ONLINE APPENDIX

Connecting Disconnected Financial Markets? By Milena Wittwer

Appendix A shows all proofs regarding equilibrium existence and features.

Appendix B has the proofs of all irrelevance theorems.

Appendix C gives all proofs that compare welfare across market structures.

Appendix D collects all proofs of the corollaries.

Appendix E derives an equilibrium for the discon. market with correlated types.

Appendix F gives more general optimality conditions for the discon. market.

A. Proofs of Equilibrium Existence and Features

A1. Proof of Proposition 1 (I) and Proposition 2

I derive the unique symmetric linear equilibrium in a guess-and-verify approach. Denote $\vec{p} = \begin{pmatrix} p_1 & p_2 \end{pmatrix}'$, $\vec{s}_i = \begin{pmatrix} s_{i,1} & s_{i,2} \end{pmatrix}'$, and guess that there is an equilibrium in which agents play

$$\vec{x}(\vec{p}, \vec{s}_i) = \vec{o} + A\vec{s}_i - C\vec{p} \tag{A1}$$

$$\text{with } A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}, \vec{o} = \begin{pmatrix} o_1 \\ o_2 \end{pmatrix}, C = \begin{pmatrix} c_1 & d_1 \\ d_2 & c_2 \end{pmatrix} \text{ pos. definite.}$$

Take the perspective of agent i and let all others play a linear strategy (A1). The agent trades against the following residual supply curve

$$\vec{\boldsymbol{RS}}(\vec{p}) = \vec{\boldsymbol{Q}} - \sum_{j \neq i} \vec{x}(\vec{p}, \vec{\boldsymbol{s}}_{\boldsymbol{j}}) = \vec{\boldsymbol{Z}} + (n-1)C\vec{p} \text{ with } \vec{\boldsymbol{Z}} \equiv \vec{\boldsymbol{Q}} - A\sum_{j \neq i} \vec{\boldsymbol{s}}_{\boldsymbol{j}} - (n-1)\vec{o}.$$

It varies only in its intercept $\vec{\boldsymbol{Z}}$. Therefore, the agent knows—for any prices $\vec{p}$ that he might bid—the amount that he wins at market clearing, $\vec{q} = \vec{RS}_i(\vec{p})$. In other words, $\vec{Z}$ is informational equivalent to $\vec{q}$ for fixed $\vec{p}$. It follows that the agent's best reply can be obtained by standard pointwise maximization by solving

$$\max_{\vec{p}} \mathbb{E}\left[U(\vec{q},\vec{s}_i) - \vec{p}'\vec{q}|\vec{q}\right] \text{ with } \vec{q} = \vec{RS}_i(\vec{p})$$

$$\Leftrightarrow \max_{\vec{p}} \left\{ (\vec{s}_i - \vec{p})'\, \vec{RS}_i(\vec{p}) - \frac{1}{2}\vec{RS}_i(\vec{p})'\Delta\vec{RS}_i(\vec{p}) \right\} \text{ with } \Delta \equiv \begin{pmatrix} \lambda & \delta \\ \delta & \lambda \end{pmatrix} \text{ by (2).}$$

An optimal price point $\vec{p}^*$ must fulfill the following necessary condition

$$(\overline{FOC}) \qquad 0 = -\,\vec{RS}_i(\vec{p}^*) + \left(\frac{\partial \vec{RS}_i(\vec{p}^*)}{\partial \vec{p}}\right)' \left(\vec{s}_i - \vec{p}^* - \Delta\vec{RS}_i(\vec{p}^*)\right)$$

and clear the market: $\vec{RS}_i(\vec{p}^*) = \vec{x}_i(\vec{p}^*, \vec{s}_i)$.
The necessary condition is sufficient because Δ is positive definite.
Rearranging $(\overline{FOC})$, agent i's best reply solves

$$\vec{p}^* = \vec{s}_i - \left(\left(\frac{1}{n-1}\right)(C')^{-1} + \Delta\right)\vec{x}_i(\vec{p}^*, \vec{s}_i) \text{ with identity matrix } I.$$

For this strategy to constitute a symmetric equilibrium it must be optimal for the agent to choose the same strategy as all others. Consequently, I can determine the equilibrium coefficients by matching coefficients of i's best reply with the choice by all others who, rearranging (A1), submit $\vec{p}^* = C^{-1}\vec{o} + C^{-1}A\vec{s}_j - C^{-1}\vec{x}(\vec{p}^*, \vec{s}_j)$. Solving the following system of equations

$$C^{-1}\vec{o} = \begin{pmatrix} 0 & 0 \end{pmatrix}' \text{ and } C^{-1}A = I \text{ and } \left(\frac{1}{n-1}\right)(C')^{-1} + \Delta = C^{-1}$$

gives a unique set of equilibrium coefficients: $C^* = A^* = \left(\frac{n-2}{n-1}\right)\Delta^{-1}$ and $\vec{o}^* = \begin{pmatrix} 0 & 0 \end{pmatrix}'$, and completes the proof of Proposition 1 (I). Inverting the demand schedules one obtains the bidding functions of Proposition 2. □

A2. Proof Proposition 1 (II)(i) and Proposition 4

Consider parameters $\{n, \lambda, \delta, \mu_{s_1}, \mu_{s_2}, \sigma_s, \rho_s, \mu_{Q_1}, \mu_{Q_2}, \sigma_Q, \rho_Q\}$, for which polynomial (P) has at least one real root ρ^* that lies in $[-1, 1]$. The goal is to show that there exists a symmetric linear BNE (Propositions 1 (II)(i)) and to derive its unique functional form (Proposition 4). In other words, I must show that there is an equilibrium in which all agents choose the following demand function

$$x_m(p_m, \vec{s}_i) = o_m + a_{m,m}s_{i,m} + a_{m,-m}s_{i,-m} - c_m p_m \text{ with } c_m > 0, \text{ for } m = 1, 2 \tag{A2}$$

and determine unique coefficients such that (A2) is an equilibrium. To do so, take the perspective of agent i and fix his type $\vec{s}_i$.

Main structure of the proof. —

- MAIN LEMMA 1 in PART 1) of the proof shows that agent i has a unique best reply given all other agents $j \neq i$ play strategy (A2) with coefficients c_1, c_2 such that $(2+c_1\lambda(n-1))(2+c_2\lambda(n-1)) - c_1c_2(n-1)^2 \neq 0$. This best reply is linear, i.e., of the same form as (A2). It constitutes a symmetric equilibrium if and only if it is optimal for the agent to choose the same coefficients as all others.
- MAIN LEMMA 2 in PART 2) I can therefore determine the equilibrium coefficients by matching coefficients of i's best reply with strategy (A2) played by all others. These equilibrium coefficients are such that $(2+c_1^*\lambda(n-1))(2+c_2^*\lambda(n-1)) - c_1^*c_2^*(n-1)^2 \neq 0$. Furthermore, there is a unique set of such coefficients. Together this implies that the determined equilibrium is the unique symmetric linear equilibrium.

For notational ease, I will drop the fixed type $\vec{s}_i$ as input argument of any function and all subscripts i. For instance, $b_{i,1}(\cdot, \vec{s}_i)$ becomes $b_1(\cdot)$, $x_{i,1}(\cdot, \vec{s}_i)$ becomes $x_1(\cdot)$ and the amount i wins at market clearing is q_m^c.

PART 1) BEST REPLY. —

MAIN LEMMA 1: *Agent i has a unique best reply to all others playing a linear strategy (A2) with coefficients* c_1, c_2 *such that* $(2 + c_1\lambda(n-1))(2 + c_2\lambda(n-1)) - c_1c_2(n-1)^2 \neq 0$.

PROOF OF LEMMA 1: The proof proceeds in five main steps:

- STEP 1. I set up the agent's optimization problem of variational calculus and simplify it.
- STEP 2. Lemma 1 gives necessary conditions for a maximum by deriving the first variation of the agent's optimization problem.
- STEP 3. Lemma 2 shows that the necessary conditions are sufficient by proving strict concavity.
- STEP 4. Lemma 3 proves that there is unique set of functions that satisfies the necessary, and that sufficient conditions for a maximum are satisfied, as long as all other agents choose coefficient c_1, c_2 such that $(2 + c_1\lambda(n-1))(2 + c_2\lambda(n-1)) - c_1c_2(n-1)^2 \neq 0$. These functions are linear.

STEP 1. — By the rules of the auction the agent can submit a pair of weakly decreasing, asymptotically linear C^2 functions $\{b_1(\cdot), b_2(\cdot)\}$. His aim is to maximize his expected total surplus from winning $\{\boldsymbol{q_1^c}, \boldsymbol{q_2^c}\}$ when offering prices $\{b_1(\boldsymbol{q_1^c}), b_2(\boldsymbol{q_2^c})\}$:

$$\max_{b_1(\cdot)\in\mathcal{B}, b_2(\cdot)\in\mathcal{B}} \mathbb{E}\left[U(\boldsymbol{q_1^c}, \boldsymbol{q_2^c}) - \sum_{m=1,2} b_m(\boldsymbol{q_m^c})\boldsymbol{q_m^c}\right] \text{ with } \boldsymbol{q_m^c} = \boldsymbol{Q_m} - \sum_{j\neq i} x_m(b_m(\boldsymbol{q_m^c}), \vec{\boldsymbol{s}}_{\boldsymbol{j}}) \tag{4}$$

where $\mathcal{B}$ denotes the set of all C^2 functions on $\mathbb{R}$ which are weakly decreasing and asymptotically linear. To solve this problem, it helps to think about how the agent chooses his optimal functions in a less abstract way. Essentially the agent chooses points on two residual supply curves which shift depending on the realizations of total supply and types of i's competitors, $\boldsymbol{RS_m}(p_m) = \boldsymbol{Q_m} - \sum_{j\neq i} x_m(p_m, \vec{\boldsymbol{s}}_{\boldsymbol{j}})$. Since all of i's competitors submit the same linear demand functions (A2) by assumption, i faces two linear residual supply curves which vary only in their intercepts $\boldsymbol{Z_1}, \boldsymbol{Z_2}$ with the quantity-axes:

$$(RS_m) \qquad RS_m(p_m, \boldsymbol{Z_m}) = \boldsymbol{Z_m} - (n-1)o_m + (n-1)c_m p_m$$

$$(Z_m) \qquad \text{with } \boldsymbol{Z_m} \equiv \boldsymbol{Q_m} - a_{m,m} \sum_{j \neq i} \boldsymbol{s_{j,m}} - a_{m,-m} \sum_{j \neq i} \boldsymbol{s_{j,-m}}.$$

As linear combinations of jointly normally distributed random variables (types and supply) these intercepts are jointly normally distributed. Furthermore, given that the residual supply curve is strictly increasing in price ($(n-1)c_m > 0$), while i can only submit decreasing functions, there is, for every quantity point q_m on any bidding function the agent may submit, a unique realization Z_m. This implies that there is a one-to-one mapping between decreasing C^2 functions $\{b_1(\cdot), b_2(\cdot)\}$ that map from quantities to prices, and decreasing C^2 functions $\{p_1(\cdot), p_2(\cdot)\}$ that map from $Z_m \in \mathbb{R}$ to prices. Rather than maximizing over $\{b_1(\cdot), b_2(\cdot)\}$, I will solve

$$(4') \qquad \max_{p_1(\cdot) \in \mathcal{B}, p_2(\cdot) \in \mathcal{B}} \mathbb{E}\left[U(\boldsymbol{q_1^c}, \boldsymbol{q_2^c}) - \sum_{m=1,2} p_m(\boldsymbol{Z_m})\boldsymbol{q_m^c}\right] \text{ with } \boldsymbol{q_m^c} = RS_m(p_m(\boldsymbol{Z_m}), \boldsymbol{Z_m}).$$

Denoting the normal density of $\boldsymbol{Z_1}, \boldsymbol{Z_2}$ by $\phi(Z_1, Z_2)$, inserting the quadratic utility function (2) and the market clearing constraint $\boldsymbol{q_m^c} = RS_m(p_m(\boldsymbol{Z_m}), \boldsymbol{Z_m})$ into the objective function, the maximization problem reads as follows:

MAXIMIZATION PROBLEM 1: *Let $\mathcal{B}$ be the set of all C^2 functions on $\mathbb{R}$ which are weakly decreasing and asymptotically linear.*

(V)

$$\max_{p_1(\cdot) \in \mathcal{B}, p_2(\cdot) \in \mathcal{B}} V(p_1(\cdot), p_2(\cdot)) = \int_{\mathbb{R}} \int_{\mathbb{R}} F(Z_1, Z_2, p_1(Z_1), p_2(Z_2)) \phi(Z_1, Z_2) dZ_1 dZ_2 \, \textit{where}$$

(F)

$$F(Z_1, Z_2, p_1(Z_1), p_2(Z_2)) = \sum_{m=1,2} (s_m - p_m(Z_m))\, RS_m(p_m(Z_m), Z_m) - \frac{1}{2}\lambda \left(RS_m(p_m(Z_m), Z_m)\right)^2 - \delta RS_1(p_1(Z_1), Z_1) RS_2(p_2(Z_2), Z_2)$$

(RS_m)

$$RS_m(p_m(Z_m), Z_m) = Z_m - (n-1)o_m + (n-1)c_m p_m(Z_m) \; \textit{for } m = 1, 2.$$

In the remainder of this section, I show that there is a unique solution to this problem. Before doing so, the following auxiliary lemma verifies that the problem is well defined. The first part shows that the expected value of the agent's total surplus never fails to exist for any functions he may submit. More specifically, it implies that the objective functional $V(p_1(\cdot), p_2(\cdot))$ converges. The second part of this lemma will be useful later on.

AUXILIARY LEMMA 1 (i): *$F(Z_1, Z_2, p_1(Z_1), p_2(Z_2))\phi(Z_1, Z_2)$ is for any functions $\{p_1(\cdot), p_2(\cdot)\} \in \mathcal{B} \times \mathcal{B}$ integrable in $\mathbb{R}^2$.*

PROOF OF AUXILIARY LEMMA 1 (i): The goal is to show that

$$\int\int_{\mathbb{R}^2} |F(Z_1, Z_2, p_1(Z_1), p_2(Z_2))|\phi(Z_1, Z_2)dZ_1dZ_2 < \infty \ \forall\{p_1(\cdot), p_2(\cdot)\} \in \mathcal{B} \times \mathcal{B}.$$

To do so, fix any $\{p_1(\cdot), p_2(\cdot)\} \in \mathcal{B}\times\mathcal{B}$. Since $F(\cdot, \cdot, \cdot, \cdot)$ is quadratic in (Z_1, Z_2, p_1, p_2) and $p_1(\cdot), p_2(\cdot)$ become linear as $|Z_1|, |Z_2| \to \infty$, function $F(\cdot, \cdot, p_1(\cdot), p_2(\cdot))$ converges to some quadratic function of (Z_1, Z_2) as $|Z_1|, |Z_2| \to \infty$. Call this function $Q(\cdot, \cdot)$. By definition of a limit we know that there exist a bounded space $\mathcal{M} = \{(Z_1, Z_2) \in \mathbb{R}^2 : |Z_1| < M_1, |Z_2| < M_2\}$ and a small positive number $\varepsilon > 0$ such that

$$|F(Z_1, Z_2, p_1(Z_1), p_2(Z_2)) - Q(Z_1, Z_2)| < \varepsilon \ \ \forall(Z_1, Z_2) \notin \mathcal{M}.$$

By the reverse triangle inequality

$$|F(Z_1, Z_2, p_1(Z_1), p_2(Z_2))| - |Q(Z_1, Z_2)| \leq |F(Z_1, Z_2, p_1(Z_1), p_2(Z_2)) - Q(Z_1, Z_2)|$$
$$\Rightarrow |F(Z_1, Z_2, p_1(Z_1), p_2(Z_2))| \leq \varepsilon + |Q(Z_1, Z_2)| \quad \forall(Z_1, Z_2) \notin \mathcal{M}.$$

Since $\phi(Z_1, Z_2) \geq 0$ and $\varepsilon > 0$

$$|F(Z_1, Z_2, p_1(Z_1), p_2(Z_2))|\phi(Z_1, Z_2) \leq |\varepsilon + Q(Z_1, Z_2)|\phi(Z_1, Z_2) \quad \forall(Z_1, Z_2) \notin \mathcal{M}.$$

Therefore

$$\int\int_{\mathbb{R}^2} |F(Z_1, Z_2, p_1(Z_1), p_2(Z_2))|\phi(Z_1, Z_2)dZ_1dZ_2 \leq \int\int_{\mathcal{M}} |F(Z_1, Z_2, p_1(Z_1), p_2(Z_2))|\phi(Z_1, Z_2)dZ_1dZ_2 + \int\int_{\mathbb{R}^2\setminus\mathcal{M}} |\varepsilon + Q(Z_1, Z_2)|\phi(Z_1, Z_2)dZ_1dZ_2.$$

The first integral goes over bounded region $\mathcal{M} \subset \mathbb{R}^2$. It takes a finite value because integrand function $F\phi$ is a continuous function of (Z_1, Z_2) that lives in $\mathbb{R}^2$. The second integral goes over an unbounded region. It takes a finite value because the exponential normal density $\phi(\cdot,\cdot)$ goes to zero much faster than any polynomial, such as $\varepsilon + Q(\cdot,\cdot)$, could grow. □

AUXILIARY LEMMA 1 (*ii*): *For any* $\{p_1(\cdot), p_2(\cdot)\} \in \mathcal{B} \times \mathcal{B}$*:*

$$\int\int_{\mathbb{R}^2} \left| \frac{\partial F(Z_1, Z_2, p_1(Z_1), p_2(Z_2))}{\partial p_m(Z_m)} \right| \phi(Z_1, Z_2) dZ_1 dZ_2 < \infty \quad \textit{for } m = 1, 2$$

$$\int\int_{\mathbb{R}^2} \left| \frac{\partial^2 F(Z_1, Z_2, p_1(Z_1), p_2(Z_2))}{\partial^2 p_m(Z_m)} \right| \phi(Z_1, Z_2) dZ_1 dZ_2 < \infty \quad \textit{for } m = 1, 2$$

$$\int\int_{\mathbb{R}^2} \left| \frac{\partial^2 F(Z_1, Z_2, p_1(Z_1), p_2(Z_2))}{\partial p_1(Z_1) \partial p_2(Z_2)} \right| \phi(Z_1, Z_2) dZ_1 dZ_2 < \infty \textit{ and similarly for } \partial p_2(Z_2) \partial p_1(Z_1).$$

PROOF OF AUXILIARY LEMMA 1 (*i*): The proofs are analogous to the above, and omitted to avoid redundancy. In essence, integrability follows from the quadratic form of $F(\cdot,\cdot,\cdot,\cdot)$ in (Z_1, Z_2, p_1, p_2) in combination with the constraint that all functions $p_1(\cdot), p_2(\cdot)$ are asymptotically linear and $\phi(\cdot,\cdot)$ being an exponential function. □

STEP 2. —

LEMMA 1 (NECESSARY CONDITIONS): *If* $\{p_1^*(\cdot), p_2^*(\cdot)\}$ *is a solution to maximization problem (1), then*

$$(NC_1) \qquad \int_{\mathbb{R}} \left(\frac{\partial F(Z_1, Z_2, p_1^*(Z_1), p_2^*(Z_2))}{\partial p_1(Z_1)} \right) \phi_{2|1}(Z_2|Z_1) dZ_2 = 0 \ \forall Z_1$$

and

$$(NC_2) \qquad \int_{\mathbb{R}} \left(\frac{\partial F(Z_1, Z_2, p_1^*(Z_1), p_2^*(Z_2))}{\partial p_2(Z_2)} \right) \phi_{1|2}(Z_1|Z_2) dZ_1 = 0 \ \forall Z_2$$

where $\phi_{-m|m}(Z_{-m}|Z_m)$ *is the conditional density of* Z_{-m} *given* Z_m *for* $m = 1, 2$.

PROOF OF LEMMA 1: Suppose that $\{p_1^*(\cdot), p_2^*(\cdot)\}$ is a solution to $\max_{p_1\in\mathcal{B},p_2\in\mathcal{B}} V(p_1(\cdot),p_2(\cdot))$ and let $\xi_1(\cdot),\xi_2(\cdot)$ be two asymptotically linear C^2 function on $\mathbb{R}$, i.e.,

$$(*)\qquad \frac{\partial \xi_m(Z_m)}{\partial Z_m} \to 0 \text{ as } |Z_m| \to \infty \text{ , for } m=1,2.$$

The "varied" function $\begin{pmatrix} p_1^*(Z_1) \\ p_2^*(Z_2) \end{pmatrix} + \varepsilon \begin{pmatrix} \xi_1(Z_1) \\ \xi_2(Z_2) \end{pmatrix}$ is, for every ε, C^2 on $\mathbb{R}^2$. Since $\{p_1^*(\cdot), p_2^*(\cdot)\}$ is a maximum,

$$\text{(A3)}\qquad I(\varepsilon) \equiv \int_{\mathbb{R}}\int_{\mathbb{R}} F\left(Z_1, Z_2, p_1^*(Z_1)+\varepsilon\xi_1(Z_1), p_2^*(Z_2)+\varepsilon\xi_2(Z_2)\right)\phi(Z_1,Z_2)dZ_1dZ_2$$

must take its maximum for $\varepsilon = 0$. By Auxiliary Lemma 1 the integrand is integrable as $p_m^*(Z_m)+\varepsilon\xi_m(Z_m)$ becomes, by constraint (*), linear as $|Z_m| \to \infty$.

The rest of the proof derives the first variation $\frac{d}{d\varepsilon}I(\varepsilon)$ and finds conditions such that $\frac{d}{d\varepsilon}I(0) = 0$ for any C^2 functions $\xi_1(\cdot),\xi_2(\cdot)$ that fulfill (*). Taking the total derivative of $I(\varepsilon)$ we obtain

$$\begin{aligned}\frac{d}{d\varepsilon}I(\varepsilon) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \xi_1(Z_1)\left(\frac{\partial F(Z_1,Z_2,p_1^*(Z_1)+\varepsilon\xi_1(Z_1),p_2^*(Z_2)+\varepsilon\xi_2(Z_2))}{\partial p_1(Z_1)}\right)\phi(Z_1,Z_2)dZ_1dZ_2\\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}} \xi_2(Z_2)\left(\frac{\partial F(Z_1,Z_2,p_1^*(Z_1)+\varepsilon\xi_1(Z_1),p_2^*(Z_2)+\varepsilon\xi_2(Z_2))}{\partial p_2(Z_2)}\right)\phi(Z_1,Z_2)dZ_1dZ_2.\end{aligned}$$

Integrability once more follows from Auxiliary Lemma 1 and constraint (*). Applying Fubini's Theorem, I can reverse the order of integration of the first integral. In addition, I use Bayes' Theorem to replace $\phi(Z_1,Z_2) = \phi_m(Z_m)\phi_{-m|m}(Z_{-m}|Z_m)$ for $m=1,2$, where $\phi_m(Z_m)$ denotes the marginal and $\phi_{-m|m}(Z_{-m}|Z_m)$ the conditional density function. This gives

$$\begin{aligned}\frac{d}{d\varepsilon}I(\varepsilon) &= \int_{\mathbb{R}} \xi_1(Z_1)\left[\int_{\mathbb{R}}\left(\frac{\partial F(Z_1,Z_2,p_1^*(Z_1)+\varepsilon\xi_1(Z_1),p_2^*(Z_2)+\varepsilon\xi_2(Z_2))}{\partial p_1(Z_1)}\right)\phi_2(Z_2|Z_1)dZ_2\right]\phi_1(Z_1)dZ_1\\ &+ \int_{\mathbb{R}} \xi_2(Z_2)\left[\int_{\mathbb{R}}\left(\frac{\partial F(Z_1,Z_2,p_1^*(Z_1)+\varepsilon\xi_1(Z_1),p_2^*(Z_2)+\varepsilon\xi_2(Z_2))}{\partial p_2(Z_2)}\right)\phi_1(Z_1|Z_2)dZ_1\right]\phi_2(Z_2)dZ_2.\end{aligned}$$

For $\frac{d}{d\varepsilon}I(0)$ to take value 0 for any C^2 functions $\xi_1(\cdot), \xi_2(\cdot)$ such that (*), conditions (NC_1) and (NC_2) must hold. □

STEP 3. —

LEMMA 2 (SUFFICIENCY): *Any pair $\{p_1^*(\cdot), p_2^*(\cdot)\} \in \mathcal{B} \times \mathcal{B}$ that satisfies the necessary conditions is a maximum.*

PROOF OF LEMMA 2: Consider a set of functions $\{p_1^*(\cdot), p_2^*(\cdot)\} \in \mathcal{B} \times \mathcal{B}$ that satisfies the necessary conditions. They constitute a maximum if

$$V(p_1^*(\cdot), p_2^*(\cdot)) \leq V(p_1(\cdot), p_2(\cdot)) \text{ for any } \{p_1(\cdot), p_2(\cdot)\} \in \mathcal{B} \times \mathcal{B}. \tag{A4}$$

To show this, I proceed in two steps. I first establish that $F(Z_1, Z_2, \cdot, \cdot)$ is for any Z_1, Z_2 strictly concave as a function of p_1, p_2. I then show that concavity implies (A4).

$F(Z_1, Z_2, \cdot, \cdot)$ is strictly concave because its hessian matrix is negative definite. To see this, compute the the partial derivatives of F w.r.t. prices:

$$\begin{aligned}
\frac{\partial^2 F(Z_1, Z_2, p_1, p_2)}{\partial^2 p_m} &= -\left(\frac{\partial RS_m(p_m, Z_m)}{\partial p_m}\right) + \left(-1 - \lambda\frac{\partial RS_m(p_m, Z_m)}{\partial p_m}\right)\left(\frac{\partial RS_m(p_m, Z_m)}{\partial p_m}\right) \\
&= -(n-1)c_m\left[2 + \lambda(n-1)c_m\right] \quad \text{for } m = 1, 2 \\
\frac{\partial^2 F(Z_1, Z_2, p_1, p_2)}{\partial p_1 p_2} &= \frac{\partial^2 F(Z_1, Z_2, p_1, p_2)}{\partial p_2 p_1} = -\delta\left(\frac{\partial RS_1(p_1, Z_1)}{\partial p_1}\right)\left(\frac{\partial RS_2(p_2, Z_2)}{\partial p_2}\right) = -\delta(n-1)^2 c_1 c_2.
\end{aligned}$$

Since $\frac{\partial^2 F(Z_1, Z_2, p_1, p_2)}{\partial^2 p_m} < 0$ given $n > 1, \lambda > 0, c_m > 0$ for $m = 1, 2$, F's hessian matrix is negative definite if

$$\begin{aligned}
&\left(\frac{\partial^2 F(Z_1, Z_2, p_1, p_2)}{\partial^2 p_1}\right)\left(\frac{\partial^2 F(Z_1, Z_2, p_1, p_2)}{\partial^2 p_2}\right) - \left(\frac{\partial^2 F(Z_1, Z_2, p_1, p_2)}{\partial p_1 p_2}\right)\left(\frac{\partial^2 F(Z_1, Z_2, p_1, p_2)}{\partial p_2 p_1}\right) > 0 \\
&\Leftrightarrow c_1 c_2 (n-1)^2\Big(4 + (n-1)(2c_1\lambda + 2c_2\lambda + c_1 c_2(n-1)(\lambda^2 - \delta^2))\Big) > 0
\end{aligned}$$

This holds given $c_m > 0$ for $m = 1, 2$ and $n > 1, \lambda > 0, |\delta| < \lambda$ by assumption.

The second step is to show that strict concavity of F in (p_1, p_2) implies (A4). For this, it is convenient to switch to the vector notation, with p and Z denoting the vectors (p_1, p_2) and (Z_1, Z_2).

Since F is strictly concave as a function of p, it is bounded from above by its first-order Taylor expansion. In particular,

$$F(Z,p) - F(Z,p^*) < \nabla_p F(Z,p^*)(p-p^*) \leq 0 \text{ for any } p \in \mathcal{B}\times\mathcal{B} \text{ at any } Z.$$

Multiply both sides by $\phi(\cdot) \geq 0$ and integrate over Z to obtain

$$\int F(Z,p)\phi(Z)dZ - \int F(Z,p^*)\phi(Z)dZ < \int \nabla_p F(Z,p^*)(p-p^*)\phi(Z)dZ \leq 0$$

It follows that $\int F(Z,p)\phi(Z)dZ < \int F(Z,p^*)\phi(Z)dZ$ for any $p \in \mathcal{B}\times\mathcal{B}$, which—by definition ($V$)—is equivalent to $V(p_1(\cdot), p_2(\cdot)) < V(p_1^*(\cdot), p_2^*(\cdot))$ for any $\{p_1(\cdot), p_2(\cdot)\} \in \mathcal{B}\times\mathcal{B}$. □

STEP 4. —

LEMMA 3 (EXISTENCE AND UNIQUENESS): *Let* $(2+c_1\lambda(n-1))(2+c_2\lambda(n-1)) - c_1c_2(n-1)^2 \neq 0$. *There is unique set of functions* $\{p_1^*(\cdot), p_2^*(\cdot)\} \in \mathcal{B}\times\mathcal{B}$ *that satisfies the necessary and sufficient conditions for a maximum.*

PROOF OF LEMMA 3: From Lemma 1 and 2 we know that there is a maximum if there is a function $\{p_1^*(\cdot), p_2^*(\cdot)\} \in \mathcal{B}\times\mathcal{B}$ such that

$$(NC_1) \qquad \int_{\mathbb{R}} \left(\frac{\partial F(Z_1, Z_2, p_1^*(Z_1), p_2^*(Z_2))}{\partial p_1(Z_1)} \right) \phi_{2|1}(Z_2|Z_1)dZ_2 = 0 \ \forall Z_1$$

and analogously for good 2. To show that there is a unique set of such functions, I simplify condition (NC_1). The simplification of (NC_2) is analogous. Computing the partial derivatives of F with respect to prices, and replacing the integral by an expectation operator (NC_1) reads

(NC_1')

$$-RS_1(p_1^*, Z_1) + \left(\frac{\partial RS_1(p_1^*, Z_1)}{\partial p_1} \right) (s_1 - p_1^* - \lambda RS_1(p_1^*, Z_1) - \delta \mathbb{E}\left[RS_2(p_2^*, \boldsymbol{Z_2}) \middle| Z_1 \right]) = 0 \ \forall Z_1$$

where p_m denotes $p_m(Z_m)$. Inserting the functional form of the residual supply (RS_m), I obtain

$$\begin{aligned}\text{(A5)} \qquad & -\left[Z_1-(n-1)o_1+(n-1)c_1p_1^*(Z_1)\right] \\ & +(n-1)c_1\left[s_1-p_1^*(Z_1)-\lambda\left[Z_1-(n-1)o_1+(n-1)c_1p_1^*(Z_1)\right]\right] \\ & -(n-1)c_1\delta\mathbb{E}[\boldsymbol{Z_2}-(n-1)o_2+(n-1)c_2p_2^*(\boldsymbol{Z_2})|Z_1]=0\ \forall Z_1.\end{aligned}$$

This equation characterizes the optimal bid price $p_1^*(Z_1)$ in auction 1 for realization Z_1, given the bidder chooses $p_2^*(\cdot)$ in auction 2. Notice that (A5) takes the following form:

$$0=A_1+C_1p_1^*(Z_1)+l_1(Z_1)+D_1\mathbb{E}[p_2^*(\boldsymbol{Z_2})|Z_1]\quad\forall Z_1$$

with (by the assumption that $\boldsymbol{Z_1},\boldsymbol{Z_2}$ are normally distributed) linear function

$$l_1(Z_1)=(n-1)^2\delta c_1o_2-[1+\lambda(n-1)c_1]Z_1-(n-1)\delta c_1\mathbb{E}[\boldsymbol{Z_2}|Z_1]$$

and constants $A_1=(n-1)o_1+(n-1)c_1[s_1+\lambda(n-1)o_1], C_1=-(n-1)c_1[2+(n-1)c_1\lambda], D_1=-(n-1)^2c_1c_2$. With the analogue holding for auction 2 we know that the solution $\{p_1^*(\cdot),p_2^*(\cdot)\}$ is characterized by the following set of equations

$$\begin{aligned}0&=A_1+C_1p_1^*(Z_1)+l_1(Z_1)+D_1\mathbb{E}[p_2^*(\boldsymbol{Z_2})|Z_1]\quad\forall Z_1\\ 0&=A_2+C_2p_2^*(Z_2)+l_2(Z_2)+D_2\mathbb{E}[p_1^*(\boldsymbol{Z_1})|Z_2]\quad\forall Z_2\end{aligned}$$

or equivalently since $C_1,C_2<0$ by $n>1,c_1>0,\lambda>0$

$$\text{(A6)}\qquad p_1^*(Z_1)=-\frac{A_1}{C_1}-\frac{l_1(Z_1)}{C_1}-\frac{D_1}{C_1}\mathbb{E}[p_2^*(\boldsymbol{Z_2})|Z_1]\ \ \forall Z_1$$

$$\text{(A7)}\qquad p_2^*(Z_2)=-\frac{A_2}{C_2}-\frac{l_2(Z_2)}{C_2}-\frac{D_2}{C_2}\mathbb{E}[p_1^*(\boldsymbol{Z_1})|Z_2]\ \ \forall Z_2.$$

To solve the system of equations (which must hold pointwise), I first insert (A7) into (A6) and rearrange to obtain an expression for $p_1^*(Z_1)$ that is independent of $p_2^*(\cdot)$ in STEP A. I then obtain an expression for $p_1^*(Z_1)$ that is independent of $p_2^*(\cdot)$ using (A7) in STEP B.

STEP A: To insert (A7) into (A6), I first develop an expression for the conditional expectation. By the law of iterative expectations

$$\begin{aligned} \mathbb{E}[p_2^*(\boldsymbol{Z_2})] &= -\frac{A_2}{C_2} - \frac{\mathbb{E}\left[l_2(\boldsymbol{Z_2})\right]}{C_2} - \frac{D_2}{C_2}\mathbb{E}[p_1^*(\boldsymbol{Z_1})] \\ \Rightarrow \mathbb{E}[p_2^*(\boldsymbol{Z_2})|Z_1] &= -\frac{A_2}{C_2} - \frac{\mathbb{E}\left[l_2(\boldsymbol{Z_2})|Z_1\right]}{C_2} - \frac{D_2}{C_2}p_1^*(Z_1). \end{aligned}$$

Inserting the last expression, (A6) rearranges to

$$\text{(A8)} \qquad \left(\frac{C_1C_2 - D_1D_2}{C_1C_2}\right)p_1^*(Z_1) = -\frac{A_1}{C_1} - \frac{l_1(Z_1)}{C_1} + \frac{D_1A_2}{C_1C_2} + \frac{D_1}{C_1C_2}\mathbb{E}\left[l_2(\boldsymbol{Z_2})|\, Z_1\right].$$

Dividing (A8) by $C_1C_2 - D_1D_2$, which by assumption $(2+c_1\lambda(n-1))(2+c_2\lambda(n-1)) - c_1c_2(n-1)^2 \neq 0 \Leftrightarrow C_1C_2 - D_1D_2 \neq 0$, gives the following formula for $p_1^*(Z_1)$:

$$\text{(A9)} \qquad p_1^*(Z_1) = \left(\frac{C_2}{C_1C_2 - D_1D_2}\right)\left[-A_1 - l_1(Z_1) + \left(\frac{D_1}{C_2}\right)\left(A_2 + \mathbb{E}\left[l_2(\boldsymbol{Z_2})|\, Z_1\right]\right)\right].$$

STEP B: Now inserting (A9) into (A7), solving for $p_2^*(Z_2)$ and simplifying gives

$$\text{(A10)} \qquad p_2^*(Z_2) = \left(\frac{C_1}{D_1D_2 - C_1C_2}\right)\left[-A_2 - l_2(Z_2) + \left(\frac{D_2}{C_1}\right)\left(A_1 + \mathbb{E}[l_1(\boldsymbol{Z_1})|Z_2]\right)\right].$$

These two functions are the only two functions that fulfill the necessary and sufficient conditions for a maximum. They are linear by the assumption that $\boldsymbol{Z_1}, \boldsymbol{Z_2}$ are normally distributed.

This completes the proof of Main Lemma 1. □

PART 2) EQUILIBRIUM. —

MAIN LEMMA 2:

(i) There exists a unique set of coefficients $\{o_1^*, o_2^*, a_{1,1}^*, a_{1,2}^*, a_{2,2}^*, a_{2,1}^*, c_1^*, c_2^*\}$ *that constitute a symmetric linear equilibrium.*

(ii) c_1^*, c_2^* *are such that* $(2 + c_1^*\lambda(n-1))(2 + c_2^*\lambda(n-1)) - c_1^*c_2^*(n-1)^2 \neq 0$.

(iii) In this equilibrium, all agents choose the bidding functions of Proposition 4.

PROOF OF LEMMA 2 (i) AND (ii): So far, I have derived a system of equations that characterizes the agent's unique best reply in terms of functions $\{p_1^*(\cdot), p_2^*(\cdot)\}$ that specify prices for realizations of Z_1 and Z_2. To solve for the equilibrium, I map $\{p_1^*(\cdot), p_2^*(\cdot)\}$ back into bidding functions over quantities, $\{b_1^*(\cdot), b_2^*(\cdot)\}$. This allows me to compare the agent's best reply with the assumed linear strategy of all other agents (A2). To do so, recall that an optimal bid price $p_1^*(Z_1)$ at a fixed point Z_1 must, for a fixed optimal choice $p_2^*(\cdot)$, satisfy:

$$0 = -RS_1(p_1^*(Z_1), Z_1) + \left(\frac{\partial RS_1(p_1^*(Z_1), Z_1)}{\partial p_1}\right)(s_1 - p_1^*(Z_1) - \lambda RS_1(p_1^*(Z_1), Z_1) - \delta\mathbb{E}[RS_2(p_2^*(\boldsymbol{Z_2}), \boldsymbol{Z_2})|Z_1]).$$

Linearity of the residual supply (RS_m) implies

$$\mathbb{E}[RS_2(p_2^*(\boldsymbol{Z_2}), \boldsymbol{Z_2})|Z_1] = \mathbb{E}[RS_2(p_2^*(\boldsymbol{Z_2}), \boldsymbol{Z_2})|RS_1(p_1^*(Z_1), Z_1)]$$

as well as $\left(\frac{\partial RS_1(p_1^*(Z_1), Z_1)}{\partial p_1}\right) = (n-1)c_1$ with analogous expressions for the other good. To obtain a characterization for $b_1^*(q_1^*)$, replace $p_1^*(Z_1)$ by $b_1^*(q_1^*)$ and $RS_m(p_m^*, Z_m)$ by q_m^* for $m = 1, 2$:

$$0 = -q_1^* + (n-1)c_1(s_1 - b_1^*(q_1^*) - \lambda q_1^* - \delta\mathbb{E}[\boldsymbol{q_2^*}|q_1^*])$$

$$\Leftrightarrow b_1^*(q_1^*) = s_1 - \left[\lambda + \frac{1}{(n-1)c_1}\right]q_1^* - \delta\mathbb{E}[\boldsymbol{q_2^*}|\, q_1^*]. \qquad \text{(A11)}$$

This equation characterizes the agent's best bidding function $b_1^*(\cdot)$ in auction 1 given his best reply in auction 2. For $b_1^*(\cdot)$ to be part of a symmetric equilibrium, it must be that everyone, including agent i himself, plays as in equilibrium in auction 2. In that case, in which all agents choose the same strategy (A2) with equilibrium coefficients $\{o_1^*, o_2^*, a_{1,1}^*, a_{1,2}^*, a_{2,2}^*, a_{2,1}^*, c_1^*, c_2^*\}$ in auction 2, the agent wins $q_2^* = \frac{1}{n} Z_2 + \left(\frac{n-1}{n}\right)[a_{2,2}^* s_2 + a_{2,1}^* s_1]$ at market clearing if Z_2 realizes. With this, it is straightforward to to compute $\mathbb{E}[\boldsymbol{q_2^*}|q_1^*]$. Since $\boldsymbol{q_1^*}$ and $\boldsymbol{q_2^*}$ are a linear transformations of normally distributed variables:

$$\begin{pmatrix} \boldsymbol{q_1^*} \\ \boldsymbol{q_2^*} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{q_1^*} \\ \mu_{q_2^*} \end{pmatrix}, \begin{pmatrix} \sigma_{q_1^*}^2 & \rho^* \sigma_{q_1^*} \sigma_{q_2^*} \\ \rho \sigma_{q_1^*}^* \sigma_{q_2^*} & \sigma_{q_2^*}^2 \end{pmatrix}\right) \text{ with}$$

(A12)

$$\mu_{q_m^*} = \left(\frac{1}{n}\right) \mu_{Q_m} + \left(\frac{n-1}{n}\right) \{a_{m,m}^*[s_m - \mu_{s_m}] + a_{m,-m}^*[s_{-m} - \mu_{s_{-m}}]\}$$

(A13)

$$\sigma_{q_m^*}^2 = \left(\frac{1}{n}\right)^2 \sigma_Q^2 + \left(\frac{n-1}{n^2}\right) \sigma_s^2 [2\rho_s a_{m,m}^* a_{m,-m}^* + (a_{m,m}^{2*} + a_{m,-m}^{2*})]$$

(A14)

$$\rho^* = \left(\frac{1}{\sigma_{q_1^*} \sigma_{q_2^*}}\right) \left\{\rho_Q \sigma_Q^2 / n^2 + (n-1)/n^2 \sigma_s^2 \left[a_{1,1}^* a_{2,1}^* + a_{1,2}^* a_{2,2}^* + \rho_s \left(a_{1,1}^* a_{2,2}^* + a_{1,2}^* a_{2,1}^*\right)\right]\right\}$$

and the conditional expectation is

$$\mathbb{E}[\boldsymbol{q_{-m}^*}|q_m^*] = \mu_{q_{-m}^*} + \rho^* \left(\frac{\sigma_{q_{-m}^*}}{\sigma_{q_m^*}}\right) \left(q_m^* - \mu_{q_m^*}\right) \text{ for } m = 1, 2.$$

Inserting the conditional expectation into condition (A11) and the analogous for $m = 2$, clustering all terms and matching coefficients with the equilibrium guess (A2), which expressed as inverse demand reads

$$\text{(A2')} \qquad b_m(q_m, \vec{s}_i) = \left(\frac{o_m^*}{c_m^*}\right) + \left(\frac{a_{m,m}^*}{c_m^*}\right) s_{i,m} + \left(\frac{a_{m,-m}^*}{c_m^*}\right) s_{i,-m} - \left(\frac{1}{c_m^*}\right) q_m,$$

gives the following implicit characterization of equilibrium coefficients for $m = 1, 2$:

(A15)

$$c_m^* = \left(\frac{n-2}{n-1}\right)\left(\frac{1}{\lambda + \delta\rho^*\left(\frac{\sigma_{q_{-m}^*}}{\sigma_{q_m^*}}\right)}\right)$$

(A16)

$$a_{m,m}^* = -\left(\frac{(n-2)n\left(n\lambda\left(\frac{\sigma_{q_{-m}^*}}{\sigma_{q_m^*}}\right) + 2\delta\rho^*\right)}{(n-1)(-\lambda^2 n^2 \sigma_{q_1^*}\sigma_{q_2^*} + \delta^2((n-2)^2 - 4\rho^{2*})\sigma_{q_1^*}\sigma_{q_2^*} - 2\delta\lambda n\rho^*(\sigma_{q_1^*}^2 + \sigma_{q_2^*}^2))}\right)\sigma_{q_m^*}^2$$

(A17)

$$a_{m,-m}^* = \left(\frac{\delta(n-2)^2 n}{(n-1)(-\lambda^2 n^2 \sigma_{q_1^*}\sigma_{q_2^*} + \delta^2((n-2)^2 - 4\rho^{2*})\sigma_{q_1^*}\sigma_{q_2^*} - 2\delta\lambda n\rho^*(\sigma_{q_1^*}^2 + \sigma_{q_2^*}^2))}\right)\sigma_{q_1^*}\sigma_{q_2^*}$$

(A18)

$$\begin{aligned} o_m^* = \frac{\delta}{n}\Big[&\left[\rho^* \frac{\sigma_{q_{-m}^*}}{\sigma_{q_m^*}}\mu_{Q_m} - \mu_{Q_{-m}}\right] \\ &+ (n-1)\left[\left[a_{-m,m}^* - a_{m,m}^*\rho^* \frac{\sigma_{q_{-m}^*}}{\sigma_{q_m^*}}\right]\mu_{s_m} - \left[a_{m,-m}^*\rho^* \frac{\sigma_{q_{-m}^*}}{\sigma_{q_1^*}} - a_{-m,-m}^*\right]\mu_{s_{-m}}\right]\Big] c_m^* \end{aligned}$$

with equilibrium variances $\sigma_{q_1^*}, \sigma_{q_2^*}$ and correlation ρ^* solving (A13) for $m = 1, 2$ and (A14). Tedious algebraic manipulations show that $\sigma_{q_1^*} = \sigma_{q_2^*}$ at the solution.

With identical variances, the equilibrium coefficients simplify to

(A19)

$$c_1^* = c_2^* = \left(\frac{n-2}{n-1}\right)\left(\frac{1}{\lambda + \delta\rho^*}\right)$$

(A20)

$$a_{1,1}^* = a_{2,2}^* \equiv a_1^* = \left(\frac{n(\lambda n + 2\delta\rho^*)(\lambda + \delta\rho^*)}{n^2(\lambda^2 - \delta^2) + 4n\delta(\delta + \lambda\rho^*) - 4\delta^2(1 - \rho^{*2})}\right)c_m^*$$

(A21)

$$a_{1,2}^* = a_{2,1}^* \equiv a_2^* = -\left(\frac{n(n-2)\delta(\lambda + \delta\rho^*)}{n^2(\lambda^2 - \delta^2) + 4n\delta(\delta + \lambda\rho^*) - 4\delta^2(1 - \rho^{*2})}\right)c_m^*$$

(A22)

$$o_m^* = \delta\left(\frac{1}{n}\right)\left[\left[\rho^*\mu_{Q_m} - \mu_{Q_{-m}}\right] + (n-1)\left[[a_2^* - a_1^*\rho^*]\mu_{s_m} - [a_2^*\rho^* - a_1^*]\mu_{s_{-m}}\right]\right]c_m^*$$

with

$$\rho^* = \frac{\rho_Q\sigma_Q^2 + (n-1)\sigma_s^2\left[2a_1^*a_2^* + \rho_s\left(a_1^{2*} + a_2^{2*}\right)\right]}{\sigma_Q^2 + (n-1)\sigma_s^2[2a_1^*a_2^*\rho_s + (a_1^{2*} + a_2^{2*})]}. \tag{A23}$$

The equilibrium correlation ρ^* is well-defined because the distribution of both winning quantities can be shown to be non-degenerate, i.e., $\sigma_{q_1^*} = \sigma_{q_2^*} > 0$. Inserting the expressions for a_1^*, a_2^* into (A23) and rearranging reveals that ρ^* can be expressed as a root of polynomial

$$\begin{aligned} P(\rho) \equiv (\rho - \rho_Q)\sigma_Q^2(n-1)(\lambda n - \delta(n - 2(1+\rho)))^2(\lambda n + \delta(n - 2(1-\rho)))^2 \\ -\sigma_s^2(n-2)^2n^2\big(\delta^2[\rho_s((n-2)^2 + 4(n-1)\rho^2) - \rho(n^2 - 4 + 4\rho^2)] \\ +2\delta\lambda n(2 - 2\rho^2 - n(1 - \rho\rho_s)) + \lambda^2n^2(\rho_s - \rho)\big). \end{aligned} \tag{P}$$

By assumption of the proposition, we are considering only parameters for which ρ^* solves $P(\rho^*) = 0$ lies in $[-1, 1]$. The latter condition guarantees that $c_1^*, c_2^* > 0$ as was assumed in the equilibrium guess. The equilibrium function of Proposition 4 can be obtained by inserting the equilibrium coefficients characterized in (A19), (A20), (A21), (A22) into (A2') and simplifying. □

PROOF OF LEMMA 2 (*iii*): To show that the condition is satisfied, I proceed by contradiction. Assume

$$(2 + c_1^*\lambda(n-1))(2 + c_2^*\lambda(n-1)) - c_1^*c_2^*(n-1)^2 = 0. \tag{A24}$$

Insert the equilibrium coefficient for c_2^* given by (A15), i.e., $c_2^* = \left(\frac{n-2}{n-1}\right)\left(1/(\lambda + \delta\rho^*\left(\frac{\sigma_{q_2}^*}{\sigma_{q_1}^*}\right))\right)$ into (A24) and solve for c_1^*. One obtains

$$c_1^* = \left(\frac{1}{n-1}\right)\left(\frac{-2\left(2\delta\rho^* + \lambda n\left(\frac{\sigma_{q_2^*}}{\sigma_{q_1^*}}\right)\right)}{2\delta\lambda\rho^* + (2 + (\lambda^2 - 1)n)\left(\frac{\sigma_{q_2^*}}{\sigma_{q_1^*}}\right)}\right).$$

This gives a contradiction, because the expression is different from the equilibrium coefficient that holds in any linear symmetric equilibrium given by (A15), that is, $c_1^* = \left(\frac{n-2}{n-1}\right)\left(1/(\lambda + \delta\rho^*\left(\frac{\sigma^*_{q_1}}{\sigma^*_{q_2}}\right))\right)$.
This completes the proof of Main Lemma 2, and with it the proof of Propositions 1 (II)(i) and 4. □

A3. Proposition 1 (II)(ii) and Proposition 3

When ($\rho_s = \rho_Q = \pm 1$) or ($\rho_s = \pm 1$ and $\sigma_Q = 0$) the correlation of equilibrium winning quantities ρ^* as defined in (A23) is perfect, i.e., $\rho^* = \pm 1$. Inserting $\rho^* = \pm 1$ into the equilibrium coefficients (A19), (A20),(A21), (A22) and simplifying shows that

$$\text{(A25)} \qquad b^*_m(q_m, s_{i,m}) = s_{i,m} - \left(\frac{n-1}{n-2}\right)(\lambda + \delta\rho^*)\, q_m + \delta\left(\frac{1}{n}\right)(\rho^*\mu_{Q_m} - \mu_{Q_{-m}})$$

is the bidding function in the disconnected market. Both conditions of equilibrium existence are satisfied: $\rho^* \in [-1, 1]$ and $\lambda + \delta\rho^* > 0$ by assumption $|\delta| < \lambda$. With this one can derive the function that is displayed in Proposition 3 by inserting the bidding function (5) of Propositions 2 into (A25). □

B. Proofs of the Irrelevance Theorems

B1. Irrelevance Theorem 1

PROOFS of THEOREM 1 (i) AND (ii): Let $n < \infty$. To show that allocations of quantities coincide iff ($\rho_s = \rho_Q = \pm 1$) or ($\rho_s = \pm 1$ and $\sigma_Q = 0$) and that the associated clearing prices coincide iff $\mu_{Q_1} = \rho_Q \mu_{Q_2}$, I will compare individual winning quantities and the associated market clearing prices across market structures. Building on Propositions 2 and 4, I first compute $q^*_{i,m}, p^*_m$ and $\bar{q}^*_{i,m}, \bar{p}^*_m$ (Lemma 4). I then derive two Auxiliary Lemmas 2 and 3 that allow me to prove the "if"- direction of the theorem, followed by the "only if"- direction.

LEMMA 4: *(i) In the connected market each agent trades*

(B1)
$$\bar{q}^*_{i,m} = \left(\frac{n-2}{n-1}\right)\left[\frac{\lambda\left[s_{i,m} - \frac{1}{n}\sum_i s_{i,m}\right] - \delta\left[s_{i,-m} - \frac{1}{n}\sum_i s_{i,-m}\right]}{\lambda^2 - \delta^2}\right] + \frac{Q_m}{n}.$$

The markets clear at

(B2)
$$\bar{p}^*_m = \frac{1}{n}\sum_i s_{i,m} - \left(\frac{n-1}{n-2}\right)\frac{1}{n}\left[\lambda Q_m + \delta Q_{-m}\right].$$

(ii) In the disconnected market each agent trades

(B3)
$$q^*_{i,m} = \left(\frac{n-2}{n-1}\right)\left[\frac{(\lambda n + 2\delta\rho^*)\left[s_{i,m} - \frac{1}{n}\sum_i s_{i,m}\right] - (n-2)\delta\left[s_{i,-m} - \frac{1}{n}\sum_i s_{i,-m}\right]}{\frac{1}{n}[\lambda n - \delta(n - 2(1+\rho^*))][\lambda n + \delta(n - 2(1-\rho^*))]}\right] + \frac{Q_m}{n}.$$

The markets clear at

(B4)
$$p^*_m = C_m + (\lambda + \delta\rho^*)\left[\frac{(\lambda n + 2\delta\rho^*)\sum_i s_{i,m} - \delta(n-2)\sum_i s_{i,-m}}{[\lambda n - \delta(n - 2(1+\rho^*))][\lambda n + \delta(n - 2(1-\rho^*))]} - \left(\frac{n-1}{n-2}\right)\frac{Q_m}{n}\right]$$

with C_m as defined in Proposition 4.

PROOF OF LEMMA 4: (i) The clearing price can be computed by expressing the bidding functions as demands and solving $Q_1 = \sum_i \bar{x}^*_1(\bar{p}^*_1, \bar{p}^*_2, \vec{s}_i)$, $Q_2 = \sum_i \bar{x}^*_2(\bar{p}^*_2, \bar{p}^*_1, \vec{s}_i)$. Plugging $\bar{p}^*_1, \bar{p}^*_2$ into the demand functions and simplifying gives $\bar{q}^*_{i,m} \equiv \bar{x}^*_m(\bar{p}^*_1, \bar{p}^*_2, \vec{s}_i)$. (ii) The proof for the disconnected market is analogous with the difference that each market now clears when $Q_m = \sum_i x^*_m(p^*_m, \vec{s}_i)$. □

In oder to compare quantities and prices under perfect correlation, I will make use of two Auxiliary Lemmas.

AUXILIARY LEMMA 2: $(\rho_s = \rho_Q = 1)$ *or* $(\rho_s = 1$ *and* $\sigma_Q = 0) \Leftrightarrow \rho^* = \pm 1.$

PROOF OF AUXILIARY LEMMA 2: By definition (A23), $(\rho_s = \rho_Q = 1)$ or $(\rho_s = 1$ and $\sigma_Q = 0)$ gives rise to $\rho^* = 1$. The analogous is true for -1. To prove the other direction solve $P(\pm 1) = 0$ for ρ_s to obtain

$$\rho_s = \pm 1 \pm (1 \pm \rho_Q) \left(\frac{n-1}{n-2}\right) \left(\frac{\delta(n-4) \pm \lambda n}{n}\right)^2 \left(\frac{\sigma_Q^2}{\sigma_s^2}\right).$$

Since $\delta(n-4) \neq \pm\lambda n$ by $\lambda > 0, |\delta| < \lambda$ and $n > 2$, this equation can only hold if $(\rho_s = \rho_Q = \pm 1)$ or $(\rho_s = \pm 1$ and $\sigma_Q = 0)$. □

AUXILIARY LEMMA 3: *Let $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ be two continuous random with expectations μ_{X_1}, μ_{X_2} and variances $\sigma_{X_1}, \sigma_{X_2}$. If $\boldsymbol{X_1}$ is perfectly correlated with $\boldsymbol{X_2}$, $|\rho| = 1$, then $\boldsymbol{X_1} = \mu_{X_1} + \rho \left(\frac{\sigma_{X_1}}{\sigma_{X_2}}\right) [\boldsymbol{X_2} - \mu_{X_2}]$ with probability one.*

PROOF OF AUXILIARY LEMMA 3: Let $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ be two continuous random with expectations μ_{X_1}, μ_{X_2} and variances $\sigma_{X_1}, \sigma_{X_2}$. $\boldsymbol{X_1}$ can be approximated by a linear function $\boldsymbol{X_1} \approx \alpha^* \boldsymbol{X_2} + \beta^*$, where $\alpha^* = \rho\sigma_X/\sigma_Y$ and $\beta^* = \mu_{X_1} - \alpha^*\mu_{X_2}$ minimize the mean-square error $MSE(\alpha, \beta) = \mathbb{E}\left[(\boldsymbol{X_1} - (\alpha\boldsymbol{X_2} + \beta))^2\right]$. Evaluating the mean-squared error at α^* and β^*: $MSE(\alpha^*, \beta^*) = \sigma_{X_1}^2(1-\rho^2)$ we see that an exact fit is achieved if $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ are perfectly correlated, i.e., $\boldsymbol{X_1} = \alpha^*\boldsymbol{X_2} + \beta^* = \mu_{X_1} + \rho\sigma_{X_1}/\sigma_{X_2} [\boldsymbol{X_2} - \mu_{X_2}]$. □

PROOF OF THE "IF"-DIRECTION OF THEOREM 1 $(i), (ii)$: I let $(\rho_s = \rho_Q = \pm 1)$ or $(\rho_s = \pm 1$ and $\sigma_Q = 0)$ and simplify the equilibrium winning quantities and clearing prices of Lemma 4 using Auxiliary Lemmas 2 and 3 applied to the random types and total supply quantities (in case supply is random), i.e., $\boldsymbol{X_m} = \boldsymbol{s_{i,m}}$ and $\boldsymbol{X_m} = \boldsymbol{Q_m}$ for $m = 1, 2$. The following corollary summarizes.

COROLLARY 8: *When $|\rho^*| = 1$, the equilibrium quantities simplify to*

$$\bar{q}^*_{i,m} = q^*_{i,m} = \left(\frac{n-2}{n-1}\right)\left(\frac{1}{\lambda+\delta\rho^*}\right)\left[s_{i,m} - \frac{1}{n}\sum_i s_{i,m}\right] + \frac{Q_m}{n}. \tag{B5}$$

The clearing prices become

$$p^*_m = \frac{1}{n}\sum_i s_{i,m} - \left(\frac{n-1}{n-2}\right)(\lambda+\delta\rho^*)\frac{Q_m}{n} + \frac{\delta}{n}\left[\rho^*\mu_{Q_m} - \mu_{Q_{-m}}\right]. \tag{B6}$$

$$\bar{p}^*_m = \frac{1}{n}\sum_i s_{i,m} - \left(\frac{n-1}{n-2}\right)(\lambda+\delta\rho^*)\frac{Q_m}{n} + \frac{\delta}{n}\left(\frac{n-1}{n-2}\right)\left[\rho^*\mu_{Q_m} - \mu_{Q_{-m}}\right]. \tag{B7}$$

They coincide iff $\mu_{Q_{-m}} = \rho^*\mu_{Q_m}$.

PROOF OF COROLLARY 8: (a) Let $\sigma_Q > 0$ and $|\rho^*| = 1$, i.e., $\rho^* = \pm 1$. By Auxiliary Lemma 2, $\rho^* = \rho_s = \rho_Q = \pm 1$. Given perfect correlation of types (of agent i) and total supply, it follows from Auxiliary Lemma 3 that

$$s_{i,-m} = \mu_{s_{-m}} - \rho^*\mu_{s_m} + \rho^* s_{i,m} \quad \text{and} \tag{B8}$$

$$Q_{-m} = \mu_{Q_{-m}} - \rho^*\mu_{Q_m} + \rho^* Q_m. \tag{B9}$$

Using (B8) to substitute out for $s_{i,-m}$ in $\bar{q}^*_{i,m}$ and $q^*_{i,m}$ of Lemma 4 gives (B5). In the expression of the clearing prices, I substitute out for Q_{-m} using (B9) in $\bar{p}^*_m$, and $s_{i,-m}$ in p^*_m using (B8).

(b) If $\sigma_Q = 0$ and $|\rho^*| = 1$, the same argument applies for types. Supply is no longer random. Instead of (B9) use $\mu_{Q_1} = Q_1$ and $\mu_{Q_2} = Q_2$ when reformulating the prices. □

PROOF OF THE "ONLY IF"- DIRECTION OF THEOREM 1 $(i), (ii)$: By contraposition, I assume $|\rho^*_s| \neq 1$ or $|\rho_Q| \neq 1$ if $\sigma_Q > 0$ and $|\rho^*_s| \neq 1$ if $\sigma_Q = 0$, and show that $q^*_{i,m} \neq \bar{q}^*_{i,m}$ and $p^*_m \neq \bar{p}^*_m$ with positive probability. According to Auxiliary Lemma 2 $|\rho_s| \neq 1$ or $|\rho_Q| \neq 1 \Rightarrow |\rho^*| \neq 1$ given $\sigma_Q > 0$ and $|\rho_s| \neq 1 \Rightarrow |\rho^*| \neq 1$ if $\sigma_Q = 0$. It directly follows from Lemma 4 that $\bar{q}^*_{i,m} \neq q^*_{i,m}$. To see this, compare the coefficients of $\left[s_{i,m} - \frac{1}{n}\sum_i s_{i,m}\right]$ and $\left[s_{i,-m} - \frac{1}{n}\sum_i s_{i,-m}\right]$ across market structures. You will see that they coincide iff $\rho^* = \pm 1$. Market prices differ with positive probability because they depend on different realizations of the random factors: $\bar{p}_m^*$ depends on $s_{i,m}$ of all i and $\{Q_1, Q_2\}$, and p_m^* depends on both types of all i and only Q_m. □

PROOFS OF THEOREM 1 (*iii*): To prove convergence, it suffices to take $\rho^* \to \pm 1$ in Lemma 4. The allocations converge to the allocations with perfect correlation in Corollary 8. □

B2. Irrelevance Theorem 2

To prove the theorem it suffices to take the expectation of winning quantities and clearing prices of Lemma 4. □

B3. Irrelevance Theorem 3

Equivalence of allocations and prices as $n \to \infty$ follows from Corollary 9.

COROLLARY 9: *Let $n \to \infty$. In the limit, the agent trades*

$$\bar{q}_{i,m}^* = q_{i,m}^* = \left(\frac{1}{\lambda^2 - \delta^2}\right) \left\{\lambda \left[s_{i,m} - \mu_{s_m}\right] - \delta \left[s_{i,-m} - \mu_{s_{-m}}\right]\right\} \tag{B10}$$

under either market structure. The markets clear at

$$\bar{p}_m^* = p_m^* = \mu_{s_m}. \tag{B11}$$

PROOF OF COROLLARY 9: The statement follows from Lemma 4 as $n \to \infty$ where ρ^* defined in (A14) goes to $\rho_s(\lambda^2 + \delta^2) - 2\delta\lambda((\lambda^2 + \delta^2) - 2\delta\lambda\rho_s)^{-1}$ and $\frac{1}{n}\sum_i s_{i,m} \to \mu_{s_m}$ by the law of large numbers. □

C. Proofs regarding Welfare

C1. Proof of Examples 1 and 2

Let $\mu_{Q_1} = \mu_{Q_2} = \sigma_Q = \rho_Q = 0$ and $\mu_{s_1} = \mu_{s_2} = \frac{1}{n}\sum_i s_{i,1} = \frac{1}{n}\sum_i s_{i,2} \equiv \mu_s$. Insert the equilibrium quantities and prices of Lemma 4 into the utility function and total payment to obtain

$$\begin{aligned}
\overline{U}_i^* &= \left(\frac{n-2}{n-1}\right)\left(\frac{1}{n-1}\right)\left(\frac{1}{\delta+\lambda}\right)[n(\mu_s+s)-2\mu_s](s-\mu_s)\\
\overline{TP}_i^* &= \left(\frac{n-2}{n-1}\right)\left(\frac{1}{n-1}\right)\left(\frac{1}{\delta+\lambda}\right)2\mu_s(n-1)(s-\mu_s)\\
\overline{TS}_i^* = \overline{U}_i^* - \overline{TP}_i^* &= \left(\frac{n-2}{n-1}\right)\left(\frac{n}{n-1}\right)\left(\frac{1}{\lambda+\delta}\right)(\mu_s-s)^2\\
U_i^* &= \left(\frac{n-2}{n-1}\right)\left(\frac{n}{n-1}\right)\left(\frac{(\delta+\lambda)n(n-2)\mu_s+(\lambda n^2+\delta((n-2)^2+4(n-1)\rho^*)])s}{(\lambda n+\delta(n-2(1-\rho^*)))^2}\right)(s-\mu_s)\\
TP_i^* &= \left(\frac{n-2}{n-1}\right)\left(\frac{2\mu_s n}{\lambda n+\delta(n-2(1-\rho^*)))}\right)(s-\mu_s)\\
TS_i^* = U_i^* - TP_i^* &= \left(\frac{n-2}{n-1}\right)\left(\frac{n}{n-1}\right)\left(\frac{\lambda n^2+\delta((n-2)^2+4(n-1)\rho^*)}{(\lambda n+\delta(n-2(1-\rho^*)))^2}\right)(\mu_s-s)^2.
\end{aligned}$$

Taking the difference $TS_i^* - \overline{TS}_i^*$ concludes the proof. The proof for Example 2 is analogous. □

C2. Proof of Example 3

Let $\mu_{Q_1} = \mu_{Q_2} = \sigma_Q = \rho_Q = 0$ and $\mu_{s_1} = \mu_{s_2} = \frac{1}{n}\sum_i s_{i,1} = \frac{1}{n}\sum_i s_{i,2} \equiv 0$. By Lemma 4, the clearing prices, and with them the total payments made by all agents are 0. Therefore, an agent's total surplus is equal to the utility he achieves from the amounts he wins, $TS_i = U(s_i, q_{i,1}, q_{i,2}) = \sum_m \left[s_{i,m}q_{i,m} - \frac{\lambda}{2}q_{i,m}^2\right] - \delta q_{i,1}q_{i,2}$, which is maximized at

$$q_{i,m}^e = a_1^e s_{i,m} + a_2^e s_{i,-m} \text{ with } a_1^e = \left(\frac{\lambda}{\lambda^2-\delta^2}\right)s_{i,m} \text{ and } a_2^e = \left(\frac{-\delta}{\lambda^2-\delta^2}\right)s_{i,-m}.$$

Due to strategic bid shading the agent does not win this amount in equilibrium, but

$$\begin{aligned}
q_{i,m}^* &= a_1^* s_{i,m} + a_2^* s_{i,-m} \quad \text{in the disconnected market}\\
\overline{q}_{i,m}^* &= \overline{a}_1^* s_{i,m} + \overline{a}_2^* s_{i,-m} \quad \text{in the connected market}
\end{aligned}$$

with $a_1^* = n(n-2)(\lambda n + 2\delta\rho^*)[(n-1)(\lambda n - \delta(n-2(1+\rho^*)))(\lambda n + \delta(n-2(1-\rho^*)))]^{-1}$

$\overline{a}_1^* = \lambda(n-2)[(n-1)(\lambda^2 - \delta^2)]^{-1}$

$a_2^* = -n(n-2)^2\delta[(n-1)(\lambda n - \delta(n-2(1+\rho^*)))(\lambda n + \delta(n-2(1-\rho^*)))]^{-1}$

$\overline{a}_2^* = \delta(n-2)[(n-1)(\lambda^2 - \delta^2)]^{-1}$.

Since (given $n > 2, \rho^* \in (-1,1), \lambda > 0, \delta < |\lambda|$)

$$\overline{a}_1^* \ \& \ a_1^* < a_1^e \quad \text{and} \quad (\overline{a}_2^* \ \& \ a_2^* > a_2^e \text{ if } \delta > 0 \text{ and } < \text{ if } \delta < 0)$$

each agent comes closer to his optimal allocation $q_{i,m}^e$ as the coefficient multiplying $s_{i,m}$ increases or the one multiplying $s_{i,-m}$ decreases when $\delta > 0$. The opposite holds for $\delta < 0$. Therefore, the connected market brings higher total surplus for each individual agent if

$$\overline{a}_1^* > a_1^* \quad \text{and} \quad (a_2^* > \overline{a}_2^* \text{ if } \delta > 0 \text{ and } < \text{ if } \delta < 0).$$

The second condition is always satisfied. Only the first, $\overline{a}_1^* > a_1^*$, must bind. It is equivalent to

$$-(\delta^2+\lambda^2)n + \sqrt{\delta^4 n^2 + \lambda^4 n^2 + 2\delta^2\lambda^2(8+(n-8)n)} < 4\delta\lambda\rho^*$$
$$\text{or } -(\delta^2+\lambda^2)n - \sqrt{\delta^4 n^2 + \lambda^4 n^2 + 2\delta^2\lambda^2(8+(n-8)n)} > 4\delta\lambda\rho^*$$

With $\rho^- \equiv \frac{-(\delta^2+\lambda^2)n - \sqrt{\delta^4 n^2 + \lambda^4 n^2 + 2\delta^2\lambda^2(8+(n-8)n)}}{4\delta\lambda}$ and $\rho^+ \equiv \frac{-(\delta^2+\lambda^2)n + \sqrt{\delta^4 n^2 + \lambda^4 n^2 + 2\delta^2\lambda^2(8+(n-8)n)}}{4\delta\lambda}$ the condition becomes $\rho^- < \rho^* < \rho^+$ for $\delta > 0$ and $\rho^+ < \rho^* < \rho^-$ for $\delta < 0$. □

C3. Proof of Proposition 5 and 6

To prove the propositions, I first compute the expectation of

$$TS^* = \sum_i U(q^*_{i,1}, q^*_{i,2}, \vec{s}_i) - \sum_m p^*_m q^*_{i,m}$$
$$\text{with } U(q^*_{i,1}, q^*_{i,2}, \vec{s}_i) = \sum_m [s_{i,m} q^*_{i,m} - \frac{\lambda}{2}(q^{*2}_{i,m})] - \delta q^*_{i,1} q^*_{i,2} \tag{2}$$

under both market structures. I will derive the expressions for the more general environment in which noise traders may play a role, i.e., allowing for $\sigma_Q \geq 0, \rho_Q \neq 0, \mu_{Q_1} \neq 0$ and $\mu_{Q_2} \neq 0$.

To compute the expectations of the equilibrium quantities and clearing prices, I rely on Lemma 4, according to which

$$q^*_{i,m} = a^*_1 \left[s_{i,m} - \frac{1}{n} \sum_i s_{i,m} \right] + a^*_2 \left[s_{i,-m} - \frac{1}{n} \sum_i s_{i,-m} \right] + \frac{1}{n} Q_m$$
$$p^*_m = C_m + (\lambda + \delta\rho^*) \left(\frac{n-1}{n-2} \right) \left[a^*_1 \frac{1}{n} \sum_i s_{i,m} + a^*_2 \frac{1}{n} \sum_i s_{i,-m} - \frac{Q_m}{n} \right] \text{ for } m = 1, 2$$

with ρ^* and C_m as specified in Propositions 1 and 4, respectively, and

$$\bar{q}^*_{i,m} = \bar{a}^*_1 \left[s_{i,m} - \frac{1}{n} \sum_i s_{i,m} \right] + \bar{a}^*_2 \left[s_{i,-m} - \frac{1}{n} \sum_i s_{i,-m} \right] + \frac{1}{n} Q_m$$
$$\bar{p}^*_m = \frac{1}{n} \sum_i s_{i,m} - \left(\frac{n-1}{n-2} \right) \frac{1}{n} (\lambda Q_m + \delta Q_{-m}) \text{ for } m = 1, 2$$

where

$$a^*_1 \equiv \left(\frac{n-2}{n-1} \right) \left[\frac{(\lambda n + 2\delta\rho^*)}{\frac{1}{n}[\lambda n - \delta(n - 2(1+\rho^*))][\lambda n + \delta(n - 2(1-\rho^*))]} \right]$$
$$a^*_2 \equiv \left(\frac{n-2}{n-1} \right) \left[\frac{-(n-2)\delta}{\frac{1}{n}[\lambda n - \delta(n - 2(1+\rho^*))][\lambda n + \delta(n - 2(1-\rho^*))]} \right]$$
$$\bar{a}^*_1 \equiv \left(\frac{n-2}{n-1} \right) \left[\frac{\lambda}{\lambda^2 - \delta^2} \right] \text{ and } \bar{a}^*_2 \equiv \left(\frac{n-2}{n-1} \right) \left[\frac{-\delta}{\lambda^2 - \delta^2} \right].$$

To derive the expected utility at market clearing, I first compute the ingredients of the utility function under both market structures by relying on the assumptions that supply by noise traders is independent of types and types are independent across agents.

$$\begin{aligned}
\mathbb{E}[\boldsymbol{s_{i,m} q^*_{i,m}}] &= \frac{1}{n}[[a_1^* + a_2^*\rho_s](n-1)\sigma_s^2 + \mu_{s_m}\mu_{Q_m}] \\
\mathbb{E}[\boldsymbol{q^{*2}_{i,m}}] &= \frac{1}{n^2}[\mu_{Q_1}^2 + \sigma_Q^2 + (n-1)n(a_1^{*2} + a_2^{*2} + 2a_1^*a_2^*\rho_s)\sigma_s^2] \\
\mathbb{E}[\boldsymbol{q^*_{i,1} q^*_{i,2}}] &= \frac{1}{n^2}[\mu_{Q_1}\mu_{Q_2} + \rho_Q\sigma_Q^2 + n(n-1)(2a_1^*a_2^* + (a_1^{*2} + a_2^{*2})\rho_s)\sigma_s^2]
\end{aligned}$$

Given functional form (2), an agents expects to earn the following utility

$$\begin{aligned}
\text{(C1)} \quad \mathbb{E}[U(\boldsymbol{q^*_{i,1}}, \boldsymbol{q^*_{i,2}}, \boldsymbol{\vec{s}_i})] = \frac{1}{n^2}\big\{ & n[\mu_{Q_1}\mu_{s_1} + \mu_{Q_2}\mu_{s_2} + 2(n-1)(a_1^* + a_2^*\rho_s)\sigma_s^2] \\
& - \lambda\left(\mu_{Q_1}^2/2 + \mu_{Q_2}^2/2 + \sigma_Q^2 + (n-1)n(a_1^{*2} + a_2^{*2} + 2a_1^*a_2^*\rho_s)\sigma_s^2\right) \\
& - \delta\left(\mu_{Q_1}\mu_{Q_2} + \rho_Q\sigma_Q^2 + (n-1)n(2a_1^*a_2^* + (a_1^{*2} + a_2^{*2})\rho_s)\sigma_s^2\right)\big\}
\end{aligned}$$

in the disconnected market. The expression for the connected market is analogous with $\bar{a}_1^*, \bar{a}_2^*$ replacing a_1^*, a_2^*.

Proof of Proposition 5. — With noise traders, that is, when $\mu_{Q_1} = \mu_{Q_2} = \sigma_Q$, the total payments made by all strategic agents must sum to 0 for each good. Therefore, $TS^* = U^*$ and $\overline{TS}^* = \overline{U}^*$ as defined in Definition 4. Consequently,

$$\mathbb{E}[\boldsymbol{TS^*}] = (n-1)\sigma_s^2\left\{2(a_1^* + a_2^*\rho_s) - \lambda(a_1^{*2} + a_2^{*2} + 2a_1^*a_2^*\rho_s)\sigma_s^2 - \delta(2a_1^*a_2^* + (a_1^{*2}a_2^{*2})\rho_s)\right\}$$

in the disconnected market, and analogously in the connected market.
Inserting the a^* coefficients, replacing ρ_s by the implied correlation of clearing prices in the disconnected market ρ^* s.t. $P(\rho^*) = 0$, and taking the differences, I obtain the statement of the proposition. The sign of f_1 is implied by the restrictions that are imposed on the parameters, $n > 2, |\delta| < \lambda|, \rho^* \in (-1, 1)$. □

Proof of Proposition 6. — Without noise traders, the expected total payments of strategic agents typically do not sum to 0. They are

$$\mathbb{E}[\overline{\boldsymbol{TP}}^{\boldsymbol{*}}] = \sum_{m=1}^{2} \left[\mu_{Q_m}\mu_{s_m} - \lambda\left(\frac{n-1}{n-2}\right)\frac{\mu_{Q_m}^2}{n}\right] - 2\delta\left(\frac{n-1}{n-2}\right)\frac{\mu_{Q_1}\mu_{Q_2}}{n} - 2(\lambda+\delta\rho^*)\left(\frac{n-1}{n-2}\right)\frac{\sigma_Q^2}{n}$$
$$\mathbb{E}[\boldsymbol{TP}^{\boldsymbol{*}}] = \mathbb{E}[\overline{\boldsymbol{TP}}^{\boldsymbol{*}}] - \frac{\delta}{n(n-2)}\left[\rho^*(\mu_{Q_1}^2 + \mu_{Q_2}^2) - 2\mu_{Q_1}\mu_{Q_2} + 2(n-1)(\rho^* - \rho_Q)\sigma_Q^2\right].$$

In the connected market, the total surplus simplifies to

$$\mathbb{E}[\overline{\boldsymbol{TS}}^{\boldsymbol{*}}] = \left(\frac{\lambda\mu_{Q_1}^2 + \lambda\mu_{Q_2}^2 + 2\delta\mu_{Q_1}\mu_{Q_2}}{2(n-2)}\right) + \left(\frac{\lambda+\delta\rho_Q}{n-2}\right)\sigma_Q^2 + n\left(\frac{n-2}{n-1}\right)\left(\frac{\lambda-\delta\rho_s}{\lambda^2-\delta^2}\right)\sigma_s^2.$$

In the disconnected market, the expression does not simplify as nicely. The reason is that the expected utility (C1) is messy when inserting the equilibrium coefficients a_1^*, a_2^*. Putting all together, taking differences, $\mathbb{E}[\boldsymbol{TS}^{\boldsymbol{*}}] - \mathbb{E}[\overline{\boldsymbol{TS}}^{\boldsymbol{*}}]$, and simplifying, one obtains the statement of the proposition. □

D. Proof of Corollaries

D1. Corollary 1

The statement follows from Corollary 8 when computing $p_m^* - \bar{p}_m^*$. □

D2. Corollary 2

Corollary 2 follows from comparing the bidding functions of Propositions 2 and 4 when taking expectations, prior to drawing types. □

D3. Corollary 3

The corollary follows from Lemma 4 when computing $p_m^* - \bar{p}_m^*$ and taking expectations. □

D4. Corollary 4

The proof of the corollary relies on two lemmas that specify the optimality conditions for equilibrium bids. For both, I use the inverted residual supply curve, denoted $p_m^{RS}(\cdot)$ in the disconnected, and $\bar{p}_{i,m}^{RS}(\cdot,\cdot)$ in the connected market. Formally, $p_m^{RS}(\cdot)$ solves $Q_m = q_m + \sum_{j\neq i} x_{j,m}^*(p_{i,m}^{RS}(q_m), \vec{s}_j)$ for each q_m and all realization of the random variables. The analogous is true for $\bar{p}_{i,m}^{RS}(\cdot,\cdot)$. Further, I use $TP(p_1, p_2, q_1, q_2) \equiv \sum_{m=1}^{2} p_m q_m$ to refer to the total payment of an agent.

LEMMA 5: *A linear BNE with bidding functions $\vec{b}_i^*(\cdot, \vec{s}_i)$ that are strictly decreasing in $\vec{q}$ must satisfy $\vec{b}_i^*(\vec{q}, \vec{s}_i) = \vec{p}_i^{RS}(\vec{q})$ and*

$$\text{(D1)} \qquad \left[\frac{\partial U(q_1, q_2, \vec{s}_i)}{\partial q_m}\right] = \left[\frac{\partial TP(\bar{p}_{i,1}^{RS}(q_1, q_2), \bar{p}_{i,2}^{RS}(q_2, q_1), q_1, q_2)}{\partial q_m}\right] \quad \forall q_1, q_2.$$

PROOF OF LEMMA 5:[21] For a given set of realizations of types and total supply (if random), agent i chooses a quantity so as to $\max_{\vec{q}} \{U(\vec{q}, \vec{s}_i) - \vec{p}'\vec{q}\}$ with $\vec{p} = \vec{p}^{RS}(\vec{q})$. The necessary condition rearranges to

$$\left(\frac{\partial U(\vec{q}, \vec{s}_i)}{\partial \vec{q}}\right)' = \vec{p}^{RS}(\vec{q}) + \left(\frac{\partial \vec{p}^{RS}(\vec{q})}{\partial \vec{q}}\right)' \vec{q} \quad \text{at } \vec{p}^{RS}(\vec{q}) = \vec{b}^*(\vec{q}, \vec{s}_i).$$

For $m = 1$, the optimality condition reads

$$\left(\frac{\partial U(q_1, q_2, \vec{s}_i)}{\partial q_1}\right) = \bar{p}_1^{RS}(q_1, q_2) + q_1 \left(\frac{\partial \bar{p}_1^{RS}(q_1, q_2)}{\partial q_1}\right) + q_2 \left(\frac{\partial \bar{p}_2^{RS}(q_2, q_1)}{\partial q_1}\right) \quad \text{at } \vec{p}^{RS}(\vec{q}) = \vec{b}^*(\vec{q}, \vec{s}_i).$$

The RHS is the marginal payment for good 1. The analogous holds for $m = 2$. □

[21] The lemma follows from the necessary condition ($\overline{FOC}$) for a linear (ex post) equilibrium that I derived in the proof of Propositions 1 (I) and 2 (p. 1). However, the condition must be inverted into the quantity-price space.

LEMMA 6: *A linear BNE with bidding functions $b^*_{i,m}(\cdot,\vec{s}_i)$ that are strictly decreasing in q_m must satisfy $b^*_{i,m}(q_m,\vec{s}_i)=p^{RS}_{i,m}(q_m)$ and*

$$\mathbb{E}\left[\frac{\partial U(q_m,\boldsymbol{q^*_{i,-m}},\vec{s}_i)}{\partial q_m}\middle| q_m\right]=\mathbb{E}\left[\frac{\partial TP(p^{RS}_{i,m}(q_m),\boldsymbol{p^*_{-m}},q_m,\boldsymbol{q^*_{i,-m}})}{\partial q_m}\middle| q_m\right]\forall q_m.$$

PROOF OF LEMMA 6: The lemma follows from Lemma 8 (ii) in the Online Appendix according to which

$$\mathbb{E}\left[\frac{\partial U(q_1,\boldsymbol{q^*_{i,2}},\vec{s}_i)}{\partial q_1}\middle| q_1\right]=b^*_1(q_1,\vec{s}_i)+q_1\left(\frac{\partial RS_1(b^*_1(q_1,\vec{s}_i))}{\partial p_1}\right)^{-1}$$

for good 1 and analogously for good 2. As Lemma 6 is stated using the residual supply curve in the quantity-price space $p^{RS}_1(\cdot)$ and not the price-quantity space $RS_1(\cdot)$, we must only replace $\left(\frac{\partial RS_1(p_1)}{\partial p_1}\right)^{-1}=\left(\frac{\partial p^{RS}_1(q_1)}{\partial q_1}\right)$. The RHS is by the definition $TP(p^{RS}_1(q_1),\boldsymbol{p^*_2},q_1,\boldsymbol{q^*_{i,2}})\equiv q_1p^{RS}_1(q_1)+\boldsymbol{q^*_{i,2}p^*_2}$ equivalent to $\mathbb{E}\left[\frac{\partial TP(p^{RS}_1(q_1),\boldsymbol{p^*_2},q_1,\boldsymbol{q^*_{i,2}})}{\partial q_1}\middle| q_1\right]$ at $p^{RS}_1(q_1)=b^*_1(q_1,\vec{s}_i)$. □

PROOF OF COROLLARY 4: Assume that all other agents than i play linear equilibrium strategies for $m=1,2$:

$$x^*_m(p_m,\vec{s}_i)=o^*_m+a^*_{m,m}s_{i,m}+a^*_{m,-m}s_{i,-m}-c^*_mp_m \quad \text{in the disconnected market}$$

$$\bar{x}^*_m(p_m,p_{-m},\vec{s}_i)=\bar{o}^*_m+\bar{a}^*_{m,m}s_{i,m}+\bar{a}^*_{m,-m}s_{i,-m}-\bar{c}^*_mp_m-\bar{e}^*_mp_{-m} \quad \text{in the connected market}$$

with $c^*_m>0,\bar{c}^*_m>0$, $\bar{c}^*_1\bar{c}^*_2-\bar{e}^*_1\bar{e}^*_2>0$. Deriving the partial derivatives of the residual supply curves in Lemmas 6 and 5, and sending $n\to\infty$ completes the proof. □

D5. Corollary 5

Let $n<\infty$ and $\rho^*=\pm1$.

(i) Quantities are identical by Theorem 1. It follows that $U^*=\overline{U}^*$.

(ii) Clearing prices, and with it the total surplus might differ. Taking differences, I obtain $TS^* - \overline{TS}^* = \sum_m \overline{TP}^*_m - \sum_m TP^*_m$. The ranking between TS^* and $\overline{TS}^*$ then follows from $\sum_m \overline{TP}^*_m - \sum_m TP^*_m$ in Corollary 6 (ii). □

D6. Corollary 6

Let $n < \infty$ and $\rho^* = \pm 1$.

(i) By definition, $TP^*_m \equiv \sum_i p^*_m q^*_{i,m} = p^*_m \sum_i q^*_{i,m}$. Since quantities coincide across market structures, the ranking of total payments follows from the implied ranking of clearing prices in Corollary 1.

(ii) $\sum_m \overline{TP}^*_m - \sum_m TP^*_m \equiv \sum_m \bar{p}^*_m \sum_i \bar{q}^*_{i,m} - \sum_m p^*_m \sum_i q^*_{i,m} = \sum_m [\bar{p}^*_m - p^*_m] Q_m$, by market clearing. From Corollary 1 we know that the difference in clearing prices is $\bar{p}^*_m - p^*_m = +\left(\frac{\delta}{n}\right)\left(\frac{1}{n-2}\right)(\rho^* \mu_{Q_m} - \mu_{Q_{-m}})$. With this, the difference in total payments becomes (D2):

$$\text{(D2)} \qquad \sum_m \overline{TP}^*_m - \sum_m TP^*_m = \begin{cases} +\left(\frac{1}{n-2}\right)\left(\frac{\delta}{n}\right)[\mu_{Q_2} - \mu_{Q_1}]^2 & \text{for } \rho^* = +1 \\ -\left(\frac{1}{n-2}\right)\left(\frac{\delta}{n}\right)[\mu_{Q_2} + \mu_{Q_1}]^2 & \text{for } \rho^* = -1. \end{cases}$$

This is immediate when $\sigma_Q = 0$ where $Q_1 = \mu_{Q_1}$ and $Q_2 = \mu_{Q_2}$. For $\sigma > 0$ use $Q_2 = \mu_{Q_2} - \rho^* \mu_{Q_1} + \rho^* Q_1$ which holds by Auxiliary Lemma 2 given $\rho^* = \pm 1$. Going through all cases of ρ^* and δ in (D2) completes the proof. □

D7. Corollary 7

To show that both market structures approach the fully efficient allocation as $n \to \infty$, Lemma 7 first derive this allocation. Sending $n \to \infty$ and comparing the limit with Corollary 9 completes the proof.

LEMMA 7: *The fully efficient allocation, i.e.,* $\{q^e_{i,1}, q^e_{i,2}\}^n_{i=1}$ *that maximizes* $\sum_i U(q_{i,1}, q_{i,2}, \vec{s}_i)$ *such that* $\sum_i q_{i,1} = Q_1$ *and* $\sum_i q_{i,2} = Q_2$*, is for* $m = 1, 2$

$$q^e_{i,m} = \begin{cases} \left(\frac{\lambda}{\lambda^2-\delta^2}\right)\left[s_{i,m} - \frac{1}{n}\sum_i s_{i,m}\right] - \left(\frac{\delta}{\lambda^2-\delta^2}\right)\left[s_{i,-m} - \frac{1}{n}\sum_i s_{i,-m}\right] + \frac{1}{n}Q_m & \textit{for } n < \infty \\ \left(\frac{1}{\lambda^2-\delta^2}\right)\left[\lambda\left[s_{i,m} \ - \mu_{s_m}\right] - \delta\left[s_{i,-m} - \mu_{s_{-m}}\right]\right] & \textit{as } n \to \infty. \end{cases}$$

The associated clearing prices are

$$p^e_m = \begin{cases} \frac{1}{n}\sum_i s_{i,m} - \frac{1}{n}[\lambda Q_m + \delta Q_{-m}] & \textit{for } n < \infty \\ \mu_{s_m} & \textit{as } n \to \infty. \end{cases}$$

PROOF OF LEMMA 7: With $U(q_{i,1}, q_{i,2}, \vec{s}_i) \overset{(2)}{=} \sum_{m=1,2}\left\{s_{i,m}q_{i,m} - \frac{\lambda}{2}q^2_{i,m}\right\} - \delta q_{i,1}q_{i,2}$, and denoting the Lagrange multipliers by γ_1, γ_2, the fully efficient allocation is characterized by

$$\text{(D3)} \qquad s_{i,1} - \lambda q^e_{i,1} \ - \delta q^e_{i,2} - \gamma_1 = 0 \text{ and } s_{i,2} - \lambda q^e_{i,2} \ - \delta q^e_{i,1} - \gamma_2 = 0 \ \ \forall i$$

and the binding constraints of market clearing: $Q_1 = \sum_i q^e_{i,1}$, $Q_2 = \sum_i q^e_{i,2}$. Solving for $q^e_{i,1}$ and $q^e_{i,2}$ gives

$$q^e_{i,m} = \left(\frac{\lambda}{\lambda^2-\delta^2}\right)\left[s_{i,m} - \frac{1}{n}\sum_i s_{i,m}\right] - \left(\frac{\delta}{\lambda^2-\delta^2}\right)\left[s_{i,-m} - \frac{1}{n}\sum_i s_{i,-m}\right] + \frac{1}{n}Q_m$$

for $m = 1, 2$. By law of large numbers

$$q^e_{i,m} \to \left(\frac{1}{\lambda^2-\delta^2}\right)\left[\lambda\left[s_{i,m} \ - \mu_{s_m}\right] - \delta\left[s_{i,-m} - \mu_{s_{-m}}\right]\right] \text{ as } n \to \infty.$$

Inserting $q^e_{i,1}, q^e_{i,2}$ into (D3) and simplifying gives

$$p^e_m \to \frac{1}{n}\left[\sum_i s_{i,m} - \lambda Q_m - \delta Q_{-m}\right] \to \mu_{s_m} \text{as } n \to \infty. \qquad \square$$

E. Equilibrium with correlated types

Changes in the setting. — In this extension, I allow for correlation between types across agents. To reduce the number of parameters, I focus on a perfectly symmetric environment.[22] More specifically, I now assume for $m = 1, 2$

$$\mathbb{E}[\boldsymbol{s_{i,m}}] = \mu_{s_m} \text{ and } Var(\boldsymbol{s_{i,m}}, \boldsymbol{s_{i,m}}) = \sigma_s^2 \text{ and } Cov(\boldsymbol{s_{i,1}}, \boldsymbol{s_{i,2}}) = \rho_s \sigma_s^2 \ \ \forall i$$
$$\text{and } Cov(\boldsymbol{s_{i,m}}, \boldsymbol{s_{j,1}}) = Cov(\boldsymbol{s_{i,m}}, \boldsymbol{s_{j,2}}) = \tilde{\rho}_s \tilde{\sigma}_s^2 \ \ \forall i, j \text{ s.t. } j \neq i$$

such that $\sigma_s^2 \geq \tilde{\rho}_s \tilde{\sigma}_s^2$ and $(1+\rho_s)\sigma_s^2 + (n-2)\tilde{\rho}_s\tilde{\sigma}_s^2 \geq 0$. These conditions guarantee that the covariance matrix of all types $\begin{pmatrix} \boldsymbol{s_{1,1}} & \boldsymbol{s_{1,2}} & \ldots \boldsymbol{s_{n,1}} & \boldsymbol{s_{n,2}} \end{pmatrix}$ is positive semi-definite. All assumptions that involve the supply by noise traders remain the same.

Equilibrium refinement. — I restrict attention to linear equilibria which are symmetric across agents and goods. More precisely, equilibrium demand schedules take the following form:

$$x_m(p_m, \vec{s}_i) = o_m + a_1 s_{i,m} + a_2 s_{i,-m} - c_m p_m \text{ with } c_m > 0, \text{ for } m = 1, 2. \tag{A2}$$

In contrast to the setting with independent types, the coefficient a_1 and a_2 can no longer be good specific. I call such an equilibrium a perfectly symmetric linear equilibrium.

PROPOSITION 7: *There is an equilibrium in the disconnected market in which traders submit*

[22]The logic goes through with asymmetric (gaussian normal) information structures, but the algebra becomes extremely tedious.

$$b_m(q_m, \vec{s}_i) = \left(\alpha_1^* s_{i,1} + \alpha_2^* s_{i,2} - \left(\frac{n-1}{n-2}\right) q_m\right)(\lambda + \delta\rho^*) + C_m^*, \textit{ where}$$
$$\alpha_1^* = +\left(\frac{(\lambda n + 2\delta\rho^*)(1+\rho_s)\sigma_s^2 - \delta(n-2)(1-\rho^*)\tilde{\rho}_s\tilde{\sigma_s}^2}{\alpha_0^*}\right)$$
$$\alpha_2^* = -\left(\frac{\delta(n-2)((1+\rho_s)\sigma_s^2 - (1-\rho^*)\tilde{\rho}_s\tilde{\sigma}_s^2)}{\alpha_0^*}\right)$$
$$\alpha_0^* = \frac{1}{n}(\lambda n - \delta(n - 2(1+\rho^*)))((\lambda n + \delta(n - 2(1-\rho^*)))(1+\rho_s)\sigma_s^2 - 2\delta(n-2)(1-\rho^*)\tilde{\rho}_s\tilde{\sigma}_s^2)$$

and $C_m^* = \frac{\delta}{n}\Big((\mu_{Q_1}\rho^* - \mu_{Q_2}) + (n-2)\Big(\alpha_1^*(\mu_{s_2} - \mu_{s_1}\rho^*) + \alpha_2^*(\mu_{s_1} - \mu_{s_2}\rho^*) - \frac{(\alpha_1^*+\alpha_2^*)(\mu_{s_1}+\mu_{s_2})(1-\rho^*)\tilde{\rho}_s\tilde{\sigma}_s^2}{(1+\rho_s)\sigma_s^2}\Big)\Big)$ *if* $\rho_s \neq -1$ *and otherwise* $C_m^* = \frac{\delta}{n}\Big((\mu_{Q_1}\rho^* - \mu_{Q_2}) + (n-2)\left(\alpha_1^*(\mu_{s_2} - \mu_{s_1}\rho^*) + \alpha_2^*(\mu_{s_1} - \mu_{s_2}\rho^*)\right)\Big)$, *as long as the real root* ρ^* *of* $P(\cdot)$ *lies in* $[-1, 1]$ *with*

$$\begin{aligned}
P(\rho) = \rho\Big((1+\rho_s)\sigma_Q^2\sigma_s^2 + \frac{(n-2)^2}{n-1}\Big((1+\rho_s)(\alpha_1^{*2} + \alpha_2^{*2} + 2\alpha_1^*\alpha_2^*\rho_s)\sigma_s^4 \\
+(\alpha_1^* + \alpha_2^*)^2(n-2)(1+\rho_s)\sigma_s^2\tilde{\rho}_s\tilde{\sigma}_s^2 - 2(\alpha_1^*+\alpha_2^*)^2(n-1)\tilde{\rho}_s^2\tilde{\sigma}_s^4\Big)\Big) \\
-\frac{(n-2)^2}{n-1}\Big((1+\rho_s)(2\alpha_1^*\alpha_2^* + (\alpha_1^{*2} + \alpha_2^{*2})\rho_s)\sigma_s^4 \\
+(\alpha_1^* + \alpha_2^*)^2(n-2)(1+\rho_s)\sigma_s^2\tilde{\rho}_s\tilde{\sigma}_s^2 \\
-2(\alpha_1^* + \alpha_2^*)^2(n-1)\tilde{\rho}_s^2\tilde{\sigma}_s^4\Big) \\
-\rho_Q(1+\rho_s)\sigma_Q^2\sigma_s^2
\end{aligned}$$

if $\rho_s \neq 1$ *and otherwise*

$$\begin{aligned}
P(\rho) = \rho\Big(\sigma_Q^2 + \frac{(n-2)^2}{n-1}\Big((\alpha_1^* - \alpha_2^*)^2\sigma_s^2 + (\alpha_1^* + \alpha_2^*)^2(n-2)\tilde{\rho}_s\tilde{\sigma}_s^2\Big)\Big) \\
-\frac{(n-2)^2}{n-1}\Big((\alpha_1^* + \alpha_2^*)^2(n-2)\tilde{\rho}_s\tilde{\sigma}_s^2 - (\alpha_1^* - \alpha_1^*)^2\sigma_s^2\Big) - \rho_Q\sigma_Q^2.
\end{aligned}$$

This equilibrium is the unique perfectly symmetric linear equilibrium if there is exactly one such ρ^**. Otherwise there is one equilibrium per* ρ^**.*

REMARK: When shutting down the correlation of types across agents, i.e., $\tilde{\rho}_s = 0$, this equilibrium coincides with the equilibrium of Proposition 1 (II) whose equilibrium bidding function is displayed in Proposition 4.

PROOF OF PROPOSITION 7: The proof is analogous to the proof with independent types. Here I only highlight what changes: In determining the best reply, an agent of type $\vec{s}_i$ now conditions on observing his type.

$$\max_{p_1(\cdot)\in\mathcal{B},p_2(\cdot)\in\mathcal{B}} \mathbb{E}\left[U(\boldsymbol{q_1},\boldsymbol{q_2}) - \sum_{m=1,2} p_m(\boldsymbol{Z_m})\boldsymbol{q_m}\middle| \vec{s}_i\right] \text{ with } \boldsymbol{q_m} = RS_m(p_m(\boldsymbol{Z_m}),\boldsymbol{Z_m}).$$

As with independent types, the agent's equilibrium bid function fulfills

$$\text{(A11')} \qquad b_1^*(q_1^*) = s_1 - \left[\lambda + \frac{1}{(n-1)c_1}\right] q_1^* - \delta\mathbb{E}\left[\boldsymbol{q_2^*}\middle| q_1^*, \vec{s}_i\right].$$

I now compute the conditional expectation as follows: First, I derive the expectation $\vec{\mu}_S$ and covariance matrix Σ_S of $\sum_{j\neq i}\vec{\boldsymbol{s}}_{\boldsymbol{j}}\Big|\,\vec{s}_i \sim N(\vec{\mu}_S,\Sigma_S)$. To do so, I partition

$$\begin{pmatrix}\vec{\boldsymbol{s}}_{\boldsymbol{i}}\\ \sum_{j\neq i}\vec{\boldsymbol{s}}_{\boldsymbol{j}}\end{pmatrix} \sim N\left(\begin{pmatrix}\vec{\mu}_1\\ \vec{\mu}_2\end{pmatrix}, \begin{pmatrix}\Sigma_{1,1} & \Sigma_{1,2}\\ \Sigma_{2,1} & \Sigma_{2,2}\end{pmatrix}\right)$$

with $\vec{\mu}_1 \equiv \begin{pmatrix}\mu_{s_1}\\ \mu_{s_2}\end{pmatrix}$, $\vec{\mu}_2 \equiv (n-1)\mu_1$ and $\Sigma_{1,1} \equiv \sigma_s^2\begin{pmatrix}1 & \rho_s\\ \rho_s & 1\end{pmatrix}$, $\Sigma_{1,2} = \Sigma_{2,1} = (n-1)\tilde{\rho}_s\tilde{\sigma}_s^2\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, $\Sigma_{2,2} \equiv (n-1)\sigma_s^2\begin{pmatrix}1 & \rho_s\\ \rho_s & 1\end{pmatrix} + (n-1)(n-2)\tilde{\rho}_s\tilde{\sigma}_s^2\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ and compute

$$\vec{\mu}_S = \vec{\mu}_2 + \Sigma_{2,1}\Sigma_{1,1}^{-1}(\vec{s}_i - \vec{\mu}_1), \Sigma_S = \Sigma_{2,2} - \Sigma_{2,1}\Sigma_{1,1}^{-1}\Sigma_{1,2}.$$

Next, I determine the conditional distribution of i's winning quantities

$$\boldsymbol{q_i^*}|s_i \sim N\left(\begin{pmatrix}\mu_{q_1^*}\\ \mu_{q_2^*}\end{pmatrix}, \begin{pmatrix}\sigma_{q_1^*}^2 & \rho^*\sigma_{q_1^*}\sigma_{q_1^*}\\ \rho^*\sigma_{q_1^*}\sigma_{q_1^*} & \sigma_{q_2^*}^2\end{pmatrix}\right) \equiv \left(A\begin{pmatrix}\vec{\mu}_s\\ \vec{\mu}_Q\end{pmatrix} + \vec{d}, A\begin{pmatrix}\Sigma_s\\ \Sigma_Q\end{pmatrix}A'\right)$$

where

$$\vec{\mu}_Q = \begin{pmatrix}\mu_{Q_1}\\ \mu_{Q_2}\end{pmatrix} \text{ and } \Sigma_Q = \sigma_Q^2\begin{pmatrix}1 & \rho_Q\\ \rho_Q & 1\end{pmatrix}$$

$$A = \left(\frac{1}{n}\right)\begin{pmatrix}-a_1 & -a_2 & 1 & 0\\ -a_1 & -a_2 & 0 & 1\end{pmatrix} \text{ and } \vec{d} = \left(\frac{n-1}{n}\right)\begin{pmatrix}a_1 s_{i,1} + a_2 s_{i,2}\\ a_2 s_{i,1} + a_1 s_{i,2}\end{pmatrix}.$$

Now, I obtain the conditional expectation

$$\mathbb{E}[\boldsymbol{q^*_{-m}}|q^*_m, s_i] = \mu_{q^*_{-m}} + \rho^* \left(\frac{\sigma_{q^*_{-m}}}{\sigma_{q^*_m}}\right)\left(q^*_m - \mu_{q^*_m}\right) \text{ for } m = 1, 2.$$

Inserting the conditional expectation into the best reply and matching coefficients $\{o^*_1, o^*_2, a^*_1, a^*_2, c^*_1, c^*_2\}$ with the guess (A2) gives the following unique solution

$$\begin{aligned}
\frac{a^*_1}{c^*_m} &= \frac{(\lambda + \delta\rho^*)((\lambda n + 2\delta\rho^*)(1+\rho_s)\sigma_s^2 - \delta(n-2)(1-\rho^*)\tilde{\rho}_s\tilde{\sigma}_s^2)}{\frac{1}{n}(\lambda n - \delta(n - 2(1+\rho^*)))((\lambda n + \delta(n - 2(1-\rho^*)))(1+\rho_s)\sigma_s^2 - 2\delta(n-2)(1-\rho^*)\tilde{\rho}_s\tilde{\sigma}_s^2)}\\
\frac{a^*_2}{c^*_m} &= \frac{-(\lambda + \delta\rho^*)\delta(n-2)^2((1+\rho_s)\sigma_s^2 - (1-\rho^*)\tilde{\rho}_s\tilde{\sigma}_s^2)}{\frac{1}{n}(\lambda n - \delta(n - 2(1+\rho^*)))((\lambda n + \delta(n - 2(1-\rho^*)))(1+\rho_s)\sigma_s^2 - 2\delta(n-2)(1-\rho^*)\tilde{\rho}_s\tilde{\sigma}_s^2)}\\
c^*_m &= \left(\frac{n-2}{n-1}\right)\left(\frac{1}{\lambda + \delta\rho^*}\right)\\
\frac{o^*_m}{c^*_m} &= \begin{cases} \frac{\delta}{n}\left((\mu_{Q_1}\rho^* - \mu_{Q_2}) + (n-1)\left(a^*_1(\mu_{s_2} - \mu_{s_1}\rho^*) + a^*_2(\mu_{s_1} - \mu_{s_2}\rho^*) - \frac{(a^*_1 + a^*_2)(\mu_{s_1} + \mu_{s_2})(1-\rho^*)\tilde{\rho}_s\tilde{\sigma}_s^2}{(1+\rho_s)\sigma_s^2}\right)\right)\\ \frac{\delta}{n}\left((\mu_{Q_1}\rho^* - \mu_{Q_2}) + (n-1)\left(a^*_1(\mu_{s_2} - \mu_{s_1}\rho^*) + a^*_2(\mu_{s_1} - \mu_{s_2}\rho^*)\right)\right) \end{cases}
\end{aligned}$$

In $\frac{o^*_m}{c^*_m}$, the first case applies when $\rho_s \neq -1$, and the second when $\rho_s = -1$. The function of the proposition follows when usng $a^*_m = \alpha^*_m\left(\frac{n-2}{n-1}\right)$. This equilibrium exists, as long as the root ρ^* of the analogous polynomial (P) lies in $[-1, 1]$. □

F. Optimality conditions

In this appendix, I provide the optimality conditions that a BNE in a disconnected market must fulfill for a more general environment than in the main text. In particular, I make the following changes to the framework:

1) The utility function $U(\cdot,\cdot,\vec{s}_i)$ of any fixed $\vec{s}_i$ is twice differentiable with continuous cross-partial derivatives. I denote

$$\mu_1(q_1,q_2,\vec{s}_i) \equiv \frac{\partial U(q_1,q_2,\vec{s}_i)}{\partial q_1}, \;\; \mu_2(q_2,q_1,\vec{s}_i) \equiv \frac{\partial U(q_1,q_2,\vec{s}_i)}{\partial q_2}, \mu(q_1,q_2,\vec{s}_i) \equiv \frac{\partial^2 U(q_1,q_2,\vec{s}_i)}{\partial q_1 \partial q_2}.$$

2) Types and total supply are drawn from a distributions with differentiable distribution function and positive density. Their support may be bounded or unbounded. Proofs are shown for the case of unbounded support, the bounded case is analogous.

3) I no longer restrict attention to demand schedules which are asymptotically linear. Instead, I impose a finite lower and upper bound on how much agents may win $q_m \in [\underline{q}_m, \overline{q}_m] \subset \mathbb{R}$. This imposed bound is not needed if all random variables have bounded support.

The conditions will be stated using the joint and marginal distributions of agent i's clearing price quantities (as in Pycia and Woodward (2018)). Before formalizing them, I define these distributions. To do so, it helps to recall the definition of i's clearing price quantity in auction m, $\boldsymbol{q^c_{i,m}}$. Let all other agent's $j \neq i$ submit demand functions $\{x^*_{j,1}(\cdot,\vec{s}_j), x^*_{j,2}(\cdot,\vec{s}_j)\}$ as in equilibrium. Then i wins

$$(MC) \qquad \boldsymbol{q^c_{i,m}} = \underbrace{\boldsymbol{Q_m} - \sum_{j\neq i} x^*_{j,m}(\boldsymbol{p^c_m}, \boldsymbol{\vec{s}_j})}_{\boldsymbol{RS_{i,m}(p^c_m)}} \text{ with } \boldsymbol{p^c_m} = b_{i,m}(\boldsymbol{q^c_{i,m}}, \vec{s}_i)$$

at market clearing, when choosing $b_{i,m}(\cdot,\vec{s}_i)$. The support of the agent's clearing price quantity $[\underline{q}^c_{i,m},\overline{q}^c_{i,m}]$ depends on the price he offers for this amount. Notice, however, that there is an imposed upper and lower bound by the rules of the auction: $[\underline{q}^c_{i,m},\overline{q}^c_{i,m}]\subseteq[\underline{q}_m,\overline{q}_m]$ for any $b_{i,m}(\cdot,\vec{s}_i)$.

DEFINITION 5: *Define the joint distribution of agent i's clearing price quantities as the probability that he receives at most quantity q_1 and at most q_2 when bidding $b_{i,1}(q_1,\vec{s}_i)=p_1, b_{i,2}(q_2,\vec{s}_i)=p_2$ by*

$$G_i(q_1,q_2,p_1,p_2)\equiv \Pr\left(\boldsymbol{q^c_{i,1}}\leq q_1 \text{ and } \boldsymbol{q^c_{i,2}}\leq q_2\right). \tag{F1}$$

Analogously, define the marginal distribution of i's clearing price quantity in market $m=1,2$ by

$$G_{i,m}(q_m,p_m)=\Pr\left(\boldsymbol{q^c_{i,m}}\leq q_m\right). \tag{F2}$$

I denote the corresponding joint and marginal density functions by g_i and $g_{i,m}$ and oftentimes abbreviate $b_{i,m}(q_m,\vec{s}_i)=b_{i,m}$.

LEMMA 8: *A BNE in the disconnected market that consists of pairs of strictly decreasing, differentiable bidding functions must, for all q_m and $m=1,2$, satisfy*

$$\mathbb{E}\left[\frac{\partial U(q_m,\boldsymbol{q^*_{i,-m}},\vec{s}_i)}{\partial q_m}\middle| q_m\right]-b^*_{i,m}(q_m,\vec{s}_i)=-q_m\left[\frac{\frac{\partial G_{i,m}(q_m,b^*_{i,m}(q_m,\vec{s}_i))}{\partial q_m}}{\frac{\partial G_{i,m}(q_m,b^*_{i,m}(q_m,\vec{s}_i))}{\partial p_m}}\right]. \tag{F3}$$

(ii) When $b^{*-1}_{i,m}(\cdot,\vec{s}_i)=x^*_{i,m}(\cdot,\vec{s}_i)$ additively separates the type $\vec{s}_i$ from the quantity q_m $\forall i$, the condition simplifies to

$$\mathbb{E}\left[\frac{\partial U(q_m,\boldsymbol{q^*_{i,-m}},\vec{s}_i)}{\partial q_m}\middle| q_m\right]-b^*_{i,m}(q_m,\vec{s}_i)=+q_m\left[\frac{\partial RS_{i,m}(b^*_{i,m}(q_m,\vec{s}_i))}{\partial p_m}\right]^{-1}. \tag{F4}$$

REMARK: The proof of Lemma 8 is analogous to the proof of Lemma 1 in Wittwer (2020). In this related paper, I derive the necessary conditions for simultaneous pay-as-bid auctions in which agents can buy but not sell. There are three major differences. First, the framework here incorporates private values. In Wittwer (2020), all agents share the same type. Second, the auction here is a double auction in which agents buy and sell. Third, I consider a uniform-price auction here instead of a pay-as-bid auction. These auction formats differ in their payment rule.

$$\text{In the uniform-price auction, an agent pays: } \sum_{m=1,2} \boldsymbol{p_m^c q_{i,m}^c} \text{ with } \boldsymbol{p_m^c} = b_{i,m}(\boldsymbol{q_{i,m}^c}, \vec{s}_i).$$

$$\text{In a pay-as-bid auction, an agent pays: } \sum_{m=1,2} \int_0^{\boldsymbol{q_{i,m}^c}} b_{i,m}(q_m, \vec{s}_i) dq_m.$$

Proof of Lemma 8. — The proof involves lengthly algebraic derivations, in which one can easily get lost. To facilitate the understanding, I first lay out the core of the argument before and then carry out all mathematical details.

A. Core of the Argument. — Take the perspective of agent i, fix his type $\vec{s}_i$ and let all other agent's $j \neq i$ submit demand functions $\{x^*_{j,1}(\cdot, \vec{s}_j), x^*_{j,2}(\cdot, \vec{s}_j)\}$ as in equilibrium.

For notational ease, I will drop the fixed type $\vec{s}_i$ as an input argument of all functions and all subscripts i. For instance, $b_{i,1}(\cdot, \vec{s}_i)$ becomes $b_1(\cdot)$, $x_{i,1}(\cdot, \vec{s}_i)$ becomes $x_1(\cdot)$ and the amount i wins at market clearing becomes q^c_m.

Taking the behavior of all others as given, agent i chooses two bidding functions that maximize the following objective functional:

($\mathcal{V}$)

$$\mathcal{V}(b_1(\cdot), b_2(\cdot)) = \mathbb{E}\left[U(\boldsymbol{q_1^c}, \boldsymbol{q_2^c}) - \sum_{m=1,2} b_m(\boldsymbol{q_m^c})\boldsymbol{q_m^c}\right] \text{ with } \boldsymbol{q_m^c} = \boldsymbol{Q_m} - \sum_{j\neq i} x^*_{j,m}(b_m(\boldsymbol{q_m^c}), \boldsymbol{\vec{s}_j}).$$

Let him choose two differentiable and strictly decreasing equilibrium functions $\{b^*_1(\cdot), b^*_2(\cdot)\}$. By definition of an equilibrium, there cannot be another pair of functions different from $\{b_1^*(\cdot), b_2^*(\cdot)\}$ which generates a higher payoff for agent i.

$$(JM) \qquad \{b_1^*(\cdot), b_2^*(\cdot)\} \in \arg\max_{b_1(\cdot), b_2(\cdot)} \mathcal{V}(b_1(\cdot), b_2(\cdot))$$

Since $\{b_1^*(\cdot), b_2^*(\cdot)\}$ must be the solution to i's maximization problem, each function must solve the agent's maximization problem holding fixed the other.

$$(M) \qquad \Rightarrow b_m^*(\cdot) \in \arg\max_{b_m(\cdot)} \mathcal{O}(b_{,m}(\cdot))$$

$$(O) \qquad \text{with } \; \mathcal{O}(b_m(\cdot)) \equiv \mathcal{V}(b_m(\cdot), b_{-m}^*(\cdot)) \text{ for } m = 1 \text{ or } 2.$$

Otherwise, there would be another pair of functions that would generate a higher payoff for the agent, so that $\{b_1^*(\cdot), b_2^*(\cdot)\}$ could not be the solution of the joint maximization problem (JM). The rest of the proof derives the first-order condition of maximization problem (M). This derivation is complicated for two reasons. First, we are maximizing over a function, not variables. Second, the objective function is the expected total surplus of the agent. It non-trivially depends on the bidding function that we are trying to determine. To solve the optimization problem, I rely on techniques of calculus of variation. The first step is to express the objective function $\mathcal{O}(b_m(\cdot))$ in a format that explicitly states its dependence of the slope of the bidding function. The following auxiliary lemma summarizes.

AUXILIARY LEMMA 4: *Let $m = 1, 2$.*

$$(O) \qquad \mathcal{O}(b_m(\cdot)) = \int_{\underline{q}_m}^{\overline{q}_m} \mathcal{F}(q_m, b_m(q_m), b_m'(q_m)) dq_m \;\; \textit{with}$$

$$\mathcal{F}(q_m, b_m(q_m), b_m'(q_m)) \equiv \left[\mu_m(q_m, \overline{q}_{-m}) - b_m(q_m) - q_m b_m'(q_m)\right] [1 - G_m(q_m, b_m(q_m))]$$

$$(\mathcal{F}) \qquad - \int_{\underline{q}_{-m}}^{\overline{q}_{-m}} \mu(q_m, q_{-m})[1 - G(q_m, q_{-m}, b_m(q_m), b_{-m}^*(q_{-m}))] dq_{-m} + const$$

$\mathcal{F}(\cdot, \cdot, \cdot)$ is continuous in its three arguments and has continuous partial derivatives

with respect to the second and third.

Building on this auxiliary lemma, the the solution $b^*_m(\cdot)$ can be characterized by an Euler Equation:[23]

$$\mathcal{F}_{b_m}(q_m, b^*_m(q_m), b'^*_m(q_m)) = \frac{d}{dq_m}\mathcal{F}_{b'_m}(q_m, b^*_m(q_m), b'^*_m(q_m))$$

where $\mathcal{F}_{b_m}$, and $\mathcal{F}_{b'_m}$ denote the partial derivative of $\mathcal{F}(\cdot,\cdot,\cdot)$ w.r.t. the second and third argument. Rearranging the Euler Equation will give the optimality condition of Lemma 8 (i). It simplifies when the demand function is additively separable between types and quantity (ii).

B. Mathematical details. —

Proof of Auxiliary Lemma 4. — The proof proceeds in two steps: I first re-expresses the bidder's objective function $\mathcal{V}$. I then fix function $b_2(\cdot) = b^*_2(\cdot)$ to obtain $\mathcal{O}(b_m(\cdot))$ and show that $\mathcal{F}(\cdot,\cdot,\cdot)$ has the claimed properties.
Several times throughout the proof, I rely on the Fundamental Theorem of Calculus (FTC) and Fubini's Theorem. The FTC applies because all functions are integrable w.r.t. $q_1 \times q_2$ and all integrals take finite values. Here I am relying on the assumption that no bidder can supply or demand infinite amounts which bounds the quantity space $[\underline{q}_m, \overline{q}_m]$ for $m = 1, 2$. Fubini's Theorem holds because the functions inside the integrals are defined on the closed interval $[\underline{q}_m, \overline{q}_m]$ and because these functions are continuous given that all the functions they rely on are differentiable by assumption.

Step 1: Simplifying $\mathcal{V}$. — The simplification of i's objective involves several rounds of integration by parts. At the end, I will have expressed everything in terms of distribution functions, such as $G(q_1, q_2, b_1, b_2)$ rather than densities $g(q_1, q_2, b_1, b_2)$, where I abbreviate $b_m \equiv b_m(q_m)$. I start with the expected utility.

[23]It is known in the literature of variational calculus (e.g., Kamien and Schwartz (1993), pp. 14-16)

(a) Re-expressing the expected utility (as in Wittwer (2020)). — Using the distribution of i's clearing price quantities, the expected utility is

$$\mathbb{E}\left[U(\boldsymbol{q_1^c},\boldsymbol{q_2^c})\right]=\int_{\underline{q}_2^c}^{\overline{q}_2^c}\int_{\underline{q}_1^c}^{\overline{q}_1^c}U(q_1,q_2)g(q_1,q_2,b_1,b_2)dq_1dq_2.$$

The bounds of integration vary in i's bid choices, because the support of i's clearing price quantities depends on the price he offers for these amounts $b_m(\boldsymbol{q_m^c})$ for $m=1,2$. This is inconvenient, because we will be looking for the *optimal* bid choices. Luckily, there is a way around this complication. Since $g_m(q_m,b_m(q_m))=g(q_1,q_2,b_1(q_2),b_2(q_2))=0$ for *any* bid price offers at $q_m\notin\left[\underline{q}_m^c,\overline{q}_m^c\right]$ in either $m=1$ or 2, I can extend the bounds of the integrals to $\underline{q}_m$ and $\overline{q}_m$:

$$\mathbb{E}\left[U(\boldsymbol{q_1^c},\boldsymbol{q_2^c})\right]=\int_{\underline{q}_2}^{\overline{q}_2}\int_{\underline{q}_1}^{\overline{q}_1}U(q_1,q_2)g(q_1,q_2,b_1,b_2)dq_1dq_2.$$

The bounds of the integrals are now independent of the bid choice. With this preparatory step, we can start the simplification of the expected utility. I first integrate the inner integral by parts, taking the derivative of $U(q_1,q_2)$ and integrating $g(q_1,q_2,b_1,b_2)$ w.r.t. q_1.

$$\begin{aligned}\int_{\underline{q}_2}^{\overline{q}_2}\int_{\underline{q}_1}^{\overline{q}_1}U(q_1,q_2)g(q_1,q_2,b_1,b_2)dq_1dq_2=&\int_{\underline{q}_2}^{\overline{q}_2}\left[U(q_1,q_1)\int_{\underline{q}_1}^{q_1}g(q_1,q_2,b_1,b_2)dq_1\Bigg|_{q_1=\underline{q}_1}^{q_1=\overline{q}_1}\right]dq_2\\&-\int_{\underline{q}_2}^{\overline{q}_2}\left[\int_{\underline{q}_1}^{\overline{q}_1}\left[\mu_1(q_1,q_2)\int_{\underline{q}_1}^{q_1}g(q_1,q_2,b_1,b_2)dq_1\right]dq_1\right]dq_2\end{aligned}$$

Evaluate the first term at its bounds of integration. Since $\int_{\underline{q}_1}^{\underline{q}_1}g(q_1,q_2,b_1,b_2)=0$ and $\int_{\underline{q}_1}^{\overline{q}_1}g(q_1,q_2,b_1,b_2)=\int_{\underline{q}_1^c}^{\overline{q}_1^c}g(q_1,q_2,b_1,b_2)\ =g(q_2,b_2)$ by definition of a marginal distribution, the first expression simplifies:

$$\int_{\underline{q}_2}^{\overline{q}_2}\int_{\underline{q}_1}^{\overline{q}_1} U(q_1,q_2)g(q_1,q_2,b_1,b_2)dq_1dq_2 = \int_{\underline{q}_2}^{\overline{q}_2} U(\overline{q}_1,q_2)g_2(q_2,b_2)dq_2$$
$$-\int_{\underline{q}_2}^{\overline{q}_2}\left[\int_{\underline{q}_1}^{\overline{q}_1}\left[\mu_1(q_1,q_2)\int_{\underline{q}_1}^{q_1} g(q_1,q_2,b_1,b_2)dq_1\right]dq_1\right]dq_2.$$

I label the first term A and the second B. Consider term A and integrate by parts w.r.t. q_2.

$$A \equiv \int_{\underline{q}_2}^{\overline{q}_2} U(\overline{q}_1,q_2)g_2(q_2,b_2)dq_2 = U(\overline{q}_1,q_2)G_2(q_2,b_2)|_{\underline{q}_2}^{\overline{q}_2} - \int_{\underline{q}_2}^{\overline{q}_2}\mu_2(\overline{q}_2,q_2)G_2(q_2,b_2)dq_2$$

Since $G_2(\overline{q}_2,b_2)=1$ and $G_2(\underline{q}_2,b_2)=0$ for all b_2, this is

$$(A) \qquad A = U(\overline{q}_1,\overline{q}_2) - \int_{\underline{q}_2}^{\overline{q}_2}\mu_2(\overline{q}_2,q_2)G_2(q_2,b_2)dq_2.$$

Now consider term B. Applying Fubini's Theorem, I can revert the order of integration of the two outer integrals:

$$B \equiv \int_{\underline{q}_1}^{\overline{q}_1}\left[\int_{\underline{q}_2}^{\overline{q}_2}\left[\mu_1(q_1,q_2)\int_{\underline{q}_1}^{q_1} g(q_1,q_2,b_1,b_2)dq_1\right]dq_2\right]dq_1.$$

In the following, I simplify the inner integral corresponding to dq_2 by parts. I integrate $\int_{\underline{q}_1}^{q_1} g(q_1,q_2,b_1,b_2)dq_1$ and take the derivative of $\mu_1(q_1,q_2)$ w.r.t. q_2.

$$B = \int_{\underline{q}_1}^{\overline{q}_1}\left[\mu_1(q_1,q_2)G(q_1,q_2,b_1,b_2)|_{q_2=\underline{q}_2}^{q_2=\overline{q}_2} - \int_{\underline{q}_2}^{\overline{q}_2}\mu(q_1,q_2)G(q_1,q_2,b_1,b_2)dq_2\right]dq_1$$

Again, the first term simplifies because for any b_2, b_1, $G(q_2,\overline{q}_2,b_1,b_2)=G_1(q_1,b_1)$ and $G(q_1,\underline{q}_2,b_1,b_2)=0$. I obtain

$$(B) \quad B = \int_{\underline{q}_1}^{\overline{q}_1}\mu_1(q_1,\overline{q}_2)G_1(q_1,b_1)dq_1 - \int_{\underline{q}_2}^{\overline{q}_2}\mu(q_1,q_2)G(q_1,q_2,b_1,b_2)dq_2.$$

Combining A - B, and applying the FTC another time, the expected utility reads

$$\text{(EU)} \qquad \mathbb{E}\left[U(\boldsymbol{q_1^c}, \boldsymbol{q_2^c})\right] = + \sum_{m=1,2} \int_{\underline{q}_m}^{\overline{q}_m} \mu_m(q_m, \overline{q}_{-m})[1 - G_m(q_m, b_m)]dq_m$$
$$- \int_{\underline{q}_1}^{\overline{q}_2} \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1, q_2)[1 - G(q_1, q_2, b_1, b_2)]dq_2 dq_1 - U(\underline{q}_1, \underline{q}_2).$$

Re-expressing the expected payments. — Using the distribution of i's clearing price quantities, as above extending the bounds of the integral to $\underline{q}_m, \overline{q}_m$, the expected payment is

$$\mathbb{E}\left[B_m(\boldsymbol{q_m^c})\right] = \int_{\underline{q}_m}^{\overline{q}_m} [b_m(q_m)q_m]g_m(q_m, b_m)dq_m.$$

Integrating by parts gives

$$\mathbb{E}\left[B_m(\boldsymbol{q_m^c})\right] = [b_m(q_m)q_m]G_m(q_m, b_m)|_{\underline{q}_m}^{\overline{q}_m} - \int_{\underline{q}_m}^{\overline{q}_m} [b_m(q_m)q_m]' \, G_m(q_m, b_m)dq_m.$$

Since $G_m(\overline{q}_m, b_m) = 1, G_m(\underline{q}_m, b_m) = 0$ for all b_m, this simplifies to

$$(EB_m) \quad \mathbb{E}\left[B_m(\boldsymbol{q_m^c})\right] = \int_{\underline{q}_m}^{\overline{q}_m} \left[b_m(q_m) + q_m b_m'(q_m)\right] [1 - G_m(q_m, b_m)]dq_m.$$

→ The objective function. — Combining (EU) -$\sum_{m=1,2}$ (EB_m), $\mathcal{V}$ becomes

$$\mathcal{V}(b_1(\cdot), b_2(\cdot)) = \sum_{m=1,2} \int_{\underline{q}_m}^{\overline{q}_m} \mu_m(q_m, \overline{q}_{-m}) - \left[b_m(q_m) + q_m b_m'(q_m)\right] [1 - G_m(q_m, b_m)]dq_m$$
$$- \int_{\underline{q}_1}^{\overline{q}_1} \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1, q_2)[1 - G(q_1, q_2, b_1, b_2)]dq_2 dq_1 - U(\underline{q}_1, \underline{q}_2).$$

Step 2: Deriving $\mathcal{F}$. — For notational convenience set $m = 1, -m = 2$. The other case is analogous. Fix $b_2(\cdot) = b_2^*(\cdot)$, and recall that $\mathcal{O}(b_1(\cdot)) \equiv \mathcal{V}(b_1(\cdot), b_2^*(\cdot))$. A straightforward mathematical manipulation rearranges $\mathcal{V}$ with $b_2(\cdot) = b_2^*(\cdot)$ to

$$(O) \qquad \mathcal{O}(b_1(\cdot)) = \int_{\underline{q}_1}^{\overline{q}_1} \mathcal{F}(q_1, b_1(q_1), b_1'(q_1))dq_1$$

with

$$(\mathcal{F})\qquad \begin{aligned}\mathcal{F}(q_1, b_1(q_1), b_1'(q_1)) \equiv & [\mu_1(q_1, \overline{q}_2) - b_1(q_1) - q_1 b_1'(q_1)] [1 - G_1(q_1, b_1(q_1))] \\ & - \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1, q_2)[1 - G(q_1, q_2, b_1(q_1), b_2(q_2))]dq_2 + const\end{aligned}$$

and $const \equiv \left[\frac{1}{\overline{q}_1 - \underline{q}_1}\right] \left[\int_{\underline{q}_2}^{\overline{q}_2} [\mu_2(q_2, \overline{q}_1) - b_2^*(q_2) - q_2 b_2'^*(q_2)] [1 - G_2(q_2, b_2^*(q_2))]dq_2 - U(\underline{q}_1, \underline{q}_2)\right].$

This leaves us with the claimed functional form of $\mathcal{F}(\cdot, \cdot, \cdot)$. This function is continuous in its three arguments and has continuous partial derivatives with respect to the second and third, because $b_1(\cdot), b_2(\cdot)$ and all distribution functions are differentiable, and the the utility function has continuous partial and cross-partial derivatives by assumption. □

NECESSARY CONDITION OF MAXIMIZATION PROBLEM (M). — In what follows I derive the necessary condition of

$$(M)\qquad \max_{b_1(\cdot)} \mathcal{O}(b_1(\cdot)) = \max_{b_1(\cdot)} \int_{\underline{q}_1}^{\overline{q}_1} \mathcal{F}(q_1, b_1(q_1), b_1'(q_1))dq_1.$$

The other auction, $m = 2$, is analogous. Since $\mathcal{F}$ is continuous in its three arguments and has continuous partial derivatives w.r.t. the second and third, this maximization problem is a standard problem of variational calculus. Its solution $b_1^*(\cdot) : [\underline{q}_1, \overline{q}_1] \to \mathbb{R}$ must satisfy the Euler Equation for all quantity points $q_1 \in [\underline{q}_1, \overline{q}_1]$:

$$(\text{F5})\qquad \mathcal{F}_{b_1}(q_1, b_1^*(q_1), b_1'^*(q_1)) = \frac{d}{dq_1}\mathcal{F}_{b_1'}(q_1, b_1^*(q_1), b_1'^*(q_1))$$

where $\mathcal{F}_{b_1}$, and $\mathcal{F}_{b_1'}$ denote the partial derivative of $\mathcal{F}(\cdot, \cdot, \cdot)$ w.r.t. the second and third argument. The two partial derivatives of $\mathcal{F}$ evaluated at the solution (here abbreviated by $\mathcal{F}_{b_1}$ and $\mathcal{F}_{b_1'}$) are

(F6)

$$\mathcal{F}_{b_1} = [\mu_1(q_2, \overline{q}_2) - b_1^*(q_1) - q_1 b_1'^*(q_1)]\,(-1)\left(\frac{\partial G_1(q_1, b_1^*(q_1))}{\partial b_1(q_1)}\right) - [1 - G(q_1, q_2, b_1^*(q_1), b_2^*(q_2))]$$
$$- \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1, q_2)(-1)\left(\frac{\partial G(q_1, q_2, b_1^*(q_1), b_2^*(q_2))}{\partial b_1(q_1)}\right) dq_2$$

(F7)

$$\mathcal{F}_{b_1'} = -\, q_1[1 - G_1(q_1, b_1^*(q_1))].$$

The total derivative w.r.t q_1 evaluated at the solution is therefore

(F8)

$$\frac{d}{dq_1}\mathcal{F}_{b_1'} = -\,[1 - G_1(q_1, b_1^*(q_1))] + q_1\left[\left(\frac{\partial G_1(q_1, b_1^*(q_1))}{\partial q_1}\right) + \left(\frac{\partial G_1(q_1, b_1^*(q_1))}{\partial b_1(q_1)}\right) b_1'^*(q_1)\right].$$

The Euler Equation equates (F6) = (F8). Simplifying it gives

$$-\,[\mu_1(q_1, \overline{q}_2) - b_1^*(q_1)]\left(\frac{\partial G_1(q_1, b_1^*(q_1))}{\partial b_1(q_1)}\right) + \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1, q_2)\left(\frac{\partial G(q_1, q_2, b_1^*(q_1), b_2^*(q_2))}{\partial b_1(q_1)}\right) dq_2 = q_1\left[\frac{\partial G_1(q_1, b_1^*(q_1))}{\partial q_1}\right].$$

Apply the FTC to replace $\mu_1(q_1, \overline{q}_2) = \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1, q_2) dq_2 + \mu_1(q_2, \underline{q}_2)$, the condition rearranges to

$$\mu_1(q_1, \underline{q}_2) - \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1, q_2)\left[1 - \left(\frac{\frac{\partial G(q_1, q_2, b_1^*(q_1), b_2^*(q_2))}{\partial b_1(q_1)}}{\frac{\partial G_1(q_1, b_1^*(q_1))}{\partial b_1(q_1)}}\right)\right] dq_2 - b_1^*(q_1) = -q_1\left[\frac{\frac{\partial G_1(q_1, b_1^*(q_1))}{\partial q_1}}{\frac{\partial G_1(q_1, b_1^*(q_1))}{\partial b_1(q_1)}}\right].$$

The first two terms on the LHS are nothing else than $\mathbb{E}\left[\mu_1(q_1, \boldsymbol{q_2^*}) \middle|\, q_1\right]$. To see this, let $G_{2|1}(q_2, b_2(q_2)|q_1)$ be the probability that i wins at most q_2 when submitting $p_2 = b_2(q_2)$ *conditional* on winning q_1 in auction 1. Then

$$G(q_1, q_2, b_1(q_1), b_2(q_2)) = G_{2|1}(q_2, b_2(q_2)|q_1) G_1(q_1, b_1(q_1))$$
$$\Rightarrow \left(\frac{\partial G(q_1, q_2, b_1^*(q_1), b_2^*(q_2))}{\partial b_1(q_1)}\right) = G_{2|1}(q_2, b_2^*(q_2)|q_1)\left(\frac{\partial G_1(q_1, b_1^*(q_1))}{\partial b_1(q_1)}\right).$$

The optimality condition becomes

$$\mu_1(q_1,\underline{q}_2) - \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1,q_2)\left[1 - G_{2|1}(q_2, b_2^*(q_2)|q_1)\right] dq_2 - b_1^*(q_1) = -q_1 \left[\frac{\frac{\partial G_1(q_1,b_1^*(q_1))}{\partial q_1}}{\frac{\partial G_1(q_1,b_1^*(q_1))}{\partial b_1(q_1)}}\right].$$

Integrating by parts, using $G_{2|1}(\overline{q}_2, b_2^*(\overline{q}_2)|q_1) = 1$, $G_{2|1}(\underline{q}_2, b_2^*(\underline{q}_2)|q_1) = 0$, and changing notation from $\partial b_1(q_1)$ to ∂p_1 gives condition (i) of the lemma, stated in simplified notation.

To prove the second part of the lemma, assume that equilibrium demand functions take the following additively separable form: $x_{i,1}^*(p_1, \vec{s}_i) = \eta_{i,1}^*(\vec{s}_i) + y_{i,1}^*(p_1)$ with differentiable and strictly decreasing function $y_{i,1}^*(\cdot)$. Given that all other agents $j \neq i$ choose such functions, agent i's residual supply curve only shifts randomly in its intercept with the quantity axis, $\boldsymbol{Z_1}$:

(F9) $$\boldsymbol{RS_1}(p_1) = \boldsymbol{Z_1} - \sum_{j\neq i} y_{j,1}^*(p_1) \quad \text{where} \quad \boldsymbol{Z_1} \equiv \boldsymbol{Q_1} - \sum_{j\neq i} \eta_{j,1}^*(\boldsymbol{\vec{s}_j}).$$

As in the proofs for Propositions 1 (II) and 4, there is a one-to-one mapping from how much the agent wins at market clearing and this intercept. Optimality conditions can be re-expressed in terms of the marginal density f_{Z_1} and distribution F_{Z_1} of $\boldsymbol{Z_1}$ using

$$G(q_1,p_1) \equiv \Pr(\boldsymbol{q_1^c} \leq q_1) = \Pr\left(\boldsymbol{Z_1} \leq q_1 + \sum_{j\neq i} y_{j,1}^*(p_1)\right) = F_{Z_1}\left(q_1 + \sum_{j\neq i} y_{j,1}^*(p_1)\right)$$

(F10)

$$\Rightarrow \frac{\partial G_1(q_1,p_1)}{\partial q_1} = f_{Z_1}\left(q_1 + \sum_{j\neq i} y_{j,1}^*(p_1)\right) \text{ and}$$

(F11)

$$\frac{\partial G_1(q_1,p_1)}{\partial p_1} = f_{Z_1}\left(q_1 + \sum_{j\neq i} y_{j,1}^*(p_1)\right)\left(\frac{\partial \sum_{j\neq i} y_{j,1}^*(p_1)}{\partial p_1}\right) \overset{(F9)}{=} -f_{Z_1}\left(q_1 + \sum_{j\neq i} y_{j,1}^*(p_1)\right)\left(\frac{\partial RS_1(p_1)}{\partial p_1}\right).$$

Dividing (F10) by (F11) and evaluating both expressions at $p_1 = b_1^*(q_1)$ we obtain the expression of part (ii) of the lemma again in simplified notation. □