Online Appendix A for "Acquisition, aggregation, and sharing of information in sequential-move aggregative games"

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Contents

In this document, we provide all proofs of lemmas and propositions of the main text. In Section 4, we also present the differentiated goods Stackelberg model with private information and study the robustness checks for our results in this set-up. Supplementary calculations in *Mathematica 12.1* are provided in online Appendix B, which is available at the following website: https://sites.google.com/site/cumbuleray1/research-1.

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1 Proofs of the common-value Stackelberg oligopoly model

1.1 Supplementary notation

Let $N_k = \{1, 2, ..., k\}$ denote the set of firms in a k-firm market. Thus, $N = N_n$. Let $w_1(N_k) = \frac{\sigma_\theta^2}{2b(\sigma_\epsilon^2+(1+k)\sigma_\theta^2)}$, $w_2(N_k) = \frac{2\sigma_\epsilon^2+(2k+1)\sigma_\theta^2}{2(\sigma_\epsilon^2+(k+1)\sigma_\theta^2)}$, $w_3(N_k) = \frac{\sigma_\theta^4(3\sigma_\epsilon^2+(4k+3)\sigma_\theta^2)}{8b(\sigma_\epsilon^2+\sigma_\theta^2(k+1))^2}$, and $w_4(N_k) = \frac{2\sigma_\epsilon^2+(2k+1)\sigma_\theta^2}{2(\sigma_\epsilon^2+k\sigma_\theta^2)}$. We add the "*" superscript for the corresponding equilibrium outcomes of the associated variables. The "NS", "PS", and "FS" subscripts denote the no-sharing, partial-sharing and full-sharing Stackelberg games, respectively. For example, $E(Q^*_{NS}(N_k))$ denotes the expected perfect revealing equilibrium (PRE) total output when the set of firms is N_k in the Stackelberg no-sharing game. The subscript "C" denotes the Cournot game.

1.2 Earlier results

First, we restate three results of Cumbul (2021) in Lemmas A1-A3 to prove our claims.

Lemma A1. *(Cumbul, 2021, Lemma 6)*

i) $E(Q^*_{NS}(N_k)) = (a+\bar{\theta})/b - E(q^*_{k,NS}(N_k))$ *and* $E(p^*_{SQ}(N_k)) = bE(q^*_{k,SQ}(N_k))$.

ii) $E(q^*_{k,NS}(N_k)) = \frac{(a+\bar{\theta})}{2b}\prod_{j=1}^{j=k-1} w_2(N_j)$.

iii) $\frac{\partial E(q^*_{1,NS}(N_k))}{\partial \sigma_\epsilon^2} < 0$, $\frac{\partial E(q^*_{1,NS}(N_k))}{\partial \sigma_\theta^2} > 0$, $\frac{\partial E(q^*_{k,NS}(N_k))}{\partial \sigma_\epsilon^2} > 0$, *and* $\frac{\partial E(q^*_{k,NS}(N_k))}{\partial \sigma_\theta^2} < 0$.

iv) For each $i \in N_k \setminus k$, $q^*_{i,NS}(N_k) = q^*_{i,NS}(N_{k+1})$.

Lemma A2. *(Cumbul, 2021, Lemma 10)*

i) For $i < k$, $E(\pi^*_{i,NS}(N_{k+1})) = w_2(N_k)E(\pi^*_{i,NS}(N_k))$.

ii) $E(\pi^*_{k,NS}(N_{k+1})) = 2bw_1(N_k)w_2(N_k)E(\pi^*_{k,NS}(N_k))$.

iii) $E(\pi^*_{i,NS}(N_i)) = \frac{(a+\bar{\theta})^2}{4b}\prod_{k=1}^{k=i-1} w_2^2(N_k) + \sum_{j=1}^{j=i}\big(\frac{\sigma_\epsilon^2\sigma_\theta^4}{4b(\sigma_\epsilon^2+(j-1)\sigma_\theta^2)(\sigma_\epsilon^2+j\sigma_\theta^2)}\prod_{k=j}^{k=i-1} w_2^2(N_k)\big)$.

iv) For $i < n$, $E(\pi^*_{i,NS}(N)) = E(\pi^*_{i,NS}(N_i)) \times 2bw_1(N_i)\prod_{k=i}^{k=n-1} w_2(N_k)$.

Lemma A3. *(Cumbul, 2021, Lemma 11 and Proposition 1)*

i) The expected PRE total surplus in the no-sharing Stackelberg game is

$$E(TS^*_{NS}(N)) = \underbrace{\frac{(a+\bar{\theta})^2(4-\prod_{i=1}^{i=n-1} w_2^2(N_i))}{8b}}_{=E(TS^*_{1,NS}(N))} + \underbrace{\sum_{i=1}^{i=n}\big(w_3(N_{i-1})\prod_{j=i}^{j=n-1} w_2^2(N_j)\big)}_{=E(TS^*_{2,NS}(N))}.$$

The expected PRE total profit is $E(\Pi^*_{NS}(N)) = \sum_{i\in N} E(\pi^*_{i,NS}(N))$ *and the expected PRE consumer surplus is* $E(CS^*_{NS}(N)) = E(TS^*_{NS}(N)) - E(\Pi^*_{NS}(N))$.

ii) The expected Bayesian equilibrium total surplus in the Cournot game is

$$E(TS^*_C(N)) = \underbrace{\frac{n(n+2)(a+\bar{\theta})^2}{2b(n+1)^2}}_{=E(TS^*_{1,C}(N))} + \underbrace{\frac{n\sigma_\theta^4(3\sigma_\epsilon^2+(n+2)\sigma_\theta^2)}{2b(2\sigma_\epsilon^2+\sigma_\theta^2(n+1))^2}}_{=E(TS^*_{2,C}(N))=\Theta(N)}.$$

The expected Bayesian equilibrium total profit and consumer surplus in the Cournot game are

$$E(\Pi_C^*(N)) = \underbrace{\frac{n(a+\bar{\theta})^2}{b(n+1)^2}}_{=E(\Pi_{1,C}^*(N))} + \underbrace{\frac{n\sigma_\theta^4(\sigma_\epsilon^2+\sigma_\theta^2)}{b(2\sigma_\epsilon^2+\sigma_\theta^2(n+1))^2}}_{=E(\Pi_{2,C}^*(N))}$$

and $E(CS_C^*(N)) = E(TS_C^*(N)) - E(\Pi_C^*(N))$*, respectively.*

iii) $E(TS_{2,NS}^*(N)) < E(TS_{2,C}^*(N)) = \Theta(N) = \dfrac{n\sigma_\theta^4(3\sigma_\epsilon^2+\sigma_\theta^2(n+2))}{2b(2\sigma_\epsilon^2+\sigma_\theta^2(n+1))^2}$.

1.3 Supplementary lemmas and their proofs

Next, we prove four useful lemmas to be used in the forthcoming proofs.

Lemma A4. *i) For all* $c \geq 1$,

$$E(TS_{2,NS}^*(N)) > \chi(N) = \frac{cn\sigma_\theta^4(\sigma_\epsilon^2+(n+1)\sigma_\theta^2)(2\sigma_\epsilon^2+(2n+1)\sigma_\theta^2)}{4b(2\sigma_\epsilon^2+(n+1)\sigma_\theta^2)^2(c^2\sigma_\epsilon^2+(c+n(c+1))\sigma_\theta^2)}. \tag{1}$$

ii) Take any $i, j \in N$ *such that* $i \leq j$.
$Cov(\theta, E(\theta|s_1, s_2, ..., s_i)) = Cov(E(\theta|s_1, s_2, ..., s_i), E(\theta|s_1, s_2, ..., s_j)) = i\sigma_\theta^4/(\sigma_\epsilon^2 + i\sigma_\theta^2)$.

Proof of Lemma A4: **i)** We prove the claim by induction. For $k = 1$,

$$E(TS_{2,NS}^*(N_1)) - \chi(N_1) = \frac{\sigma_\theta^4(2c(3c-1)\sigma_\epsilon^4+(6+c(6c+5))\sigma_\epsilon^2\sigma_\theta^2+6(c+1)\sigma_\theta^4)}{16b(\sigma_\epsilon^2+\sigma_\theta^2)^2(c^2\sigma_\epsilon^2+(2c+1)\sigma_\theta^2)} > 0$$

because $b > 0$ and $c \geq 1$. Suppose that $E(TS_{2,NS}^*(N_k)) > \chi(N_k)$ for some $k < n$. We want to show that $E(TS_{2,NS}^*(N_{k+1})) > \chi(N_{k+1})$. As $E(TS_{2,NS}^*(N_{k+1})) = w_2^2(N_k)E(TS_{2,NS}^*(N_k)) + w_3(N_k)$ by Lemma A3-*i*), we should show that

$$\zeta = w_2^2(N_k)E(TS_{2,NS}^*(N_k)) + w_3(N_k) - \chi(N_{k+1}) > 0. \tag{2}$$

Note that $w_2^2(N_k) > 0$. After replacing $E(TS_{2,NS}^*(N_k))$ with its lower bound $\chi(N_k)$ in (2) by using the initial supposition, we obtain the lowest possible value of ζ as

$$\begin{aligned}\underline{\zeta} &= w_2^2(N_k)\chi(N_k) + w_3(N_k) - \chi(N_{k+1}) \\ &= \frac{\sigma_\theta^4(\psi_0+\psi_1\sigma_\theta^2+\psi_2\sigma_\theta^4+\psi_3\sigma_\theta^6+\psi_4\sigma_\theta^8+\psi_5\sigma_\theta^{10}+\psi_6\sigma_\theta^{12}+\psi_7\sigma_\theta^{14})}{16b(\sigma_\epsilon^2+\sigma_\theta^2(k+1))^2(2\sigma_\epsilon^2+\sigma_\theta^2(k+1))^2(2\sigma_\epsilon^2+(k+2)\sigma_\theta^2)^2(c^2\sigma_\epsilon^2+\sigma_\theta^2(c+k(c+1)))(c^2\sigma_\epsilon^2+\sigma_\theta^2(1+2c+k(c+1)))},\end{aligned}$$

where

$$\begin{aligned}
\psi_0 &= 32c^3(3c-1)\sigma_\epsilon^{14}, \\
\psi_1 &= 16c^2\Big(4+12(k+2c^2)+c(5+2k(10c-1))\Big)\sigma_\epsilon^{12}, \\
\psi_2 &= 8\Big(c(12+c(46+c(77+75c)))+12k+2c(20+c(68+c(38+63c)))k+(12+c(16+c(84+c(9+50c))))k^2\Big)\sigma_\epsilon^{10}, \\
\psi_3 &= 4\Big(16k(1+k)(6+5k)+4c(24+k(99+k(96+25k)))+c^2(208+2k(326+5k(67+22k)))\Big)\sigma_\epsilon^8+ \\
&\quad +4\Big(c^3(275+k(518+k(265+26k)))+2c^4(57+k(145+3k(39+10k)))\Big)\sigma_\epsilon^8,
\end{aligned}$$

$$\psi_4 = 2\Big(4k(1+k)(75+2k(63+25k))+4c(75+k(370+k(548+309k+57k^2)))+2c^2(1+k)(239+k(583+2k(243+67k)))\Big)\sigma_\epsilon^6+$$
$$+2\Big(c^3(440+k(1208+k(1090+k(347+20k))))+c^4(1+k)(2+k)(42+k(81+35k))\Big)\sigma_\epsilon^6,$$
$$\psi_5 = \Big(2c(228+k(1314+k(2584+k(2251+k(881+124k)))))+2c^2(296+k(1134+k(1701+2k(635+k(240+37k)))))\Big)\sigma_\epsilon^4+$$
$$+\Big(8k(1+k)(57+k(145+3k(39+10k)))+c^3(332+k(1200+k(1570+k(883+2k(93+2k)))))+2c^4(1+k)^2(2+k)^2(3+4k)\Big)\sigma_\epsilon^4,$$
$$\psi_6 = \Big(2k(1+k)^2(2+k)(42+k(81+35k))+c(1+k)(168+k(944+k(1774+k(1441+520k+68k^2))))\Big)\sigma_\epsilon^2+$$
$$+\Big(c^2(188+k(808+k(1354+k(1136+k(515+14k(9+k))))))+c^3(1+k)(2+k)(24+k(72+k(67+18k)))\Big)\sigma_\epsilon^2,$$
$$\psi_7 = (1+k)(2+k)\Big(2k(1+k)^2(2+k)(3+4k)+c(1+k)(12+k(5+2k)(12+k(13+4k)))+c^2(12+k(40+k(42+13k)))\Big).$$

As $c, k \geq 1$, $\psi_i > 0$ for $i = 0, 1, ..., 7$. Thus, $\zeta > \underline{\zeta} > 0$, as claimed.

ii) By using Lemma 1, direct computations show that

$$\begin{aligned} Cov(\theta, E(\theta|s_1, s_2, ..., s_i)) &= Cov(\theta, \tfrac{\sigma_\epsilon^2\bar{\theta}+\sigma_\theta^2(s_1+s_2+...+s_i)}{\sigma_\epsilon^2+i\sigma_\theta^2}) \\ &= \tfrac{\sigma_\theta^2}{\sigma_\epsilon^2+i\sigma_\theta^2} Cov(\theta, s_1+s_2+...+s_i) = \tfrac{i\sigma_\theta^4}{\sigma_\epsilon^2+i\sigma_\theta^2} \end{aligned}$$

because $Cov(\theta, s_1+s_2+...+s_i) = \sum_{k=1}^{k=i} Cov(\theta, s_k) = i\sigma_\theta^2$. Similarly, for $i \leq j$,

$$\begin{aligned} &Cov(E(\theta|s_1, s_2, ..., s_i), E(\theta|s_1, s_2, ..., s_j)) = \\ &= Cov(\tfrac{\sigma_\epsilon^2\bar{\theta}+\sigma_\theta^2(s_1+s_2+...+s_i)}{\sigma_\epsilon^2+i\sigma_\theta^2}, \tfrac{\sigma_\epsilon^2\bar{\theta}+\sigma_\theta^2(s_1+s_2+...+s_j)}{\sigma_\epsilon^2+j\sigma_\theta^2}) \\ &= \tfrac{\sigma_\theta^4}{(\sigma_\epsilon^2+i\sigma_\theta^2)(\sigma_\epsilon^2+j\sigma_\theta^2)} \times \big(Var(\textstyle\sum_{k=1}^{k=i} s_k) + i(j-1)Cov(s_k, s_l)\big) \\ &= \tfrac{\sigma_\theta^4}{(\sigma_\epsilon^2+i\sigma_\theta^2)(\sigma_\epsilon^2+j\sigma_\theta^2)} \times \big(i(\sigma_\epsilon^2+\sigma_\theta^2) + i(j-1)\sigma_\theta^2\big) \\ &= \tfrac{\sigma_\theta^4}{(\sigma_\epsilon^2+i\sigma_\theta^2)(\sigma_\epsilon^2+j\sigma_\theta^2)} \times i(\sigma_\epsilon^2 + j\sigma_\theta^2) = \tfrac{i\sigma_\theta^4}{\sigma_\epsilon^2+i\sigma_\theta^2} \end{aligned}$$

because $Cov(s_k, s_l) = \sigma_\theta^2$ for $k \neq l$ and $Var(s_k) = \sigma_\epsilon^2 + \sigma_\theta^2$. □

Lemma A5. *The expected equilibrium individual profits in the partial and full sharing games are given as follows.*

i) For each $i \in N_k$, $E(\pi_{i,PS}^*(N_{k+1})) = E(\pi_{i,PS}^*(N_k))/2$.

ii) $E(\pi_{k+1,PS}^*(N_{k+1})) = \dfrac{E(\pi_{k,PS}^*(N_{k+1}))}{2} + \dfrac{\sigma_\epsilon^2\sigma_\theta^4}{4b(\sigma_\epsilon^2+k\sigma_\theta^2)(\sigma_\epsilon^2+(k+1)\sigma_\theta^2)}$.

iii) For each $i \in N$, $E(\pi_{i,PS}^*(N)) = \dfrac{(a+\bar{\theta})^2}{2^{n+i}b} + \displaystyle\sum_{j=0}^{j=i-1} \dfrac{\sigma_\epsilon^2\sigma_\theta^4}{2^{n+i-2j}b(\sigma_\epsilon^2+j\sigma_\theta^2)(\sigma_\epsilon^2+(j+1)\sigma_\theta^2)}$.

iv) For each $i \in N$, $E(\pi_{i,FS}^*(N)) = \dfrac{(a+\bar{\theta})^2}{2^{n+i}b} + \dfrac{n\sigma_\theta^4}{2^{n+i}b(\sigma_\epsilon^2+n\sigma_\theta^2)}$.

Proof of Lemma A5: **i)** Take any $i \in N_k$. We can show that

$$\begin{aligned} E(\pi_{i,PS}^*(N_{k+1})) &= E((a+\theta-bQ_{PS}^*(N_{k+1}))q_{i,PS}^*(N_{k+1})) \\ &= E((a+\theta-b\textstyle\sum_{j=1}^{j=k} q_{j,PS}^*(N_{k+1}) - bq_{k+1,PS}^*(N_{k+1}))q_{i,PS}^*(N_{k+1})) \\ &= E(\tfrac{a-b\sum_{j=1}^{j=k} q_{j,PS}^*(N_{k+1})+2\theta-E(\theta|s_1,s_2,...,s_{k+1})}{2} q_{i,PS}^*(N_{k+1})) \\ &= E(\tfrac{(a+\theta-b\sum_{j=1}^{j=k} q_{j,PS}^*(N_k))q_{i,PS}^*(N_k)}{2}) \\ &= \tfrac{E(\pi_{i,PS}^*(N_k))}{2}, \end{aligned}$$

where the first, second, and fifth equalities follow by definition. In the third equality, we plug in the

PS equilibrium quantity strategy of firm $k+1$ by (19). The fourth equality is valid as $q^*_{j,PS}(N_{k+1}) = q^*_{j,PS}(N_k)$ for each $j \in N_k$ by Lemma 4; and $E((2\theta - E(\theta|s_1, s_2, ..., s_{k+1}))q^*_{i,PS}(N_k)) = E(\theta q^*_{i,PS}(N_k))$ by Lemma A4-*ii*).

ii) By Lemma 4, the partial sharing equilibrium quantities of firms k and $k+1$ in the N_{k+1}-firm market should satisfy

$$q^*_{k,PS}(N_{k+1}) = \frac{a + 2^{k-1}E(\theta|s_1, s_2, ..., s_k) - \sum_{j=1}^{j=k-1}\left(2^{j-1}E(\theta|s_1, s_2, ..., s_j)\right)}{2^k b}$$

and

$$q^*_{k+1,PS}(N_{k+1}) = \frac{a + 2^{k}E(\theta|s_1, s_2, ..., s_{k+1}) - \sum_{j=1}^{j=k}\left(2^{j-1}E(\theta|s_1, s_2, ..., s_j)\right)}{2^{k+1} b}.$$

Accordingly, we can derive that

$$q^*_{k+1,PS}(N_{k+1}) = \frac{q^*_{k,PS}(N_{k+1})}{2} + \frac{E(\theta|s_1, s_2, ..., s_{k+1}) - E(\theta|s_1, s_2, ..., s_k)}{2b}. \quad (3)$$

Moreover, total equilibrium quantity in the partial sharing game is

$$Q^*_{PS}(N_{k+1}) = \sum_{i=1}^{i=k+1} q^*_{i,PS}(N_{k+1}) = \frac{a}{b}(1 - \frac{1}{2^{k+1}}) + \sum_{i=1}^{i=k+1} \frac{E(\theta|s_1, s_2, ..., s_i)}{b 2^{k-i+2}} \quad (4)$$

by using Lemma 4. Finally, we can show that

$$\begin{aligned}
&E(\pi^*_{k+1,PS}(N_{k+1})) = E((a + \theta - bQ^*_{PS}(N_{k+1}))q^*_{k+1,PS}(N_{k+1})) \\
&= E((a + \theta - bQ^*_{PS}(N_{k+1}))(\tfrac{q^*_{k,PS}(N_{k+1})}{2} + \tfrac{E(\theta|s_1,s_2,...,s_{k+1}) - E(\theta|s_1,s_2,...,s_k)}{2b})) \\
&= \tfrac{E(\pi^*_{k,PS}(N_{k+1}))}{2} + E\big((a + \theta - bQ^*_{PS}(N_{k+1}))\tfrac{E(\theta|s_1,s_2,...,s_{k+1}) - E(\theta|s_1,s_2,...,s_k)}{2b}\big) \\
&= \tfrac{E(\pi^*_{k,PS}(N_{k+1}))}{2} + E\big((\tfrac{a}{2^{k+1}} + \theta - \textstyle\sum_{i=1}^{i=k+1} \tfrac{E(\theta|s_1,s_2,...,s_i)}{2^{k-i+2}})\tfrac{E(\theta|s_1,s_2,...,s_{k+1}) - E(\theta|s_1,s_2,...,s_k)}{2b}\big) \\
&= \tfrac{E(\pi^*_{k,PS}(N_{k+1}))}{2} + Cov(\theta - \textstyle\sum_{i=1}^{i=k+1} \tfrac{E(\theta|s_1,s_2,...,s_i)}{2^{k-i+2}}, \tfrac{E(\theta|s_1,s_2,...,s_{k+1}) - E(\theta|s_1,s_2,...,s_k)}{2b}) \\
&= \tfrac{E(\pi^*_{k,PS}(N_{k+1}))}{2} + Cov(\theta - \tfrac{E(\theta|s_1,s_2,...,s_{k+1})}{2}, \tfrac{E(\theta|s_1,s_2,...,s_{k+1}) - E(\theta|s_1,s_2,...,s_k)}{2b}) \\
&= \tfrac{E(\pi^*_{k,PS}(N_{k+1}))}{2} + \tfrac{\sigma^2_\epsilon \sigma^4_\theta}{4b(\sigma^2_\epsilon + k\sigma^2_\theta)(\sigma^2_\epsilon + (k+1)\sigma^2_\theta)},
\end{aligned}$$

where the first and third equalities follow by definition. In the second equality, we plugged in the value of $q^*_{k+1,PS}(N_{k+1})$ from (3). In the fourth equality, we insert the value of $Q^*_{k+1,PS}(N_{k+1})$ from (4). In the fifth equality, the non-covariance terms disappear because $E(E(\theta|s_1, s_2, ..., s_{k+1})) = E(E(\theta|s_1, s_2, ..., s_k)) = \bar{\theta}$ by the law of iterated expectations. The remaining two equalities follow by Lemma A4-*ii*).

iii) When $n = 1$, there is no difference between no-sharing and sharing games. In this case, the expected monopoly profit equals

$$E(\pi^*_{1,PS}(N_1)) = E(\pi^*_{1,FS}(N_1)) = \frac{(a + \bar{\theta})^2}{4b} + \frac{\sigma^4_\theta}{4b(\sigma^2_\epsilon + \sigma^2_\theta)}. \quad (5)$$

Starting from the monopoly profit and using Lemma A5-i) and Lemma A5-ii) iteratively, we have

$$E(\pi^*_{i,PS}(N_i)) = \frac{(a+\bar{\theta})^2}{2^{2i}b} + \sum_{j=0}^{j=i-1} \frac{\sigma_\epsilon^2\sigma_\theta^4}{2^{2i-2j}b(\sigma_\epsilon^2 + j\sigma_\theta^2)(\sigma_\epsilon^2 + (j+1)\sigma_\theta^2)}. \tag{6}$$

Similarly, using Lemma A5-i) again, we can get

$$E(\pi^*_{i,PS}(N)) = \frac{E(\pi^*_{i,PS}(N_i))}{2^{n-i}} \tag{7}$$

iteratively. The result follows after plugging (6) into (7).

iv) Using Lemma 4,

$$Q^*_{FS}(N) = \sum_{i=1}^{i=n} q^*_{i,FS}(N) = \frac{a + E(\theta|s_1, s_2, ..., s_n)}{b}(1 - \frac{1}{2^n}). \tag{8}$$

One can further derive that

$$\begin{aligned}
E(\pi^*_{i,FS}(N)) &= E((a+\theta - bQ^*_{FS}(N))q^*_{i,FS}(N)) \\
&= E(a+\theta - bQ^*_{FS}(N))E(q^*_{i,FS}(N)) + Cov(a+\theta - bQ^*_{FS}(N), q^*_{i,FS}(N)) \\
&= \tfrac{(a+\bar{\theta})^2}{2^{n+i}b} + Cov(\theta - \tfrac{(2^n-1)E(\theta|s_1,s_2,...,s_n)}{2^n}, \tfrac{E(\theta|s_1,s_2,...,s_n)}{2^i b}) \\
&= \tfrac{(a+\bar{\theta})^2}{2^{n+i}b} + \tfrac{Cov(\theta, E(\theta|s_1,s_2,...,s_n))}{2^i b} - \tfrac{(2^n-1)Cov(E(\theta|s_1,s_2,...,s_n),E(\theta|s_1,s_2,...,s_n))}{2^{n+i}b} \\
&= \tfrac{(a+\bar{\theta})^2}{2^{n+i}b} + \tfrac{n\sigma_\theta^4}{2^{n+i}b(\sigma_\epsilon^2 + n\sigma_\theta^2)},
\end{aligned}$$

where the first and the second equalities are by definition. We insert the value of $Q^*_{FS}(N)$ from (8) in the third equality. We use the properties of covariance in the fourth equality. In the last step, we use Lemma A4-ii). □

Lemma A6. *The expected equilibrium total profits, total surpluses and consumer surpluses in the partial and full sharing games are, respectively, given by*

i) For each $j = \{PS, FS\}$, $E(\Pi^*_j(N)) = \sum_{i\in N} E(\pi^*_{i,j}(N))$ *and* $E(CS^*_j(N)) = E(TS^*_j(N)) - E(\Pi^*_j(N))$.

ii) $E(TS^*_{PS}(N)) = \underbrace{\frac{2^n-1}{2^n}(2 - \frac{2^n-1}{2^n})\frac{(a+\bar{\theta})^2}{2b}}_{=E(TS^*_{1,PS})} + \underbrace{\sum_{i=1}^{i=n} \frac{3i\sigma_\theta^4}{2^{2n-2i+3}b(\sigma_\epsilon^2 + i\sigma_\theta^2)}}_{=E(TS^*_{2,PS})}.$

iii) $E(TS^*_{FS}(N)) = \underbrace{\frac{2^n-1}{2^n}(2 - \frac{2^n-1}{2^n})\frac{(a+\bar{\theta})^2}{2b}}_{=E(TS^*_{1,FS})} + \underbrace{\frac{2^n-1}{2^n}(2 - \frac{2^n-1}{2^n})\frac{n\sigma_\theta^4}{2b(\sigma_\epsilon^2 + n\sigma_\theta^2)}}_{=E(TS^*_{2,FS})}.$

Proof of Lemma A6: **i)** The claims follow by definition of total profits and consumer surplus.

ii) We can derive that

$$
\begin{aligned}
&E(TS^*_{PS}) = (a+\bar{\theta})E(Q^*_{PS}) - b(E(Q^*_{PS}))^2/2 + \mathrm{Cov}(\theta, Q^*_{PS}) - b\mathrm{Var}(Q^*_{PS})/2 \\
&= \tfrac{2^n-1}{2^n}(2-\tfrac{2^n-1}{2^n})\tfrac{(a+\bar{\theta})^2}{2b} + \mathrm{Cov}(\theta, \textstyle\sum_{i=1}^{i=n}\tfrac{E(\theta|s_1,s_2,\ldots,s_i)}{b2^{n-i+1}}) - b\mathrm{Var}(\textstyle\sum_{i=1}^{i=n}\tfrac{E(\theta|s_1,s_2,\ldots,s_i)}{b2^{n-i+1}})/2 \\
&= \tfrac{2^n-1}{2^n}(2-\tfrac{2^n-1}{2^n})\tfrac{(a+\bar{\theta})^2}{2b} + \textstyle\sum_{i=1}^{i=n}\tfrac{i\sigma_\theta^4}{2^{n-i+1}b(\sigma_\epsilon^2+i\sigma_\theta^2)} - \textstyle\sum_{i=1}^{i=n}\tfrac{(2^{n-i+2}-3)i\sigma_\theta^4}{2^{2n-2i+3}b(\sigma_\epsilon^2+i\sigma_\theta^2)} \\
&= \tfrac{2^n-1}{2^n}(2-\tfrac{2^n-1}{2^n})\tfrac{(a+\bar{\theta})^2}{2b} + \textstyle\sum_{i=1}^{i=n}\tfrac{3i\sigma_\theta^4}{2^{2n-2i+3}b(\sigma_\epsilon^2+i\sigma_\theta^2)},
\end{aligned}
$$

where the first equality is by definition. Note also that $Q^*_{PS} = \frac{a}{b}(1-\frac{1}{2^n}) + \sum_{i=1}^{i=n}\frac{E(\theta|s_1,s_2,\ldots,s_i)}{b2^{n-i+1}}$ by (4) and $E(Q^*_{PS}) = (a+\bar{\theta})(1-\frac{1}{2^n})$ by Corollary 2. For the second equality, we plugged in these values of total quantity and its expected value. The third inequality follows by repeatedly using Lemma A4-*ii*) and the properties of covariance and variance. The last equality is a simplification.

iii) By using (8), Corollary 2, and Lemma A4-*ii*), similar computations show that

$$
\begin{aligned}
&E(TS^*_{FS}) = (a+\bar{\theta})E(Q^*_{FS}) - b(E(Q^*_{FS}))^2/2 + \mathrm{Cov}(\theta, Q^*_{FS}) - b\mathrm{Var}(Q^*_{FS})/2 \\
&= \tfrac{2^n-1}{2^n}(2-\tfrac{2^n-1}{2^n})\tfrac{(a+\bar{\theta})^2}{2b} + \mathrm{Cov}(\theta, \tfrac{(2^n-1)E(\theta|s_1,s_2,\ldots,s_n)}{b2^n}) - b\mathrm{Var}(\tfrac{(2^n-1)E(\theta|s_1,s_2,\ldots,s_n)}{b2^n})/2 \\
&= \tfrac{2^n-1}{2^n}(2-\tfrac{2^n-1}{2^n})\tfrac{(a+\bar{\theta})^2}{2b} + \tfrac{(2^n-1)n\sigma_\theta^4}{2^n b(\sigma_\epsilon^2+n\sigma_\theta^2)} - \tfrac{(2^n-1)^2 n\sigma_\theta^4}{2^{2n+1}b(\sigma_\epsilon^2+n\sigma_\theta^2)} \\
&= \tfrac{2^n-1}{2^n}(2-\tfrac{2^n-1}{2^n})\tfrac{(a+\bar{\theta})^2}{2b} + \tfrac{2^n-1}{2^n}(2-\tfrac{2^n-1}{2^n})\tfrac{n\sigma_\theta^4}{2b(\sigma_\epsilon^2+n\sigma_\theta^2)}.
\end{aligned}
$$

□

Lemma A7. *We claim that for $k \geq 2$, $W(N_k) > Y(N_k) > Z(N_k)$, where*

$$
\begin{aligned}
W(N_k) &= \frac{\sigma_\theta^4\left(\sigma_\theta^2+\sigma_\epsilon^2\right)\left((2+k(2+k(k+2)))\sigma_\theta^4 + 4(2+k(k+2))\sigma_\theta^2\sigma_\epsilon^2 + 4(k+2)\sigma_\epsilon^4\right)}{2b\left(2\sigma_\epsilon^2+(k+1)\sigma_\theta^2\right)^2\left(2\sigma_\epsilon^2+(k+2)\sigma_\theta^2\right)^2}, \\
Y(N_k) &= \sum_{j=0}^{j=k}\frac{\sigma_\epsilon^2\sigma_\theta^4}{b2^{2(k-j+1)}\left(\sigma_\epsilon^2+j\sigma_\theta^2\right)\left(\sigma_\epsilon^2+(j+1)\sigma_\theta^2\right)}, \\
Z(N_k) &= \frac{\sigma_\theta^4((2^{k+1}+k-1)\sigma_\epsilon^2+k(k+1)\sigma_\theta^2)}{b2^{2k+2}\left(\sigma_\epsilon^2+k\sigma_\theta^2\right)\left(\sigma_\epsilon^2+(k+1)\sigma_\theta^2\right)}.
\end{aligned}
$$

Proof of Lemma A7: For $k=2$,

$$
W(N_2) - Y(N_2) = \frac{\sigma_\theta^4\left(44\sigma_\epsilon^{10} + 408\sigma_\theta^2\sigma_\epsilon^8 + 1295\sigma_\theta^4\sigma_\epsilon^6 + 1917\sigma_\theta^6\sigma_\epsilon^4 + 1400\sigma_\theta^8\sigma_\epsilon^2 + 420\sigma_\theta^{10}\right)}{64b\left(\sigma_\epsilon^2+\sigma_\theta^2\right)\left(\sigma_\epsilon^2+3\sigma_\theta^2\right)\left(\sigma_\epsilon^2+2\sigma_\theta^2\right)^2(2\sigma_\epsilon^2+3\sigma_\theta^2)^2} > 0
$$

and

$$
Y(N_2) - Z(N_2) = \frac{3\sigma_\epsilon^2\sigma_\theta^4(2\sigma_\epsilon^2+3\sigma_\theta^2)}{32b(\sigma_\epsilon^2+\sigma_\theta^2)(\sigma_\epsilon^2+2\sigma_\theta^2)(\sigma_\epsilon^2+3\sigma_\theta^2)} > 0.
$$

Now suppose that $W(N_k) > Y(N_k) > Z(N_k)$ for some $k \geq 2$. We want to show that $W(N_{k+1}) > Y(N_{k+1}) > Z(N_{k+1})$. We have that

$$
\begin{gathered}
W(N_{k+1}) - W(N_k)/4 > b\sigma_\epsilon^2 w_1(N_k)w_1(N_{k+1}) > Z(N_{k+1}) - Z(N_k)/4 \Rightarrow \\
W(N_{k+1}) - \tfrac{W(N_k)}{4} + \tfrac{W(N_k)}{4} > \tfrac{Y(N_k)}{4} + b\sigma_\epsilon^2 w_1(N_k)w_1(N_{k+1}) > Z(N_{k+1}) - \tfrac{Z(N_k)}{4} + \tfrac{Z(N_k)}{4} \\
\Rightarrow W(N_{k+1}) > Y(N_{k+1}) > Z(N_{k+1}),
\end{gathered}
$$

where the first implication follows by our initial supposition and the last implication is by definition. The inequalities in the first line should hold because at $k > 2$,

$$\begin{aligned}&b\sigma_\epsilon^2 w_1(N_k)w_1(N_{k+1}) - Z(N_{k+1}) + \frac{Z(N_k)}{4}\\&= \frac{\sigma_\epsilon^2\sigma_\theta^4((2^{k+1}(2^{k+1}-1)-1)\sigma_\epsilon^2 + (4(2^k-1)+k(2^{k+1}(2^{k+1}-1)-1))\sigma_\theta^2)}{b2^{2k+4}(\sigma_\epsilon^2+k\sigma_\theta^2)(\sigma_\epsilon^2+(k+1)\sigma_\theta^2)(\sigma_\epsilon^2+(k+2)\sigma_\theta^2)} > 0\end{aligned}$$

and

$$\begin{aligned}&W(N_{k+1}) - W(N_k)/4 - b\sigma_\epsilon^2 w_1(N_k)w_1(N_{k+1})\\&= \frac{\sigma_\theta^4\big(16(3k+2)\sigma_\epsilon^{14}+16(16+3k(11+4k))\sigma_\theta^2\sigma_\epsilon^{12}+f_1\sigma_\theta^4\sigma_\epsilon^{10}+f_2\sigma_\theta^6\sigma_\epsilon^8+f_3\sigma_\theta^8\sigma_\epsilon^6+f_4\sigma_\theta^{10}\sigma_\epsilon^4+f_5\sigma_\theta^{12}\sigma_\epsilon^2+f_6\sigma_\theta^{14}\big)}{8b(\sigma_\epsilon^2+\sigma_\theta^2(k+1))(2\sigma_\epsilon^2+\sigma_\theta^2(k+1))^2(\sigma_\epsilon^2+\sigma_\theta^2(k+2))(2\sigma_\epsilon^2+\sigma_\theta^2(k+2))^2(2\sigma_\epsilon^2+\sigma_\theta^2(k+3))^2} > 0,\end{aligned}$$

where f_i, $i = 1, 2, ..., 6$ are positive because

$$\begin{aligned}f_1 &= 4(196+3k(175+2k(61+13k))),\\f_2 &= 4(312+k(1042+k(1059+k(439+66k)))),\\f_3 &= 1138+k(4642+k(6196+3k(1255+k(364+41k)))),\\f_4 &= 2(k+2)(150+k(670+3k(302+k(179+5k(10+k))))),\\f_5 &= 170+k(1040+k(2102+3k(693+k(383+k(120+k(k+19)))))),\\f_6 &= (k+1)(k+2)(10+k(62+k(11+3k)(8+k(k+3)))).\end{aligned}$$

Altogether, it should hold that $W(N_{k+1}) > Y(N_{k+1}) > Z(N_{k+1})$, as desired. □

1.4 Proofs of Lemmas 1-4 and Propositions 1-4 of the base model

Proofs of Lemma 1 and Lemma 2: The proofs follow from Cumbul (2021). □

Proof of Proposition 1: **i)** By Lemma A1-*i*) and Lemma A1-*ii*),

$$E(Q_{NS}^*(N_{k+1})) - E(Q_{NS}^*(N_k)) = \frac{\sigma_\theta^2 E(q_{k,NS}^*(N_k))}{2(\sigma_\epsilon^2+\sigma_\theta^2(k+1))}.$$

As $E(q_{k,NS}^*(N_k)) > 0$ by Lemma A1-*ii*), $E(Q_{NS}^*(N_{k+1})) > E(Q_{NS}^*(N_k))$. As $p = a+\theta-bQ$, it also follows that $E(p_{NS}^*(N_{k+1})) < E(p_{NS}^*(N_k))$.

Finally, using Lemma A3-*i*), $E(TS_{NS}^*(N_k)) = E(TS_{1,NS}^*(N_k)) + E(TS_{2,NS}^*(N_k))$. By definition,

$$E(TS_{1,NS}^*(N_{k+1})) - E(TS_{1,NS}^*(N_k)) = \frac{(a+\bar{\theta})^2(1-w_2^2(N_k))\prod_{j=1}^{j=k-1} w_2^2(N_j)}{8b},$$

which is positive because $1 - w_2^2(N_k) = \frac{\sigma_\theta^2(4\sigma_\epsilon^2+(3+4k)\sigma_\theta^2)}{4(\sigma_\epsilon^2+\sigma_\theta^2(k+1))^2} > 0$. Using Lemma A3-*i*),

$$E(TS_{2,NS}^*(N_{k+1})) - E(TS_{2,NS}^*(N_k)) = w_3(N_k) - (1-w_2^2(N_k))E(TS_{2,NS}^*(N_k)). \tag{9}$$

As $1 - w_2^2(N_k) > 0$, the difference in (9) is decreasing in $E(TS_{2,NS}^*(N_k))$. When $E(TS_{2,NS}^*(N_k))$

gets its upper bound of $\Theta(N_k)$ by Lemma A3-*iii*), (9) reduces to

$$E(TS^*_{2,NS}(N_{k+1}) - TS^*_{2,NS}(N_k)) = \frac{\sigma_\theta^4(12\sigma_\epsilon^6 + 8(3+2k)\sigma_\epsilon^4\sigma_\theta^2 + (15+k(17+3k))\sigma_\epsilon^2\sigma_\theta^4 + (3+4k)\sigma_\theta^6)}{8b(\sigma_\epsilon^2+\sigma_\theta^2(k+1))^2(2\sigma_\epsilon^2+\sigma_\theta^2(k+1))^2},$$

which is positive. Altogether, $E(TS^*_{NS}(N_{k+1})) > E(TS^*_{NS}(N_k))$, as claimed.

ii) As $n \to \infty$, the expected PRE total output level converges to its perfectly competitive level of $(a+\bar{\theta})/b$, which is smaller than the monopoly output level of $(a+\bar{\theta})/(2b)$ by part *iii*). As the first best level of total profit occurs at the monopoly profit level and the expected total profit is concave in $E(Q)$, the expected PRE total profit does not converge to its first best level.

Now take any $i \in N_k \setminus k$. Using Lemma A2-*i*) and noting that $1 - w_2(N_k) = \sigma_\theta^2/(2(\sigma_\epsilon^2 + \sigma_\theta^2(k+1))) > 0$ and $E(\pi^*_{i,NS}(N_k)) > 0$,

$$E(\pi^*_{i,NS}(N_k)) - E(\pi^*_{i,NS}(N_{k+1})) = (1 - w_2(N_k))E(\pi^*_{i,NS}(N_k)) > 0. \tag{10}$$

Moreover, iterative calculations show that

$$\lim_{l\to\infty} E(\pi_i^*(N_{k+l})) = E(\pi_i^*(N_k)) \lim_{l\to\infty} \prod_{j=k}^{j=k+l-1} w_2(N_j) = 0$$

because $\lim_{n\to\infty}(\prod_{k=1}^{k=n-1} w_2(N_k)) = 0$ by our next proof in part *iii*). Similarly, $E(\pi^*_{k,NS}(N_k)) - E(\pi^*_{k,NS}(N_{k+1})) = (1 - 2bw_1(N_k)w_2(N_k))E(\pi^*_{k,NS}(N_k)) > 0$ because

$$1 - 2bw_1(N_k)w_2(N_k) = \frac{2\sigma_\epsilon^4 + (1+2k(1+k))\sigma_\theta^4 + 2\sigma_\epsilon^2\sigma_\theta^2(2k+1)}{2(\sigma_\epsilon^2+\sigma_\theta^2(k+1))^2} > 0.$$

In a similar manner, using Lemma A2 iteratively,

$$\lim_{l\to\infty} E(\pi_k(N_{k+l})) = 2bw_1(N_k)E(\pi_k(N_k)) \lim_{l\to\infty} \prod_{j=k}^{j=k+l-1} w_2(N_j) = 0.$$

iii) *(Total output and price)* By Lemma A1-*ii*),

$$\lim_{k\to\infty} E(q^*_{k,NS}(N_k)) = \frac{(a+\bar{\theta})}{2b} \times \lim_{k\to\infty}\left(\prod_{j=1}^{j=k-1} \frac{2\sigma_\epsilon^2 + (2j+1)\sigma_\theta^2}{2(\sigma_\epsilon^2 + (j+1)\sigma_\theta^2)}\right). \tag{11}$$

As $\sigma_\epsilon^2, \sigma_\theta^2 \in \mathbb{R}_+$, there exists $f \in \mathbb{N}$ such that $f\sigma_\theta^2 > \sigma_\epsilon^2$ by the Archimedean property. Moreover, $\partial w_2(N_j)/\partial\sigma_\epsilon^2 = \sigma_\theta^2/(2(\sigma_\epsilon^2 + \sigma_\theta^2(j+1))^2) > 0$. Therefore, replacing σ_ϵ^2 with $f\sigma_\theta^2$ in the second multiplicative term of (11) increases it and it becomes

$$\lim_{y\to\infty} \frac{2 \times 4 \times 6 \times ... \times (2f+2)}{1 \times 3 \times 5 \times ... \times (2f+1)} \times \frac{(2y-1)!!}{(2y)!!}, \tag{12}$$

where $y = f + k$, $(2y-1)!! = \prod_{i=1}^{i=y}(2i-1)$ and $(2y)!! = \prod_{i=1}^{i=y}(2i)$. As f is a finite natural number, the first multiplicative term in (12) is finite, greater than 1, and independent of k. To conclude the proof, it is then sufficient to show that $\lim_{y\to\infty}\frac{(2y-1)!!}{(2y)!!} = 0$. Note that

$$\lim_{y\to\infty}\frac{(2y-1)!!}{(2y)!!} = \lim_{y\to\infty}\frac{(2y)!}{((2y)!!)^2} = \lim_{y\to\infty}\frac{(2y)!}{4^y * (y!)^2}. \tag{13}$$

Using Stirling's approximation, $y! \approx (\frac{y}{e})^y\sqrt{2\pi y}$ and $(2y)! \approx (\frac{2y}{e})^{2y}\sqrt{4\pi y}$. So, (13) simplifies to

$$\lim_{y\to\infty}\frac{(2y-1)!!}{(2y)!!} = \lim_{y\to\infty}\frac{1}{\sqrt{\pi y}} = 0. \tag{14}$$

As $E(Q^*_{NS}(N_k)) = (a+\bar{\theta})/b - E(q^*_{k,NS}(N_k))$ by Lemma A1-i), $\lim_{k\to\infty} E(Q^*_{NS}(N_k)) = (a + \bar{\theta})/b - \lim_{k\to\infty} E(q^*_{k,NS}(N_k)) = (a+\bar{\theta})/b$. Moreover, as $p = a + \theta - bQ$,

$$\lim_{k\to\infty} E(p^*_{NS}(N_k)) = a + \bar{\theta} - b\lim_{k\to\infty} E(Q^*_{NS}(N_k)) = 0. \tag{15}$$

(Total Surplus) Recall by Lemma *A3-i*) that $E(TS^*_{NS}(N)) = E(TS^*_{1,NS}(N)) + E(TS^*_{2,NS}(N))$. As $\lim_{n\to\infty}(\prod_{k=1}^{k=n-1} w_2(N_k)) = 0$ by the above proof,

$$\lim_{n\to\infty} E(TS^*_{1,NS}(N)) = \frac{(a+\bar{\theta})^2(4 - \lim_{n\to\infty}\prod_{k=1}^{n-1} w_2^2(N_k))}{8b} = \frac{(a+\bar{\theta})^2}{2b}.$$

Moreover, as $E(TS^*_{2,NS}(N)) > \chi(N)$ for any finite n by Lemma A4-i), for all $c \in \mathbb{N}_+$,

$$\lim_{n\to\infty} E(TS^*_{2,NS}(N)) \geq \lim_{n\to\infty}\chi(N) = \frac{c\sigma_\theta^2}{2b(c+1)}. \tag{16}$$

Similarly, $E(TS^*_{2,NS}(N)) < \Theta(N) = \frac{n\sigma_\theta^4(3\sigma_\epsilon^2+\sigma_\theta^2(n+2))}{2b(2\sigma_\epsilon^2+\sigma_\theta^2(n+1))^2}$ for any finite n by Lemma A3-iii). Thus,

$$\lim_{n\to\infty} E(TS^*_{2,NS}(N)) \leq \lim_{n\to\infty}\Theta(N) = \frac{\sigma_\theta^2}{2b} \tag{17}$$

by continuity. Combining (16) and (17),

$$\frac{\sigma_\theta^2}{2b} \geq \lim_{n\to\infty} E(TS^*_{2,NS}(N)) \geq \frac{c\sigma_\theta^2}{2b(c+1)}.$$

Moreover, for sufficiently high natural number c, $\frac{c\sigma_\theta^2}{2b(c+1)} \approx \frac{\sigma_\theta^2}{2b}$. Therefore, by the Sandwich theorem, $\lim_{n\to\infty} E(TS^*_{2,NS}(N)) \approx \frac{\sigma_\theta^2}{2b}$, as desired. □

Proof of Lemma 3: Using the Cournot equilibrium quantities provided by Vives (2011, Prop S1), the expected equilibrium total surplus equals

$$E(\hat{TS}_C) = \frac{n(2\lambda + b(n+2))(a+\bar{\theta})^2}{2(2\lambda + b(n+1))^2} + \frac{n\sigma_\theta^4(b(3\sigma_\epsilon^2 + (n+2)\sigma_\theta^2) + 2\lambda(\sigma_\epsilon^2+\sigma_\theta^2))}{2(2\lambda(\sigma_\epsilon^2+\sigma_\theta^2) + b(2\sigma_\epsilon^2 + \sigma_\theta^2(n+1)))^2}. \tag{18}$$

By simple algebra, $\lim_{n\to\infty} E(\hat{TS}_C) = ((a+\bar{\theta})^2 + \sigma_\theta^2)/(2b)$ at any $\lambda \geq 0$. □

Proof of Proposition 2: **i)** (Expected Profits and $\sigma_\epsilon^2, \sigma_\theta^2$) First, for firm 1, we can show that

$$\begin{aligned}
&E(\pi^*_{1,NS}(N)) = E(\pi^*_{1,NS}(N_1)) \times 2bw_1(N_1)\prod_{k=1}^{k=n-1} w_2(N_k)\\
&= (\tfrac{(a+\bar{\theta})^2}{4b} + \tfrac{\sigma_\theta^4}{4b(\sigma_\epsilon^2+\sigma_\theta^2)})\tfrac{\sigma_\theta^2}{\sigma_\epsilon^2+2\sigma_\theta^2}\prod_{k=1}^{k=n-1}\tfrac{(2\sigma_\epsilon^2+(2k+1)\sigma_\theta^2)}{2(\sigma_\epsilon^2+(k+1)\sigma_\theta^2)}\\
&= \tfrac{(a+\bar{\theta})^2}{4b}\underbrace{\frac{\sigma_\theta^2(2\sigma_\epsilon^2+3\sigma_\theta^2)}{(2\sigma_\epsilon^2+4\sigma_\theta^2)(\sigma_\epsilon^2+n\sigma_\theta^2)}\prod_{k=2}^{k=n-1} w_4(N_k)}_{=\varphi(N)} + \tfrac{\sigma_\theta^6}{4b(\sigma_\epsilon^2+2\sigma_\theta^2)(\sigma_\epsilon^2+n\sigma_\theta^2)}\prod_{k=1}^{k=n-1} w_4(N_k).
\end{aligned}$$

The first equality is due to Lemma A2-*iv*) for $i=1$. For the second equality, we substitute the values of $w_1(N_k)$, $w_2(N_k)$, and $E(\pi^*_{1,NS}(N_1))$, where the latter is from (5). The third equality follows after rearranging the equation.

Note that $\frac{\partial w_4(N_k)}{\partial\sigma_\epsilon^2} = -\frac{\sigma_\theta^2}{2(\sigma_\epsilon^2+k\sigma_\theta^2)^2} < 0$. Thus, it is sufficient to show that $\frac{\partial\varphi(N)}{\partial\sigma_\epsilon^2} < 0$ from the above derivation. For $n = 2,3,4$,

$$\begin{aligned}
\frac{\partial\varphi(N_2)}{\partial\sigma_\epsilon^2} &= -\frac{\sigma_\theta^2(\sigma_\epsilon^2+\sigma_\theta^2)}{(\sigma_\epsilon^2+2\sigma_\theta^2)^3} < 0,\\
\frac{\partial\varphi(N_3)}{\partial\sigma_\epsilon^2} &= -\frac{\sigma_\theta^2(4\sigma_\epsilon^6+24\sigma_\epsilon^4\sigma_\theta^2+45\sigma_\epsilon^2\sigma_\theta^4+24\sigma_\theta^6)}{4(\sigma_\epsilon^2+2\sigma_\theta^2)^3(\sigma_\epsilon^2+3\sigma_\theta^2)^2} < 0,\\
\frac{\partial\varphi(N_4)}{\partial\sigma_\epsilon^2} &= -\frac{\sigma_\theta^2(8\sigma_\epsilon^{10}+104\sigma_\epsilon^8\sigma_\theta^2+526\sigma_\epsilon^6\sigma_\theta^4+1276\sigma_\epsilon^4\sigma_\theta^6+1449\sigma_\epsilon^2\sigma_\theta^8+582\sigma_\theta^{10})}{8(\sigma_\epsilon^2+2\sigma_\theta^2)^3(\sigma_\epsilon^2+3\sigma_\theta^2)^2(\sigma_\epsilon^2+4\sigma_\theta^2)^2} < 0.
\end{aligned}$$

Now suppose $n > 4$. Rearranging $\varphi(N)$ yields that

$$\varphi(N) = \underbrace{\frac{(2\sigma_\epsilon^2+3\sigma_\theta^2)\prod_{k=2}^{k=4} w_4(N_k)}{2\sigma_\epsilon^2+4\sigma_\theta^2}}_{=\vartheta}\frac{\sigma_\theta^2}{\sigma_\epsilon^2+n\sigma_\theta^2}\prod_{k=5}^{k=n-1} w_4(N_k).$$

We know that the terms except ϑ is decreasing in σ_ϵ^2. We can also show that

$$\frac{\partial\vartheta}{\partial\sigma_\epsilon^2} = -\frac{16\sigma_\epsilon^{10}\sigma_\theta^2+208\sigma_\epsilon^8\sigma_\theta^4+1016\sigma_\epsilon^6\sigma_\theta^6+2228\sigma_\epsilon^4\sigma_\theta^8+1929\sigma_\epsilon^2\sigma_\theta^{10}+198\sigma_\theta^{12}}{16(\sigma_\epsilon^2+2\sigma_\theta^2)^3(\sigma_\epsilon^2+3\sigma_\theta^2)^2(\sigma_\epsilon^2+4\sigma_\theta^2)^2} < 0.$$

Thus, we proved that $\frac{\partial E(\pi^*_{1,NS}(N))}{\partial\sigma_\epsilon^2} < 0$. In order to see $\frac{\partial E(\pi^*_{1,NS}(N))}{\partial\sigma_\theta^2} > 0$, we note that $\frac{\partial w_4(N_k)}{\partial\sigma_\theta^2} = -\frac{\partial w_4(N_k)}{\partial\sigma_\epsilon^2}\frac{\sigma_\epsilon^2}{\sigma_\theta^2} > 0$, $\frac{\partial\varphi(N_i)}{\partial\sigma_\theta^2} = -\frac{\partial\varphi(N_i)}{\partial\sigma_\epsilon^2}\frac{\sigma_\epsilon^2}{\sigma_\theta^2} > 0$ for $i = 2,3,4$, and $\frac{\partial\vartheta}{\partial\sigma_\theta^2} = -\frac{\partial\vartheta}{\partial\sigma_\epsilon^2}\frac{\sigma_\epsilon^2}{\sigma_\theta^2} > 0$. Furthermore, for $h_1 = \frac{\sigma_\theta^6}{4b(\sigma_\epsilon^2+2\sigma_\theta^2)(\sigma_\epsilon^2+n\sigma_\theta^2)}$ and $h_2 = \frac{\sigma_\theta^2}{\sigma_\epsilon^2+n\sigma_\theta^2}$, $\frac{\partial h_1}{\partial\sigma_\theta^2} = \frac{\sigma_\theta^4(3\sigma_\epsilon^4+2(n+2)\sigma_\epsilon^2\sigma_\theta^2+2n\sigma_\theta^4)}{4b(\sigma_\epsilon^2+2\sigma_\theta^2)^2(\sigma_\epsilon^2+n\sigma_\theta^2)^2} > 0$ and $\frac{\partial h_2}{\partial\sigma_\theta^2} = \frac{\sigma_\epsilon^2}{(\sigma_\epsilon^2+n\sigma_\theta^2)^2} > 0$. These computations are sufficient to conclude that $\frac{\partial E(\pi^*_{1,NS}(N))}{\partial\sigma_\theta^2} > 0$.

Lastly, using Lemma A2-*iii*) and Lemma A2-*iv*),

$$\begin{aligned}
\frac{\partial E(\pi^*_{2,NS}(N_2))}{\partial\sigma_\epsilon^2} &= \frac{\sigma_\theta^2(2\sigma_\epsilon^2+3\sigma_\theta^2)(a+\bar{\theta})^2}{8b(\sigma_\epsilon^2+2\sigma_\theta^2)^3} + \frac{\sigma_\theta^4(-8\sigma_\epsilon^6-24\sigma_\epsilon^4\sigma_\theta^2-15\sigma_\epsilon^2\sigma_\theta^4+4\sigma_\theta^6)}{16b(\sigma_\epsilon^2+\sigma_\theta^2)^2(\sigma_\epsilon^2+2\sigma_\theta^2)^3},\\
\frac{\partial E(\pi^*_{2,NS}(N_3))}{\partial\sigma_\epsilon^2} &= \frac{\sigma_\theta^2(2\sigma_\epsilon^2+3\sigma_\theta^2)(-2\sigma_\epsilon^6-9\sigma_\epsilon^4\sigma_\theta^2-9\sigma_\epsilon^2\sigma_\theta^4+3\sigma_\theta^6)(a+\bar{\theta})^2}{16b(\sigma_\epsilon^2+2\sigma_\theta^2)^3(\sigma_\epsilon^2+3\sigma_\theta^2)^3}\\
&\quad+\frac{-\sigma_\theta^6(32\sigma_\epsilon^{10}+256\sigma_\epsilon^8\sigma_\theta^2+760\sigma_\epsilon^6\sigma_\theta^4+991\sigma_\epsilon^4\sigma_\theta^6+485\sigma_\epsilon^2\sigma_\theta^8+12\sigma_\theta^{10})}{32b(\sigma_\epsilon^2+\sigma_\theta^2)^2(\sigma_\epsilon^2+2\sigma_\theta^2)^3(\sigma_\epsilon^2+3\sigma_\theta^2)^3},
\end{aligned}$$

where both partial derivatives may be positive or negative. A similar conclusion is valid for the relationship of these profits with σ_θ^2.

ii) (Total Output and $\sigma_\epsilon^2, \sigma_\theta^2$) It follows from Lemma A1-i) that $\frac{\partial E(Q_{NS}^*(N))}{\partial \sigma_j^2} = -\frac{\partial E(q_{n,NS}^*(N))}{\partial \sigma_j^2}$ for $j = \theta, \epsilon$. By Lemma A1-iii), $\frac{\partial E(q_{n,NS}^*(N))}{\partial \sigma_\epsilon^2} > 0$ and $\frac{\partial E(q_{n,NS}^*(N))}{\partial \sigma_\theta^2} < 0$, and therefore, $\frac{\partial E(Q_{NS}^*(N))}{\partial \sigma_\epsilon^2} < 0$ and $\frac{\partial E(Q_{NS}^*(N))}{\partial \sigma_\theta^2} > 0$.

iii) (Total Surplus and σ_ϵ^2) We claim that $\frac{\partial E(TS_{NS}^*(N))}{\partial \sigma_\epsilon^2} < 0$. Using Lemma A3, $E(TS_{NS}^*(N)) = E(TS_{1,NS}^*(N)) + E(TS_{2,NS}^*(N))$. The proof is done in two steps.

STEP 1: We first claim that the total derivative $\frac{dE(TS_{1,NS}^*(N))}{d\sigma_\epsilon^2} < 0$. This derivative equals

$$\begin{aligned}\frac{dE(TS_{1,NS}^*(N))}{d\sigma_\epsilon^2} &= \frac{\partial E(TS_{1,NS}^*(N))}{\partial E(Q_{NS}^*(N))} \times \frac{\partial E(Q_{NS}^*(N))}{\partial \sigma_\epsilon^2} \\ &= (a + \bar{\theta} - bE(Q_{NS}^*(N))) \times \frac{\partial E(Q_{NS}^*(N))}{\partial \sigma_\epsilon^2}.\end{aligned}$$

As $E(p_{NS}^*(N)) = a + \bar{\theta} - bE(Q_{NS}^*(N)) = bE(q_{NS}^*(N)) > 0$ by Lemma A1, the claim follows after noting that $\partial E(Q_{NS}^*(N))/\partial \sigma_\epsilon^2 < 0$ by part i).

STEP 2: We finally claim that $\partial E(TS_{2,NS}^*(N_k))/\partial \sigma_\epsilon^2 < 0$. For $k = 2$,

$$\frac{\partial E(TS_{2,NS}(N_2))}{\partial \sigma_\epsilon^2} = -\frac{\sigma_\theta^4(24\sigma_\epsilon^6 + 104\sigma_\epsilon^4\sigma_\theta^2 + 145\sigma_\epsilon^2\sigma_\theta^4 + 68\sigma_\theta^6)}{32b(\sigma_\epsilon^2 + \sigma_\theta^2)^2(\sigma_\epsilon^2 + 2\sigma_\theta^2)^3} < 0$$

by Lemma A3-i). Now suppose that $\frac{\partial E(TS_{2,NS}(N_k))}{\partial \sigma_\epsilon^2} < 0$ for some $k < n$. It must then be shown that $\frac{\partial E(TS_{2,NS}(N_{k+1}))}{\partial \sigma_\epsilon^2} < 0$ holds. Remark that $E(TS_{2,NS}(N_{k+1})) = w_2^2(N_k) \times E(TS_{2,NS}(N_k)) + w_3(N_k)$ by Lemma A3-i). Therefore,

$$\frac{\partial E(TS_{2,NS}(N_{k+1}))}{\partial \sigma_\epsilon^2} = 2w_2(N_k)\frac{\partial w_2(N_k)}{\partial \sigma_\epsilon^2}E(TS_{2,NS}(N_k)) + \frac{\partial w_3(N_k)}{\partial \sigma_\epsilon^2} + w_2^2(N_k)\frac{\partial E(TS_{2,NS}(N_k))}{\partial \sigma_\epsilon^2}.$$

By the initial supposition, $\frac{\partial E(TS_{2,NS}(N_k))}{\partial \sigma_\epsilon^2} < 0$ holds, which implies $w_2^2(N_k) \times \frac{\partial E(TS_{2,NS}(N_k))}{\partial \sigma_\epsilon^2} < 0$. Hence, it will be sufficient to show that the sign of

$$\mho(N_k) = 2w_2(N_k)\frac{\partial w_2(N_k)}{\partial \sigma_\epsilon^2} \times E(TS_{2,NS}(N_k)) + \frac{\partial w_3(N_k)}{\partial \sigma_\epsilon^2}$$

is negative to conclude that $\frac{\partial E(TS_{2,NS}(N_{k+1}))}{\partial \sigma_\epsilon^2} < 0$. As $w_2(N_k) > 0$ and $\frac{\partial w_2(N_k)}{\partial \sigma_\epsilon^2} = \frac{\sigma_\theta^2}{2(\sigma_\epsilon^2 + \sigma_\theta^2(k+1))^2} > 0$, $\mho(N_k)$ is increasing in $E(TS_{2,NS}(N_k))$. Therefore $\mho(N_k)$ is maximized when $E(TS_{2,NS}(N_k))$ gets its upper bound of $\Theta(N_k)$ by Lemma A3-ii). In this case, it reduces to

$$\mho(N_k) = -\frac{\sigma_\theta^4\left(12\sigma_\epsilon^6 + 4(6+5k)\sigma_\epsilon^4\sigma_\theta^2 + (15 + k(24+7k))\sigma_\epsilon^2\sigma_\theta^4 + (3 + k(7 + k(3+k)))\sigma_\theta^6\right)}{8b(\sigma_\epsilon^2 + (k+1)\sigma_\theta^2)^3(2\sigma_\epsilon^2 + (k+1)\sigma_\theta^2)^2},$$

which is negative. This finding proves that $\frac{\partial E(TS_{2,NS})(N_{k+1})}{\partial \sigma_\epsilon^2} < 0$, as desired.

iii) (Total Surplus and σ_θ^2) We subsequently claim that $\frac{\partial E(TS_{NS}^*(N))}{\partial \sigma_\theta^2} > 0$. Analogous calculations to part ii) shows that $\frac{\partial E(TS^*_{1,NS}(N))}{\partial \sigma^2_\theta} > 0$. To see $\frac{\partial E(TS^*_{2,NS}(N_k))}{\partial \sigma^2_\theta} > 0$, first note that for $k = 2$,

$$\frac{\partial E(TS^*_{2,NS}(N_2))}{\partial \sigma^2_\theta} = \frac{\sigma^2_\theta(6\sigma^4_\epsilon + 15\sigma^2_\epsilon\sigma^2_\theta + 10\sigma^4_\theta)(8\sigma^4_\epsilon + 22\sigma^2_\epsilon\sigma^2_\theta + 11\sigma^4_\theta)}{32b(\sigma^2_\epsilon + \sigma^2_\theta)^2(\sigma^2_\epsilon + 2\sigma^2_\theta)^3} > 0.$$

Now suppose that $\frac{\partial E(TS^*_{2,NS}(N_k))}{\partial \sigma^2_\theta} < 0$ for some $k < n$. We will show that $\frac{\partial E(TS^*_{2,NS}(N_{k+1}))}{\partial \sigma^2_\theta} < 0$. As $E(TS^*_{2,NS}(N_{k+1})) = w^2_2(N_k) \times E(TS^*_{2,NS}(N_k)) + w_3(N_k)$,

$$\frac{\partial E(TS^*_{2,NS}(N_{k+1}))}{\partial \sigma^2_\theta} = 2w_2(N_k)\frac{\partial w_2(N_k)}{\partial \sigma^2_\theta}E(TS^*_{2,NS}(N_k)) + \frac{\partial w_3(N_k)}{\partial \sigma^2_\theta} + w^2_2(N_k)\frac{\partial E(TS^*_{2,NS}(N_k))}{\partial \sigma^2_\theta}.$$

As $w^2_2(N_k)\frac{\partial E(TS^*_{2,NS}(N_k))}{\partial \sigma^2_\theta} > 0$ by the initial supposition, it is adequate to show that

$$\Xi(N_k) = 2w_2(N_k)\frac{\partial w_2(N_k)}{\partial \sigma^2_\theta}E(TS^*_{2,NS}(N_k)) + \frac{\partial w_3(N_k)}{\partial \sigma^2_\theta} > 0.$$

$\Xi(N_k)$ is minimized when $E(TS^*_{2,NS}(N_k))$ gets its maximum value of $\Theta(N_k)$ as $w_2(N_k) > 0$ and $\partial w_2(N_k)/\partial\sigma^2_\theta = -\sigma^2_\epsilon/(2(\sigma^2_\epsilon + (k+1)\sigma^2_\theta)^2) < 0$. For $E(TS^*_{2,NS}(N_k)) = \Theta(N_k)$,

$$\Xi(N_k) = \frac{\sigma^2_\theta\left(24\sigma^8_\epsilon + 60(k+1)\sigma^6_\epsilon\sigma^2_\theta + 2(27+k(55+27k))\sigma^4_\epsilon\sigma^4_\theta + (21+k(66+k(67+24k)))\sigma^2_\epsilon\sigma^6_\theta + (k+1)^3(3+4k)\sigma^8_\theta\right)}{8b(\sigma^2_\epsilon + (k+1)\sigma^2_\theta)^3(2\sigma^2_\epsilon + (k+1)\sigma^2_\theta)^2},$$

which is positive. This result shows that $\partial E(TS^*_{2,NS}(N_{k+1}))/\partial\sigma^2_\theta > 0$. □

Proof of Lemma 4: The last-follower aims to maximize

$$\max_{q_n} E(\pi_n|s_1, s_2, ..., s_n) = E((a + \theta - bq_n - b\sum_{i=1}^{i=n-1} q_i)q_n|s_1, s_2, ..., s_n).$$

The first order condition (FOC) boils down to

$$q^*_{n,PS} = \frac{a - b\sum_{i=1}^{i=n-1} q_i + E(\theta|s_1, s_2, ..., s_n)}{2b}. \tag{19}$$

Similarly, firm $n-1$ maximizes its expected profit $\max_{q_{n-1}} E(\pi_{n-1}|s_1, s_2, ..., s_{n-1})$:

$$\max_{q_{n-1}} E((a + \theta - bq_{n-1} - b\sum_{i=1}^{i=n-2} q_i - bq^*_{n,PS})q_{n-1}|s_1, s_2, ..., s_{n-1}).$$

After substituting (19) into this maximization problem, the FOC reduces to

$$q^*_{n-1,PS} = \frac{a - b\sum_{i=1}^{i=n-2} q_i + E(\theta|s_1, s_2, ..., s_{n-1})}{2b}.$$

Iterative calculations show that firm i's best response is

$$q^*_{i,PS} = \frac{a - b\sum_{i=1}^{i=i-1} q_i + E(\theta|s_1, s_2, ..., s_i)}{2b}$$

and the leader's equilibrium output is $q^*_{1,PS} = \frac{a+E(\theta|s_1)}{2b}$. The partial sharing equilibrium outputs of firms can be iteratively derived from these derivations. The full sharing equilibrium outputs can be similarly derived; and therefore, the proof is skipped. □

Proof of Proposition 3: **i)** Gal-Or (1987) shows that for $n = 2$, $E(\pi^*_{1,NS}) < E(\pi^*_{1,PS})$. For $n > 2$ and for each $i < n$, we claim that $E(\pi^*_{i,NS}) < E(\pi^*_{i,PS})$ if $\sigma^2_\epsilon/\sigma^2_\theta \to \infty$, and $E(\pi^*_{i,PS}) < E(\pi^*_{i,NS})$ if $\sigma^2_\epsilon/\sigma^2_\theta \to 0$. Let $H_j = \frac{\sigma^2_\epsilon\sigma^4_\theta}{4b(\sigma^2_\epsilon+(j-1)\sigma^2_\theta)(\sigma^2_\epsilon+j\sigma^2_\theta)}$. First, we show that

$$\begin{aligned}\lim_{\sigma^2_\epsilon\to 0} E(\pi^*_{i,NS}(N_i)) &= \lim_{\sigma^2_\epsilon\to 0}\big(\tfrac{(a+\bar\theta)^2}{4b}\textstyle\prod_{k=1}^{k=i-1} w^2_2(N_k) + \sum_{j=1}^{j=i}\big(H_j\prod_{k=j}^{k=i-1} w^2_2(N_k)\big)\big)\\ &= \tfrac{(a+\bar\theta)^2+\sigma^2_\theta}{4b}\lim_{\sigma^2_\epsilon\to 0}\textstyle\prod_{k=1}^{k=i-1} w^2_2(N_k) + \lim_{\sigma^2_\epsilon\to 0}\sum_{j=2}^{j=i}\big(H_j\prod_{k=j}^{k=i-1} w^2_2(N_k)\big)\\ &= \tfrac{(a+\bar\theta)^2+\sigma^2_\theta}{4b}\textstyle\prod_{k=1}^{k=i-1}(\frac{2k+1}{2k+2})^2,\end{aligned}$$

where the first equality is by Lemma A2-*iii*). The second equality is because for $j = 1$, $\lim_{\sigma^2_\epsilon\to 0} H_j = \frac{\sigma^2_\theta}{4b}$. As $\lim_{\sigma^2_\epsilon\to 0} w_2(N_k) = \frac{2k+1}{2k+2}$, and for $j > 1$, $\lim_{\sigma^2_\epsilon\to 0} H_j = 0$, the last equality should hold. Similarly,

$$\begin{aligned}\lim_{\sigma^2_\epsilon\to\infty} E(\pi^*_{i,NS}(N_i)) &= \lim_{\sigma^2_\epsilon\to\infty}\big(\tfrac{(a+\bar\theta)^2}{4b}\textstyle\prod_{k=1}^{k=i-1} w^2_2(N_k)\big)\\ &\quad+\lim_{\sigma^2_\epsilon\to\infty}\big(\textstyle\sum_{j=1}^{j=i}\big(H_j\prod_{k=j}^{k=i-1} w^2_2(N_k)\big)\big) = (a+\bar\theta)^2/(4b),\end{aligned}$$

where the first equality is by Lemma A2-*iv*). The second equality is a consequence of $\lim_{\sigma^2_\epsilon\to\infty} w^2_2(N_k) = 1$, and for $j \geq 1$, $\lim_{\sigma^2_\epsilon\to\infty} H_j = 0$. Using the above derivations, for each $i < n$,

$$\begin{aligned}\lim_{\sigma^2_\epsilon\to 0} E(\pi^*_{i,NS}(N)) &= \lim_{\sigma^2_\epsilon\to 0} E(\pi^*_{i,NS}(N_i))\times\lim_{\sigma^2_\epsilon\to 0}\big(2bw_1(N_i)\big)\times\lim_{\sigma^2_\epsilon\to 0}\textstyle\prod_{k=i}^{k=n-1} w_2(N_k)\\ &= \tfrac{(a+\bar\theta)^2+\sigma^2_\theta}{4b}\textstyle\prod_{k=1}^{k=i-1}(\frac{2k+1}{2k+2})^2\times\frac{1}{i+1}\times\prod_{k=i}^{k=n-1}(\frac{2k+1}{2k+2})\\ &= \tfrac{(a+\bar\theta)^2+\sigma^2_\theta}{4b(i+1)}\textstyle\prod_{k=1}^{k=n-1}(\frac{2k+1}{2k+2})\prod_{k=1}^{k=i-1}(\frac{2k+1}{2k+2}),\\ \lim_{\sigma^2_\epsilon\to\infty} E(\pi^*_{i,NS}(N)) &= \lim_{\sigma^2_\epsilon\to\infty} E(\pi^*_{i,NS}(N_i))\times\lim_{\sigma^2_\epsilon\to\infty}\big(2bw_1(N_i)\big)\times\lim_{\sigma^2_\epsilon\to\infty}\textstyle\prod_{k=i}^{k=n-1} w_2(N_k)\\ &= \tfrac{(a+\bar\theta)^2}{4b}\times 0\times 1 = 0.\end{aligned}$$

Following similar steps to the above proofs,

$$\begin{aligned}\lim_{\sigma^2_\epsilon\to 0} E(\pi^*_{i,PS}(N)) &= \lim_{\sigma^2_\epsilon\to 0}\big(\tfrac{(a+\bar\theta)^2}{2^{n+i}b} + \textstyle\sum_{j=0}^{j=i-1}\frac{\sigma^2_\epsilon\sigma^4_\theta}{2^{n+i-2j}b(\sigma^2_\epsilon+j\sigma^2_\theta)(\sigma^2_\epsilon+(j+1)\sigma^2_\theta)}\big)\\ &= \tfrac{(a+\bar\theta)^2+\sigma^2_\theta}{2^{n+i}b} + \lim_{\sigma^2_\epsilon\to 0}\big(\textstyle\sum_{j=1}^{j=i-1}\frac{\sigma^2_\epsilon\sigma^4_\theta}{2^{n+i-2j}b(\sigma^2_\epsilon+j\sigma^2_\theta)(\sigma^2_\epsilon+(j+1)\sigma^2_\theta)}\big)\\ &= \tfrac{(a+\bar\theta)^2+\sigma^2_\theta}{2^{n+i}b},\\ \lim_{\sigma^2_\epsilon\to\infty} E(\pi^*_{i,PS}(N)) &= \lim_{\sigma^2_\epsilon\to\infty}\big(\tfrac{(a+\bar\theta)^2}{2^{n+i}b} + \textstyle\sum_{j=0}^{j=i-1}\frac{\sigma^2_\epsilon\sigma^4_\theta}{2^{n+i-2j}b(\sigma^2_\epsilon+j\sigma^2_\theta)(\sigma^2_\epsilon+(j+1)\sigma^2_\theta)}\big)\\ &= \tfrac{(a+\bar\theta)^2}{2^{n+i}b}.\end{aligned}$$

It is clear that $\lim_{\sigma^2_\epsilon\to\infty} E(\pi^*_{i,PS}(N)) = \frac{(a+\bar\theta)^2}{2^{n+i}b} > \lim_{\sigma^2_\epsilon\to\infty} E(\pi^*_{i,NS}(N)) = 0$. We next show that for $n \geq 3$ and $i < n$,

$$\lim_{\sigma^2_\epsilon\to 0}\frac{E(\pi^*_{i,NS}(N))}{E(\pi^*_{i,PS}(N))} = \frac{1}{i+1}\prod_{k=1}^{k=n-1}\frac{2k+1}{k+1}\prod_{k=1}^{k=i-1}\frac{2k+1}{k+1} > 1,$$

where the equality is due to the above findings. $\lim_{\sigma_\epsilon^2 \to 0} \frac{E(\pi^*_{1,NS}(N_3))}{E(\pi^*_{1,PS}(N_3))} = \lim_{\sigma_\epsilon^2 \to 0} \frac{E(\pi^*_{2,NS}(N_3))}{E(\pi^*_{2,PS}(N_3))} = \frac{5}{4} > 1$ at $n = 3$. Now suppose $n \geq 4$. It is clear that $\prod_{k=1}^{k=i-1} \frac{2k+1}{k+1} > 1$. As $\frac{1}{i+1}$ gets its minimum at $i = n-1$, it is sufficient to show that $\psi_n = \prod_{k=1}^{k=n-1} \frac{2k+1}{k+1} > n$ for $n \geq 4$. For $n = 4$, $\psi_4 = \frac{35}{8} > 4$. Suppose it holds that $\psi_n > n$ for some $n \geq 4$. We want to show that $\psi_{n+1} > n+1$, which requires $\frac{2n+1}{n+1}\psi_n > n+1$. As $\psi_n > n$ by the initial supposition, it is sufficient to show that $\frac{n(2n+1)}{n+1} > n+1$. This inequality should hold because $n^2 > n+1$ at $n \geq 4$. This finding concludes the induction step.

In addition, $\lim_{\sigma_\theta^2 \to 0} E(\pi^*_{i,PS}(N)) = \frac{(a+\bar{\theta})^2}{2^{n+i}b} > \lim_{\sigma_\theta^2 \to 0} E(\pi^*_{i,NS}(N)) = 0$ by analogous calculations to the above. Moreover, $\lim_{\sigma_\epsilon^2 \to 0} \frac{E(\pi^*_{i,NS}(N))}{E(\pi^*_{i,PS}(N))} = \lim_{\sigma_\theta^2 \to \infty} \frac{E(\pi^*_{i,NS}(N))}{E(\pi^*_{i,PS}(N))} > 1$ by the above finding.

ii) Using Lemma A5-*iii*) and Lemma A5-*iv*), direct computations show that for $n \geq 2$,

$$E(\pi^*_{1,FS}(N)) - E(\pi^*_{1,PS}(N)) = \frac{\sigma_\epsilon^2 \sigma_\theta^4 (n-1)}{2^{n+1} b(\sigma_\epsilon^2 + \sigma_\theta^2)(\sigma_\epsilon^2 + n\sigma_\theta^2)} > 0.$$

Lastly, we show that $E(\pi^*_{n,FS}(N)) < E(\pi^*_{n,PS}(N)) < E(\pi^*_{n,NS}(N))$ by induction. For $n = 2$, $E(\pi^*_{2,PS}(N_2)) - E(\pi^*_{2,FS}(N_2)) = \frac{3\sigma_\epsilon^2\sigma_\theta^4}{16b(\sigma_\epsilon^2+\sigma_\theta^2)(\sigma_\epsilon^2+2\sigma_\theta^2)} > 0$ and $E(\pi^*_{2,NS}(N_2)) - E(\pi^*_{2,PS}(N_2)) = \frac{(3\sigma_\epsilon^2+5\sigma_\theta^2)(\sigma_\theta^4+(\sigma_\epsilon^2+\sigma_\theta^2)(a+\bar{\theta})^2)}{16b(\sigma_\epsilon^2+2\sigma_\theta^2)^2} > 0$ by Lemmas A2 and A5.

Suppose that $E(\pi^*_{k,FS}(N_k)) < E(\pi^*_{k,PS}(N_k)) < E(\pi^*_{k,NS}(N_k))$ for some $k < n$. We want to show that $E(\pi^*_{k+1,FS}(N_{k+1})) < E(\pi^*_{k+1,PS}(N_{k+1})) < E(\pi^*_{k+1,NS}(N_{k+1}))$. Using Lemmas A2-*iii*), A5-*iii*), and A5-*iv*), this claim is equivalent to

$$\frac{E(\pi^*_{k,FS}(N_k))}{4} + \frac{\Lambda(N_k)}{2^{2k}} < \frac{E(\pi^*_{k,PS}(N_k))}{4} + \Lambda(N_k) < w_2^2(N_k)E(\pi^*_{k,NS}(N_k)) + \Lambda(N_k),$$

where $\Lambda(N_k) = \frac{\sigma_\epsilon^2\sigma_\theta^4}{4b(\sigma_\epsilon^2+k\sigma_\theta^2)(\sigma_\epsilon^2+(k+1)\sigma_\theta^2)}$. As $E(\pi^*_{k,FS}(N_k)) < E(\pi^*_{k,PS}(N_k))$ by the initial supposition and $\frac{\Lambda(N_k)}{2^{2k}} < \Lambda(N_k)$, the left inequality holds. Similarly, the right inequality should hold because $E(\pi^*_{k,PS}(N_k)) < E(\pi^*_{k,NS}(N_k))$ by the initial supposition and $w_2^2(N_k) - \frac{1}{4} = \frac{(\sigma_\epsilon^2+k\sigma_\theta^2)(3\sigma_\epsilon^2+(2+3k)\sigma_\theta^2)}{4(\sigma_\epsilon^2+(k+1)\sigma_\theta^2)^2} > 0$. Our induction proof is now complete. □

Proof of Proposition 4: **i) (Total Output Rankings)** We can use Corollary 2 to show that $E(Q^*_{PS}(N)) = E(Q^*_{FS}(N)) = \sum_{i=1}^{i=n} \frac{a+\bar{\theta}}{2^i b}$. Using simple algebra,

$$\sum_{i=1}^{i=n} \frac{a+\bar{\theta}}{2^i b} = \frac{(2^n-1)(a+\bar{\theta})}{2^n b}.$$

Moreover, $E(Q^*_{NS}(N)) < E(Q^*_C(N)) = \frac{n(a+\bar{\theta})}{b(n+1)}$ by the proof of Proposition 1 of Cumbul (2021). Therefore, it is sufficient to show that $E(Q^*_{FS}(N)) > \frac{n(a+\bar{\theta})}{b(n+1)}$ or $\frac{2^n-1}{2^n} > \frac{n}{n+1}$. Rearranging terms yields $2^n > n+1$. By induction, one can show that this inequality holds for $n \geq 2$. Altogether, $E(Q^*_{FS}(N)) = E(Q^*_{PS}(N)) > E(Q^*_C(N)) > E(Q^*_{NS}(N))$.

ii) (Total Surplus Rankings) The expected total surplus can be divided into two as

$$E(TS) = \underbrace{(a+\bar{\theta})E(Q) - b(E(Q))^2/2}_{=E(TS_1)} + \underbrace{\text{Cov}(\theta, Q) - b\text{Var}(Q)/2}_{=E(TS_2)}.$$

We prove the proposition in two steps.

STEP 1: We claim that $E(TS^*_{1,FS}(N)) = E(TS^*_{1,PS}(N)) > E(TS^*_{1,C}(N)) > E(TS^*_{1,NS}(N))$. Let $g(E(Q)) = (a+\bar{\theta})E(Q) - b(E(Q))^2/2$. Note that $\frac{\partial g(.)}{\partial E(Q)} = (a+\bar{\theta}) - bE(Q)$ and $\frac{\partial^2 g(.)}{\partial^2 E(Q)} = -b < 0$. Therefore, $g(.)$ is concave in $E(Q)$ and it is maximized at $E(Q) = \frac{a+\bar{\theta}}{b}$. $E(Q^*_{FS}(N)) = E(Q^*_{PS}(N)) > E(Q^*_{C}(N)) > E(Q^*_{NS}(N))$ by part i). To conclude the proof, it is sufficient to show that $\frac{a+\bar{\theta}}{b} \geq E(Q^*_{FS}(N)) = \frac{(2^n-1)(a+\bar{\theta})}{2^n b}$ by the concavity of $g(.)$. This should hold because $2^n - 1 < 2^n$.

STEP 2: We claim that $E(TS^*_{2,FS}(N)) > E(TS^*_{2,PS}(N)) > E(TS^*_{2,C}(N)) > E(TS^*_{2,NS}(N))$. It holds that $E(TS^*_{2,C}(N)) > E(TS^*_{2,NS}(N))$ by Proposition 1 of Cumbul (2021). Next, we will show that $E(TS^*_{2,FS}(N)) > E(TS^*_{2,PS}(N)) > E(TS^*_{2,C}(N))$ by using the induction method. For $k=2$,

$$E(TS^*_{2,PS}(N_2)) - E(TS^*_{2,C}(N_2)) = \frac{\sigma_\theta^4(12\sigma_\epsilon^6 + 28\sigma_\epsilon^4\sigma_\theta^2 + 27\sigma_\epsilon^2\sigma_\theta^4 + 14\sigma_\theta^6)}{32b(\sigma_\epsilon^2+\sigma_\theta^2)(\sigma_\epsilon^2+2\sigma_\theta^2)(2\sigma_\epsilon^2+3\sigma_\theta^2)^2} > 0$$

and

$$E(TS^*_{2,FS}(N_2)) - E(TS^*_{2,PS}(N_2)) = \frac{3\sigma_\epsilon^2\sigma_\theta^4}{32b(\sigma_\epsilon^2+\sigma_\theta^2)(\sigma_\epsilon^2+2\sigma_\theta^2)} > 0.$$

by Lemmas A3-ii) and A6. Suppose it holds that $E(TS^*_{2,FS}(N_k)) > E(TS^*_{2,PS}(N_k)) > E(TS^*_{2,C}(N_k))$ for some $k < n$. We want to prove that $E(TS^*_{2,FS}(N_{k+1})) > E(TS^*_{2,PS}(N_{k+1})) > E(TS^*_{2,C}(N_{k+1}))$. As $E(TS^*_{2,PS}(N_{k+1})) = \frac{E(TS^*_{2,PS}(N_k))}{4} + \frac{3(k+1)\sigma_\theta^4}{8b(\sigma_\epsilon^2+(k+1)\sigma_\theta^2)}$ by Lemma A6-ii), it is sufficient to show that

$$\Upsilon_1 = E(TS^*_{2,FS}(N_{k+1})) - \frac{E(TS^*_{2,FS}(N_k))}{4} > \frac{3(k+1)\sigma_\theta^4}{8b(\sigma_\epsilon^2+(k+1)\sigma_\theta^2)}$$

and

$$\frac{3(k+1)\sigma_\theta^4}{8b(\sigma_\epsilon^2+(k+1)\sigma_\theta^2)} > \Upsilon_2 = E(TS^*_{2,C}(N_{k+1})) - \frac{E(TS^*_{2,C}(N_k))}{4}$$

by using the initial supposition. By using Lemmas A3-ii), A6-ii), and A6-iii), we can show that

$$\Upsilon_1 - \frac{3(k+1)\sigma_\theta^4}{8b(\sigma_\epsilon^2+(k+1)\sigma_\theta^2)} = \frac{(4^k-1)\sigma_\epsilon^2\sigma_\theta^4}{2^{2k+3}b(\sigma_\epsilon^2+k\sigma_\theta^2)(\sigma_\epsilon^2+(k+1)\sigma_\theta^2)} > 0$$

and

$$\frac{3(k+1)\sigma_\theta^4}{8b(\sigma_\epsilon^2+(k+1)\sigma_\theta^2)} - \Upsilon_2 = \frac{\sigma_\theta^4 k(12\sigma_\epsilon^8+4\sigma_\epsilon^6(7+3k)\sigma_\theta^2+\sigma_\epsilon^4(28+3k(8+k))\sigma_\theta^4+\sigma_\epsilon^2(16+5k(4+k))\sigma_\theta^6+(1+k)(4+3k)\sigma_\theta^8)}{8b(\sigma_\epsilon^2+(k+1)\sigma_\theta^2)(2\sigma_\epsilon^2+(k+1)\sigma_\theta^2)^2(2\sigma_\epsilon^2+(k+2)\sigma_\theta^2)^2} > 0.$$

iii) (Total Profit Rankings) In general, the expected total profit reduces to

$$E(\Pi) = E(pQ) = \underbrace{(a+\bar{\theta})E(Q) - b(E(Q))^2}_{=E(\Pi_1)} + \underbrace{\text{Cov}(\theta, Q) - b\text{Var}(Q)}_{=E(\Pi_2)}. \tag{20}$$

Note that Cumbul (2021, Proposition 1) shows that $E(\Pi^*_C(N)) < E(\Pi^*_{NS}(N))$. We prove that $E(\Pi^*_{FS}(N)) < E(\Pi^*_{PS}(N)) < E(\Pi^*_C(N))$ in two steps.

STEP 1: First, we claim that $E(\Pi^*_{1,FS}(N)) = E(\Pi^*_{1,PS}(N)) < E(\Pi^*_{1,C}(N))$. It holds that

$E(\Pi^*_{1,FS}(N)) = E(\Pi^*_{1,PS}(N))$ because $E(Q^*_{FS}(N)) = E(Q^*_{PS}(N))$ by part i). Lastly, we show that $(a+\bar{\theta})E(Q^*_C(N)) - b(E(Q^*_C(N)))^2 > (a+\bar{\theta})E(Q^*_{FS}(N)) - b(E(Q^*_{FS}(N)))^2$ to conclude the proof. To see this, let $f(E(Q)) = (a+\bar{\theta})E(Q) - b(E(Q))^2$. Note that $\frac{\partial f(.)}{\partial E(Q)} = (a+\bar{\theta}) - 2bE(Q)$ and $\frac{\partial^2 f(.)}{\partial^2 E(Q)} = -2b < 0$. Therefore, $f(.)$ is concave in $E(Q)$ and it is maximized at $E(Q) = (a+\bar{\theta})/(2b)$. By part i), $E(Q^*_{FS}(N)) > E(Q^*_C(N))$. It is then sufficient to show that $\frac{a+\bar{\theta}}{2b} \leq E(Q^*_C(N)) = \frac{n(a+\bar{\theta})}{b(n+1)}$ by the concavity of $f(.)$. This should hold at $n > 1$.

STEP 2: We now claim that $E(\Pi^*_{2,C}(N)) > E(\Pi^*_{2,PS}(N)) > E(\Pi^*_{2,FS}(N))$, where

$$E(\Pi^*_{2,C}(N_k)) = U(N_k) = \frac{k\sigma^4_\theta\left(\sigma^2_\theta+\sigma^2_\epsilon\right)}{b\left(2\sigma^2_\epsilon+(k+1)\sigma^2_\theta\right)^2},$$

$$E(\Pi^*_{2,PS}(N_k)) = X(N_k) = \sum_{i=1}^{i=k}\sum_{j=0}^{j=i-1} \frac{\sigma^2_\epsilon\sigma^4_\theta}{b2^{i-2j+k}\left(\sigma^2_\epsilon + j\sigma^2_\theta\right)\left(\sigma^2_\epsilon + (j+1)\sigma^2_\theta\right)},$$

$$E(\Pi^*_{2,FS}(N_k)) = T(N_k) = \sum_{i=1}^{k} \frac{k\sigma^4_\theta}{b2^{i+k}\left(\sigma^2_\epsilon+k\sigma^2_\theta\right)}$$

using Lemmas A3-ii), A5-iii), A5-iv), and A6-i). For $k=2$, $X(N_2) - T(N_2) = \frac{\sigma^2_\epsilon\sigma^4_\theta}{16b(\sigma^2_\epsilon+\sigma^2_\theta)(\sigma^2_\epsilon+2\sigma^2_\theta)}$ and $U(N_2) - X(N_2) = \frac{\sigma^4_\theta\left(4\sigma^6_\epsilon+20\sigma^2_\theta\sigma^4_\epsilon+25\sigma^4_\theta\sigma^2_\epsilon+10\sigma^6_\theta\right)}{16b(\sigma^2_\epsilon+\sigma^2_\theta)(\sigma^2_\epsilon+2\sigma^2_\theta)(2\sigma^2_\epsilon+3\sigma^2_\theta)^2}$ are both positive. Now, suppose that $U(N_k) > X(N_k) > T(N_k)$ for some $k < n$. We want to show that $U(N_{k+1}) > X(N_{k+1}) > T(N_{k+1})$. We have that

$$\begin{gathered} W(N_k) > Y(N_k) > Z(N_k) \Rightarrow \\ W(N_k) + \frac{U(N_k)}{2} > Y(N_k) + \frac{X(N_k)}{2} > Z(N_k) + \frac{T(N_k)}{2} \\ \Rightarrow U(N_{k+1}) > X(N_{k+1}) > T(N_{k+1}), \end{gathered}$$

where the first line inequalities follow by Lemma A7, the first implication is a result of our initial supposition, and the second implication follows because $U(N_{k+1}) = W(N_k) + U(N_k)/2$, $X(N_{k+1}) = Y(N_k) + X(N_k)/2$, and $T(N_{k+1}) = Z(N_k) + T(N_k)/2$ by definition. Our induction proof is now complete. □

2 Proofs of n_1-leaders and n_2-followers model of Section II

Lemma A8. *Provided that $b + \lambda + bn_2B_1 > 0$, there exists a unique linear PRE, where each leader $i \in N_1$ and each follower $j \in N_2$ produce $\hat{q}_{i,SQ} = A_0 + A_1(s_i - \bar{\theta})$ and $\hat{q}_{j,SQ} = B_0 + B_1\sum_{i\in N_1}\hat{q}_{i,SQ} + B_2(h_j - \bar{\theta})$, respectively, where $A_0 = B_0 + G_1G_7(a+\bar{\theta})/G_5$, $A_1 = \sigma^2_\theta G_3/(G_2G_4)$, $B_0 = \sigma^2_{\epsilon_1}(a+\bar{\theta})G_6/G_5$, $B_1 = (b+2\lambda)G_1/G_3$, $B_2 = \sigma^2_{\epsilon_1}\sigma^2_\theta/G_2$, where*

$$\begin{aligned} &G_0 = (b+2\lambda)(2\lambda + b(n_1+n_2+1)) - b^2n_1n_2, \quad G_1 = 2\sigma^2_{\epsilon_2}(b+\lambda)\left(\sigma^2_\theta + \sigma^2_{\epsilon_1}\right) - b\sigma^2_\theta\sigma^2_{\epsilon_1}, \\ &G_2 = 2\sigma^2_{\epsilon_2}(b+\lambda)\left(\sigma^2_{\epsilon_1} + n_1\sigma^2_\theta\right) + \sigma^2_\theta\sigma^2_{\epsilon_1}(2\lambda + b(n_2+1)), \quad G_3 = G_2(b+2\lambda) - bn_2(G_1 + b\sigma^2_\theta\sigma^2_{\epsilon_1}), \\ &G_4 = G_8(\sigma^2_\theta + \sigma^2_{\epsilon_1}) - b(n_1-1)(b+2\lambda)\sigma^2_{\epsilon_1}, \quad G_5 = G_2G_8 - 2b^2n_2(n_1-1)(b+\lambda)\sigma^2_{\epsilon_1}\sigma^2_{\epsilon_2}, \\ &G_6 = G_8\sigma^2_\theta - 2(n_1-1)(b+\lambda)(b+2\lambda)\sigma^2_{\epsilon_2}, \quad G_7 = 2\lambda n_1 + b(n_1 - n_2), \quad G_8 = G_0 + b^2n_2(n_1-1). \end{aligned}$$

When $n_1 = b = 1$, $n_2 = n$, and $\lambda = 0$, this result coincides with a result in Nakamura (2015).

Proof of Lemma A8: Let the sum of leaders' signals be $S_L = \sum_{i\in N_1} s_i$. For $Q_L = \sum_{i\in N_1} q_i$, let $q_i = A_0 + A_1(s_i - \bar{\theta})$ and $q_j = B_0 + B_1 Q_L + B_2(h_j - \bar{\theta})$ in a linear PRE for some constants $A_0, A_1, B_0, B_1, B_2 \in \mathbb{R}$. Our aim is to derive these five constants.

STEP 1: First, we derive the conditional expectations $E(\theta|h_j, s_1, s_2, ..., s_{n_1})$ and $E(h_k|h_j, s_1, s_2, ..., s_{n_1})$ for some $k, j \in N_2$, $k \neq j$. As the values are common and the conditional expectations are linear, we have $E(\theta|h_j, s_1, s_2, ..., s_{n_1}) = E(\theta|h_j, S_L)$ and $E(h_k|h_j, s_1, s_2, ..., s_{n_1}) = E(h_k|h_j, S_L)$. By the projection theorem for the multivariate normally distributed random variables, for $t \in \{\theta, h_k\}$,

$$E(t|S_L, h_j) = E(t) + \Big(\text{Cov}(t, S_L), \text{Cov}(t, h_j)\Big) \cdot \begin{pmatrix} \text{Var}(S_L) & \text{Cov}(S_L, h_j) \\ \text{Cov}(h_j, S_L) & \text{Var}(h_j) \end{pmatrix}^{-1} \cdot \begin{pmatrix} S_L - E(S_L) \\ h_j - E(h_j) \end{pmatrix}.$$

For each $i \in N_1$ and $j \in N_2$, $s_i = \theta + \epsilon_i$, $h_j = \theta + \epsilon_j$, $E(\theta) = E(s_i) = E(h_j) = \bar{\theta}$, $Var(\epsilon_i) = \sigma^2_{\epsilon_1}$, $Var(\epsilon_j) = \sigma^2_{\epsilon_2}$, and $E(\epsilon_k) = \text{Cov}(\epsilon_i, \epsilon_j) = \text{Cov}(h_j, \epsilon_i) = \text{Cov}(\theta, \epsilon_k) = \text{Cov}(s_i, \epsilon_j) = 0$ for $k \in \{i, j\}$ by assumption. Thus, $E(S_L) = n_1\bar{\theta}$, $\text{Cov}(\theta, S_L) = \text{Cov}(h_j, S_L) = n_1\sigma^2_\theta$, $\text{Cov}(\theta, h_j) = \sigma^2_\theta$, $\text{Var}(S_L) = n_1(\sigma^2_{\epsilon_1} + n_1\sigma^2_\theta)$ and $\text{Var}(h_j) = \sigma^2_\theta + \sigma^2_{\epsilon_2}$. Inserting these derivations into the above expression,

$$E(\theta|S_L, h_j) = E(h_k|S_L, h_j) = \bar{\theta} + \frac{\sigma^2_\theta(\sigma^2_{\epsilon_1}(h_j - \bar{\theta}) + \sigma^2_{\epsilon_2}(S_L - n_1\bar{\theta}))}{\sigma^2_\theta\sigma^2_{\epsilon_1} + \sigma^2_{\epsilon_2}(\sigma^2_{\epsilon_1} + n_1\sigma^2_\theta)}. \tag{21}$$

Similar calculations to the above yield that for $i, l \in N_1$, $j \in N_2$, and $t \in \{\theta, s_l, h_j\}$ such that $l \neq i$,

$$E(\theta|s_i) = E(s_l|s_i) = E(h_j|s_i) = E(t) + Cov(t, s_i)(Var(s_i))^{-1}(s_i - E(s_i)) = \bar{\theta} + \frac{\sigma^2_\theta(s_i - \bar{\theta})}{\sigma^2_\theta + \sigma^2_{\epsilon_1}}. \tag{22}$$

STEP 2: Consider any follower firm $j \in N_2$. It maximizes

$$\max_{q_j} E(\pi_j|I_j) = \max_{q_j} E((a + \theta - bQ)q_j - \lambda q_j^2|I_j), \tag{23}$$

where $I_j = h_j, q_1, q_2, ..., q_{n_1}$ is the information set of follower j. Thus, the FOC from (23) yields the best response of any follower j to the changes in the total quantity of leaders (Q_L) as

$$\begin{aligned} q_j(Q_L) &= B_0 + B_1 Q_L + B_2(h_j - \bar{\theta}) \\ &= \frac{a + E(\theta|I_j) - b\big(B_0(n_2 - 1) + (1 + B_1(n_2 - 1))Q_L + B_2(n_2 - 1)(E(h_k|I_j) - \bar{\theta})\big)}{2(b + \lambda)}, \end{aligned} \tag{24}$$

where $k \in N_2$, $k \neq j$, by the symmetry among followers. In a PRE, the follower firm j learns the sum of the signals of its leaders from the leaders' output choices. As for each leader $i \in N_1$, $q_i = A_0 + A_1(s_i - \bar{\theta})$, $Q_L = n_1 A_0 - n_1 A_1\bar{\theta} + A_1 \sum_{i \in N_1} s_i$. Thus, each follower infers that

$$S_L = \sum_{i \in N_1} s_i = \frac{Q_L + n_1 A_1\bar{\theta} - n_1 A_0}{A_1}. \tag{25}$$

Any leader firm $i \in N_1$ solves $\max_{q_i} E(\pi_i|s_i) = E(q_i(a + \theta - bQ) - \lambda q_i^2|s_i)$, where

$$Q = \sum_{l \in N_1} q_l + \sum_{j \in N_2} q_j(Q_L) = (q_i + \sum_{l \in N_1 \setminus i} q_l)(1 + n_2 B_1) + n_2(B_0 - B_2\bar{\theta}) + B_2 \sum_{j \in N_2} h_j$$

by the symmetry among followers. Thus, the FOC from the leader i's problem simplifies to

$$\begin{aligned} q_i &= A_0 + A_1(s_i - \bar{\theta}) \\ &= \frac{a + E(\theta|s_i) - b\big(n_2(B_0 - B_2\bar{\theta} + B_2E(h_j|s_i)) + (n_1-1)(1+n_2B_1)(A_0 - A_1\bar{\theta} + A_1E(s_l|s_i))\big)}{2(b+\lambda+bn_2B_1)}, \end{aligned} \tag{26}$$

where $l \in N_1$ and $l \neq i$, by the symmetry among leaders. Moreover, for $k \in N_2$ such that $k \neq j$, $E(\theta|I_j) = E(h_k|I_j) = E(\theta|S_L, h_j) = E(h_k|S_L, h_j)$ by Step 1, where S_L is given by (25). After substituting (22) into (26) and (21) into (24) using (25), we obtain five equations with five unknowns A_0, A_1, B_0, B_1 and B_2. We find the constants by solving these five equations simultaneously as stated in the lemma. The second-order conditions hold because for each $i \in N_1$, $\partial^2\pi_i((q_k)_{k\in N_1}, (q_j(Q_L))_{j\in N_2})/\partial^2 q_i = -2(b+\lambda+bn_2B_1) < 0$ and for each $j \in N_2$, $\partial^2\pi_j/\partial^2 q_j = -2(b+\lambda) < 0$ because $b+\lambda+bn_2B_1 > 0$, $b > 0$, and $\lambda \geq 0$ by assumption. □

Proposition 9. *i) As $n_1 \to \infty$ or $n_2 \to \infty$, the expected PRE total output, consumer surplus and total surplus approach their first best efficient levels of $(a+\bar{\theta})/b$, $((a+\bar{\theta})^2 + \sigma_\theta^2)/(2b)$, and $((a+\bar{\theta})^2 + \sigma_\theta^2)/(2b)$, respectively, and the expected total profit approaches 0 at any $\lambda \geq 0$.*

ii) Suppose that the demand is replicated and it is given by $p_{n_1+n_2} = a + \theta - BQ/(n_1+n_2)$. As $n_1 \to \infty$, the expected PRE consumer and total surpluses over n_1+n_2 converge to their second best efficient levels of $\frac{B(a+\bar{\theta})^2}{2(B+2\lambda)^2} + \frac{B\sigma_\theta^6}{2(2\lambda\sigma_{\epsilon_1}^2+(B+2\lambda)\sigma_\theta^2)^2}$ and $\frac{(a+\bar{\theta})^2}{2(B+2\lambda)} + \frac{\sigma_\theta^4}{2(2\lambda\sigma_{\epsilon_1}^2+(B+2\lambda)\sigma_\theta^2)}$, respectively, if $\lambda > 0$ and $\sigma_{\epsilon_1}^2 > 0$.

Proof of Proposition 9: Using $\hat{q}_{i,SQ} = A_0 + A_1(s_i - \bar{\theta})$ and $\hat{q}_{j,SQ} = B_0 + B_1\hat{Q}_{SQ} + B_2(h_j - \bar{\theta})$ from Lemma A8,

$$\begin{aligned} &\hat{Q}_{SQ} = \textstyle\sum_{i\in N_1}\hat{q}_{i,SQ} + \sum_{j\in N_2}\hat{q}_{j,SQ}, \qquad E(\hat{CS}_{SQ}) = bE(\hat{Q}_{SQ}^2)/2, \\ &E(\hat{\Pi}_{SQ}) = E((a+\theta-b\hat{Q}_{SQ})\hat{Q}_{SQ}) - \lambda(\textstyle\sum_{i\in N_1}E(\hat{q}_{i,SQ}^2) + \sum_{j\in N_2}E(\hat{q}_{j,SQ}^2)), \\ &E(\hat{TS}_{SQ}) = (a+\theta)E(\hat{Q}_{SQ}) - bE(\hat{Q}_{SQ}^2)/2 - \lambda(\textstyle\sum_{i\in N_1}E(\hat{q}_{i,SQ}^2) + \sum_{j\in N_2}E(\hat{q}_{j,SQ}^2)). \end{aligned} \tag{27}$$

To find the covariance parts of expected total profit, consumer surplus, and total surplus, we let $a = \bar{\theta} = 0$, $E(s_i^2) = \sigma_{\epsilon_1}^2 + \sigma_\theta^2$, $E(h_j^2) = \sigma_{\epsilon_2}^2 + \sigma_\theta^2$, $E(S_L^2) = n_1(\sigma_{\epsilon_1}^2 + n_1\sigma_\theta^2)$, $E(H_F^2) = n_2(\sigma_{\epsilon_2}^2 + n_2\sigma_\theta^2)$, $E(S_LH_F) = n_1n_2\sigma_\theta^2$, $E(\theta S_L) = E(h_jS_L) = n_1\sigma_\theta^2$, $E(\theta H_F) = n_2\sigma_\theta^2$, and $E(h_js_i) = E(\theta h_j) = E(\theta s_i) = \sigma_\theta^2$ for $S_L = \sum_{i\in N_1}s_i$ and $H_F = \sum_{j\in N_2}h_j$ in their definitions. The multiple-leader and multiple-follower large markets aggregate information efficiently because at any $\lambda \geq 0$, $i \in \{1,2\}$,

$$\lim_{n_i\to\infty}E(\hat{Q}_{SQ}) = \frac{a+\bar{\theta}}{b} \quad \text{and} \quad \lim_{n_i\to\infty}E(\hat{CS}_{SQ}) = \lim_{n_i\to\infty}E(\hat{TS}_{SQ}) = \frac{(a+\bar{\theta})^2+\sigma_\theta^2}{2b}.$$

Thus, $\lim_{n_i\to\infty}E(\hat{\Pi}_{SQ}) = \lim_{n_i\to\infty}E(\hat{TS}_{SQ} - \hat{CS}_{SQ}) = 0$. We now assume that the demand is replicated and let $p_{n_1+n_2} = a+\theta-BQ/(n_1+n_2)$. After replacing b with $B/(n_1+n_2)$ in the equilibrium outcomes of the non-replicated demand model, we obtain the expected PRE total profit, consumer surplus and total surplus in the replicated demand market as $E(\hat{\Pi}_{SQ,R}) = E(\hat{\Pi}_{SQ})|_{b=B/(n_1+n_2)}$, $E(\hat{CS}_{SQ,R}) = E(\hat{CS}_{SQ})|_{b=B/(n_1+n_2)}$ and $E(\hat{TS}_{SQ,R}) = E(\hat{TS}_{SQ})|_{b=B/(n_1+n_2)}$. As $n_1 \to \infty$, expected consumer and total surpluses over n_1+n_2 converge to their second best efficient levels

(Vives, 1988, Table I) when $\lambda > 0$ and $\sigma_{\epsilon_1}^2 > 0$ because

$$\begin{aligned}
\lim_{n_1\to\infty} \frac{E(\hat{CS}_{SQ,R})}{n_1+n_2} &= \frac{B(a+\bar{\theta})^2}{2(B+2\lambda)^2} + \frac{B\sigma_\theta^6}{2((B+2\lambda)\sigma_\theta^2 + 2\lambda\sigma_{\epsilon_1}^2)^2},\\
\lim_{n_1\to\infty} \frac{E(\hat{TS}_{SQ,R})}{n_1+n_2} &= \frac{(a+\bar{\theta})^2}{2(B+2\lambda)} + \frac{\sigma_\theta^4}{2(B\sigma_\theta^2 + 2\lambda(\sigma_{\epsilon_1}^2+\sigma_\theta^2))},\\
\lim_{n_1\to\infty} \frac{E(\hat{\Pi}_{SQ,R})}{n_1+n_2} &= \frac{\lambda(a+\bar{\theta})^2}{(B+2\lambda)^2} + \frac{\lambda\sigma_\theta^4(\sigma_\theta^2+\sigma_{\epsilon_1}^2)}{((B+2\lambda)\sigma_\theta^2 + 2\lambda\sigma_{\epsilon_1}^2)^2}.
\end{aligned}$$

Lastly, the limit values can be smaller or greater than the above ones when n_2 approaches ∞.

$$\begin{aligned}
\lim_{n_2\to\infty} \frac{E(\hat{CS}_{SQ,R})}{n_1+n_2} &= \frac{B(a+\bar{\theta})^2}{2(B+2\lambda)^2} + \frac{B\sigma_\theta^4(4\lambda^2 n_1\sigma_{\epsilon_1}^2\sigma_{\epsilon_2}^4 + \sigma_\theta^2((B+2\lambda)\sigma_{\epsilon_1}^2 + 2\lambda n_1\sigma_{\epsilon_2}^2)^2)}{2(B+2\lambda)^2((B+2\lambda)\sigma_\theta^2\sigma_{\epsilon_1}^2 + 2\lambda\sigma_{\epsilon_2}^2(n_1\sigma_\theta^2+\sigma_{\epsilon_1}^2))^2},\\
\lim_{n_2\to\infty} \frac{E(\hat{TS}_{SQ,R})}{n_1+n_2} &= \frac{(a+\bar{\theta})^2}{2(B+2\lambda)} + \frac{\sigma_\theta^4((B+2\lambda)\sigma_{\epsilon_1}^2 + 2\lambda n_1\sigma_{\epsilon_2}^2)}{2(B+2\lambda)((B+2\lambda)\sigma_\theta^2\sigma_{\epsilon_1}^2 + 2\lambda\sigma_{\epsilon_2}^2(n_1\sigma_\theta^2+\sigma_{\epsilon_1}^2))},\\
\lim_{n_2\to\infty} \frac{E(\hat{\Pi}_{SQ,R})}{n_1+n_2} &= \frac{\lambda(a+\bar{\theta})^2}{(B+2\lambda)^2} + \frac{\lambda\sigma_\theta^4(\sigma_\theta^2((B+2\lambda)\sigma_{\epsilon_1}^2+2\lambda n_1\sigma_{\epsilon_2}^2)^2+(B+2\lambda)^2\sigma_{\epsilon_2}^2\sigma_{\epsilon_1}^4+4\lambda^2 n_1\sigma_{\epsilon_2}^4\sigma_{\epsilon_1}^2)}{(B+2\lambda)^2((B+2\lambda)\sigma_\theta^2\sigma_{\epsilon_1}^2+2\lambda\sigma_{\epsilon_2}^2(n_1\sigma_\theta^2+\sigma_{\epsilon_1}^2))^2}.
\end{aligned}$$

The supportive calculations in *Mathematica 12.1* can be found in online Appendix B. □

3 Proofs of the private value model of Section IV

Lemma A9. *We have that*

$$\begin{aligned}
E(\theta_2|s_1,s_2,z) &= \bar{\theta} + \frac{\Delta_2\sigma_{\epsilon_2}^2(s_1-\bar{\theta}) + \Delta_3(s_2-\bar{\theta}) + 2\Delta_4\sigma_{\epsilon_2}^2(z-\bar{\theta})}{\Delta_6},\\
E(\theta_1|s_1,z) &= \bar{\theta} + \frac{\sigma_\theta^2(\Delta_1(s_1-\bar{\theta}) + 2\sigma_{\epsilon_1}^2(1+\rho)(z-\bar{\theta}))}{\Delta_4+\Delta_5},\\
E(s_2|s_1,z) &= \bar{\theta} + \frac{\Delta_2(s_1-\bar{\theta}) + 2\Delta_4(z-\bar{\theta})}{\Delta_4+\Delta_5},
\end{aligned}$$

where

$$\begin{aligned}
&\Delta_1 = \left(1-\rho^2\right)\sigma_\theta^2 + 4\sigma_{\epsilon_z}^2, \quad \Delta_2 = \sigma_\theta^2\left(4\rho\sigma_{\epsilon_z}^2 - \left(1-\rho^2\right)\sigma_\theta^2\right), \quad \Delta_3 = \sigma_\theta^2\left(\Delta_1\sigma_{\epsilon_1}^2 + 4\left(1-\rho^2\right)\sigma_\theta^2\sigma_{\epsilon_z}^2\right),\\
&\Delta_4 = \sigma_\theta^2(1+\rho)\left((1-\rho)\sigma_\theta^2 + \sigma_{\epsilon_1}^2\right), \ \Delta_5 = 4\sigma_{\epsilon_z}^2\left(\sigma_\theta^2+\sigma_{\epsilon_1}^2\right) + (1+\rho)\sigma_\theta^2\sigma_{\epsilon_1}^2, \ \Delta_6 = \Delta_3 + (\Delta_4+\Delta_5)\sigma_{\epsilon_2}^2.
\end{aligned}$$

Proof of Lemma A9: The proof of this lemma can be done similarly to the proof of Lemma 1 of Gal-Or (1987). Alternatively, one can apply the projection theorem to find the conditional expectations for the normally distributed random variables as Vives (2011) or Rostek and Weretka (2012). Supplementary calculations in *Mathematica 12.1* are provided in online Appendix B. □

Lemma A10. $\Delta_2 \gtreqless 0$ *if* $\rho \gtreqless \bar{\rho} = (\sqrt{\sigma_\theta^4 + 4\sigma_{\epsilon_z}^4} - 2\sigma_{\epsilon_z}^2)/\sigma_\theta^2$, *and* $f(\rho) = 2-\rho(4-\rho) \gtreqless 0$ *if* $\rho \lesseqgtr 2-\sqrt{2}$.

Proof of Lemma A10: Δ_2 is quadratic and convex in ρ because $\partial^2\Delta_2/\partial^2\rho = 2\sigma_\theta^4 > 0$. Moreover, two roots of Δ_2 in ρ are $\bar{\rho} = (\sqrt{\sigma_\theta^4 + 4\sigma_{\epsilon_z}^4} - 2\sigma_{\epsilon_z}^2)/\sigma_\theta^2$ and $\underline{\rho} = -(\sqrt{\sigma_\theta^4 + 4\sigma_{\epsilon_z}^4} + 2\sigma_{\epsilon_z}^2)/\sigma_\theta^2$. Clearly,

$\underline{\rho} < -1$ at $\sigma^2_{\epsilon_z} > 0$, and it is not binding. The binding root $\bar{\rho} \in (0,1)$ because $\bar{\rho} > 0$ and $(\sigma^2_\theta + 2\sigma^2_{\epsilon_z})^2 - (\sigma^4_\theta + 4\sigma^4_{\epsilon_z}) = 4\sigma^2_\theta\sigma^2_{\epsilon_z} > 0$. So, the claim goes through using the convexity of Δ_2 in ρ.

Similarly, $f(\rho) = 2 - \rho(4-\rho)$ is quadratic and convex in ρ because $\partial^2 f(\rho)/\partial^2\rho = 2 > 0$. Two roots of f in ρ are $\rho_1 = 2 - \sqrt{2}$ and $\rho_2 = 2 + \sqrt{2}$. Note that $\rho_1 \in (0,1)$ and $\rho_2 > 1$. Only the first root is binding and the claim is valid using the convexity of f in ρ. □

Proof of Lemma 5: **i)** Let $q_1 = \xi_0 + \xi_1(s_1 - \bar{\theta}) + \xi_2(z - \bar{\theta})$ and $q_2 = \psi_0 + \psi_1(s_2 - \bar{\theta}) + \psi_2 q_1 + \psi_3(z - \bar{\theta})$ in the PRE of the no-sharing game. Using backwards induction, the follower firm maximizes

$$\max_{q_2} E(\pi_2|s_2, q_1, z) = E((a + \omega\theta_2 - b(q_1 + q_2))q_2|s_2, q_1, z). \tag{28}$$

The FOC provides the best response of the follower to the changes in q_1 as

$$q_2(q_1) = \psi_0 + \psi_1(s_2 - \bar{\theta}) + \psi_2 q_1 + \psi_3(z - \bar{\theta}) = \frac{a - bq_1 + \omega E(\theta_2|s_2, q_1, z)}{2b}. \tag{29}$$

In a PRE, the follower perfectly infers the private signal of the leader from the leader's strategy $q_1 = \xi_0 + \xi_1(s_1 - \bar{\theta}) + \xi_2(z - \bar{\theta})$ as $s_1 = (q_1 - \xi_0 + \xi_1\bar{\theta} - \xi_2(z - \bar{\theta}))/\xi_1$, which implies that

$$E(\theta_2|s_2, q_1, z) = E(\theta_2|s_2, s_1 = \frac{q_1 - \xi_0 + \xi_1\bar{\theta} - \xi_2(z - \bar{\theta})}{\xi_1}, z). \tag{30}$$

In the first stage, the leader maximizes its expected profit after inserting $q_2(q_1)$ from above

$$\max_{q_1} E(\pi_1(q_1, q_2(q_1))|s_1, z) = E\big(q_1(a + \omega\theta_1 - b(q_1 + \psi_0 + \psi_1(s_2 - \bar{\theta}) + \psi_2 q_1 + \psi_3(z - \bar{\theta})))|s_1, z\big).$$

The FOC is

$$q_1 = \xi_0 + \xi_1(s_1 - \bar{\theta}) + \xi_2(z - \bar{\theta}) = \frac{a + \omega E(\theta_1|s_1, z) - b(\psi_0 - \bar{\theta}\psi_1 + \psi_1 E(s_2|s_1, z) + \psi_3(z - \bar{\theta}))}{2b(1 + \psi_2)}. \tag{31}$$

First, we substitute $E(\theta_2|s_1, s_2, z)$ from Lemma A9 into (29) in light of (30). Second, we substitute $E(\theta_1|s_1, z)$ and $E(s_2|s_1, z)$ from Lemma A9 into (31). Then, we get 7 equations with 7 unknowns by (29) and (31). By solving them, we derive the constants $\xi_0, \xi_1, \xi_2, \psi_0, \psi_1, \psi_2$, and ψ_3 as stated in the lemma. The second-order conditions would hold because $\partial^2 E(\pi_2(q_1, q_2))/\partial^2 q_2 = -2b < 0$ and

$$\frac{\partial^2 E(\pi_1(q_1, q_2(q_1)))}{\partial^2 q_1} = -2b(1 + \psi_2) = -\frac{\sigma^2_\theta\left(\Delta_3(\Delta_1 + \Delta_7) + 2\Delta_1(\Delta_4 + \Delta_5)\sigma^2_{\epsilon_2}\right)}{2\Delta_6(\Delta_4 + \Delta_5)(\xi_1/\omega)} < 0$$

because $\Delta_1, \Delta_3, \Delta_5, \Delta_6 > 0$, $\Delta_4, \Delta_7 \geq 0$, $\xi_1/\omega > 0$ at $\rho \in [-1, 1]$ and $b > 0$. The expected PRE price and output levels are derived after letting $E(s_i) = E(z) = \bar{\theta}$ in the equilibrium outcomes.

ii) Let $q_1 = D_0 + D_1(s_1 - \bar{\theta}) + D_2(z - \bar{\theta})$ and $q_2 = E_0 + E_1(s_1 - \bar{\theta}) + E_2(s_2 - \bar{\theta}) + E_3 q_1 + E_4(z - \bar{\theta})$ in the equilibrium of the partial-sharing game, where only the leader firm partially shares its private information (s_1) with the follower firm. Using backwards induction, the follower firm maximizes

$$\max_{q_2} E(\pi_2|s_1, s_2, q_1, z) = E((a + \omega\theta_2 - b(q_1 + q_2))q_2|s_1, s_2, q_1, z). \tag{32}$$

The best response of the follower follows from its FOC.

$$q_2(q_1) = E_0 + E_1(s_1 - \bar{\theta}) + E_2(s_2 - \bar{\theta}) + E_3 q_1 + E_4(z - \bar{\theta}) = \frac{a - bq_1 + \omega E(\theta_2|s_1, s_2, q_1, z)}{2b}. \tag{33}$$

As the information of the leader remains the same between no-sharing and partial-sharing regimes, the FOC of the leader is similar to (31) and it becomes

$$\begin{aligned} q_1 &= D_0 + D_1(s_1 - \bar{\theta}) + D_2(z - \bar{\theta}) \\ &= \frac{a + \omega E(\theta_1|s_1, z) - b(E_0 - \bar{\theta}(E_1 + E_2 + E_4) + E_1 s_1 + E_2 E(s_2|s_1, z) + E_4 z)}{2b(1 + E_3)}. \end{aligned} \tag{34}$$

We first substitute $E(\theta_2|s_1, s_2, z)$ from Lemma A9 into (33). Second, we substitute $E(\theta_1|s_1, z)$ and $E(s_2|s_1, z)$ from Lemma A9 into (34). Then, we get 8 equations with 8 constants by (33) and (34). By solving them, we derive the constants as stated in the lemma. The second order conditions would hold because $\partial^2 E(\pi_2(q_1, q_2))/\partial^2 q_2 = -2b < 0$ and $\partial^2 E(\pi_1(q_1, q_2(q_1)))/\partial^2 q_1 = -2b(1+E_3) = -b < 0$. □

Proof of Proposition 5: The total output, the profit of firm i, consumer surplus, and total surplus are defined by $Q = q_1 + q_2$, $\pi_i = (a - bQ + \omega\theta_i)q_i$, $CS = bQ^2/2$, and $TS = CS + \pi_1 + \pi_2 = aQ - bQ^2/2 + \omega(\theta_1 q_1 + \theta_2 q_2)$, respectively. We can derive the non-covariance parts of the expected equilibrium outcomes in no-sharing and partial-sharing regimes after letting $E(\theta_i) = E(s_i) = E(z) = \bar{\theta}$, where $z = (\theta_1 + \theta_2)/2 + \epsilon_z$. To find the covariance parts of equilibrium outcomes, we let $a = \bar{\theta} = 0$, $E(\theta_1 s_1) = \sigma_\theta^2$, $E(\theta_1 z) = E(s_1 z) = E(s_2 z) = \sigma_\theta^2(1+\rho)/2$, $E(\theta_1 s_2) = E(s_1 s_2) = \rho\sigma_\theta^2$, $E(s_1^2) = \sigma_\theta^2 + \sigma_{\epsilon_1}^2$, $E(s_2^2) = \sigma_\theta^2 + \sigma_{\epsilon_2}^2$, and $E(z^2) = \sigma_\theta^2(1+\rho)/2 + \sigma_{\epsilon_z}^2$ according to our information structure. First, let

$$\Delta_7 = 2(1-\rho)\left(2\sigma_{\epsilon_z}^2 + \sigma_\theta^2(1+\rho)\right) \qquad \text{and} \qquad \Delta_8 = (1+\rho)^2\left(\sigma_{\epsilon_1}^2 + \sigma_\theta^2(1-\rho)^2\right).$$

i) Using Lemma 5, the subtraction of expected equilibrium total output, consumer surplus and total surplus in the no sharing game from the partial sharing game are

$$\begin{aligned} &E(Q_{PS}^{**}) - E(Q_{NS}^{**}) = \frac{\sigma_{\epsilon_2}^2(\Delta_4 + \Delta_5)\left(a + \omega\bar{\theta}\right)}{4b\sigma_\theta^2\Delta_6(\Delta_1 + \Delta_7)} \times \Delta_2, \\ &E(CS_{PS}^{**}) - E(CS_{NS}^{**}) = \Big(\frac{\sigma_{\epsilon_2}^2(\Delta_4+\Delta_5)\left(a+\omega\bar{\theta}\right)^2\left(6\Delta_3(\Delta_1+\Delta_7)+(5\Delta_1+7\Delta_7)(\Delta_4+\Delta_5)\sigma_{\epsilon_2}^2\right)}{32b\sigma_\theta^2\Delta_6^2(\Delta_1+\Delta_7)^2} \\ &\quad + \frac{\omega^2\sigma_\theta^2\sigma_{\epsilon_2}^2\left(\sigma_{\epsilon_2}^2(\Delta_4+\Delta_5)\left((1+\rho)^2(5\Delta_1+7\Delta_7)\sigma_{\epsilon_1}^2+F_1\right)+2\Delta_3(\Delta_1+\Delta_7)\left(3\left(\Delta_1\rho^2+\Delta_8\right)+16\left(1-\rho^2\right)\sigma_{\epsilon_z}^2\right)\right)}{32b\Delta_6^2(\Delta_1+\Delta_7)^2}\Big) \times \Delta_2, \\ &E(TS_{PS}^{**}) - E(TS_{NS}^{**}) = \Big(\frac{\sigma_{\epsilon_2}^2(\Delta_4+\Delta_5)\left(a+\omega\bar{\theta}\right)^2\left(2\Delta_3(\Delta_1+\Delta_7)+(3\Delta_1+\Delta_7)(\Delta_4+\Delta_5)\sigma_{\epsilon_2}^2\right)}{32b\sigma_\theta^2\Delta_6^2(\Delta_1+\Delta_7)^2} + \\ &\frac{\omega^2\sigma_\theta^2\sigma_{\epsilon_2}^2\left(\sigma_{\epsilon_2}^2(\Delta_4+\Delta_5)\left((1+\rho)^2(3\Delta_1+\Delta_7)\sigma_{\epsilon_1}^2+F_2\right)+2\Delta_3(\Delta_1+\Delta_7)\left(4(1-\rho)(2\Delta_1+\Delta_7)+\Delta_1\rho^2+\Delta_8\right)\right)}{32b\Delta_6^2(\Delta_1+\Delta_7)^2}\Big) \times \Delta_2, \end{aligned}$$

respectively, where

$$\begin{aligned} &F_1 = (23-4\rho)(1-\rho^2)^2\sigma_\theta^4 + 4(1-\rho^2)(24+(1-\rho)(16+\rho))\sigma_\theta^2\sigma_{\epsilon_z}^2 + 16(4+\rho)(2-\rho)^2\sigma_{\epsilon_z}^4, \\ &F_2 = (97-92\rho)(1-\rho^2)^2\sigma_\theta^4 + 4(8+71(1-\rho)+57(1-\rho)^2)(1-\rho^2)\sigma_\theta^2\sigma_{\epsilon_z}^2 + 48(2-\rho)^2(4-3\rho)\sigma_{\epsilon_z}^4. \end{aligned}$$

Note that $a + \omega\bar{\theta} > 0$ by assumption, $b, \Delta_1, \Delta_5, \Delta_6, F_1, F_2 > 0$ and $\Delta_4, \Delta_7, \Delta_8 \geq 0$ at $\rho \in [-1, 1]$. Accordingly, $E(Q_{PS}^{**}) \gtreqless E(Q_{NS}^{**})$, $E(CS_{PS}^{**}) \gtreqless E(CS_{NS}^{**})$, or $E(TS_{PS}^{**}) \gtreqless E(TS_{NS}^{**})$ if $\Delta_2 \gtreqless 0$. The proof follows because $\Delta_2 \gtrless 0$ if $\rho \gtrless \bar{\rho}$ by Lemma A10. Moreover, the leader's expected profit always increases after it shares its private information with the follower because

$$E(\pi_{1,PS}^{**}) - E(\pi_{1,NS}^{**}) = \frac{1}{8b}\left(\frac{\sigma_{\epsilon_2}^2 \Delta_2(\Delta_4 + \Delta_5)\left(a + \omega\bar{\theta}\right)}{\Delta_6(\Delta_1 + \Delta_7)\sigma_\theta^2}\right)^2 + \frac{\Delta_2^2\omega^2\sigma_{\epsilon_2}^4(\Delta_4 + \Delta_5)\left(\Delta_1(2-\rho)^2 + \Delta_8\right)}{8b\Delta_6^2(\Delta_1 + \Delta_7)^2}$$

is positive because $b, \Delta_1, \Delta_5, \Delta_6 > 0$ and $\Delta_4, \Delta_7, \Delta_8 \geq 0$ at $\rho \in [-1, 1]$.

Finally, let $J = (E(\pi_{2,PS}^{**}) - E(\pi_{2,NS}^{**}))/\Delta_2$ and $H = (E(\Pi_{PS}^{**}) - E(\Pi_{NS}^{**}))/\Delta_2$. We show below that $J < 0$ at $a \geq 5\sigma_\theta - \omega\bar{\theta}$ and $H < 0$ at $a \geq 4\sigma_\theta - \omega\bar{\theta}$. Thus, $\text{sign}\{E(\pi_{2,PS}^{**}) - E(\pi_{2,NS}^{**})\} = \text{sign}\{E(\Pi_{PS}^{**}) - E(\Pi_{NS}^{**})\} = \text{sign}\{-\Delta_2\}$, and the expected profit of follower and expected total profit are smaller (greater) in the partial-sharing game than in the no-sharing game at sufficiently large a if $\Delta_2 > 0$ ($\Delta_2 < 0$) or if $\rho > \bar{\rho}$ ($\rho < \bar{\rho}$).

Using Lemma 5 and the definitions of the profit formulas, it follows that

$$\frac{\partial J}{\partial a} = -\frac{2\sigma_{\epsilon_2}^2(a + \omega\bar{\theta})(\Delta_4 + \Delta_5)\left(2\Delta_3(\Delta_1 + \Delta_7) + (3\Delta_1 + \Delta_7)(\Delta_4 + \Delta_5)\sigma_{\epsilon_2}^2\right)}{16b\sigma_\theta^2\Delta_6^2(\Delta_1 + \Delta_7)^2} < 0$$

and

$$\frac{\partial H}{\partial a} = -\frac{2\sigma_{\epsilon_2}^2(a + \omega\bar{\theta})(\Delta_4 + \Delta_5)\left(2\Delta_3(\Delta_1 + \Delta_7) + (\Delta_1 + 3\Delta_7)(\Delta_4 + \Delta_5)\sigma_{\epsilon_2}^2\right)}{16b\sigma_\theta^2\Delta_6^2(\Delta_1 + \Delta_7)^2} < 0$$

because $a > -\omega\bar{\theta}$, $b, \Delta_1, \Delta_3, \Delta_5 > 0$ and $\Delta_4, \Delta_7 \geq 0$ at $\rho \in [-1, 1]$. Moreover, when $a = 5\sigma_\theta - \omega\bar{\theta}$,

$$J(a = 5\sigma_\theta - \omega\bar{\theta}) = -\frac{\sigma_{\epsilon_2}^2\left(2\Delta_3(\Delta_1 + \Delta_7)g_2 + \sigma_{\epsilon_2}^2(\Delta_4 + \Delta_5)\left(2\sigma_\theta^2 g_3 + (3\Delta_1 + \Delta_7)\sigma_{\epsilon_1}^2 g_1\right)\right)}{16b\Delta_6^2(\Delta_1 + \Delta_7)^2}$$

and when $a = 4\sigma_\theta - \omega\bar{\theta}$,

$$H(a = 4\sigma_\theta - \omega\bar{\theta}) = -\frac{\sigma_{\epsilon_2}^2\left(2\Delta_3(\Delta_1 + \Delta_7)\left(\sigma_{\epsilon_1}^2 g_4 + \sigma_\theta^2 g_5\right) + \sigma_{\epsilon_2}^2(\Delta_4 + \Delta_5)\left(g_6 + (\Delta_1 + 3\Delta_7)\sigma_{\epsilon_1}^2 g_4\right)\right)}{16b\Delta_6^2(\Delta_1 + \Delta_7)^2}.$$

Both of these derivations are negative because $b, \Delta_1, \Delta_3, \Delta_5, g_1, g_2, g_3, g_4, g_5, g_6 > 0$ and $\Delta_4, \Delta_7 \geq 0$ at $\rho \in [-1, 1]$, where

$$\begin{aligned}
&g_1 = 25\Delta_1 + 26(1+\rho)^2\sigma_\theta^2, \qquad g_2 = 2\Delta_1(9 + 4\rho)\sigma_\theta^2 + g_1\sigma_{\epsilon_1}^2 + 12\left(1 - \rho^2\right)\sigma_\theta^2\sigma_{\epsilon_z}^2,\\
&g_3 = \left(1 - \rho^2\right)\left(13\sigma_\theta^2\left(1 - \rho^2\right)\left(6\sigma_{\epsilon_z}^2 + (3 + 2\rho)\sigma_\theta^2\right) + 52(5 + 3\rho)\sigma_\theta^2\sigma_{\epsilon_z}^2 + (200 + 56(1-\rho))\sigma_{\epsilon_z}^4\right)\\
&\qquad +208(2+\rho)\sigma_{\epsilon_z}^4,\\
&g_4 = 16\Delta_1 + 17(1+\rho)^2\sigma_\theta^2, \qquad g_5 = 12\left(1 - \rho^2\right)\sigma_{\epsilon_z}^2 + \Delta_1(9 + 8\rho),\\
&g_6 = 16\left(17 + 7(1-\rho) + (19 + 5(1-\rho))\left(1 - \rho^2\right)\right)\sigma_\theta^2\sigma_{\epsilon_z}^4\\
&\qquad +\sigma_\theta^4\left(1 - \rho^2\right)\left(\sigma_\theta^2\left(1 - \rho^2\right)(31 + 44(1+\rho)) + 4\sigma_{\epsilon_z}^2\left(62 + 37(1+\rho) + 29\left(1 - \rho^2\right)\right)\right).
\end{aligned}$$

As $\partial J/\partial a < 0$ and $\partial H/\partial a < 0$, $J < 0$ at $a \geq 5\sigma_\theta - \omega\bar{\theta}$ and $H < 0$ at $a \geq 4\sigma_\theta - \omega\bar{\theta}$, as claimed.

ii) Using Lemma 5 and the associated expected profit formulas, it follows that as $b > 0$,

$$\frac{\partial E(\pi_{1,PS}^{**})}{\partial\sigma_{\epsilon_1}^2} = -\frac{\omega^2(\Delta_1 + \Delta_7)^2\sigma_\theta^4}{8b(\Delta_4 + \Delta_5)^2} < 0 \text{ and } \frac{\partial E(\pi_{2,PS}^{**})}{\partial\sigma_{\epsilon_2}^2} = -\frac{\omega^2\sigma_\theta^4(\Delta_1\sigma_{\epsilon_1}^2 + 4\sigma_\theta^2\sigma_{\epsilon_z}^2(1-\rho^2))^2}{4b\Delta_6^2} < 0.$$

iii) Using Lemma 5, we can derive that

$$
\begin{aligned}
\frac{\partial E(q_{1,NS}^{**})}{\partial \sigma_{\epsilon_1}^2} &= -2\frac{\partial E(q_{2,NS}^{**})}{\partial \sigma_{\epsilon_1}^2} = \frac{\Delta_2^3 \sigma_{\epsilon_2}^2 \left(a+\omega\bar{\theta}\right)}{2b\sigma_\theta^2 \Delta_6^2(\Delta_1+\Delta_7)}, \\
\frac{\partial E(q_{1,NS}^{**})}{\partial \sigma_{\epsilon_2}^2} &= -2\frac{\partial E(q_{2,NS}^{**})}{\partial \sigma_{\epsilon_2}^2} = -\frac{\Delta_2\Delta_3(\Delta_4+\Delta_5)\left(a+\omega\bar{\theta}\right)}{2b\sigma_\theta^2 \Delta_6^2(\Delta_1+\Delta_7)}, \\
\frac{\partial E(Q_{NS}^{**})}{\partial \sigma_{\epsilon_1}^2} &= \frac{\Delta_2^3 \sigma_{\epsilon_2}^2 \left(a+\omega\bar{\theta}\right)}{4b\sigma_\theta^2 \Delta_6^2(\Delta_1+\Delta_7)}, \\
\frac{\partial E(Q_{NS}^{**})}{\partial \sigma_{\epsilon_2}^2} &= -\frac{\Delta_2\Delta_3(\Delta_4+\Delta_5)\left(a+\omega\bar{\theta}\right)}{4b\sigma_\theta^2 \Delta_6^2(\Delta_1+\Delta_7)}.
\end{aligned}
\tag{35}
$$

As $a > -\omega\bar{\theta}$, $b, \Delta_1, \Delta_3, \Delta_5 > 0$ and $\Delta_4, \Delta_7 \geq 0$ at $\rho \in [-1,1]$, the above derivations reveal that $\frac{\partial E(q_{1,NS}^{**})}{\partial \sigma_{\epsilon_1}^2} \gtreqless 0$, $\frac{\partial E(q_{2,NS}^{**})}{\partial \sigma_{\epsilon_1}^2} \lesseqgtr 0$, $\frac{\partial E(q_{1,NS}^{**})}{\partial \sigma_{\epsilon_2}^2} \lesseqgtr 0$, $\frac{\partial E(q_{2,NS}^{**})}{\partial \sigma_{\epsilon_2}^2} \gtreqless 0$, $\frac{\partial E(Q_{NS}^{**})}{\partial \sigma_{\epsilon_1}^2} \gtreqless 0$ and $\frac{\partial E(Q_{NS}^{**})}{\partial \sigma_{\epsilon_2}^2} \lesseqgtr 0$ if $\Delta_2 \gtreqless 0$ or $\rho \gtreqless \bar{\rho}$ by Lemma A10.

iv) First define

$$
\begin{aligned}
U = {} & \sigma_\theta^4\big(4\Delta_1\Delta_7\sigma_{\epsilon_1}^2 + \sigma_{\epsilon_1}^4(1+\rho)\left(\Delta_1+3\Delta_7+4(4-3\rho)\sigma_{\epsilon_z}^2\right) \\
& +\sigma_\theta^2(1-\rho)(3\left(1-\rho^2\right)^2\sigma_\theta^4 + 8(2-\rho)\left(1-\rho^2\right)\sigma_\theta^2\sigma_{\epsilon_z}^2 + 16\left(2+\rho^2\right)\sigma_{\epsilon_z}^4)\big).
\end{aligned}
$$

We can show that

$$
\begin{aligned}
& \frac{\partial E(q_{1,NS}^{**})}{\partial \sigma_{\epsilon_z}^2} = 2\frac{\partial E(Q_{NS}^{**})}{\partial \sigma_{\epsilon_z}^2}, \qquad \frac{\partial E(q_{2,NS}^{**})}{\partial \sigma_{\epsilon_z}^2} = -\frac{\partial E(Q_{NS}^{**})}{\partial \sigma_{\epsilon_z}^2}, \qquad \text{and} \\
& \frac{\partial E(Q_{NS}^{**})}{\partial \sigma_{\epsilon_z}^2} = -\frac{\sigma_\theta^2\sigma_{\epsilon_2}^2\left(a+\omega\bar{\theta}\right)(1+\rho)^2\left((1-\rho)\left(2\sigma_{\epsilon_2}^2(\Delta_4+\Delta_5)^2+U\right)+16\sigma_\theta^2\sigma_{\epsilon_1}^4\sigma_{\epsilon_z}^4(2-\rho(4-\rho))\right)}{b\Delta_6^2(\Delta_1+\Delta_7)^2}.
\end{aligned}
$$

It is clear that $\frac{\partial E(q_{1,NS}^{**})}{\partial \sigma_{\epsilon_z}^2} = \frac{\partial E(q_{2,NS}^{**})}{\partial \sigma_{\epsilon_z}^2} = \frac{\partial E(Q_{NS}^{**})}{\partial \sigma_{\epsilon_z}^2} = 0$ if $\rho = -1$. As $a > -\omega\bar{\theta}$, $b, \Delta_1, \Delta_5, U > 0$, $\Delta_4, \Delta_7 \geq 0$ and $2-\rho(4-\rho) > 0$ if $\rho \leq 2-\sqrt{2}$ by Lemma A10, we can see from the above derivations that $\frac{\partial E(q_{1,NS}^{**})}{\partial \sigma_{\epsilon_z}^2} < 0$, $\frac{\partial E(Q_{NS}^{**})}{\partial \sigma_{\epsilon_z}^2} < 0$ and $\frac{\partial E(q_{2,NS}^{**})}{\partial \sigma_{\epsilon_z}^2} > 0$ at $\rho \in (-1, 2-\sqrt{2}]$. The signs of these partial derivatives are reversed at $\rho = 1$ because $\frac{\partial E(Q_{NS}^{**})}{\partial \sigma_{\epsilon_z}^2} = \frac{64\sigma_{\epsilon_2}^2\sigma_{\epsilon_1}^4\sigma_{\epsilon_z}^4\sigma_\theta^4(a+\omega\bar{\theta})}{b(\Delta_1\Delta_6)^2} > 0$.

□

Lemma A11. *Let* $\rho = 1$, $\delta_1 = \sigma_{\epsilon_1}^2\sigma_{\epsilon_2}^2\left(\sigma_\theta^2+\sigma_{\epsilon_z}^2\right) + \sigma_\theta^2\left(\sigma_{\epsilon_1}^2+\sigma_{\epsilon_2}^2\right)\sigma_{\epsilon_z}^2$, $\delta_2 = \sigma_\theta^2\sigma_{\epsilon_z}^2 + \sigma_{\epsilon_1}^2\left(\sigma_\theta^2+\sigma_{\epsilon_z}^2\right)$, $\delta_3 = \sigma_\theta^2\sigma_{\epsilon_z}^2 + \sigma_{\epsilon_2}^2\left(\sigma_\theta^2+\sigma_{\epsilon_z}^2\right)$, $\delta_4 = \delta_1 + \sigma_{\epsilon_2}^2\delta_2$, *and* $\delta_5 = \delta_2(a+\omega\bar{\theta})^2 + \omega^2\sigma_\theta^4(\sigma_{\epsilon_1}^2+\sigma_{\epsilon_z}^2)$.

i) $E(q_{1,NS}^{**}) = \dfrac{\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2\sigma_\theta^2(a+\omega\bar{\theta})}{2b\delta_1}$, $E(q_{2,NS}^{**}) = \dfrac{\delta_4(a+\omega\bar{\theta})}{4b\delta_1}$, *and* $E(\Pi_{NS}^{**}) = \sum_{i=1,2} E(\pi_{i,NS}^{**})$.

ii) $E(\pi_{1,NS}^{**}) = \dfrac{\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2\sigma_\theta^2\delta_4\delta_5}{8b\delta_1^2\delta_2}$ *and* $E(\pi_{2,NS}^{**}) = \dfrac{4\omega^2\sigma_{\epsilon_1}^4\sigma_{\epsilon_z}^4\sigma_\theta^4\delta_1 + \delta_4^2\delta_5}{16b\delta_1^2\delta_2}$.

iii) $E(CS_{NS}^{**}) = \dfrac{4\omega^2\sigma_{\epsilon_1}^4\sigma_{\epsilon_z}^4\sigma_\theta^4\delta_1 + (\delta_4+2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2\sigma_\theta^2)^2\delta_5}{32b\delta_1^2\delta_2}$ *and* $E(TS_{NS}^{**}) = E(CS_{NS}^{**}) + E(\Pi_{NS}^{**})$.

Proof of Lemma A11: When $\rho = 1$, $\theta_1 = \theta_2 = \theta$. The expected profit of firm i, consumer surplus, total surplus are calculated from $E(\pi_{i,NS}^{**}) = E((a+\omega\theta - b(q_{1,NS}^{**}+q_{2,NS}^{**}))q_{i,NS}^{**})$, $E(CS_{NS}^{**}) = bE((q_{1,NS}^{**}+q_{2,NS}^{**})^2)/2$ and $E(TS_{NS}^{**}) = E(\pi_{1,NS}^{**}) + E(\pi_{2,NS}^{**}) + E(CS_{NS}^{**})$, respectively. Furthermore, we use $E(\theta) = E(s_i) = \bar{\theta}$, $E(\theta s_j) = E(s_i s_j) = E(s_i z) = \sigma_\theta^2 + \bar{\theta}^2$, $E(s_i^2) = \sigma_{\epsilon_i}^2 + \sigma_\theta^2 + \bar{\theta}^2$ and $E(z^2) = \sigma_{\epsilon_z}^2 + \sigma_\theta^2 + \bar{\theta}^2$ in these calculations in light of our assumptions to prove the stated results. □

Proof of Proposition 6: First, let

$\mu_1 = -\omega^2\sigma_\theta^6\sigma_{\epsilon_1}^6\sigma_{\epsilon_z}^6,\ \mu_2 = -3\delta_2\omega^2\sigma_\theta^4\sigma_{\epsilon_1}^4\sigma_{\epsilon_z}^4,\ \mu_3 = 2\delta_2^2(\delta_5 - \omega^2\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2),\ \mu_4 = \delta_2(2\delta_5 - \omega^2\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2),$
$\mu_5 = \sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2(\delta_5 - 2\omega^2\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2),\quad \mu_6 = \mu_3/\delta_2,\quad \mu_7 = -\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2\mu_4,\quad \mu_8 = -4\delta_2^2\omega^2\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2$
$\mu_9 = -4\delta_2^3\omega^2,\ \mu_{10} = 3\delta_2(2\delta_5\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2 - 3\omega^2\sigma_\theta^4\sigma_{\epsilon_1}^4\sigma_{\epsilon_z}^4),$
$Y_1 = \mu_1 + \mu_2\sigma_{\epsilon_2}^2 + \mu_3\sigma_{\epsilon_2}^4,\quad Y_2 = \mu_2/\delta_2 + \mu_4\sigma_{\epsilon_2}^2 + 2\delta_2^2\omega^2\sigma_{\epsilon_2}^4,\quad Y_3 = \mu_5 + \mu_6\sigma_{\epsilon_2}^2,$
$Y_4 = -\mu_1 + \mu_7\sigma_{\epsilon_2}^2 + \mu_8\sigma_{\epsilon_2}^4 + \mu_9\sigma_{\epsilon_2}^6,\quad Y_5 = 5\mu_1 + \mu_{10}\sigma_{\epsilon_2}^2 + 4\delta_2^2\mu_5\sigma_{\epsilon_2}^4/(\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2) + \mu_9\sigma_{\epsilon_2}^6,$
$Y_6 = 3\mu_1 + (2\mu_7 + \mu_{10})\sigma_{\epsilon_2}^2 + (2\mu_8 + 4\delta_2^2\mu_5/(\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2))\sigma_{\epsilon_2}^4 + 3\mu_9\sigma_{\epsilon_2}^6.$

i) Using Lemma A11, for each $h \in \{q_{1,NS}^{**}, q_{2,NS}^{**}, \pi_{1,NS}^{**}, \pi_{2,NS}^{**}, CS_{NS}^{**}, TS_{NS}^{**}\}$, $\partial E(h)/\partial\sigma_{\epsilon_1}^2 = (\sigma_{\epsilon_z}^4/\sigma_{\epsilon_1}^4) \times \partial E(h)/\partial\sigma_{\epsilon_z}^2$. The claim follows from these derivations.

ii) Using Lemma A11, we have that

$$\frac{\partial E(\pi_{1,NS}^{**})}{\partial\sigma_{\epsilon_1}^2} = \frac{\sigma_\theta^4\sigma_{\epsilon_z}^4 Y_1}{8b\delta_1^3\delta_2^2} \quad \text{and} \quad \frac{\partial E(\pi_{2,NS}^{**})}{\partial\sigma_{\epsilon_1}^2} = -\frac{\sigma_\theta^4\sigma_{\epsilon_z}^4(2\delta_2\sigma_{\epsilon_2}^2 + \sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2)Y_2}{16b\delta_1^3\delta_2^2}. \tag{36}$$

As $b, \delta_1, \delta_2 > 0$, $\text{sign}\{\partial E(\pi_{1,NS}^{**})/\partial\sigma_{\epsilon_1}^2\} = \text{sign}\{Y_1\}$ and $\text{sign}\{\partial E(\pi_{2,NS}^{**})/\partial\sigma_{\epsilon_1}^2\} = \text{sign}\{-Y_2\}$. As $Y_1 = \mu_1 + \mu_2\sigma_{\epsilon_2}^2 + \mu_3\sigma_{\epsilon_2}^4$ and $Y_2 = \mu_2/\delta_2 + \mu_4\sigma_{\epsilon_2}^2 + 2\delta_2^2\omega^2\sigma_{\epsilon_2}^4$, Y_1 and Y_2 are quadratic in $\sigma_{\epsilon_2}^2$.

As $\delta_2 > 0$, it is clear that $\mu_1, \mu_2 < 0$. Moreover, μ_3 is increasing in a when a obtains its lowest possible value of $\sigma_\theta - \omega\bar{\theta}$ by assumption, μ_3 reduces to

$$\mu_3(a = \sigma_\theta - \omega\bar{\theta}) = 2\delta_2^2(\sigma_\theta^4(1+\omega^2)(\sigma_{\epsilon_z}^2 + \sigma_{\epsilon_1}^2) + (1-\omega^2)\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2), \tag{37}$$

which is positive because $\omega^2 = 1$ as $\omega \in \{-1, 1\}$. In summary, $Y_1 = \mu_1 + \mu_2\sigma_{\epsilon_2}^2 + \mu_3\sigma_{\epsilon_2}^4$, $\mu_1 < 0$, $\mu_2 < 0$, and $\mu_3 > 0$. Y_1 has two real roots in $\sigma_{\epsilon_2}^2$ and by the Descartes' rule of signs, one root of Y_1 is positive (call it $\bar{\sigma}_{\epsilon_2}^2$) and the other root is negative and it is not binding. As $\sigma_{\epsilon_2}^2 > 0$, $\delta_1 > 0$ and $Y_1(\sigma_{\epsilon_2}^2 = 0) = \mu_1 < 0$, $\partial E(\pi_{1,NS})/\partial\sigma_{\epsilon_1}^2 \gtreqless 0$ if $Y_1 \gtreqless 0$ or $\sigma_{\epsilon_2}^2 \gtreqless \bar{\sigma}_{\epsilon_2}^2$, as claimed.

Similarly, $\mu_2/\delta_2 < 0$ and $2\delta_2^2\omega^2 > 0$ because $\delta_2 > 0$. The number of sign changes in the sequence of quadratic polynomial $Y_2(\sigma_{\epsilon_2}^2)$'s coefficients change exactly once irrespective of μ_4 being positive or negative. By the Descartes' rule of signs, there is exactly one positive real root of Y_2 in $\sigma_{\epsilon_2}^2$. As $Y_2(\sigma_{\epsilon_2}^2 = 0) = \mu_2/\delta_2 < 0$, $\partial E(\pi_{2,NS}^{**})/\partial\sigma_{\epsilon_1}^2 > 0$ if and only if $\sigma_{\epsilon_2}^2$ is sufficiently low (or $\tau_{\epsilon_2} = 1/\sigma_{\epsilon_2}^2$ is sufficiently high). By part i), $\text{sign}\{E(\pi_{i,NS}^{**})/\partial\sigma_{\epsilon_1}^2\} = \text{sign}\{E(\pi_{i,NS}^{**})/\partial\sigma_{\epsilon_z}^2\}$ for $i = 1, 2$.

iii) Using Lemma A11,

$$\frac{\partial E(\pi_{2,NS}^{**})}{\partial\sigma_{\epsilon_2}^2} = \frac{\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2 Y_3}{8b\delta_1^3} \tag{38}$$

As $b, \delta_1 > 0$, $\text{sign}\{E(\pi_{2,NS}^{**})/\partial\sigma_{\epsilon_2}^2\} = \text{sign}\{Y_3\}$. Moreover, $Y_3 = \mu_5 + \mu_6\sigma_{\epsilon_2}^2$ has one real root in $\sigma_{\epsilon_2}^2$,

$$\mu_5(a = 2\sigma_\theta - \omega\bar{\theta}) = \sigma_\theta^4\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2(2\delta_2 + 3\sigma_\theta^2(\sigma_{\epsilon_z}^2 + \sigma_{\epsilon_1}^2)) \text{ and } \mu_6(a = \sigma_\theta - \omega\bar{\theta}) = 4\delta_2\sigma_\theta^4(\sigma_{\epsilon_z}^2 + \sigma_{\epsilon_1}^2)$$

are both positive. As μ_5 and μ_6 are increasing in a, $\mu_5 > 0$ when $a \geq 2\sigma_\theta - \omega\bar{\theta}$ and $\mu_6 > 0$ when $a \geq \sigma_\theta - \omega\bar{\theta}$. These findings imply that if $a \geq 2\sigma_\theta - \omega\bar{\theta}$, then $Y_3 > 0$ and $E(\pi_{2,NS}^{**})/\partial\sigma_{\epsilon_2}^2 > 0$.

However, if $a = \sigma_\theta - \omega\bar{\theta}$ and $\sigma_\theta^2 < \sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2/(2(\sigma_{\epsilon_z}^2 + \sigma_{\epsilon_1}^2))$, $Y_3(\sigma_{\epsilon_2}^2 = 0) = \mu_5 < 0$ and we can show that

$$\frac{\partial E(\pi_{2,NS}^{**})}{\partial \sigma_{\epsilon_2}^2} \lesseqgtr 0 \Rightarrow Y_3 \lesseqgtr 0 \Rightarrow \sigma_{\epsilon_2}^2 \lesseqgtr \frac{\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2(\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2 - 2\sigma_\theta^2(\sigma_{\epsilon_z}^2 + \sigma_{\epsilon_1}^2))}{4\delta_2(\sigma_{\epsilon_z}^2 + \sigma_{\epsilon_1}^2)}. \tag{39}$$

iv) Using Lemma A11, direct computations show that

$$\begin{aligned}
\frac{\partial E(\pi_{1,NS}^{**})}{\partial \sigma_{\epsilon_2}^2} &= -\frac{\delta_2\delta_5\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_2}^2\sigma_{\epsilon_z}^2}{4b\delta_1^3} < 0,\\
\frac{\partial E(CS_{NS}^{**})}{\partial \sigma_{\epsilon_2}^2} &= -\frac{\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2(2\delta_1\delta_2((a+\omega\bar{\theta})^2 + \omega^2\sigma_\theta^2) + \delta_5\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2)}{16b\delta_1^3} < 0,\\
\frac{\partial E(TS_{NS}^{**})}{\partial \sigma_{\epsilon_2}^2} &= -\frac{\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2(2\delta_2\delta_5\sigma_{\epsilon_2}^2 + \sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2(6\delta_1\omega^2 + \delta_5))}{16b\delta_1^3} < 0,
\end{aligned} \tag{40}$$

because $a + \omega\bar{\theta} > 0$ by assumption and $b, \delta_1, \delta_2, \delta_5 > 0$. Similarly, we have

$$\frac{\partial E(\Pi_{NS}^{**})}{\partial \sigma_{\epsilon_2}^2} = \frac{\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2(\mu_5 - 2\delta_2\omega^2\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_2}^2\sigma_{\epsilon_z}^2)}{8b\delta_1^3} \quad \text{and} \quad \frac{\partial E(\Pi_{NS}^{**})}{\partial \sigma_{\epsilon_1}^2} = \frac{\sigma_\theta^4\sigma_{\epsilon_z}^4 Y_4}{16b\delta_1^3\delta_2^2} \tag{41}$$

by Lemma A11. Note that $\mu_5 > 0$ when $a \geq 2\sigma_\theta - \omega\bar{\theta}$ by our proof in part *iii*). Thus, when $a \geq 2\sigma_\theta - \omega\bar{\theta}$, we have that $\partial E(\Pi_{NS})/\partial\sigma_{\epsilon_2}^2 \gtreqless 0$ if $\sigma_{\epsilon_2}^2 \lesseqgtr \mu_5/(2\delta_2\omega^2\sigma_\theta^2\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2)$ by (41). Moreover, $\text{sign}\{E(\Pi_{NS}^{**})/\partial\sigma_{\epsilon_1}^2\} = \text{sign}\{Y_4\}$ because $b, \delta_1, \delta_2 > 0$. As $Y_4 = -\mu_1 + \mu_7\sigma_{\epsilon_2}^2 + \mu_8\sigma_{\epsilon_2}^4 + \mu_9\sigma_{\epsilon_2}^6$, $-\mu_1 > 0$, $\mu_8 < 0$ and $\mu_9 < 0$, the number of sign changes in the sequence of polynomial $Y_4(\sigma_{\epsilon_2}^2)$'s coefficients change exactly once irrespective of μ_7 being positive or negative. By the Descartes' rule of signs, there is exactly one positive real root of Y_4 in $\sigma_{\epsilon_2}^2$. As $Y_4(\sigma_{\epsilon_2}^2 = 0) = -\mu_1 > 0$, $\partial E(\Pi_{NS}^{**})/\partial\sigma_{\epsilon_1}^2 > 0$ if and only if $\sigma_{\epsilon_2}^2$ is sufficiently low. By part i), $\text{sign}\{E(\Pi_{NS}^{**})/\partial\sigma_{\epsilon_1}^2\} = \text{sign}\{E(\Pi_{NS}^{**})/\partial\sigma_{\epsilon_z}^2\}$.

v) Using Lemma A11, we can compute that

$$\frac{\partial E(CS_{NS}^{**})}{\partial \sigma_{\epsilon_1}^2} = \frac{\sigma_\theta^4\sigma_{\epsilon_z}^4 Y_5}{32b\delta_1^3\delta_2^2} \quad \text{and} \quad \frac{\partial E(TS_{NS}^{**})}{\partial \sigma_{\epsilon_1}^2} = \frac{\sigma_\theta^4\sigma_{\epsilon_z}^4 Y_6}{32b\delta_1^3\delta_2^2}. \tag{42}$$

Moreover, using $b, \delta_1, \delta_2 > 0$ and our finding in part *i*), $\text{sign}\{E(CS_{NS}^{**})/\partial\sigma_{\epsilon_1}^2\} = \text{sign}\{E(CS_{NS}^{**})/\partial\sigma_{\epsilon_z}^2\}$ $= \text{sign}\{Y_5\}$ and $\text{sign}\{E(TS_{NS}^{**})/\partial\sigma_{\epsilon_1}^2\} = \text{sign}\{E(TS_{NS}^{**})/\partial\sigma_{\epsilon_z}^2\} = \text{sign}\{Y_6\}$.

Note that $\mu_5 > 0$ at $a \geq 2\sigma_\theta - \omega\bar{\theta}$ by part iii), $\mu_1 < 0$ and $\mu_9 < 0$. Thus, irrespective of the sign of μ_{10}, the number of sign changes in the sequence of polynomial $Y_5(\sigma_{\epsilon_2}^2)$'s coefficients change is two. So, Y_5 has zero or two positive real roots in $\sigma_{\epsilon_2}^2$ by the Descartes' rule of signs. To see that it has two positive real roots at $a \geq 2\sigma_\theta - \omega\bar{\theta}$, $\lim_{\sigma_{\epsilon_2}^2 \to 0} \partial E(CS_{NS}^{**})/\partial\sigma_{\epsilon_1}^2 = 1.25 \lim_{\sigma_{\epsilon_2}^2 \to \infty} \partial E(CS_{NS}^{**})/\partial\sigma_{\epsilon_1}^2 = -5\omega^2\sigma_\theta^4\sigma_{\epsilon_z}^4/(32b\delta_2^2) < 0$ and when $\sigma_{\epsilon_2}^2 = \sigma_\theta^2$, $a = 2\sigma_\theta - \omega\bar{\theta}$, and $\omega \in \{-1, 1\}$,

$$\frac{\partial E(CS_{NS}^{**})}{\partial \sigma_{\epsilon_1}^2} = \frac{14\sigma_{\epsilon_1}^6\sigma_{\epsilon_z}^6 + 69\sigma_\theta^2\sigma_{\epsilon_1}^4\sigma_{\epsilon_z}^4(\sigma_{\epsilon_1}^2 + \sigma_{\epsilon_z}^2) + 66\sigma_\theta^4\sigma_{\epsilon_1}^2\sigma_{\epsilon_z}^2(\sigma_{\epsilon_1}^2 + \sigma_{\epsilon_z}^2)^2 + 16\sigma_\theta^6(\sigma_{\epsilon_1}^2 + \sigma_{\epsilon_z}^2)^3}{32b\sigma_\theta^{-4}\sigma_{\epsilon_z}^{-4}\delta_2^2(\sigma_\theta^2\sigma_{\epsilon_1}^2 + \sigma_{\epsilon_z}^2(\sigma_\theta^2 + 2\sigma_{\epsilon_1}^2))^3} > 0.$$

Since $\partial E(CS_{NS}^{**})/\partial a > 0$, $\partial E(CS_{NS}^{**})/\partial\sigma_{\epsilon_1}^2 > 0$ at $\sigma_{\epsilon_2}^2 = \sigma_\theta^2$ and $a \geq 2\sigma_\theta - \omega\bar{\theta}$. Thus, $\partial E(CS_{NS}^{**})/\partial\sigma_{\epsilon_1}^2$ < 0 if $\tau_{\epsilon_2} = 1/\sigma_{\epsilon_2}^2$ is sufficiently low or high and it becomes positive if τ_{ϵ_2} takes intermediate values.

Lastly, $\mu_1 < 0$, $\mu_9 < 0$, and $g = 2\mu_7 + \mu_{10} > 0$ at $a \geq 2\sigma_\theta - \omega\bar{\theta}$ as we show below. Thus, the number of sign changes in the sequence of polynomial $Y_6(\sigma^2_{\epsilon_2})$'s coefficients change is two. So, Y_6 has zero or two positive real roots in $\sigma^2_{\epsilon_2}$ by the Descartes' rule of signs. To see that it has two positive real roots at $a \geq 3\sigma_\theta - \omega\bar{\theta}$, $\lim_{\sigma^2_{\epsilon_2}\to 0} \partial E(TS^{**}_{NS})/\partial\sigma^2_{\epsilon_1} = 0.25 \lim_{\sigma^2_{\epsilon_2}\to 0} \partial E(TS^{**}_{NS})/\partial\sigma^2_{\epsilon_1} = -3\omega^2\sigma^4_\theta\sigma^4_{\epsilon_z}/(32b\delta^2_2) < 0$, and when $\sigma^2_{\epsilon_2} = \sigma^2_\theta$, $a = 3\sigma_\theta - \omega\bar{\theta}$, and $\omega \in \{-1, 1\}$,

$$\frac{\partial E(TS^{**}_{NS})}{\partial\sigma^2_{\epsilon_1}} = \frac{16\sigma^6_{\epsilon_1}\sigma^6_{\epsilon_z} + 84\sigma^4_\theta\sigma^2_{\epsilon_1}\sigma^2_{\epsilon_z}(\sigma^2_{\epsilon_1} + \sigma^2_{\epsilon_z})^2 + 75\sigma^2_\theta\sigma^4_{\epsilon_1}\sigma^4_{\epsilon_z}(\sigma^2_{\epsilon_1} + \sigma^2_{\epsilon_z}) + 28\sigma^6_\theta(\sigma^2_{\epsilon_1} + \sigma^2_{\epsilon_z})^3}{32b\sigma_\theta^{-4}\sigma_{\epsilon_z}^{-4}\delta^2_2(\sigma^2_\theta\sigma^2_{\epsilon_1} + \sigma^2_{\epsilon_z}(\sigma^2_\theta + 2\sigma^2_{\epsilon_1}))^3} > 0.$$

Since $\partial E(TS^{**}_{NS})/\partial a > 0$, $\partial E(TS^{**}_{NS})/\partial\sigma^2_{\epsilon_1} > 0$ at $\sigma^2_{\epsilon_2} = \sigma^2_\theta$ and $a \geq 3\sigma_\theta - \omega\bar{\theta}$. The same result holds at $a \geq 2\sigma_\theta - \omega\bar{\theta}$ at sufficiently large σ^2_θ. So, $\partial E(TS^{**}_{NS})/\partial\sigma^2_{\epsilon_1} < 0$ if $\tau_{\epsilon_2} = 1/\sigma^2_{\epsilon_2}$ is sufficiently low or high and it is positive if τ_{ϵ_2} takes intermediate values. Lastly, since g increases in a and

$$g(a = 2\sigma_\theta - \omega\bar{\theta}) = \delta_2\sigma^4_\theta\sigma^2_{\epsilon_1}\sigma^2_{\epsilon_z}((8 - 7\omega^2)\sigma^2_{\epsilon_1}\sigma^2_{\epsilon_z} + 2\sigma^2_\theta(4 + \omega^2)(\sigma^2_{\epsilon_z} + \sigma^2_{\epsilon_1})) > 0, \tag{43}$$

it holds that $g = 2\mu_7 + \mu_{10} > 0$ at $a \geq 2\sigma_\theta - \omega\bar{\theta}$, as claimed above. □

4 Differentiated goods Stackelberg model of Section IV

Let $n = 2$. Each firm i produces a differentiated product at a price level of p_i and at a production level of q_i. We consider the following linear inverse demand curve

$$p_i = a + \theta - bq_i - b\lambda q_j, \qquad i \neq j, \quad i, j \in \{1, 2\}, \tag{44}$$

where θ is a random variable with mean $\bar{\theta} \geq 0$ and variance σ^2_θ, $a > 0$ is the observed demand parameter, $-b < 0$ is the known slope parameter, and $\lambda \in [0, 1]$ is an inverse measure of product differentiation in the market. When $\lambda = 1$, products are perfect substitutes and no longer differentiable as in the main text. On the other hand, when $\lambda = 0$, products are unrelated. We normalize the unit cost of production to zero. The remaining assumptions of our base model in Section I of the main text are valid. Consumer surplus is given by $CS = (a + \theta)(q_1 + q_2) - \frac{b(q_1^2+q_2^2)}{2} - b\lambda q_1 q_2 - \sum_{j=1}^{2} p_j q_j$.

In the Stackelberg model, firm 1 first chooses its optimal quantity level q_1 after observing its private signal s_1. The follower firm 2 conditions its optimal quantity level on both its private signal s_2 and on the quantity level q_1. Cumbul (2021) derives the unique linear PRE in this model.

Lemma A12. *(Cumbul, 2021) In the two-player differentiated Stackelberg (DS) quantity-setting game, the unique linear PRE quantities of the leader and the follower are given, respectively, by*

$$q^*_{1,DS}(s_1) = \frac{(2\sigma^2_\epsilon(1 - \lambda) + \sigma^2_\theta(4 - 3\lambda))((a + \bar{\theta})\sigma^2_\epsilon + (a + s_1)\sigma^2_\theta)}{2b(\sigma^2_\epsilon + \sigma^2_\theta)(\sigma^2_\epsilon + 2\sigma^2_\theta)(2 - \lambda^2)},$$

$$q^*_{2,DS}(s_2, q^*_{1,DS}) = \frac{(a + s_2)\sigma^2_\theta}{2b(\sigma^2_\epsilon + 2\sigma^2_\theta)} + q^*_{1,DS}\left(\frac{(2 - \lambda^2)(\sigma^2_\epsilon + \sigma^2_\theta)}{2(1 - \lambda)\sigma^2_\epsilon + (4 - 3\lambda)\sigma^2_\theta} - \frac{\lambda}{2}\right).$$

The equilibrium quantity of the follower is a strategic complement to the leader's quantity for any $\lambda \in [0,1]$. When $\lambda = 0$, each firm produces the monopoly outcome, i.e., $E(q^*_{1,DS}) = E(q^*_{2,DS}) = (a+\bar{\theta})/(2b)$. When $\lambda = 1$, this lemma becomes a special case of Lemma 2 of the main text at $n = 2$.

Next, we provide our results in our horizontally differentiated goods model.

Proposition 10. *Let $\lambda \in (0,1]$. Whereas the expected production and profit of the leader, total output, consumer surplus, and total surplus increase, the expected production of the follower decreases with more precise information and higher prior uncertainty. Moreover, the expected total profit increases with more precise information and higher prior uncertainty if $\sigma^2_\epsilon/\sigma^2_\theta > (2-\lambda^2)/(2\lambda(1-\lambda))$.*

Thus, our comparative statics in Proposition 2, which were done for $\lambda = 1$ and $n \geq 2$, are robust at any $\lambda \in (0,1]$ and $n = 2$.

Proof of Proposition 10: **i)** The PRE outcomes are provided in Lemma B11 of online Appendix B. These values are from Cumbul (2021). Using this lemma,

$$
\begin{aligned}
&\tfrac{\partial E(q^*_{1,DS})}{\partial\sigma^2_\epsilon} = -\tfrac{\lambda\sigma^2_\theta(a+\bar{\theta})}{2b(2-\lambda^2)(\sigma^2_\epsilon+2\sigma^2_\theta)^2} < 0, \quad \tfrac{\partial E(q^*_{2,DS})}{\partial\sigma^2_\epsilon} = \tfrac{\lambda^2\sigma^2_\theta(a+\bar{\theta})}{4b(2-\lambda^2)(\sigma^2_\epsilon+2\sigma^2_\theta)^2} > 0,\\
&\tfrac{\partial E(Q^*_{DS})}{\partial\sigma^2_\epsilon} = -\tfrac{\lambda\sigma^2_\theta(2-\lambda)(a+\bar{\theta})}{4b(2-\lambda^2)(\sigma^2_\epsilon+2\sigma^2_\theta)^2} < 0,\\
&\tfrac{\partial E(\Pi^*_{DS})}{\partial\sigma^2_\epsilon} = -\tfrac{\lambda^2\sigma^2_\theta(2\lambda(1-\lambda)\sigma^2_\epsilon-(2-\lambda)^2\sigma^2_\theta)(a+\bar{\theta})^2}{8b(2-\lambda^2)^2(\sigma^2_\epsilon+2\sigma^2_\theta)^3} - \tfrac{\sigma^4_\theta(8y_1\sigma^6_\epsilon+16y_2\sigma^4_\epsilon\sigma^2_\theta+2y_3\sigma^2_\epsilon\sigma^4_\theta+8y_4\sigma^6_\theta)}{32b(2-\lambda^2)^2(\sigma^2_\epsilon+\sigma^2_\theta)^2(\sigma^2_\epsilon+2\sigma^2_\theta)^3},\\
&\tfrac{\partial E(\pi^*_{1,DS})}{\partial\sigma^2_\epsilon} = -W_0 - \tfrac{\sigma^4_\theta(4(1-\lambda)\sigma^6_\epsilon+4(6(1-\lambda)+\lambda^2)\sigma^4_\epsilon\sigma^2_\theta+(48(1-\lambda)+11\lambda^2)\sigma^2_\epsilon\sigma^4_\theta+8(2-\lambda)^2\sigma^6_\theta)}{8b(2-\lambda^2)(\sigma^2_\epsilon+\sigma^2_\theta)^2(\sigma^2_\epsilon+2\sigma^2_\theta)^3} < 0,\\
&\tfrac{\partial E(CS^*_{DS})}{\partial\sigma^2_\epsilon} = -W_1 - \tfrac{\sigma^4_\theta(4y_5\sigma^6_\epsilon+8y_6\sigma^4_\epsilon\sigma^2_\theta+y_7\sigma^2_\epsilon\sigma^4_\theta+4y_8\sigma^6_\theta)}{32b(2-\lambda^2)^2(\sigma^2_\epsilon+\sigma^2_\theta)^2(\sigma^2_\epsilon+2\sigma^2_\theta)^3} < 0,\\
&\tfrac{\partial E(TS^*_{DS})}{\partial\sigma^2_\epsilon} = -W_2 - \tfrac{\sigma^4_\theta(4(2y_1+y_5)\sigma^6_\epsilon+8(2y_2+y_6)\sigma^4_\epsilon\sigma^2_\theta+(2y_3+y_7)\sigma^2_\epsilon\sigma^4_\theta+4(2y_4+y_8)\sigma^6_\theta)}{32b(2-\lambda^2)^2(\sigma^2_\epsilon+\sigma^2_\theta)^2(\sigma^2_\epsilon+2\sigma^2_\theta)^3} < 0,
\end{aligned}
\tag{45}
$$

where

$$
\begin{aligned}
W_0 &= \tfrac{2\sigma^2_\theta\lambda^2(\sigma^2_\epsilon+\sigma^2_\theta)^3(a+\bar{\theta})^2}{8b(2-\lambda^2)(\sigma^2_\epsilon+\sigma^2_\theta)^2(\sigma^2_\epsilon+2\sigma^2_\theta)^3},\\
W_1 &= \tfrac{\lambda\sigma^2_\theta((6-4\lambda+(1-\lambda)^2(2+4\lambda))\sigma^2_\epsilon+(2-\lambda)(5+(1-\lambda)(3+5\lambda))\sigma^2_\theta)(a+\bar{\theta})^2}{16b(2-\lambda^2)^2(\sigma^2_\epsilon+2\sigma^2_\theta)^3},\\
W_2 &= \tfrac{\lambda\sigma^2_\theta((2(2(1-\lambda)+2-\lambda^2))\sigma^2_\epsilon+(4-\lambda^2)(4-3\lambda)\sigma^2_\theta)(a+\bar{\theta})^2}{16b(2-\lambda^2)^2(\sigma^2_\epsilon+2\sigma^2_\theta)^3}
\end{aligned}
$$

and

$$
\begin{aligned}
&y_1 = 2+8(1-\lambda)+(1-\lambda)^2(2+\lambda(4+\lambda)), \quad \& \quad y_2 = 4+17(1-\lambda)+7(1-\lambda)^2(1+\lambda),\\
&y_3 = 37+176(1-\lambda)+(1-\lambda)^2(139+(70-27\lambda)\lambda), \quad \& \quad y_4 = (2-\lambda)(3+15(1-\lambda)+(1-\lambda)^2(6+7\lambda)),\\
&y_5 = 2+10(1-\lambda)+\lambda(6+\lambda)(1-\lambda)^2, \quad \& \quad y_6 = 5+23(1-\lambda)+3\lambda(5+\lambda)(1-\lambda)^2,\\
&y_7 = 43+28\lambda+(1-\lambda^2)(162+108(1-\lambda)+39(1-\lambda^2)), \quad \& \quad y_8 = 5+6\lambda+(1-\lambda^2)(24+14(1-\lambda)+5(1-\lambda^2)).
\end{aligned}
$$

As $b, a+\bar{\theta}, \sigma^2_\epsilon, \sigma^2_\theta > 0$, $y_i > 0$ for each $i = \{1,2,..,8\}$, and $\lambda \in (0,1]$, the above inequalities follow. Moreover, $\partial E(\Pi^*_{DS})/\partial\sigma^2_\epsilon < 0$ if its first summation term is negative, or if $\sigma^2_\epsilon/\sigma^2_\theta > (2-\lambda^2)/(2\lambda(1-\lambda))$.

Using our above findings and Lemma A12, for each $l \in \{q^*_{1,DS}, q^*_{2,DS}, Q^*_{DS}\}$, $\partial E(l)/\partial\sigma^2_\theta = -\sigma^2_\epsilon/\sigma^2_\theta \times \partial E(l)/\partial\sigma^2_\epsilon > 0$. For $l \in \{\pi^*_{1,DS}, \Pi^*_{DS}, CS^*_{DS}, TS^*_{DS}\}$, let $E(\hat{l}) = E(l) \mid_{a=\bar{\theta}=0}$ be the covariance parts of the related expected equilibrium outcomes. Thus, for each $l \in \{\pi^*_{1,DS}, CS^*_{DS}, TS^*_{DS}\}$, $\partial E(l)/\partial\sigma^2_\theta = E(\hat{l})/\sigma^2_\theta - \sigma^2_\epsilon/\sigma^2_\theta \times \partial E(l)/\partial\sigma^2_\epsilon > 0$ because $\partial E(l)/\partial\sigma^2_\epsilon < 0$ by our above findings and $E(\hat{l}) > 0$. Lastly, $\partial E(\Pi^*_{DS})/\partial\sigma^2_\theta = E(\hat{\Pi}^*_{DS})/\sigma^2_\theta - \sigma^2_\epsilon/\sigma^2_\theta \times \partial E(\Pi^*_{DS})/\partial\sigma^2_\epsilon > 0$ if $\sigma^2_\epsilon/\sigma^2_\theta > K = (2-\lambda^2)/(2\lambda(1-\lambda))$ because $\partial E(\Pi^*_{DS})/\partial\sigma^2_\epsilon < 0$ if $\sigma^2_\epsilon/\sigma^2_\theta > K$ and $E(\hat{\Pi}^*_{DS}) > 0$. □

5 Proofs of the supply chain model of Section VI

Proof of Lemma 6: Let $\tilde{w}_{NS} = \eta_0 + \eta_1 s_0 + \eta_2 z$ and $\tilde{q}_i(\tilde{w}_{NS}) = \eta_3 + \eta_4 s_i + \eta_5 \tilde{w}_{NS} + \eta_6 z$ be the PRE wholesale price of the manufacturer and output of retailer $i \in N$, respectively. We use the backwards induction method to solve for the seven constants, $\eta_0, \eta_1, ..., \eta_6$. The retailer i maximizes

$$\max_{q_i} E(\pi_{R_i}|s_i, w, z) = E((a + \theta - bQ - w)q_i - \lambda q_i^2|s_i, w, z). \tag{46}$$

The FOC gives the best response of any retailer i to the changes in w as

$$q_i(w) = \eta_3 + \eta_4 s_i + \eta_5 w + \eta_6 z = \frac{a+E(\theta|s_i,w,z)-w-b\sum_{j\in N, j\neq i}(\eta_3+\eta_4 E(s_j|s_i,w,z)+\eta_5 w+\eta_6 z)}{2(b+\lambda)}, \tag{47}$$

In a PRE, the retailer perfectly infers the private signal of the manufacturer from the manufacturer's strategy, $w = \eta_0 + \eta_1 s_0 + \eta_2 z$, as $s_0 = (w - \eta_0 - \eta_2 z)/\eta_1$, which implies that

$$\begin{aligned} E(s_j|s_i, w, z) &= E(s_j|s_i, s_0 = (w - \eta_0 - \eta_2 z)/\eta_1, z), \\ E(\theta|s_i, w, z) &= E(\theta|s_i, s_0 = (w - \eta_0 - \eta_2 z)/\eta_1, z). \end{aligned} \tag{48}$$

Similarly, the manufacturer chooses w to maximize

$$\max_{w} E(\pi_{M_0}|s_0, z) = E(w\sum_{i\in N} q_i(w)|s_0, z) = E\big(w(n(\eta_3 + \eta_5 w + \eta_6 z) + \eta_4 \sum_{i\in N} s_i)|s_0, z\big). \tag{49}$$

The FOC is

$$w = \eta_0 + \eta_1 s_0 + \eta_2 z = \frac{-((\eta_3 + \eta_6 z)n + \eta_4 \sum_{i\in N} E(s_i|s_0, z))}{2n\eta_5}. \tag{50}$$

Moreover, by Lemma A9 at $\rho = 1$, for $j \in N$, $j \neq i$,

$$\begin{aligned} E(\theta|s_i, s_0, z) = E(s_j|s_i, s_0, z) &= \frac{\sigma_{\epsilon_m}^2 \sigma_{\epsilon_r}^2 \sigma_\theta^2 z + \sigma_{\epsilon_z}^2(\sigma_\theta^2(\sigma_{\epsilon_r}^2 s_0 + \sigma_{\epsilon_m}^2 s_i) + \sigma_{\epsilon_m}^2 \sigma_{\epsilon_r}^2 \bar{\theta})}{\sigma_{\epsilon_m}^2 \sigma_{\epsilon_r}^2(\sigma_{\epsilon_z}^2 + \sigma_\theta^2) + \sigma_{\epsilon_z}^2 \sigma_\theta^2(\sigma_{\epsilon_m}^2 + \sigma_{\epsilon_r}^2)}, \\ E(s_i|s_0, z) &= \frac{\sigma_\theta^2(\sigma_{\epsilon_z}^2 s_0 + \sigma_{\epsilon_m}^2 z) + \sigma_{\epsilon_z}^2 \sigma_{\epsilon_m}^2 \bar{\theta}}{\sigma_{\epsilon_m}^2 \sigma_\theta^2 + \sigma_{\epsilon_z}^2(\sigma_{\epsilon_m}^2 + \sigma_\theta^2)}. \end{aligned} \tag{51}$$

We first substitute the first two conditional expectations in (51) into (47) by using (48). Second, we substitute the third conditional expectation in (51) into (50). Then, we get seven equations with seven unknowns from (47) and (50). By solving them, one can then derive the equilibrium constants η_i, $i = 0, 1, ..., 6$ as stated in the lemma. The second order conditions would hold because $\frac{\partial^2 E(\pi_{R_i})}{\partial^2 q_i} = -2(b + \lambda) < 0$ and $\frac{\partial^2 E(\pi_{M_0}(w, q_i(w))}{\partial^2 w} = 2n\eta_5 < 0$ as $\eta_5 < 0$ for $n, \sigma_{\epsilon_z}^2, \sigma_{\epsilon_r}^2, \sigma_{\epsilon_m}^2, \sigma_\theta^2, b > 0$ and $\lambda \geq 0$. □

Proof of Lemma 7: Note that $L_2 = L_1 + 2\sigma_{\epsilon_r}^2(b + \lambda)L_3$, $L_1 = 2\sigma_{\epsilon_r}^2(b + \lambda)L_3 + \sigma_{\epsilon_z}^2 \sigma_{\epsilon_m}^2 \sigma_\theta^2(2\lambda + b(n + 1))$, and $L_3 = \sigma_{\epsilon_z}^2 \sigma_{\epsilon_m}^2 + (\sigma_{\epsilon_z}^2 + \sigma_{\epsilon_m}^2)\sigma_\theta^2$ by definition in Lemma 6. It follows from the text that the wholesale price signaling effect (SE) is measured by $SE \equiv \eta_5 + \frac{1}{2\lambda+b(n+1)} = \frac{4\sigma_{\epsilon_r}^2 L_3(b+\lambda)}{L_2(2\lambda+b(n+1))}$, which is positive as $b, L_2, L_3 > 0$ and $\lambda \geq 0$. The claims in the lemma follows because

$$\frac{\partial SE}{\partial \sigma^2_{\epsilon_m}} = \frac{\sigma^4_{\epsilon_z}}{\sigma^4_{\epsilon_m}} \frac{\partial SE}{\partial \sigma^2_{\epsilon_z}} = \frac{\sigma^4_{\epsilon_\theta}}{\sigma^4_{\epsilon_m}} \frac{\partial SE}{\partial \sigma^2_{\epsilon_\theta}} = -\frac{4\sigma^2_{\epsilon_r}\sigma^4_{\epsilon_z}\sigma^4_\theta(b+\lambda)}{L_2^2} < 0,$$
$$\frac{\partial SE}{\partial n} = \frac{-8b\sigma^2_{\epsilon_r} L_1 L_3 (b+\lambda)}{L_2^2(2\lambda + b(n+1))^2} < 0, \quad \text{and} \quad \frac{\partial SE}{\partial \sigma^2_{\epsilon_r}} = \frac{4\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_{\epsilon_\theta} L_3(b+\lambda)}{L_2^2} > 0.$$

□

In the following lemma, we derive the equilibrium outcomes in the no-sharing supply chain model.

Lemma A13.

i) $E(\tilde{Q}_{NS}) = nE(\tilde{q}_{i,NS}) = \dfrac{n\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_\theta(a+\bar{\theta})}{2L_1}$ *and* $E(\tilde{w}_{NS}) = \dfrac{L_2(a+\bar{\theta})}{2L_1}$.

ii) $E(\tilde{\pi}_{M_0,NS}) = \dfrac{n\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_\theta L_2}{4L_1^2}\Big((a+\bar{\theta})^2 + \dfrac{\sigma^4_\theta(\sigma^2_{\epsilon_z}+\sigma^2_{\epsilon_m})}{L_3}\Big)$.

iii) $E(\tilde{\pi}_{R_i,NS}) = \dfrac{\sigma^4_{\epsilon_z}\sigma^4_{\epsilon_m}\sigma^4_\theta(b+\lambda)}{4L_1^2}\Big((a+\bar{\theta})^2 + 4\sigma^2_{\epsilon_r} + \dfrac{\sigma^2_\theta(4\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m} + (\sigma^2_{\epsilon_z}+\sigma^2_{\epsilon_m})\sigma^2_\theta)}{L_3}\Big)$.

iv) $E(\tilde{CS}_{NS}) = \dfrac{bn\sigma^4_{\epsilon_z}\sigma^4_{\epsilon_m}\sigma^4_\theta}{8L_1^2}\Big(n(a+\bar{\theta})^2 + 4\sigma^2_{\epsilon_r} + \dfrac{n\sigma^2_\theta(4\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m} + (\sigma^2_{\epsilon_z}+\sigma^2_{\epsilon_m})\sigma^2_\theta)}{L_3}\Big)$.

v) $E(\tilde{TS}_{NS}) = E(\tilde{\pi}_{M_0,NS}) + nE(\tilde{\pi}_{R_i,NS}) + E(\tilde{CS}_{NS})$.

Proof of Lemma A13: By definition, the PRE outcomes are $\tilde{Q}_{NS} = \sum_{i\in N}\tilde{q}_{i,NS} = n(\eta_3 + \eta_5\tilde{w}_{NS} + \eta_6 z) + \eta_4 S_R$, $S_R = \sum_{i\in N} s_i$, $\tilde{p}_{NS} = a + \theta - b\tilde{Q}_{NS}$, $E(\tilde{\pi}_{R_i,NS}) = E((\tilde{p}_{NS} - \tilde{w}_{NS})\tilde{q}_{i,NS} - \lambda\tilde{q}^2_{i,NS})$, $E(\tilde{\pi}_{M_0,NS}) = E(\tilde{w}_{NS}\tilde{Q}_{NS})$ and $E(\tilde{CS}_{NS}) = bE((\tilde{Q}_{NS})^2)/2$, and $E(\tilde{TS}_{NS})$ is given by v). We obtain the stated results by using these definitions, Lemma 6 of the main text, $E(s_i) = E(z) = \bar{\theta}$, for $i \neq j$, $i, j = 0, 1, ..., n$, $E(s_i s_j) = E(s_i\theta) = E(s_i z) = E(\theta z) = \sigma^2_\theta + \bar{\theta}^2$, $E(z^2) = \sigma^2_{\epsilon_z} + \sigma^2_\theta + \bar{\theta}^2$, $E(s_0^2) = \sigma^2_{\epsilon_m} + \sigma^2_\theta + \bar{\theta}^2$, for retailer $i \in N$, $E(s_i^2) = \sigma^2_{\epsilon_r} + \sigma^2_\theta + \bar{\theta}^2$, $\text{Cov}(s_0, S_R) = \text{Cov}(z, S_R) = n\sigma^2_\theta$, $\text{Var}(S_R) = n(n\sigma^2_\theta + \sigma^2_{\epsilon_r})$, and $\text{Cov}(s_i, S_R) = n\sigma^2_\theta + \sigma^2_{\epsilon_r}$. The auxiliary calculations in *Mathematica 12.1* are in online Appendix B. □

Proof of Proposition 7: **i)** As we show in online Appendix B, the full information equilibrium production level of each Cournot retailer is $q_{i,FI} = (a+\theta)/(4\lambda + 2b(n+1))$. By the symmetry among retailers, expected equilibrium total surplus in full information regime becomes

$$E(TS_{FI}) = \frac{n(6\lambda + b(4+3n))((a+\bar{\theta})^2 + \sigma^2_\theta)}{8(2\lambda + b(n+1))^2}. \tag{52}$$

Similarly, when the retailers produce identical outputs, the full information first efficient level is the level that maximizes total surplus. At the first best level, each firm produces $(a+\theta)/(2\lambda + bn)$ and the expected total surplus is

$$E(TS_{FB}) = \frac{n((a+\bar{\theta})^2 + \sigma^2_\theta)}{4\lambda + 2bn}. \tag{53}$$

Using Lemma A13-v), as n approaches ∞, the expected PRE total surplus ($E(\tilde{TS}_{NS})$) in our incomplete information no-sharing game converges to

$$\lim_{n\to\infty} E(\tilde{TS}_{NS}) = \frac{3(a+\bar{\theta})^2}{8b} + \frac{\sigma^2_\theta(4\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m} + 3\sigma^2_\theta(\sigma^2_{\epsilon_z} + \sigma^2_{\epsilon_m}))}{8bL_3}. \tag{54}$$

Using (52) and (53), $\lim_{n\to\infty} E(TS_{FB}) = \frac{(a+\bar{\theta})^2+\sigma_\theta^2}{2b}$ and $\lim_{n\to\infty} E(TS_{FI}) = \frac{3(a+\bar{\theta})^2+\sigma_\theta^2}{8b}$. Thus,

$$\lim_{n\to\infty}(E(\tilde{TS}_{NS}) - E(TS_{FI})) = \frac{\sigma_{\epsilon_z}^2\sigma_{\epsilon_m}^2\sigma_\theta^2}{8bL_3} > 0 \tag{55}$$

and

$$\lim_{n\to\infty}(E(TS_{FB} - E(\tilde{TS}_{NS}))) = \frac{(a+\bar{\theta})^2}{8b} + \frac{(\sigma_{\epsilon_z}^2+\sigma_{\epsilon_m}^2)\sigma_\theta^4}{8bL_3} > 0 \tag{56}$$

because $b, \sigma_{\epsilon_z}^2, \sigma_{\epsilon_m}^2, \sigma_\theta^2, L_3 > 0$.

ii) By Proposition 8-*iv*), $\partial E(\tilde{TS}_{NS})/\partial\sigma_{\epsilon_r}^2 < 0$. As $\partial E(\tilde{TS}_{FI})/\partial\sigma_{\epsilon_r}^2 = 0$, it becomes a corollary that $(\partial E(\tilde{TS}_{FI} - \tilde{TS}_{NS}))/\partial\sigma_{\epsilon_r}^2 > 0$. Moreover,

$$\begin{aligned}\lim_{\sigma_{\epsilon_r}^2\to 0} E(\tilde{TS}_{FI} - \tilde{TS}_{NS}) &= -\frac{\sigma_{\epsilon_z}^2\sigma_{\epsilon_m}^2 n\sigma_\theta^2(2\lambda+b(n+4))}{8L_3(2\lambda+b(n+1))^2} < 0\\ \lim_{\sigma_{\epsilon_r}^2\to\infty} E(\tilde{TS}_{FI} - \tilde{TS}_{NS}) &= \frac{n(6\lambda+b(3n+4))((a+\bar{\theta})^2+\sigma_\theta^2)}{8(2\lambda+b(n+1))^2} > 0.\end{aligned} \tag{57}$$

Altogether, $E(\tilde{TS}_{FI}) < E(\tilde{TS}_{NS})$ if and only if $\sigma_{\epsilon_r}^2 \in [0, \hat{\sigma}_{\epsilon_r}^2)$ for a unique $\hat{\sigma}_{\epsilon_r}^2 > 0$; that is, the precision of the retailers' signals is sufficiently high. □

When the precision of retailer 1's signal, say $1/\sigma_{\epsilon_{r_1}}^2$, is different than the precision levels of the remaining $n-1$ retailers, say $1/\sigma_{\epsilon_{r_2}}^2$, the equilibrium strategies of the retailers involve asymmetries as shown by the next lemma. This lemma will be used in the proof of Proposition 8-*iii*).

Lemma A14. *In the no-sharing game with asymmetrically informed retailers, the unique linear PRE strategies of the manufacturer, retailer 1, and any retailer* $j \neq 1$ *are given by* $\widehat{w}_{NS} = T_0 + T_1 s_0 + T_2 z$, $\widehat{q}_{1,NS} = T_3 + T_4 s_1 + T_5\widehat{w}_{NS} + T_6 z$, *and* $\widehat{q}_{j,NS} = T_7 + T_8 s_j + T_9\widehat{w}_{NS} + T_{10} z$, *respectively, where* $T_0 = \frac{(aL_3+\bar{\theta}\sigma_{\epsilon_m}^2\sigma_{\epsilon_z}^2)(H_3+H_4+H_5)}{H_3L_3}$, $T_1 = \frac{\sigma_{\epsilon_z}^2\sigma_\theta^2 T_0}{aL_3+\bar{\theta}\sigma_{\epsilon_m}^2\sigma_{\epsilon_z}^2}$, $T_2 = \frac{\sigma_{\epsilon_m}^2 T_1}{\sigma_{\epsilon_z}^2}$, $T_3 = aT_4 = \frac{H_2T_7}{H_1}$, $T_5 = \frac{n(H_4+H_5)}{(H_3+H_4+H_5)(2\lambda+b(n+1))} - T_9(n-1)$, $T_9 = \frac{H_4+H_5}{(H_3+H_4+H_5)(2\lambda+b(n+1))} - \frac{4L_3\sigma_{\epsilon_z}^2\sigma_{\epsilon_m}^2\sigma_\theta^2(b+\lambda)(\sigma_{\epsilon_{r_1}}^2-\sigma_{\epsilon_{r_2}}^2)}{H_3+H_4+H_5}$, $T_6 = T_{10} = 0$, *and* $T_8 = \frac{T_7}{a} = \frac{2n\sigma_{\epsilon_z}^2\sigma_{\epsilon_m}^2\sigma_\theta^2 H_1}{H_3}$, *where*

$$\begin{aligned}
H_i &= 2L_3(b+\lambda)\sigma_{\epsilon_{r_i}}^2 + (b+2\lambda)\sigma_\theta^2\sigma_{\epsilon_m}^2\sigma_{\epsilon_z}^2, \quad \text{for } i=1,2,\\
H_3 &= 4n(b+\lambda)\big(L_3\sigma_{\epsilon_{r_1}}^2 + \sigma_{\epsilon_z}^2\sigma_{\epsilon_m}^2\sigma_\theta^2\big)H_2 + 2b\sigma_{\epsilon_z}^2\sigma_{\epsilon_m}^2\sigma_\theta^2 n(n-1)H_1,\\
H_4 &= -2\sigma_{\epsilon_m}^2\sigma_{\epsilon_z}^2\sigma_\theta^2(2\lambda+b(n+1))((n-1)\sigma_{\epsilon_{r_1}}^2 + \sigma_{\epsilon_{r_2}}^2)(bL_3 + \lambda\sigma_{\epsilon_m}^2(\sigma_{\epsilon_z}^2+\sigma_\theta^2)),\\
H_5 &= -\sigma_{\epsilon_m}^2\sigma_{\epsilon_z}^4\sigma_\theta^4(2\lambda+b(n+1))\big(b\sigma_{\epsilon_m}^2 n + 2\lambda(\sigma_{\epsilon_m}^2 n + \sigma_{\epsilon_{r_1}}^2(n-1) + \sigma_{\epsilon_{r_2}}^2)\big).
\end{aligned}$$

Proof of Lemma A14: The proof of this lemma is done in online Appendix B by following similar steps to the proof of Lemma 6. The second-order conditions would hold because $\frac{\partial^2\pi_{R_i}}{\partial^2 q_i} = -2(b+\lambda) < 0$ and $\frac{\partial^2\pi_{M_0}}{\partial^2 q_i} = 2(T_5 + T_9(n-1)) = \frac{2n(H_4+H_5)}{(H_3+H_4+H_5)(2\lambda+b(n+1))} < 0$ as $H_4+H_5 < 0$, $H_3+H_4+H_5 > 0$, $b > 0$, $\lambda \geq 0$, and $n \geq 1$. When $\sigma_{\epsilon_{r_1}}^2 = \sigma_{\epsilon_{r_2}}^2 = \sigma_{\epsilon_r}^2$ or $n = 1$, this lemma coincides with Lemma 5-*i*) of the main text. □

Proof of Proposition 8: **i)** Using Lemma A13, straightforward calculations show that for each $h \in \{\tilde{q}_{i,NS}, \tilde{w}_{NS}, \tilde{\pi}_{R_i,NS}, \tilde{\pi}_{M_0,NS}, \tilde{CS}_{NS}, \tilde{TS}_{NS}\}$, $\frac{\partial E(h)}{\partial\sigma_{\epsilon_m}^2} = \frac{\sigma_{\epsilon_z}^4}{\sigma_{\epsilon_m}^4}\frac{\partial E(h)}{\partial\sigma_{\epsilon_z}^2}$. The claim follows from these derivations.

ii) Using Lemma A13-*ii*), $\partial E(\tilde{\pi}_{M_0,NS})/\partial\sigma^2_{\epsilon_m} = n\sigma^4_\theta\sigma^4_{\epsilon_z}Z_1/(4L_1^3L_3^2)$, where

$$X_0 = -(2\lambda + b(n+1))^2\sigma^6_\theta\sigma^6_{\epsilon_m}\sigma^6_{\epsilon_z}, \qquad X_1 = -6L_3(b+\lambda)(2\lambda + b(n+1))\sigma^4_\theta\sigma^4_{\epsilon_m}\sigma^4_{\epsilon_z}$$
$$X_2 = 8L_3^2(b+\lambda)^2(L_3(a+\bar{\theta})^2 + \sigma^2_\theta(\sigma^2_\theta(\sigma^2_{\epsilon_m} + \sigma^2_{\epsilon_z}) - \sigma^2_{\epsilon_m}\sigma^2_{\epsilon_z})), \quad \text{and} \quad Z_1 = X_0 + X_1\sigma^2_{\epsilon_r} + X_2\sigma^4_{\epsilon_r}.$$

Note that $\text{sign}\{\partial E(\tilde{\pi}_{M_0,NS})/\partial\sigma^2_{\epsilon_m}\} = \text{sign}\{Z_1\}$ because $L_1, L_3 > 0$. Moreover, $X_0 < 0$ and $X_1 < 0$ because $b > 0$, $\lambda \geq 0$, and $n \geq 1$ by assumption. Here, $X_2 > 0$ because X_2 is increasing in a, $a \geq \sigma_\theta - \bar{\theta}$ by assumption, and $X_2(a = \sigma_\theta - \bar{\theta}) = 16L_3^2\sigma^4_\theta(b+\lambda)^2(\sigma^2_{\epsilon_m} + \sigma^2_{\epsilon_z}) > 0$. Thus, the number of sign changes in the sequence of quadratic polynomial $Z_1(\sigma^2_{\epsilon_r})$'s coefficients change once. By the Descartes' rule of signs, there is one positive real root of Z_1 in $\sigma^2_{\epsilon_r}$. As $Z_2(\sigma^2_{\epsilon_r} = 0) = X_0 < 0$, $\partial E\pi_{M_0,NS}/\partial\sigma^2_{\epsilon_m} > 0$ if and only if $\sigma^2_{\epsilon_r}$ is sufficiently high (or $\tau_{\epsilon_r} = 1/\sigma^2_{\epsilon_r}$ is sufficiently low).

iii) Without loss of generality, suppose *only* retailer 1 acquires better information. Thus, the equilibrium strategies of the retailers become asymmetric and the unique PRE strategies of firms are given by Lemma A14. By using this lemma, for $i \neq j$, $i, j = 1, 2, ..., n$, the partial derivative of the PRE expected profit of retailer 1 evaluated at the symmetric precision levels is

$$\frac{\partial E(\widehat{\pi}_{R_1,NS})}{\partial\sigma^2_{\epsilon_{r_1}}}\Big|_{\sigma^2_{\epsilon_{r_2}}=\sigma^2_{\epsilon_{r_1}}} = \frac{-\sigma^4_{\epsilon_z}\sigma^4_{\epsilon_m}\sigma^4_\theta(b+\lambda)\big(L_3(b+\lambda)(a+\bar{\theta})^2 + H_6 + 2n(n-1)b^2\sigma^4_{\epsilon_z}\sigma^4_{\epsilon_m}\sigma^4_\theta/H_1\big)}{n(H_1 + bn\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_\theta)^3},$$

which is negative because $b, L_3, H_1, H_6 > 0$, $n \geq 1$, and $\lambda \geq 0$, where

$$H_6 = 2nL_3(b+\lambda)\sigma^2_{\epsilon_{r_1}} + n(2\lambda + b(n+1))\sigma^2_\theta\sigma^2_{\epsilon_m}\sigma^2_{\epsilon_z} + (b+\lambda)(\sigma^2_{\epsilon_m} + \sigma^2_{\epsilon_z})\sigma^4_\theta.$$

This finding proves the claim. Finally, when $n = 1$, this derivative naturally coincides with the partial derivative of $E(\tilde{\pi}_{R_1,NS})$ with respect to $\sigma^2_{\epsilon_r}$ from Lemma A13-*iii*).

iv) It follows by part *i*) that for each $h \in \{Q, CS, TS\}$, $\text{sign}\{\frac{\partial E(\tilde{h}_{NS})}{\partial\sigma^2_{\epsilon_z}}\} = \text{sign}\{\frac{\partial E(\tilde{h}_{NS})}{\partial\sigma^2_{\epsilon_m}}\}$. By using Lemma A13, direct computations show that

$$\begin{aligned}
\frac{\partial E(\tilde{Q}_{NS})}{\partial\sigma^2_{\epsilon_r}} &= \frac{-n\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_\theta L_3(b+\lambda)(a+\bar{\theta})}{L_1^2} < 0,\\
\frac{\partial E(\tilde{CS}_{NS})}{\partial\sigma^2_{\epsilon_r}} &= \frac{-bn\sigma^4_{\epsilon_z}\sigma^4_{\epsilon_m}\sigma^4_\theta\Big(nL_3(b+\lambda)(a+\bar{\theta})^2 + L_1 + 2\sigma^2_\theta\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}(n-1)(b+2\lambda) + n\sigma^4_\theta(b+\lambda)(\sigma^2_{\epsilon_z} + \sigma^2_{\epsilon_m})\Big)}{2L_1^3} < 0,\\
\frac{\partial E(\tilde{TS}_{NS})}{\partial\sigma^2_{\epsilon_r}} &= \frac{-n\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_\theta\Big(L_3(b+\lambda)(L_2 + b\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_\theta)(a+\bar{\theta})^2 + \sigma^2_\theta X_3\Big)}{2L_1^3} < 0,\\
\frac{\partial E(\tilde{Q}_{NS})}{\partial\sigma^2_{\epsilon_m}} &= \frac{n(b+\lambda)(a+\bar{\theta})\sigma^2_{\epsilon_r}\sigma^4_\theta\sigma^4_{\epsilon_z}}{L_1^2} > 0,\\
\frac{\partial E(\tilde{CS}_{NS})}{\partial\sigma^2_{\epsilon_m}} &= \frac{\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_r}\sigma^2_\theta}{L_3\sigma^2_{\epsilon_m}} \times \frac{-\partial E(\tilde{CS}_{NS})}{\partial\sigma^2_{\epsilon_r}} + \frac{bn\sigma^2_{\epsilon_m}\sigma^6_{\epsilon_z}\sigma^6_\theta(4\sigma^2_{\epsilon_r}L_3 + 3n\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_\theta)}{8L_1^2L_3^2} > 0,\\
\frac{\partial E(\tilde{TS}_{NS})}{\partial\sigma^2_{\epsilon_m}} &= \frac{\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_r}\sigma^2_\theta}{L_3\sigma^2_{\epsilon_m}} \times \frac{-\partial E(\tilde{TS}_{NS})}{\partial\sigma^2_{\epsilon_r}} + \frac{n\sigma^2_{\epsilon_m}\sigma^6_{\epsilon_z}\sigma^6_\theta(4b\sigma^2_{\epsilon_r}L_3 + \sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_\theta(2\lambda + b(n+4)))}{8L_1^2L_3^2} > 0,\\
\frac{\partial E(\tilde{Q}_{NS})}{\partial\sigma^2_{\epsilon_\theta}} &= \frac{\sigma^4_{\epsilon_m}}{\sigma^4_\theta} \times \frac{\partial E(\tilde{Q}_{NS})}{\partial\sigma^2_{\epsilon_m}} > 0,\\
\frac{\partial E(\tilde{CS}_{NS})}{\partial\sigma^2_{\epsilon_\theta}} &= \frac{\sigma^4_{\epsilon_m}}{\sigma^4_\theta} \times \frac{\partial E(\tilde{CS}_{NS})}{\partial\sigma^2_{\epsilon_m}} + \frac{bn^2\sigma^4_{\epsilon_z}\sigma^4_{\epsilon_m}\sigma^4_\theta}{8L_1^2} > 0,\\
\frac{\partial E(\tilde{TS}_{NS})}{\partial\sigma^2_{\epsilon_\theta}} &= \frac{\sigma^4_{\epsilon_m}}{\sigma^4_\theta} \times \frac{\partial E(\tilde{TS}_{NS})}{\partial\sigma^2_{\epsilon_m}} + \frac{n\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_\theta(2L_2 + \sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_\theta(2\lambda + b(n+2)))}{8L_1^2} > 0,
\end{aligned}$$

where

$$X_3 = \sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}L_1(b+2\lambda) + 2\sigma^2_\theta(\sigma^2_{\epsilon_z}+\sigma^2_{\epsilon_m})(b+\lambda)^2(2\sigma^2_{\epsilon_r}L_3 + \sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_\theta) + b\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}(b+\lambda)(4\sigma^2_{\epsilon_r}L_3 + \sigma^2_\theta(4\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m} + n\sigma^2_\theta(\sigma^2_{\epsilon_z}+\sigma^2_{\epsilon_m}))).$$

The above inequalities follow because $L_1, L_2, L_3, X_1, X_3, a, b, \sigma^2_{\epsilon_z}, \sigma^2_{\epsilon_m}, \sigma^2_{\epsilon_r} > 0$, $n \geq 1$, $\bar{\theta} \geq 0$, and $\lambda \geq 0$. □

Proof of Lemma 8: Let $\tilde{w}_{PS} = \varsigma_0 + \varsigma_1 s_0 + \varsigma_2 z$ and $\tilde{q}_{i,PS} = \varsigma_3 + \varsigma_4 s_i + \varsigma_5 \tilde{w}_{PS} + \varsigma_6 z + \varsigma_7 s_0$ be the linear equilibrium strategies of the manufacturer and the retailer in the partial-sharing game for some constants $\varsigma_0, \varsigma_1, ..., \varsigma_7 \in \mathbb{R}$. Our aim is to solve for these 8 constants. Retailer $i \in N$ maximizes

$$\max_{q_i} E(\pi_{R_i}|s_i, w, z, s_0) = E((a + \theta - bQ - w)q_i - \lambda q_i^2 | s_i, w, z, s_0). \tag{58}$$

The FOC implies the best response of any retailer i to the changes in w as

$$\begin{aligned} q_i(w) &= \varsigma_3 + \varsigma_4 s_i + \varsigma_5 w + \varsigma_6 z + \varsigma_7 s_0 \\ &= \frac{a + E(\theta|s_i, w, z, s_0) - w - b\sum_{j\neq i, j\in N}(\varsigma_3 + \varsigma_4 E(s_j|s_i, w, z, s_0) + \varsigma_5 w + \varsigma_6 z + \varsigma_7 s_0)}{2(b+\lambda)}. \end{aligned} \tag{59}$$

The supplier solves $\max_w E(w\sum_{i\in N} q_i(w)) = \max_w E(w\sum_{i\in N}(\varsigma_3 + \varsigma_4 s_i + \varsigma_5 w + \varsigma_6 z + \varsigma_7 s_0)|s_0, z)$. The FOC gives

$$w = \varsigma_0 + \varsigma_1 s_0 + \varsigma_2 z = \frac{-((\varsigma_3 + \varsigma_6 z + \varsigma_7 s_0)n + \varsigma_4 \sum_{i\in N} E(s_i|s_0, z))}{2n\varsigma_5}. \tag{60}$$

Note that $E(\theta|s_i, w, z, s_0) = E(\theta|s_i, z, s_0)$ and $E(s_j|s_i, w, z, s_0) = E(s_j|s_i, z, s_0)$ because s_0 has already shared to the retailers. First, we substitute the conditional expectations, $E(\theta|s_i, s_0, z)$ and $E(s_j|s_i, s_0, z)$ from (51) into (59). Second, we substitute the value of $E(s_i|s_0, z)$ from (51) into (60). Then, we get eight equations with eight unknowns from (59) and (60). By solving them, one can then derive the equilibrium constants ς_i, $i = 0, 1, ..., 7$ as stated in the lemma. The second order conditions hold because $\frac{\partial^2 E(\pi_{R_i})}{\partial^2 q_i} = -2(b+\lambda) < 0$ and $\frac{\partial^2 E(\pi_{M_0}(w, q_i(w)))}{\partial^2 w} = 2n\varsigma_5 < 0$ as $\varsigma_5 < 0$ for $n, \sigma^2_{\epsilon_z}, \sigma^2_{\epsilon_r}, \sigma^2_{\epsilon_m}, \sigma^2_\theta, b > 0$ and $\lambda \geq 0$. □

Lemma A15. *i)* $E(\tilde{Q}_{PS}) = nE(\tilde{q}_{i,PS}) = \frac{n(a+\bar{\theta})}{2(2\lambda + b(n+1))}$ *and* $E(\tilde{w}_{PS}) = \frac{a+\bar{\theta}}{2}$.

ii) $E(\tilde{\pi}_{M_0,PS}) = \frac{n}{4(2\lambda+b(n+1))}\left((a+\bar{\theta})^2 + \frac{\sigma^4_\theta(\sigma^2_{\epsilon_z}+\sigma^2_{\epsilon_m})}{L_3}\right)$,

iii) $E(\tilde{\pi}_{R_i,PS}) = E(\tilde{\pi}_{R_i,NS}) + \frac{L_3\sigma^2_{\epsilon_r}(b+\lambda)^2(L_3\sigma^2_{\epsilon_r}(b+\lambda)+\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_\theta(2\lambda+b(n+1)))(a+\bar{\theta})^2}{L_1^2(2\lambda+b(n+1))^2}\left(1+\frac{\sigma^4_\theta(\sigma^2_{\epsilon_z}+\sigma^2_{\epsilon_m})}{L_3(a+\bar{\theta})^2}\right)$,

iv) $E(\tilde{CS}_{PS}) = E(\tilde{CS}_{NS}) + \frac{bn^2}{2(b+\lambda)} \times E(\tilde{\pi}_{R_i,PS} - \tilde{\pi}_{R_i,NS})$.

v) $E(\tilde{TS}_{PS}) = E(\tilde{CS}_{PS}) + E(\tilde{\pi}_{M_0,PS}) + nE(\tilde{\pi}_{R_i,PS})$.

Proof of Lemma A15: The proof of this lemma can be done similarly to the proof of Lemma A13, and it is contained in online Appendix B. □

Proof of Proposition 9: **i, ii)** Using Lemmas A13 and A15, one can obtain

$$
\begin{aligned}
&E(\tilde{w}_{PS} - \tilde{w}_{NS}) = \frac{-L_3\sigma^2_{\epsilon_r}(b+\lambda)(a+\bar{\theta})}{L_1} < 0,\\
&E(\tilde{Q}_{PS} - \tilde{Q}_{NS}) = \frac{nL_3\sigma^2_{\epsilon_r}(b+\lambda)(a+\bar{\theta})}{L_1(2\lambda+b(n+1))} > 0,\\
&E(\tilde{\pi}_{M_0,PS} - \tilde{\pi}_{M_0,NS}) = \frac{nL_3^2\sigma^4_{\epsilon_r}(b+\lambda)^2(a+\bar{\theta})^2}{L_1^2(2\lambda+b(n+1))}\left(1 + \frac{\sigma^4_\theta(\sigma^2_{\epsilon_z}+\sigma^2_{\epsilon_m})}{L_3(a+\bar{\theta})^2}\right) > 0,\\
&E(\tilde{\pi}_{R_i,PS} - \tilde{\pi}_{R_i,NS}) = \frac{L_3\sigma^2_{\epsilon_r}(b+\lambda)^2(L_3\sigma^2_{\epsilon_r}(b+\lambda)+\sigma^2_{\epsilon_z}\sigma^2_{\epsilon_m}\sigma^2_\theta(2\lambda+b(n+1)))(a+\bar{\theta})^2}{L_1^2(2\lambda+b(n+1))^2}\left(1+\frac{\sigma^4_\theta(\sigma^2_{\epsilon_z}+\sigma^2_{\epsilon_m})}{L_3(a+\bar{\theta})^2}\right) > 0,\\
&E(\tilde{CS}_{PS} - \tilde{CS}_{NS}) = \frac{bn^2}{2(b+\lambda)} \times E(\tilde{\pi}_{R_i,PS} - \tilde{\pi}_{R_i,NS}) > 0,\\
&E(\tilde{TS}_{PS} - \tilde{TS}_{NS}) = \frac{nL_3\sigma^2_{\epsilon_r}(b+\lambda)(L_3\sigma^2_{\epsilon_r}(b+\lambda)(2\lambda+bn)+L_1(2\lambda+b(n+2)))(a+\bar{\theta})^2}{2L_1^2(2\lambda+b(n+1))^2}\left(1+\frac{\sigma^4_\theta(\sigma^2_{\epsilon_z}+\sigma^2_{\epsilon_m})}{L_3(a+\bar{\theta})^2}\right) > 0
\end{aligned}
$$

because $L_1, L_3, b > 0$, $n \geq 1$, and $\lambda \geq 0$. □

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