned} \tag{A.25}$$

where $d_{-1} \equiv 0 \equiv c_{-1}$. This equation in turn depends on the coefficients of X^2 and $\mu_t X$, i.e., k and c_t. The X^2 terms of the Bellman equation give:

$$\begin{aligned} k &= -\frac{1}{2}((1+\alpha)a^2 + \alpha - 2a\alpha) + \beta(ka^2) \\ \Rightarrow k &= -\frac{1}{2}\left(\frac{(1+\alpha)a^2 + \alpha - 2a\alpha}{1-\beta a^2}\right), \end{aligned} \tag{A.26}$$

which is a known quantity. As a check, another way to compute k is to use the envelope theorem result:

$$\begin{aligned} \frac{\partial V}{\partial X} &= \alpha(\pi(X,Y) - X) \\ &= \alpha((a-1)X + \sum_t b_t \mu_t) \end{aligned}$$

Integrating this, we should find that

$$k = \alpha\frac{a-1}{2}.$$

Using (A.4) it can be shown that these two formulae for k agree, and we thus use the second, simpler, expression.

Equating coefficients of the $\mu_t X$ terms in the Bellman equation gives:

$$\begin{aligned} c_t &= -\frac{1}{2}((1+\alpha)2ab_t - 2a\delta_{t,0} - 2\alpha b_t) + \beta(2kab_t + ac_{t-1}(1-\delta_{t,0})) \\ &= (\alpha - a(1+\alpha) + 2\beta ka)b_t + a\delta_{t,0} + a\beta c_{t-1} \end{aligned}$$

Consider the factor in front of b_t in this expression. Substituting $k = \frac{\alpha}{2}(a-1)$ into this factor we see that it is equal to

$$\alpha\beta a^2 - a(1+\alpha(1+\beta)) + \alpha$$

But from the definition of a in (A.4) this expression is identically zero. Thus c_t satisfies

$$c_t = a\delta_{t,0} + a\beta c_{t-1}$$

where $c_{-1} = 0$. Thus, we conclude that

$$c_t = a(a\beta)^t \tag{A.27}$$

for all $t \geq 0$.

Equation (A.25) thus becomes:

$$\begin{aligned} d_t &= \left(\alpha\beta\frac{a-1}{2} - \frac{1}{2}(\alpha+1)\right)b_t^2 + \left(b_0 - \frac{1}{2}\right)\delta_{t,0} + (a\beta)^t b_t + \beta d_{t-1} \\ &= -\frac{1}{2}(1+\alpha+\alpha\beta(1-a))b_t^2 + (a\beta)^t b_t + (b_0 - \frac{1}{2})\delta_{t,0} + \beta d_{t-1} \end{aligned}$$

From (A.8) and the definition of Λ in (A.9) we have

$$b_t = \frac{1}{\Lambda}\left(\frac{\alpha\beta}{\Lambda}\right)^t.$$

Thus

$$
\begin{aligned}
d_0 &= -\frac{\Lambda}{2} b_0^2 + b_0 - \frac{1}{2} \\
&= -\frac{\Lambda}{2} (\frac{1}{\Lambda})^2 + \frac{1}{\Lambda} - \frac{1}{2} \\
&= \frac{1}{2} (\frac{1}{\Lambda} - 1)
\end{aligned}
$$

Also for $t \geq 1$:

$$
\begin{aligned}
d_t &= -\frac{1}{2\Lambda} \left(\frac{\alpha\beta}{\Lambda} \right)^{2t} + (a\beta)^t \frac{1}{\Lambda} \left(\frac{\alpha\beta}{\Lambda} \right)^t + \beta d_{t-1} \\
&= -\frac{1}{2\Lambda} \left(\frac{\alpha\beta}{\Lambda} \right)^2 \left(\frac{\alpha\beta}{\Lambda} \right)^{2(t-1)} + \frac{1}{\Lambda} \frac{a\alpha\beta^2}{\Lambda} \left(\frac{a\alpha\beta^2}{\Lambda} \right)^{t-1} + \beta d_{t-1}
\end{aligned}
$$

This is a non-homogeneous first order difference equation. The solution for $t \geq 1$ is

$$
\begin{aligned}
d_t &= \beta^t d_0 - \frac{1}{2\Lambda} \left(\frac{\alpha\beta}{\Lambda} \right)^2 \sum_{k=0}^{t-1} \beta^{t-k-1} \left(\frac{\alpha\beta}{\Lambda} \right)^{2k} + \frac{1}{\Lambda} \frac{a\alpha\beta^2}{\Lambda} \sum_{k=0}^{t-1} \beta^{t-k-1} \left(\frac{a\alpha\beta^2}{\Lambda} \right)^k \\
&= \beta^t d_0 - \frac{1}{2\Lambda} \left(\frac{\alpha\beta}{\Lambda} \right)^2 \beta^{t-1} \sum_{k=0}^{t-1} \left(\frac{\alpha^2\beta}{\Lambda^2} \right)^k + \frac{1}{\Lambda} \frac{a\alpha\beta^2}{\Lambda} \beta^{t-1} \sum_{k=0}^{t-1} \left(\frac{a\alpha\beta}{\Lambda} \right)^k \\
&= \beta^t d_0 - \frac{1}{2\Lambda} \left(\frac{\alpha\beta}{\Lambda} \right)^2 \beta^{t-1} \left(\frac{1 - (\alpha^2\beta/\Lambda^2)^t}{1 - \alpha^2\beta/\Lambda^2} \right) + \frac{1}{\Lambda} \frac{a\alpha\beta}{\Lambda} \beta^t \frac{1 - \left(\frac{a\alpha\beta}{\Lambda} \right)^t}{1 - \frac{a\alpha\beta}{\Lambda}} \\
&= \beta^t \left(\frac{1}{2\Lambda} - \frac{1}{2} - \frac{1}{2\Lambda} \frac{\alpha^2\beta}{\Lambda^2} \left(\frac{1 - (\alpha^2\beta/\Lambda^2)^t}{1 - \alpha^2\beta/\Lambda^2} \right) + \frac{1}{\Lambda} \frac{a\alpha\beta}{\Lambda} \frac{1 - \left(\frac{a\alpha\beta}{\Lambda} \right)^t}{1 - \frac{a\alpha\beta}{\Lambda}} \right)
\end{aligned}
$$

Since $\Lambda = \frac{\alpha}{a}$ we see that

$$
\frac{\alpha^2\beta}{\Lambda^2} = \frac{a\alpha\beta}{\Lambda} = a^2\beta.
$$

Thus the solution for d_t simplifies to

$$
d_t = \frac{1}{2} \beta^t \left[\frac{a}{\alpha} - 1 + \frac{a}{\alpha} \frac{a^2\beta}{1 - a^2\beta} (1 - (a^2\beta)^t) \right]. \tag{A.28}
$$

We can use this solution for d_t, and the matrix $\mathbf{f}$ to find an explicit solution for the coefficients $f_{i,t}$. Let $i = t - \nu$ where $1 \leq \nu < t$. Then the matrix $\mathbf{f}$ shows that

$$\begin{aligned} f_{t-\nu,t} &= \beta^{t-\nu}(\beta d_{\nu-1} - d_\nu) \\ &= \beta^{t-\nu}\left[\beta\left(\frac{1}{2}\beta^{\nu-1}\left[\frac{a}{\alpha} - 1 + \frac{a}{\alpha}\frac{a^2\beta}{1-a^2\beta}(1-(a^2\beta)^{\nu-1})\right]\right) - \frac{1}{2}\beta^\nu\left[\frac{a}{\alpha} - 1 + \frac{a}{\alpha}\frac{a^2\beta}{1-a^2\beta}(1-(a^2\beta)^{\nu})\right]\right] \\ &= -\frac{1}{2}\beta^{t-\nu}\frac{a}{\alpha}(a\beta)^{2\nu} \end{aligned}$$

For $t = i \geq 1$, $\mathbf{f}$ gives

$$\begin{aligned} f_{t,t} &= -\beta^t(d_0 + \frac{1}{2}) \\ &= -\beta^t\left[\frac{1}{2}(\frac{a}{\alpha} - 1) + \frac{1}{2}\right] \\ &= -\frac{1}{2}\beta^t\frac{a}{\alpha}. \end{aligned}$$

Thus we conclude that for any $t \geq 1$, $1 \leq i \leq t$,

$$f_{i,t} = -\frac{1}{2}\frac{a}{\alpha}\beta^i(a\beta)^{2(t-i)}.$$

The values of $f_{0,t}$ and $f_{i,0}$ can be read directly off the matrix $\mathbf{f}$. Summarizing these results, the terms of interest to us are given by:

$$T(\vec{\tau}) = \sum_{t=1}^{\infty}\sum_{i=1}^{t}\frac{f_{i,t}}{\lambda_t + h_{i,t}} \tag{A.29}$$

where for $t \geq 1, 1 \leq i \leq t$,

$$h_{i,t} = \sum_{k=t+1-i}^{t}\tau_k, \tag{A.30}$$

$$f_{i,t} = -\frac{1}{2}\frac{a}{\alpha}\beta^t(a^2\beta)^{t-i}. \tag{A.31}$$

Using the definition of $F_t^k(\vec{\lambda})$ as the $(t+1)$-th element of the k-th iterate of F (i.e., $F^{(k)}(\vec{\lambda})$), where $F(\vec{\lambda})$ is given by (13), we can reorder the terms of the sum in our expression for

$T(\vec{\tau})$ to see that

$$\begin{aligned}
T(\vec{\tau}) =& -\frac{1}{2}\frac{a\beta}{\alpha}\left[\left(\frac{1}{\lambda_1+\tau_1}+(a^2\beta^2)\frac{1}{\lambda_2+\tau_2}+(a^2\beta^2)^2\frac{1}{\lambda_3+\tau_3}+\dots\right)\right.\\
&+\beta\left(\frac{1}{(\lambda_2+\tau_2)+\tau_1}+(a^2\beta^2)\frac{1}{(\lambda_3+\tau_3)+\tau_2}+(a^2\beta^2)^2\frac{1}{(\lambda_4+\tau_4)+\tau_3}+\dots\right)\\
&\left.+\beta^2\left(\frac{1}{(\lambda_3+\tau_3+\tau_2)+\tau_1}+(a^2\beta^2)\frac{1}{(\lambda_4+\tau_4+\tau_3)+\tau_2}+(a^2\beta^2)^2\frac{1}{(\lambda_5+\tau_5+\tau_4)+\tau_3}+\dots\right)+\dots\right]\\
=& -\frac{1}{2}\frac{a\beta}{\alpha}\left[\left(\frac{1}{F_0(\vec{\lambda})}+(a^2\beta^2)\frac{1}{F_1(\vec{\lambda})}+(a^2\beta^2)^2\frac{1}{F_2(\vec{\lambda})}+\dots\right)\right.\\
&+\beta\left(\frac{1}{F_0^2(\vec{\lambda})}+(a^2\beta^2)\frac{1}{F_1^2(\vec{\lambda})}+(a^2\beta^2)^2\frac{1}{F_2^2(\vec{\lambda})}+\dots\right)\\
&\left.+\beta^2\left(\frac{1}{F_0^3(\vec{\lambda})}+(a^2\beta^2)\frac{1}{F_1^3(\vec{\lambda})}+(a^2\beta^2)^2\frac{1}{F_2^3(\vec{\lambda})}+\dots\right)+\dots\right]\\
=& -\frac{1}{2}b_0\left[\sum_{k=1}^{\infty}\beta^k\sum_{t=0}^{\infty}\left(\frac{b_t}{b_0}\right)^2\frac{1}{F_t^k(\vec{\lambda})}\right]. \qquad \text{(A.32)}
\end{aligned}$$

where in the last line we've used the solution for b_t in (A.8). This is the expression stated in the proposition.

In the process of solving for the parameters that enter the term T we solved for k, c_t, d_t, $f_{i,t}$ and $h_{i,t}$. To show that our guess for the value function does indeed yield the solution, we now derive expressions for the final outstanding coefficients of the value function, $D_{t,p}$. From the Bellman equation we see that

$$D_{t,p} = -\frac{1}{2}\left[(1+\alpha)2b_tb_p-\delta_{t,0}2b_p\right]+\beta\Big[2kb_tb_p+(1-\delta_{t,0})c_{t-1}b_p+b_tc_{p-1}+(1-\delta_{t,0})D_{t-1,p-1}\Big]\ .$$

For $t=0$, we find

$$D_{0,p} = A(a\beta)^p \quad , \quad A=\frac{a}{\alpha}\ .$$

For $t\geq 1$,

$$D_{t,p} = A(a\beta)^{t+p}+\beta D_{t-1,p-1} \quad , \quad A=\frac{a}{\alpha}\ .$$

The recursive equation

$$y(m,n) = A\xi^{m+n}+By(m-1,n-1) \quad , \ m<n$$

has the solution

$$y(m,n) = A\xi^{m+n}\frac{1-\left(\frac{B}{\xi^2}\right)^m}{1-\frac{B}{\xi^2}} + B^m y(0,n-m)\ .$$

Applying this general formula with $\xi = a\beta$ leads to

$$D_{t,p} = \frac{a}{\alpha}(a\beta)^{t+p}\frac{1-(a^2\beta)^{-t}}{1-(a^2\beta)^{-1}} + \frac{a}{\alpha}\beta^t(a\beta)^{p-t}\ .$$

Thus we have found unique solutions for all the free coefficients of our guess for the value function, confirming that the initial guess does indeed yield the solution.

E Proof of Proposition 3

From the proof of Proposition 2 we have

$$\frac{dV}{d\tau_m} = \frac{dT}{d\tau_m} = -\sum_{i=1}^{\infty}\sum_{t=1}^{\infty}\frac{f_{i,t}}{(\lambda_t + h_{i,t})^2}\frac{dh_{i,t}}{d\tau_m}.$$

From (A.19),

$$\frac{dh_{i,t}}{d\tau_m} = \begin{cases} 1 & t \geq m \text{ and } t \geq i \geq t+1-m \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\frac{dV}{d\tau_m} = -\sum_{t=m}^{\infty}\sum_{i=t+1-m}^{t}\frac{f_{i,t}}{(\lambda_t + \sum_{k=t+1-i}^{t}\tau_k)^2} \tag{A.33}$$

Evaluate this quantity at $\tau_t = 0$ for all t:

$$\left.\frac{dV}{d\tau_m}\right|_0 = -\sum_{t=m}^{\infty}\frac{1}{\lambda_t^2}\sum_{i=t+1-m}^{t} f_{i,t}$$

Let $t = m+k$ where $k \geq 0$, and consider the sum $\sum_{i=t+1-m}^{t} f_{i,t} = \sum_{i=k+1}^{m+k} f_{i,m+k}$. This sum is equivalent to starting at diagonal element $m+k+1, m+k+1$ of the matrix $\mathbf{f}$ in (A.23), and summing the m terms above this diagonal element (including the diagonal). Reading off the matrix, we see that this sum simplifies to:

$$\sum_{i=k+1}^{m+k} f_{i,m+k} = -\beta^{k+1}d_{m-1} - \frac{1}{2}\beta^{m+k}.$$

and hence

$$\left.\frac{dV}{d\tau_m}\right|_0 = -\sum_{t=m}^{\infty} \frac{1}{\lambda_t^2} \sum_{i=t+1-m}^{t} f_{i,t}$$
$$= \left(d_{m-1} \sum_{k=0}^{\infty} \frac{\beta^{k+1}}{\lambda_{m+k}^2} + \frac{1}{2}\beta^m \sum_{k=0}^{\infty} \frac{\beta^k}{\lambda_{m+k}^2} \right).$$

Using the definition of $g(m)$ this expression becomes

$$\left.\frac{dV}{d\tau_m}\right|_0 = \beta g(m) \left(d_{m-1} + \frac{1}{2}\beta^{m-1} \right).$$

From (A.28) we have

$$d_{m-1} + \frac{1}{2}\beta^{m-1} = \beta^{m-1} \left(\frac{a}{\alpha} + \frac{a}{\alpha} \frac{a^2\beta}{1-a^2\beta} (1 - (a^2\beta)^{m-1}) \right)$$
$$= \frac{a}{\alpha}\beta^{m-1} \left(1 + \frac{a^2\beta}{1-a^2\beta} (1 - (a^2\beta)^{m-1}) \right)$$
$$= \frac{a}{\alpha}\beta^{m-1} \left[\frac{1-(a^2\beta)^m}{1-a^2\beta} \right].$$

The result follows.

F Behaviour of R_m

When $\phi < \beta$, it is obvious from (21) that R_m is increasing in m. We thus focus on the case $\phi > \beta$. From the formula (21), and the requirement $\phi > \beta$, it is clear that $\lim_{m\to\infty} R_m = 0$. Here we show that R_m is either monotonically decreasing in m, or has a unique global maximum for some $m \geq 2$, and characterize the parameter ranges where these two behaviours occur.

The fact that R_m has at most one maximum at $m \geq 2$ can be shown by treating m as a continuous variable. Then R_m has a stationary point iff $\frac{d}{dm}R_m = 0$, which a little algebra shows occurs if

$$(a^2\beta)^m \ln\left(\frac{a^2\beta^2}{\phi}\right) = \ln\left(\frac{\beta}{\phi}\right). \tag{A.34}$$

This condition has at most one solution for $m \geq 1$. Since $R_m > 0$ for all m, $R_1 = 1$, $\lim_{m\to\infty} R_m = 0$, and dR_m/dm changes sign at most once, R_m cannot have a local minimum. Thus R_m must be either monotonically declining, or be unimodal with a global maximum at some $m \geq 2$.

It is simple to determine conditions under which these different qualitative behaviours occur. Since if R_m is not monotonically declining it must be unimodal, the condition $R_2 > R_1 = 1$ is both necessary and sufficient for R_m to be unimodal. A little algebra shows that $R_2 > 1 \iff \Gamma \equiv a^2\beta^2 + \beta - \phi > 0$. Since $a = 0$ at $\alpha = 0$, we know $\Gamma = \beta - \phi < 0$ when $\alpha = 0$. Also, since a is increasing in α, so is Γ. Combining these facts we see that the set of parameters values for which $\Gamma > 0$ must either be empty, or of the form $\alpha > \hat{\alpha}(\beta, \phi)$, where $\hat{\alpha}(\beta, \phi)$ is some critical value of α at which $\Gamma = 0$. Solving the condition $\Gamma = 0$ for α, we find two solutions:

$$\alpha_1 = \frac{(\phi - \beta)(1 + \beta) + \phi\sqrt{\phi - \beta}}{\beta^2 + (\phi - \beta)^2 - (\phi - \beta)(1 + \beta^2)} \quad , \quad \alpha_2 = \frac{(\phi - \beta)(1 + \beta) - \phi\sqrt{\phi - \beta}}{\beta^2 + (\phi - \beta)^2 - (\phi - \beta)(1 + \beta^2)} \ .$$

α_2 is negative for all β and $\phi \in [\beta, 1]$ so we conclude that

$$\hat{\alpha}(\beta, \phi) = \frac{(\phi - \beta)(1 + \beta) + \phi\sqrt{\phi - \beta}}{\beta^2 + (\phi - \beta)^2 - (\phi - \beta)(1 + \beta^2)} \ . \tag{A.35}$$

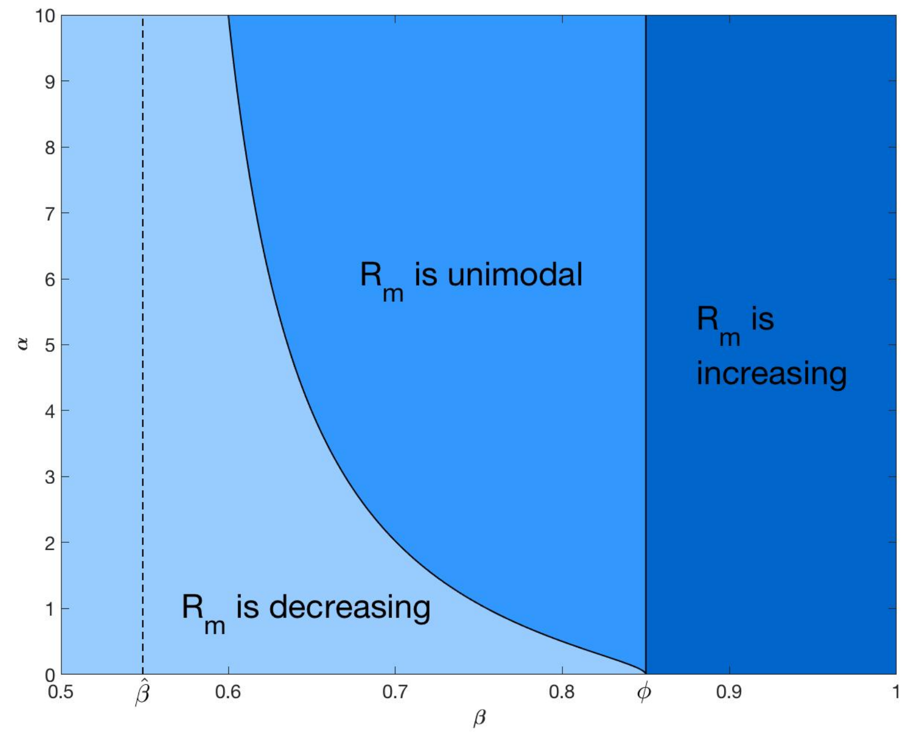
Observe that $\hat{\alpha}(0, \phi) = \frac{\phi(1+\sqrt{\phi})}{\phi(\phi-1)} < 0$ so Γ is negative at $\beta = 0$ irrespective of α. To find the conditions on β under which $\hat{\alpha}(\beta, \phi) \geq 0$ we solve $\hat{\alpha}(\beta, \phi) = 0$ for β, finding the following three roots:

$$\beta_1 = \phi \ , \ \beta_2 = -\frac{1 + \sqrt{1 + 4\phi}}{2} \ , \ \beta_3 = \frac{\sqrt{1 + 4\phi} - 1}{2} \ .$$

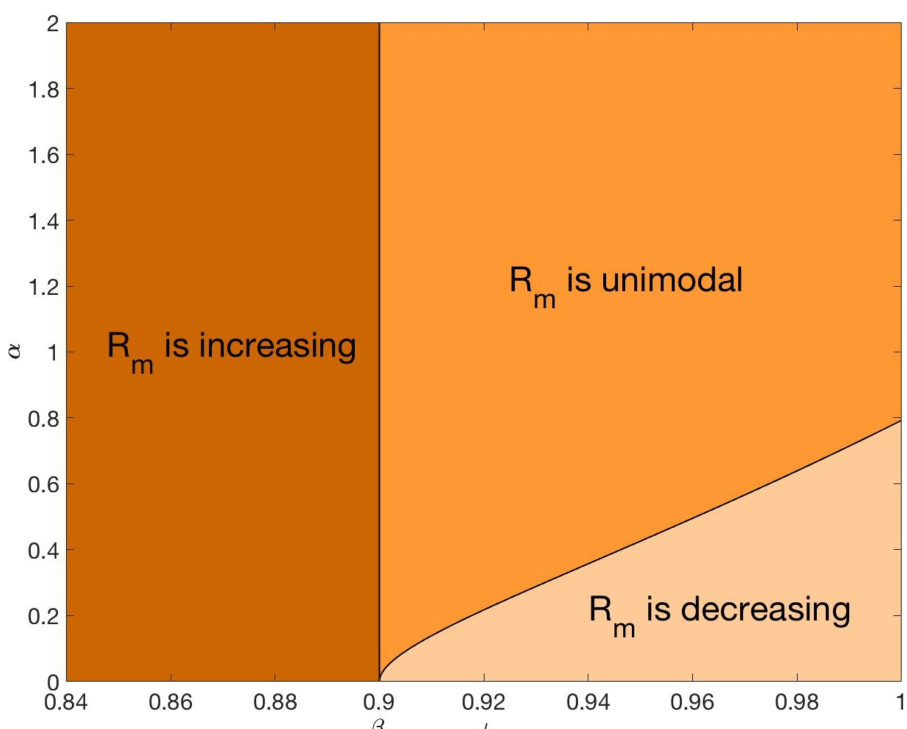
β_1 violates the condition $\beta < \phi$, β_2 is always negative, but $\beta_3 < \phi$ which makes the latter the relevant critical level of β at which $\hat{\alpha}(\beta, \phi) \geq 0$. Thus we define the critical value of β as

$$\hat{\beta}(\phi) = \frac{\sqrt{1 + 4\phi} - 1}{2} \ . \tag{A.36}$$

Thus, when $\beta \in [\hat{\beta}, \phi)$, R_m has a maximum at some $m > 1$ if $\alpha > \hat{\alpha}$, otherwise R_m is decreasing. Figure F.2 below demonstrates these results graphically.



(a) $\phi = 1$



(b) $\beta = 0.9$

Figure F.2: Qualitative behaviour of R_m in different regions of parameter space.

G Uniqueness of the solution to the predictability allocation problem in Eq. 22

Define

$$\begin{aligned} P(\vec{\lambda}) &= -\left(\frac{1}{\lambda_0} + (a^2\beta^2)\frac{1}{\lambda_1} + (a^2\beta^2)^2\frac{1}{\lambda_2} + \dots\right) \\ &= -\sum_{k=0}^{\infty}(a\beta)^{2k}\frac{1}{\lambda_k}, \end{aligned}$$

and assume that

$$\lim_{k\to\infty}\frac{\lambda_{k+1}}{\lambda_k} > (a\beta)^2$$

so that $P(\vec{\lambda})$ converges.

From (14) we see that finding the optimal vector $\vec{\tau}$ in (22) is equivalent to solving the following deterministic dynamic programming problem:

$$Q(\vec{\lambda}) = \max_{\vec{\tau}} P(\vec{\lambda}) + \beta Q(F(\vec{\lambda}))$$

where

$$F(\vec{\lambda}) = \Delta(\vec{\lambda}) + \vec{\tau} \tag{A.37}$$

as in (13), the operator Δ is defined in (12), and $\vec{\tau}$ satisfies the additional constraint

$$\sum_{m=1}^{\infty}\tau_m = B. \tag{A.38}$$

$P(\vec{\lambda})$ is a strictly concave function of $\vec{\lambda}$, the 'state equations' (A.37) are concave functions of the states and controls, and the auxiliary constraint (A.38) is also concave. Thus $\vec{\tau}$ lies in a convex set, and standard results (see e.g. **?**) imply that this optimization problem has a unique solution.

H Sensitivity analysis for Figures 3 and 4

Figures F.3 and F.4 below represents the outcome of a calculation identical to that in Fig. 3, but for $\lambda_0/B = 1/50$ and $\lambda_0/B = 50$ respectively. Fig. F.3 simply demonstrates that the rate of decline of the prior with the time horizon has no effect on budget allocations when λ_0/B is small. In Fig. F.4 predictions are marginal relative to the prior, making interactions between lead times unimportant. To a good approximation then, the value function is linear in forecast precisions in this case, as discussed in Proposition 3. Thus, when $\phi > \beta$, we expect the entire budget to be allocated to the most valuable forecast lead time (i.e., the value of m for which R_m in (21) is maximised). The bottom panel of Fig. F.4 confirms this expectation. When $\phi < \beta$ however, the marginal analysis in Proposition 3 shows that the value of a marginal unit of predictability is increasing in lead time m – there is no 'most valuable' lead time. Since the agent cannot allocate her entire budget to infinite lead times, and λ_0/B is large, but not infinite in Fig. F.4 (so forecasts are only approximately marginal), interaction effects are still at work in this case, and lead to the spread out peaks in the top panel of Fig. F.4. Notice however that these peaks place more weight on the long run than the analysis for $\lambda_0/B = 1$ in Fig. 3, indicating that first order effects are more important in this case than in Fig. 3, as we would expect when choosing a very large value of λ_0/B. We emphasise however that $\lambda_0/B = 50$ is an unrealistically large value. As discussed in the text, priors and forecasts usually have roughly the same precisions in practice, so the results in Fig. F.4 grossly underestimate the importance of the interactions between lead times.

In addition, Fig. F.5 presents results analogous to those in Fig. 4, for the lower value $\beta = 0.95$.

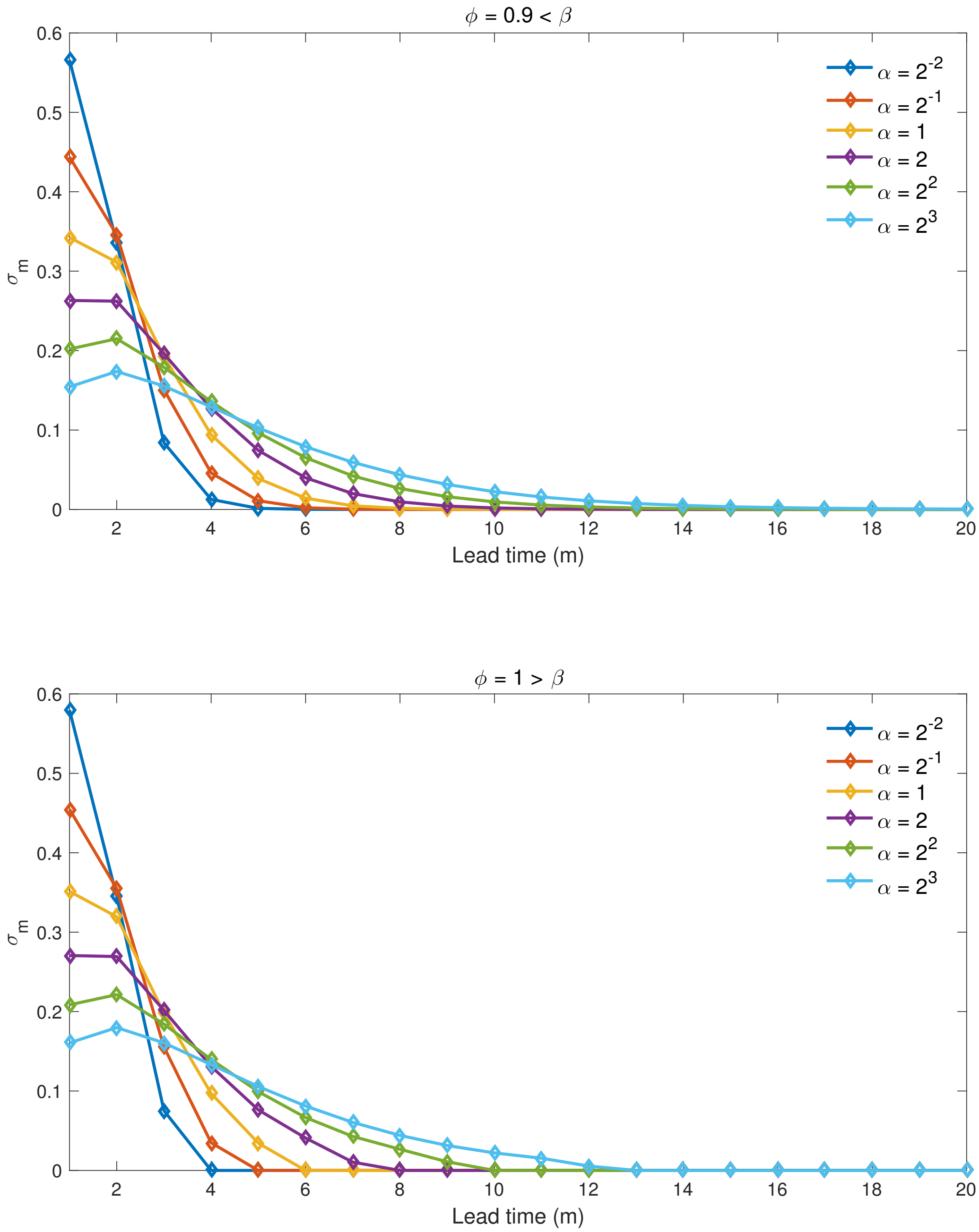


Figure F.3: Budget share σ_m allocated to lead time m in the optimization problem in (22). $\beta = 0.95$, $\frac{\lambda_0}{B} = 1/50$.

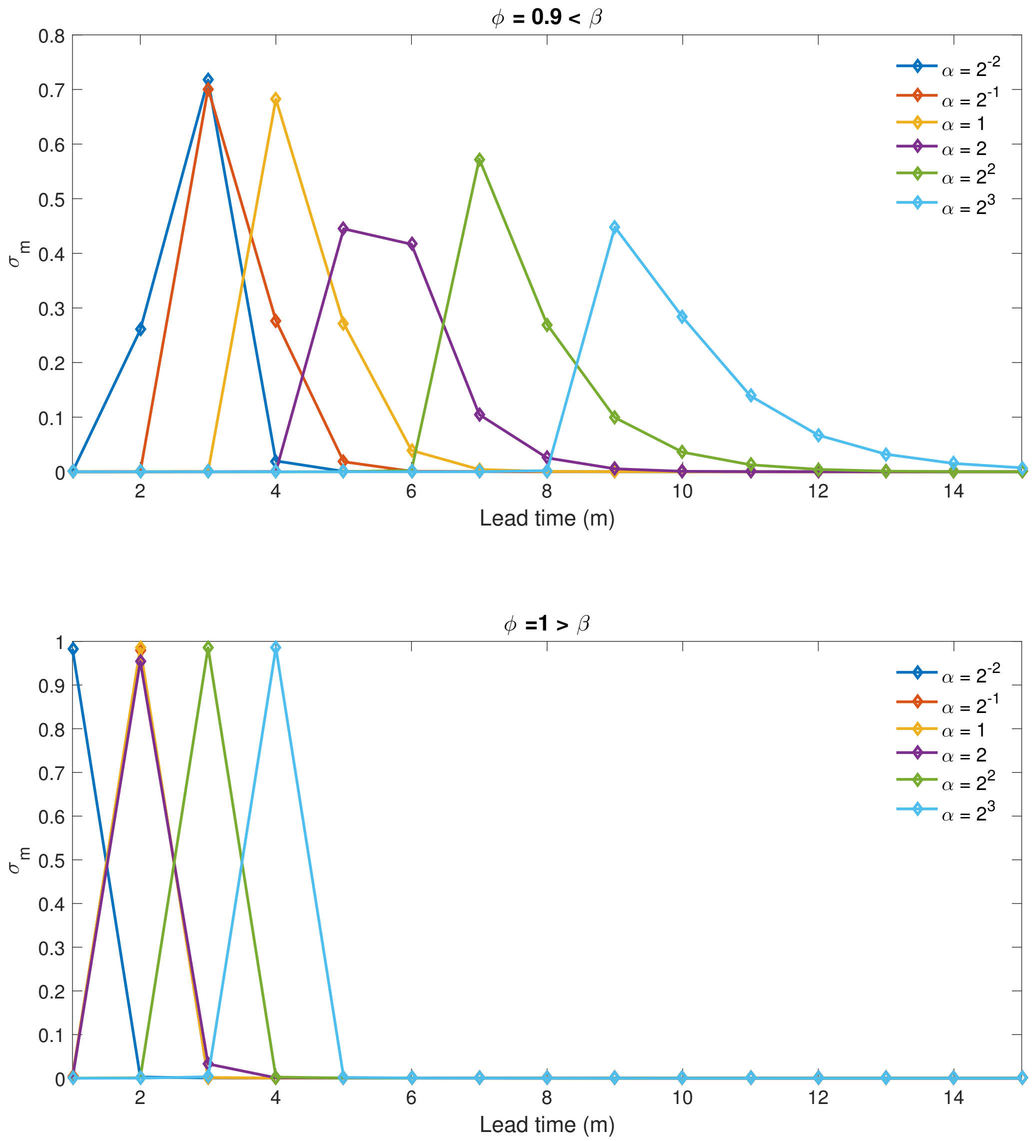


Figure F.4: Budget share σ_m allocated to lead time m in the optimization problem in (22). $\beta = 0.95$, $\frac{\lambda_0}{B} = 50$.

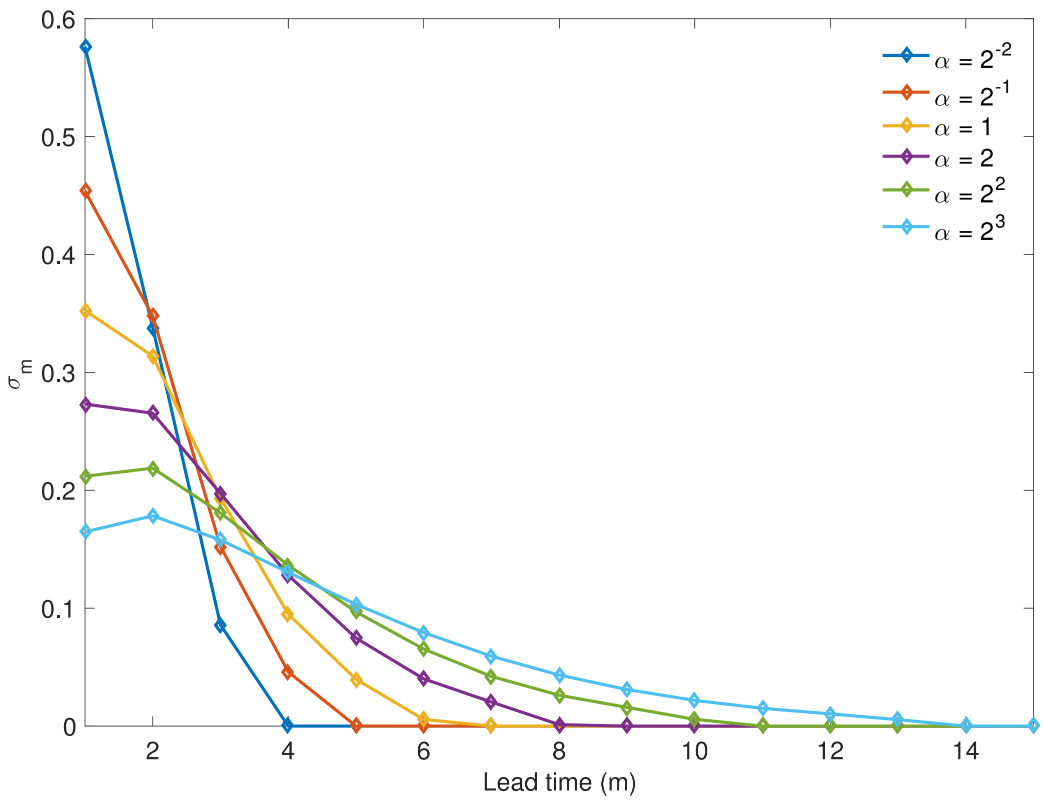


Figure F.5: Budget share σ_m allocated to lead time m in the optimisation problem in (22), when $\beta = 0.95$, $\frac{\lambda_0}{B} \to 0$. This figure illustrates the 'pure' effect of substitution between lead times when priors play no role in the analysis.