Online Appendix
to
"A Theory of Stability in Matching with Incomplete Information"

Yi-Chun Chen and Gaoji Hu

Formal analysis for Section IV

In this online appendix, we formally state and prove the results that are briefly discussed in Section IV. Throughout, we fix a potential blocking combination $(i, j; p)$ and let the partition profile be Π.

First, in Appendix B.B1, we formally elaborate why we shall pay specific attention to the *uniform* N case, the *supported* N case and the *inseparable* N case, which we will refer as the fundamental sources of consideration refinements. Then we formally define sophisticated blocking in Appendix B.B2. Related results will be stated in Appendix B.B3 and proved in Appendix B.B4.

B1. Fundamental sources of consideration refinements

Suppose for now that we do not have a formulation of consideration refinement yet, and we are about to define some consideration correspondences that specifies for each agent at each type assignment a set of considered type assignments. Denote the consideration correspondences to be defined by $C_i^\dagger(\mathbf{t})$ and $C_j^\dagger(\mathbf{t})$, where $\mathbf{t} \in T$.

For convenience, here we restate the upper-bound constraint (12) and the lower-bound constraint (13) introduced in Section IV.B:

$$(12) \qquad C_i^\dagger(\mathbf{t}) \subset \Pi_i(\mathbf{t}) \quad \text{and} \quad C_j^\dagger(\mathbf{t}) \subset \Pi_j(\mathbf{t}), \quad \text{for all } \mathbf{t} \in T.$$

$$(13) \qquad \mathbf{t}' \in C_k^\dagger(\mathbf{t}) \text{ implies } [\Pi_i \vee \Pi_j](\mathbf{t}') \subset C_k^\dagger(\mathbf{t}), \quad \text{for } k = i, j \text{ and all } \mathbf{t}, \mathbf{t}' \in T.$$

Based on these natural restrictions and upon nothing else, we proceed to figure out in which cases worker i (firm j) can refine his (her) consideration. Indeed, taking into account the possibility that refinements may lead to further refinements, we shall focus on cases which initiate consideration refinements, instead of defining $C_i^\dagger(\mathbf{t})$ and $C_j^\dagger(\mathbf{t})$ directly. And the initiating cases will actually pin down $C_i^\dagger(\mathbf{t})$ and $C_j^\dagger(\mathbf{t})$.

We now identify all fundamental sources of consideration refinements, which could serve as the starting points of an iteration process, like (5)-(6), that will precisely define the consideration correspondences $C_i^\dagger$ and $C_j^\dagger$. Given the symmetry between i and j, we will focus on worker i's point of view: When should worker i exclude $\mathbf{t} \in \Pi_i(\mathbf{t})$ from consideration? As has been discussed, we shall classify the cases only according to properties (12) and (13) and, of course, common knowledge of the model.

Consider the following cases:

1) *For every* $\mathbf{t}' \in \Pi_j(\mathbf{t})$*, we have* $\chi_j(\mathbf{t}') = N$*.*[26]

   In this case, whatever $C_j^\dagger(\mathbf{t})$ is, $C_j^\dagger(\mathbf{t})$ can only be empty or contain just N's by (12) (abusing terminology). The former subcase, as well as similar situations below with empty $C_j^\dagger(\mathbf{t})$, involves more properties of $C_j^\dagger$ other than (12) and (13), which makes the case not fundamental. In the latter subcase, agent i should exclude $\mathbf{t}$. This is the source of uniform N's.

2) *For every* $\mathbf{t}' \in \Pi_j(\mathbf{t})$*, we have* $\chi_j(\mathbf{t}') = Y$*.*

   Now, $C_j^\dagger(\mathbf{t})$ is empty or contains only Y's by (12). The former subcase is not fundamental. In the latter subcase, agent i has no hope to exclude $\mathbf{t}$ from consideration, and, in fact, has to consider $\mathbf{t}$.

3) *For some* $\mathbf{t}' \in \Pi_j(\mathbf{t})$*, we have* $\chi_j(\mathbf{t}') = N$*; and for some* $\mathbf{t}'' \in \Pi_j(\mathbf{t})$*, we have* $\chi_j(\mathbf{t}'') = Y$*.*

   Consider the following mutually exclusive subcases:

   a) *There is* $\mathbf{t}''' \in \Pi_j(\mathbf{t})$ *such that* $\chi_j(\mathbf{t}''') \neq \chi_j(\mathbf{t})$ *and* $\mathbf{t}''' \in \Pi_i(\mathbf{t})$*.*

      Whatever $C_j^\dagger(\mathbf{t})$ is, either it does not contain $\mathbf{t}$ (not a fundamental case because it involves further discussion of $C_j^\dagger$ other than (12) and (13)) or we have $\mathbf{t}, \mathbf{t}''' \in C_j^\dagger(\mathbf{t})$ (this is true by (13) and $\mathbf{t}, \mathbf{t}''' \in [\Pi_i \vee \Pi_j](\mathbf{t})$). In the latter subcase, firm j will definitely object the new partnership, and agent i should ignore $\mathbf{t}$. This is the source of inseparable N's.

   b) *For any* $\mathbf{t}''' \in \Pi_j(\mathbf{t})$ *such that* $\chi_j(\mathbf{t}''') \neq \chi_j(\mathbf{t})$*, we have* $\mathbf{t}''' \notin \Pi_i(\mathbf{t})$*.*

      For worker i to ignore $\mathbf{t}$, we must have $C_j^\dagger(\mathbf{t})$ containing at least one N. Pick an arbitrary $\mathbf{t}' \in \Pi_j(\mathbf{t})$ such that $\chi_j(\mathbf{t}') = N$. Consider the following cases which may result in $\mathbf{t}' \in C_j^\dagger(\mathbf{t})$:

      i) *All* Y*'s (and maybe some* N*'s) will be ignored by agent* j*, so that* $C_j^\dagger(\mathbf{t})$ *could at most contain only* N*'s.*

         This is not a fundamental case.

      ii) *For every* $\mathbf{t}'' \in \Pi_i(\mathbf{t}')$*, we have* $\chi_i(\mathbf{t}'') = Y$*.*

         In this case, whatever $C_i^\dagger(\mathbf{t}')$ is, either it is empty (not a fundamental case) or it must contain only Y's by (12). Suppose the latter happens. Then, firm j will have to consider $\mathbf{t}'$, i.e., $\mathbf{t}' \in C_j^\dagger(\mathbf{t})$, due to worker i's definite willingness to participate in the new partnership. This is the source of supported N's.

      iii) *For every* $\mathbf{t}'' \in \Pi_i(\mathbf{t}')$*, we have* $\chi_i(\mathbf{t}'') = N$*.*

         In this case, whatever $C_i^\dagger(\mathbf{t}')$ is, either it is empty (not a fundamental case) or it must contain only N's by (12). Suppose the latter happens. Then, firm j will not consider $\mathbf{t}'$, i.e., $\mathbf{t}' \notin C_j^\dagger(\mathbf{t})$.

[26] Recall that the indicator functions χ_i and χ_j are defined by (3)-(4) in Section II.B.

iv) *For some* $\mathbf{t}'' \in \Pi_i(\mathbf{t}')$, *we have* $\chi_i(\mathbf{t}'') = Y$*; and for some* $\mathbf{t}''' \in \Pi_i(\mathbf{t}')$, *we have* $\chi_i(\mathbf{t}''') = N$.
Now we are faced with another question of what $C_i^\dagger(\mathbf{t}')$ shall be, just as we started with. Naturally, instead of starting another round of discussion, we shall stop here and define $C_i^\dagger$ and $C_j^\dagger$ iteratively using the fundamental sources of consideration refinement which we just identified.

To sum up, there are only three fundamental sources of consideration refinements, uniform N's, inseparable N's and supported N's, exactly as we have introduced. Within them, from agent i's perspective, uniform N's are of order zero in the sense that it is identified using just the information of Π_j (and restriction (12)); whereas inseparable N's and supported N's are of order one in the sense that they are identified using both the information of Π_j and the information of Π_i (and restriction (12) or (13)). Clearly, higher order reasoning will be taken account in the iterative consideration refinement process.

B2. Definition of sophisticated blocking

To define sophisticated blocking, we first describe how agents' sophisticated consideration correspondences are formed.

We first demonstrate that agents' willingness/unwillingness (i.e., the indicator functions defined in (3)-(4)) may be adjusted to reflect inseparable N's or supported N's, which helps us to build the discussion upon our uniform-N benchmark in Section II.B. To wit, consider the inseparable N (right panel) and the supported N (left panel) in the figure below:

Agent	$\{Y$	$N\}$	$\cdots$	Agent	$\{Y$	$N\}$	$\cdots$
Potential partner	$\dots$	$\{Y$	$Y\}$	Potential partner	$\{Y$	N	$Y\}$

In the left panel, since the agent will either consider neither of Y and N, or definitely consider the supported N, the potential partner can treat the agent's Y as an N because it is tied with a supported N in the agent's consideration. In the right panel, since the agent will either consider neither of Y and N, or definitely consider both together, again, the potential partner can treat the agent's Y as an N because it is tied with an inseparable N in the agent's consideration. Therefore, in terms of consideration refinement, the information conveyed by the inseparable N and the supported N above can be reflected in the following figure, where the hypothetical willingness/unwillingness of the agent is adjusted:
These adjustments turn the two kinds of sophisticated N's into the familiar uniform N, with which we could build our analysis on the benchmark in Section II.B.

Agent	$\{N$	$N\}$	$\cdots$	Agent	$\{N$	$N\}$	$\cdots$
Potential partner	$\ldots$	$\{Y$	$Y\}$	Potential partner	$\{Y$	N	$Y\}$

Naturally, as in Section II.B, an agent only considers type assignments such that the potential partner does not have uniform N (up to adjustment). Set $\chi^0 = \chi$, where χ is defined in (3)-(4), and $C^0 = \Pi$. Agents' consideration correspondences are defined as the limit of C_k^l in the following double-iteration process:

Indicator function adjustment. For all $l \geq 1$ and all $\mathbf{t}' \in T$,

$$\chi_i^l(\mathbf{t}') := \begin{cases} N & \text{if either } \exists \mathbf{t}'' \in C_i^{l-1}(\mathbf{t}') \text{ s.t. } \chi_i^{l-1}(\mathbf{t}'') \neq \chi_i^{l-1}(\mathbf{t}') \text{ and } \mathbf{t}', \mathbf{t}'' \in C_j^{l-1}(\mathbf{t}') \\ & \text{or } \exists \mathbf{t}'' \in C_i^{l-1}(\mathbf{t}') \text{ s.t. } \chi_i^{l-1}(\mathbf{t}'') = N \text{ and } \forall \mathbf{t}''' \in C_j^{l-1}(\mathbf{t}''), \chi_j^{l-1}(\mathbf{t}''') = Y, \\ \chi_i^{l-1}(\mathbf{t}') & \text{otherwise;} \end{cases}$$

$$\chi_j^l(\mathbf{t}') := \begin{cases} N & \text{if either } \exists \mathbf{t}'' \in C_j^{l-1}(\mathbf{t}') \text{ s.t. } \chi_j^{l-1}(\mathbf{t}'') \neq \chi_j^{l-1}(\mathbf{t}') \text{ and } \mathbf{t}', \mathbf{t}'' \in C_i^{l-1}(\mathbf{t}') \\ & \text{or } \exists \mathbf{t}'' \in C_j^{l-1}(\mathbf{t}') \text{ s.t. } \chi_j^{l-1}(\mathbf{t}'') = N \text{ and } \forall \mathbf{t}''' \in C_i^{l-1}(\mathbf{t}''), \chi_i^{l-1}(\mathbf{t}''') = Y, \\ \chi_j^{l-1}(\mathbf{t}') & \text{otherwise.} \end{cases}$$

Sophisticated consideration refinement. For all $l \geq 1$ and all $\mathbf{t}' \in T$,

$$\text{(B1)} \qquad C_i^l(\mathbf{t}') := \left\{ \mathbf{t}'' \in \Pi_i(\mathbf{t}') : \exists \mathbf{t}''' \in C_j^{l-1}(\mathbf{t}'') \text{ s.t. } \chi_j^l(\mathbf{t}''') = Y \right\}$$

$$\text{(B2)} \qquad C_j^l(\mathbf{t}') := \left\{ \mathbf{t}'' \in \Pi_j(\mathbf{t}') : \exists \mathbf{t}''' \in C_i^{l-1}(\mathbf{t}'') \text{ s.t. } \chi_i^l(\mathbf{t}''') = Y \right\}.$$

The following lemma verifies that the consideration correspondence is monotonically decreasing in l.

LEMMA 5: *For $k = i, j$ and each $\mathbf{t}' \in T$, $C_k^l(\mathbf{t}')$ is decreasing in l w.r.t. set inclusion.*

Since the set T is finite, there exists some l^* such that $C_i^l(\mathbf{t}') = C_i^\infty(\mathbf{t}')$ and $C_j^l(\mathbf{t}') = C_j^\infty(\mathbf{t}')$ for all $l \geq l^*$ and all $\mathbf{t}' \in T$. We say a state is sophisticatedly blocked by a combination $(i, j; p)$ if both agents in it have higher rematching payoffs under every type assignment that is sophisticatedly considered at the true one.

DEFINITION 6: *A state $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ is **sophisticatedly blocked** by $(i, j; p)$ if $C_i^\infty(\mathbf{t}) \neq \emptyset$, $C_j^\infty(\mathbf{t}) \neq \emptyset$ and*

$$\nu_{\mathbf{t}'(i),\mathbf{t}'(j)} + p > \nu_{\mathbf{t}'(i),\mathbf{t}'(\mu(i))} + \mathbf{p}_{i,\mu(i)} \textit{ for all } \mathbf{t}' \in C_i^\infty(\mathbf{t}) \textit{ and}$$
$$\phi_{\mathbf{t}'(i),\mathbf{t}'(j)} - p > \phi_{\mathbf{t}'(\mu(j)),\mathbf{t}'(j)} - \mathbf{p}_{\mu(j),j} \textit{ for all } \mathbf{t}' \in C_j^\infty(\mathbf{t}).$$

B3. Additional results for Section IV

In this subsection, we list properties and connections of different blocking and stability notions. To facilitate comparison, we also include the complete-information blocking/stability and the results already discussed in Section IV sometimes.

The following proposition examines IT and IM for all blocking notions.

PROPOSITION 4: *IT and IM of blocking notions are summarized as follows:*

Blocking Notions	*Properties*	
	IT	*IM*
Complete-information blocking	*satisfied*	*not applicable*
Sophisticated blocking	*satisfied*	*not satisfied*
Level-l blocking ($l \in \mathbb{N}$)	*satisfied*	*satisfied*

The following proposition ranks blocking notions.

PROPOSITION 5: *The following statements are true:*

(i) *For every $l \in \mathbb{N}$, if a state is level-l blocked, then it is level-$(l+1)$ blocked.*

(ii) *If a state is (level-∞) blocked, then it is sophisticatedly blocked.*

(iii) *If a state is sophisticatedly blocked, then it is complete-information blocked.*

Moreover, none of the converse is true.

Denote by $\mathcal{B}$ the set of blocking combinations. Then an immediate corollary of Proposition 5 is the following: Fix an arbitrary state, we have

$$\mathcal{B}^0 \subset \mathcal{B}^1 \subset \cdots \subset \mathcal{B}^\infty \subset \mathcal{B}^{sophisticated} \subset \mathcal{B}^{complete-information}.$$

For each of the blocking notions, we have a corresponding stability notion. More precisely, we take individual rationality as in Definition 1. The way to formulate information stability for all blocking notions is the identical to that of Section II.C up to notional replacement. The following proposition says that the set of stable states is nonempty for each of the stability notions.

PROPOSITION 6: *The sets of level-l stable states with $l \in \mathbb{N}$ and sophisticatedly stable states are all nonempty. Particularly, for any $\mathbf{t} \in T$, let $(\mu, \mathbf{p})$ be a complete-information stable allocation. Then*

(i) *for any partition profile Π, $(\mu, \mathbf{p}, \mathbf{t}, H^\infty_{\mu,\mathbf{p}}(\Pi))$ is level-l stable, where the $H_{\mu,\mathbf{p}}(\cdot)$ operator is defined by level-l blocking; and*

(ii) *for the complete-information partition profile Π, $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ is sophisticatedly stable.*

Now we are ready to rank the stability notions.

PROPOSITION 7: *The following statements are true:*

(i) *For every $l \in \mathbb{N}$, a state is (essentially) level-l stable if it is level-$(l+1)$ stable.*

(ii) *A state is (essentially level-∞) stable if it is sophisticatedly stable.*

(iii) *A state is (essentially) sophisticatedly stable if it is complete-information stable.*

Moreover, none of the converse is true.

Denote by $\mathcal{S}$ the set of (essentially) stable states. Then an immediate corollary of Proposition 7 is the following set-inclusion relation:

$$\mathcal{S}^0 \supset \mathcal{S}^1 \supset \cdots \supset \mathcal{S}^\infty \supset \mathcal{S}^{sophisticate} \supset \mathcal{S}^{complete-information}.$$

We close this subsection by establishing the equivalence between naive blocking and sophisticated blocking, which will imply the equivalence with blocking by Theorem 4. Equivalence between blocking notions implies equivalence between stability notions. The conditions to guarantee equivalence are exactly the ones C1-C4 we introduced in Section IV.A.

PROPOSITION 8: *Under C1-C4, the following statements are equivalent:*

(i) $(\mu, \mathbf{w}, \mathbf{f}, \Pi)$ *is level-l blocked for some $l \in \mathbb{N}$.*

(ii) $(\mu, \mathbf{w}, \mathbf{f}, \Pi)$ *is sophisticatedly blocked.*

An immediate corollary of Proposition 8 is that under C1-C4, the stability notions are all equivalent.

B4. Proofs of Propositions 4-8

PROOF OF PROPOSITION 4:

It follows from the proof of Theorem 2 that every level-l blocking satisfies both properties. Given Theorem 2 and Example 3 in Section IV.C, we only need to prove that sophisticated blocking satisfies IT.

We prove SIT, i.e., $\chi_i(\mathbf{t}') = Y$ and $\chi_j(\mathbf{t}') = Y$ for all $\mathbf{t}' \in [\Pi_i \vee \Pi_j](\mathbf{t})$, by contradiction. The proof will be similar to that of Lemma 3, which uses measurability (Lemma 2). We omit the establishment of measurability in the corrent context and refers directly to Lemma 2, as the extension is straightforward without changing the statement. However, we present the rest of the proof completely here, instead of just discussing the difference, to avoid confusion.

Suppose that there exists $\mathbf{t}' \in [\Pi_i \vee \Pi_j](\mathbf{t})$ such that $\chi_i(\mathbf{t}') = N$. Since $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ is blocked by $(i, j; p)$, we know that

$$\chi_i(\mathbf{t}'') = Y \text{ for every } \mathbf{t}'' \in C_i^{l^*}(\mathbf{t}).$$

Therefore, measurability (Lemma 2) implies

$$C_i^{l^*}(\mathbf{t}) \cap [\Pi_i \vee \Pi_j](\mathbf{t}) = \emptyset. \tag{B3}$$

By the iteration of consideration (B1)-(B2), (B3) is true only if one of the following two cases happens (for agent j at round $l^* - 1$):

(a) $C_j^{l^*-1}(\mathbf{t}) \neq \emptyset$ and for every $\mathbf{t}''' \in C_j^{l^*-1}(\mathbf{t})$, we have $\chi_j^{l^*}(\mathbf{t}''') = N$.

(b) $C_j^{l^*-1}(\mathbf{t}) = \emptyset$, which implies $C_j^{l^*-1}(\mathbf{t}) \cap [\Pi_i \vee \Pi_j](\mathbf{t}) = \emptyset$.

Suppose case (a) holds. Then $C_j^{l^*}(\mathbf{t}) \subset C_j^{l^*-1}(\mathbf{t})$ (Lemma 5), implies that

$$\chi_j^{l^*}(\mathbf{t}''') = N \text{ for every } \mathbf{t}''' \in C_j^{l^*}(\mathbf{t}).$$

Since $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ is blocked by $(i, j; p)$, we have

$$\chi_j(\mathbf{t}''') = Y \text{ for every } \mathbf{t}''' \in C_j^{l^*}(\mathbf{t}).$$

Therefore, these indicators Y's under χ_j are adjusted to N's when we update the indicator functions. Measurability (Lemma 2) implies that these Y's never turn to N's by inseparable N. Hence, these Y's are adjusted to N's by some supported N, i.e., for some l, $C_j^l(\mathbf{t})$ contains some type assignment $\mathbf{t}''''$ with $\chi_j(\mathbf{t}'''') = N$ and it is supported by worker i's uniform Y's. Obviously, such a type assignment $\mathbf{t}''''$ must be considered by firm j as long as $C_j^l(\mathbf{t}) \neq \emptyset$.[27] Then we have $\mathbf{t}'''' \in C_j^{l^*}(\mathbf{t})$. However, this contradicts $\chi_j(\mathbf{t}''') = Y$ for every $\mathbf{t}''' \in C_j^{l^*}(\mathbf{t})$. Therefore, case (a) does not hold.

Suppose case (b) holds. This is true only if either case (a) or case (b) holds for agent i at round $l^* - 2$. Similar argument as in the last paragraph shows that case (a) leads to a contradiction. This continues until round zero is reached and only case (b) is possible, i.e., we have either (case (b) for agent i at round zero)

$$C_i^0(\mathbf{t}) \cap [\Pi_i \vee \Pi_j](\mathbf{t}) = \emptyset$$

or (case (b) for agent j at round zero)

$$C_j^0(\mathbf{t}) \cap [\Pi_i \vee \Pi_j](\mathbf{t}) = \emptyset.$$

[27] It is generally true that supported N type assignments must be considered as long as the corresponding consideration set is nonempty.

However, both of them bring us to a contradiction since $C^0 = \Pi$. Hence, for all $\mathbf{t}' \in [\Pi_i \vee \Pi_j](\mathbf{t})$, we have $\chi_i(\mathbf{t}') = Y$. Symmetric argument shows that $\chi_j(\mathbf{t}') = Y$ holds as well for all $\mathbf{t}' \in [\Pi_i \vee \Pi_j](\mathbf{t})$.

In the following example, IM is not satisfied by sophisticated blocking because the consideration refinement due to a supported N disappears when information becomes finer.

EXAMPLE 5: Consider a potential blocking pair for a state, where there are three possible type assignments and the agents' hypothetical willingness/unwillingness is summarized in the figure below.

$$\begin{array}{rccc} & \textit{Truth} & & \\ & \mathbf{t}^1 & \mathbf{t}^2 & \mathbf{t}^3 \\ \textit{Worker} & \{Y & N\} & \{Y\} \\ \textit{Firm} & \{Y\} & \{Y & N\} \end{array}$$

Clearly, the firm's right partition cell, which contains $\mathbf{t}^2$ and $\mathbf{t}^3$, is a case of supported N. The worker knows that the true type assignment is either $\mathbf{t}^1$ or $\mathbf{t}^2$, and he worries about $\mathbf{t}^2$, under which he would not obtain a higher payoff through rematching. However, he would think that if the true type assignment is $\mathbf{t}^2$, then the firm would know her right partition cell $\{\mathbf{t}^2, \mathbf{t}^3\}$, where the indicators should be adjusted into uniform N's due to the supported N. Therefore, the worker does not need to consider $\mathbf{t}^2$, which means that the potential blocking pair is indeed a blocking pair.

Now let us consider an alternative situation in the figure below, where only the firm's partition changes.

$$\begin{array}{rccc} & \textit{Truth} & & \\ & \mathbf{t}^1 & \mathbf{t}^2 & \mathbf{t}^3 \\ \textit{Worker} & \{Y & N\} & \{Y\} \\ \textit{Firm} & \{Y\} & \{Y\} & \{N\} \end{array}$$

Clearly, agents have more precise information as the firm's partition becomes strictly finer. However, the worker and the firm no longer constitute a blocking pair because the worker would consider $\mathbf{t}^2$ when evaluating the rematching.

PROOF OF PROPOSITION 5:

Straightforward by definitions and examples in the main text.

PROOF OF PROPOSITION 6:

Identical to that of Theorem 1 for level-l stable states; and identical to that of Lemma 4 for sophisticatedly stable states.

PROOF OF PROPOSITION 7:

We provide a unified proof for statements (i)-(iii). More precisely, let W(eak)-blocking and S(trong)-blocking be any two blocking notions among those mentioned in Proposition 5 such that if a state is W-blocked, then it is S-blocked. For example, we can have W = level-l and S = level-$(l+1)$. We proceed to show that if a state $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ is S-stable, then the state $(\mu, \mathbf{p}, \mathbf{t}, H^{W,\infty}_{\mu,\mathbf{p}}(\Pi))$ is W-stable, where $H^W_{\mu,\mathbf{p}}$ is the information refinement operator as in (11) but associated with the W-blocking notion. We use similar superscripts to differentiate notations associated with the two blocking/stability notions.

Now suppose $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ is S-stable. Then $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ is individually rational and not S-blocked. Moreover, by the definition of $\mathcal{N}^S_{\mu,\mathbf{p},\Pi}(\mathbf{t})$, we know that $(\mu, \mathbf{p}, \mathbf{t}', \Pi)$ is individually rational and not S-blocked for all $\mathbf{t}' \in \mathcal{N}^S_{\mu,\mathbf{p},\Pi}(\mathbf{t})$. By our assumption on S- and W-blocking, $(\mu, \mathbf{p}, \mathbf{t}', \Pi)$ being not S-blocked implies that $(\mu, \mathbf{p}, \mathbf{t}', \Pi)$ is not W-blocked. Therefore, we have

$$\mathcal{N}^S_{\mu,\mathbf{p},\Pi}(\mathbf{t}) \subset \mathcal{N}^W_{\mu,\mathbf{p},\Pi}(\mathbf{t}). \tag{B4}$$

Since $\Pi_k(\mathbf{t}') \subset \mathcal{N}^S_{\mu,\mathbf{p},\Pi}(\mathbf{t})$ for every $\mathbf{t}' \in \mathcal{N}^S_{\mu,\mathbf{p},\Pi}(\mathbf{t})$ and every k (by the definition of $H^S_{\mu,\mathbf{p}}(\cdot)$ and the fact that $H^S_{\mu,\mathbf{p}}(\Pi) = \Pi$), it follows from (B4) that

$$\Pi_k(\mathbf{t}') \subset \mathcal{N}^W_{\mu,\mathbf{p},\Pi}(\mathbf{t}) \text{ for all } \mathbf{t}' \in \mathcal{N}^S_{\mu,\mathbf{p},\Pi}(\mathbf{t}) \text{ and all } k \in I \cup J.$$

Therefore, we have

$$\Pi_k(\mathbf{t}') = [H^W_{\mu,\mathbf{p}}(\Pi)]_k(\mathbf{t}') \text{ for all } \mathbf{t}' \in \mathcal{N}^S_{\mu,\mathbf{p},\Pi}(\mathbf{t}) \text{ and all } k \in I \cup J. \tag{B5}$$

We claim that the state $(\mu, \mathbf{p}, \mathbf{t}', H^W_{\mu,\mathbf{p}}(\Pi))$ is not W-blocked for every $\mathbf{t}' \in \mathcal{N}^S_{\mu,\mathbf{p},\Pi}(\mathbf{t})$. Otherwise, the state $(\mu, \mathbf{p}, \mathbf{t}', \Pi)$ is S-blocked by (B5) and our assumption on S- and W-blocking (applied locally on the event $\mathcal{N}^S_{\mu,\mathbf{p},\Pi}(\mathbf{t})$), which is a contradiction. Hence, $(\mu, \mathbf{p}, \mathbf{t}', H^W_{\mu,\mathbf{p}}(\Pi))$ is not W-blocked. The individual rationality of $(\mu, \mathbf{p}, \mathbf{t}', \Pi)$ is equivalent to that of $(\mu, \mathbf{p}, \mathbf{t}', H^W_{\mu,\mathbf{p}}(\Pi))$ by (B5).

Inductively for every integer $l \geq 1$, assuming

$$\Pi_k(\mathbf{t}') = [H^{W,l}_{\mu,\mathbf{p}}(\Pi)]_k(\mathbf{t}') \text{ for all } \mathbf{t}' \in \mathcal{N}^S_{\mu,\mathbf{p},\Pi}(\mathbf{t}) \text{ and all } k \in I \cup J,$$

we have that the state $(\mu, \mathbf{p}, \mathbf{t}', H^{W,l}_{\mu,\mathbf{p}}(\Pi))$ is individually rational and not W-blocked for every $\mathbf{t}' \in \mathcal{N}^S_{\mu,\mathbf{p},\Pi}(\mathbf{t})$. Therefore,

$$\mathcal{N}^S_{\mu,\mathbf{p},\Pi}(\mathbf{t}) \subset \mathcal{N}^W_{\mu,\mathbf{p},H^{W,l}_{\mu,\mathbf{p}}(\Pi)}(\mathbf{t}). \tag{B6}$$

Hence, for every $\mathbf{t}' \in \mathcal{N}^S_{\mu,\mathbf{p},\Pi}(\mathbf{t})$ and every k, we know by the fact $H^S_{\mu,\mathbf{p}}(\Pi) = \Pi$ that

$$\Pi_k(\mathbf{t}') = [H^{W,l+1}_{\mu,\mathbf{p}}(\Pi)]_k(\mathbf{t}'),$$

which implies that $(\mu, \mathbf{p}, \mathbf{t}', H^{W,l+1}_{\mu,\mathbf{p}}(\Pi))$ is individually rational and not W-blocked. Particularly, $(\mu, \mathbf{p}, \mathbf{t}, H^{W,l+1}_{\mu,\mathbf{p}}(\Pi))$ is individually rational and not W-blocked. Since

$H^{W,l+1}_{\mu,\mathbf{p}}(\Pi)$ is weakly finer than $H^{W,l}_{\mu,\mathbf{p}}(\Pi)$, there exists l^* such that $H^{W,l^*+1}_{\mu,\mathbf{p}}(\Pi) = H^{W,l^*}_{\mu,\mathbf{p}}(\Pi)$. Therefore, the limit state $(\mu, \mathbf{p}, \mathbf{t}, H^{W,\infty}_{\mu,\mathbf{p}}(\Pi)) = (\mu, \mathbf{p}, \mathbf{t}, H^{W,l^*}_{\mu,\mathbf{p}}(\Pi))$ is W-stable.

PROOF OF PROPOSITION 8:

By Theorem 4, it suffices to show that (i, j) is a blocking pair for the state $(\mu, \mathbf{w}, \mathbf{f}, \Pi)$ if and only if it is a sophisticated one. The only-if part follows from Proposition 5. We proceed to show that if (i, j) is a sophisticated blocking pair, then it is a blocking pair.

First of all, we claim that under C4, there does not exist inseparable N. More precisely, by C4, the join partition must has a singleton cell at every type assignment, particularly at the true type profile $(\mathbf{w}, \mathbf{f})$, i.e.,

$$\text{(B7)} \qquad [\Pi_i \vee \Pi_j]\,(\mathbf{w}', \mathbf{f}') = \big\{(\mathbf{w}', \mathbf{f}')\big\} \text{ for all } (\mathbf{w}', \mathbf{f}') \in T.$$

Then, an immediate implication is that there does not exist inseparable N when the two agents update their consideration.

Now we are ready to show that if (i, j) is a sophisticated blocking pair, then it is a blocking pair. The two blocking notion differs only in the refinement of agents' consideration. Namely, sophisticated blocking takes inseparable N and supported N into account but blocking does not. Since inseparable N never happens, we only need to show that supported N, if not a uniform N, cannot differentiate the two blocking notions (of course, uniform N's cannot distinguish the two blocking notions).

Suppose $C^l_i(\mathbf{w}, \mathbf{f})$ can be refined due to agent j's supported N. More precisely, suppose that for some l, there exists $(\mathbf{w}', \mathbf{f}') \in C^l_i(\mathbf{w}, \mathbf{f}) \setminus C^{l+1}_i(\mathbf{w}, \mathbf{f})$ and some $(\mathbf{w}'', \mathbf{f}'') \in C^l_j(\mathbf{w}', \mathbf{f}')$ such that $\chi^l_j(\mathbf{w}'', \mathbf{f}'') = N$ and

$$\text{(B8)} \qquad \chi^l_i(\mathbf{w}''', \mathbf{f}''') = Y \text{ for all } (\mathbf{w}''', \mathbf{f}''') \in C^l_i(\mathbf{w}'', \mathbf{f}'').$$

By the assumption that we are in a supported N case which is not a uniform N, $C^l_j(\mathbf{w}', \mathbf{f}')$ contains both Y and N.

We claim that $(\mathbf{w}'', \mathbf{f}'') \in C^l_i(\mathbf{w}'', \mathbf{f}'')$. Otherwise, it was either ruled out by uniform N or ruled out by supported N, the latter of which implies that the indicators in $C^l_j(\mathbf{w}', \mathbf{f}')$ were all adjusted to N. Then both cases contradicts $C^l_j(\mathbf{w}', \mathbf{f}')$ containing both Y and N. Hence, $(\mathbf{w}'', \mathbf{f}'') \in C^l_i(\mathbf{w}'', \mathbf{f}'')$.

Obviously, by (B8), $(\mathbf{w}'', \mathbf{f}'') \in C^l_i(\mathbf{w}'', \mathbf{f}'')$ implies $\chi^l_i(\mathbf{w}'', \mathbf{f}'') = Y$, which in turn implies by Conditions C1-C3 that

$$\mathbf{f}''(j) > \mathbf{f}''(\mu(i)).$$

By C4, firm j knows the types of firms. Then, for each $(\mathbf{w}'''', \mathbf{f}'''') \in \Pi_j(\mathbf{w}'', \mathbf{f}'')$, we have $\mathbf{f}'''' = \mathbf{f}''$. Since $(\mathbf{w}'', \mathbf{f}'') \in C^l_j(\mathbf{w}', \mathbf{f}')$, we know that $(\mathbf{w}', \mathbf{f}') \in \Pi_j(\mathbf{w}'', \mathbf{f}'')$. As a result, $\mathbf{f}' = \mathbf{f}''$. Therefore, $\chi_i^l(\mathbf{w}', \mathbf{f}') = Y$. Since $(\mathbf{w}', \mathbf{f}') \in C_i^l(\mathbf{w}, \mathbf{f}) \setminus C_i^{l+1}(\mathbf{w}, \mathbf{f})$ is taken arbitrarily, we know that agent j's supported N only rules out Y in $C_i^l(\mathbf{w}, \mathbf{f})$.

By symmetric arguments, agent i's supported N only rules out Y in $C_j^l(\mathbf{w}, \mathbf{f})$. Since (i, j) is a sophisticated blocking pair for $(\mu, \mathbf{w}, \mathbf{f}, \Pi)$, we know that with more Y's at $(\mathbf{w}, \mathbf{f})$ for both agents, (i, j) is also a blocking pair.