Contest Design with Stochastic Performance

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Online Appendix

I. Symmetric contests with a single costly prize

This appendix considers a special case of a Separable and Monotonic contest, as defined in Section III. Specifically, it is assumed that there is a single prize, which is costly to the designer to award. The aim of the appendix is to explore how the optimal action profile and the optimal design depends on the cost. However, it is assumed for simplicity that all agents are identical and must be induced to take symmetric actions. Hence, subscripts are omitted.

Let $z \geq 0$ denote the cost of the prize to the designer. Although the feasible set is independent of z, the optimal action profile is now generally speaking both in the interior of the feasible set and sensitive to z. From (10), the prize is awarded only if the highest individual score exceeds z. Thus, there is a minimum standard q^z such that the prize is given out only if at least one agent performs above q^z. Note that q^z depends on the symmetric equilibrium action, a, and that (10) implies that $L(q^z|a) > 0$ when $a < \overline{a}^s$. It can be verified from the agents' first-order conditions that q^z is strictly decreasing in a for all $a < \overline{a}^s$.[1] Stated differently, a lower standard and a higher equilibrium action go hand in hand. This in turn means that the probability that the prize is awarded, $1 - G(q^z|a)^n$, increases when a higher action is induced.

The designer's problem is to induce an action a to maximize expected payoff

$$U_0(a, a, ..., a) - z(1 - G(q^z|a)^n).$$

[1] When $a < \overline{a}^s$, the action profile is in the interior of the feasible set. In contrast, Proposition 2 considers action profiles along the frontier of the feasible set and a minimum standard that is found where the LR is zero.

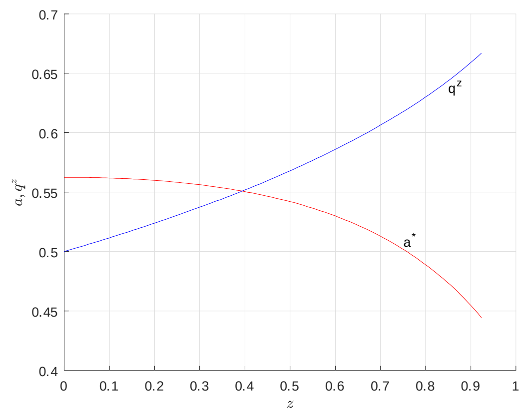

Figure 3: The equilibrium standard and equilibrium action as a function of costs.

Now consider an increase in cost z. This makes it less attractive to award the prize. Given the conclusion in the previous paragraph, this implies that the designer will induce a lower action. In summary, when the prize is costlier to the designer, she induces a lower action by imposing a higher standard. However, when the cost of the prize becomes too high, the designer is better off shutting down the contest.

As an example, consider a special case of the spanning model, with

$$G(q|a) = \sqrt{a}q^2 + \left(1 - \sqrt{a}\right) q, \; q \in [0, 1],$$

whenever $a \in [0, 1)$. Assume $v = 6$ and $c(a) = a$. If the prize is costless, or $z = 0$, the optimal minimum standard is $\widehat{q} = \frac{1}{2}$. The highest action that can be implemented is then $\overline{a} = \frac{9}{16}$.

For simplicity, assume that there is exactly one contestant or agent. Thus, the agent wins the prize if and only if his performance exceeds the minimum standard. For any given minimum standard q^z, the first-order condition implies that the agent's best response is $a = 9 \left(q^z\right)^2 (1 - q^z)^2$, which is of course maximized if $q^z = \widehat{q} = \frac{1}{2}$. However, it may be better to increase q^z in order to lower the probability that the designer has to incur the cost of awarding the prize. Whenever $q^z > \frac{1}{2}$, a and q^z move in opposite directions.

The probability that the prize is awarded is $1 - G(q^z|a) = 3 \left(q^z\right)^4 - 6 \left(q^z\right)^3 +$

$3\left(q^z\right)^2-\left(q^z\right)+1$. Assume that the designer wishes to maximize the agent's action, or $U_0(a)=a$. Then, the designer's expected payoff is

$$9\left(q^z\right)^2\left(1-q^z\right)^2-z\left(3\left(q^z\right)^4-6\left(q^z\right)^3+3\left(q^z\right)^2-\left(q^z\right)+1\right).$$

At $z=\frac{12}{13}$, this is maximized at $q^z=\frac{2}{3}$ and at exactly zero expected payoff. Thus, it is optimal for the designer to shut down the contest if $z>\frac{12}{13}$, which can be achieved by imposing a minimum standard of $q=1$. On the other hand, as long as $z\in\left(0,\frac{12}{13}\right)$, a minimum standard in the interval $\left(\frac{1}{2},\frac{2}{3}\right)$ is optimal and this minimum standard is increasing in z. Figure 3 illustrates the solution.

Note that $z=\frac{12}{13}$ is substantially higher than the action that is induced at $q^z=\frac{2}{3}$, which is $a=\frac{4}{9}$. Even though the price is extremely expensive to the designer, the contest is still profitable when z is just below $\frac{12}{13}$ because there is only a small chance that the prize must be awarded.

II. The best-shot model

This appendix complements the treatment of the best-shot model in Section IV. Among other things, the CSF is computed and the feasible set is constructed when rationing is possible. It concludes with a discussion of how to microfound the biased lottery CSF that is often used in the current literature, and whether this microfoundation is appealing or desirable.

A. Winning probabilities and CSFs

As mentioned in Section V.B, it is possible to extend the model to more than two groups of agents. The main complication is that there are now many types of "sequential" allocation rules of the kind described at the beginning of Section II.C. For example, group 1 might be "served" first, followed then by group 2 and later by group 3, while all remaining groups at the very end fight each other simultaneously if the prize is still available. A complete description of the feasible set requires one to piece together all these cases. See Kirkegaard (2020) for details.

This subsection instead considers the simplest possible case in which the allocation rule is “simultaneous”. Thus, every agent receives a score of the form $\mu_i v_i L_i(q_i|a_i)$, with $\mu_i \in (0,\infty)$ for all $i \in N$, and the agent with the highest score wins. Using the notation from Section IV.A, this means that $\tau_i \in (0,\infty)$. It turns out that τ_i is in a sense a measure of how favorable the contest is to agent i, as demonstrated in the following result.

Proposition 9 *Consider the best-shot model with an arbitrary number of agents. Fix an action profile* $\mathbf{a}^*$ *on the frontier of the feasible set in which all agents are active* ($a_i > 0$) *and which is implemented by giving each agent a scoring function* $s_i(q_i) = \mu_i v_i L_i(q_i|a_i^*)$, *with* $\mu_i \in (0,\infty)$ *for all* $i \in N$. *Then, agent* i*'s ex ante equilibrium winning probability exceeds that of agent* j *if and only if* $\tau_i > \tau_j$, *regardless of whether rationing is allowed or not.*

Proof. Note that if agents i and j perform equally well given what is expected of them – i.e. they perform at the same quantiles, or $G_i(q_i|a_i^*) = G_j(q_j|a_j^*)$ – then agent i's score beats agent j's score if $\tau_i > \tau_j$ and the likelihood-ratios are positive. However, agent's j's score is higher if the likelihood-ratios are negative.

Consequently, the result is trivial if rationing is allowed. Then, only positive likelihood-ratios have a chance of winning. Recall that agents i and j have positive likelihood-ratios with the same probability, specifically $1 - e^{-1}$. Given a performance at any fixed quantile above e^{-1}, such that $G_i(q_i|a_i^*) = G_j(q_j|a_j^*) \geq e^{-1}$, agent i outscores agent j if and only if $\tau_i > \tau_j$. Since quantiles are distributed the same way (uniformly) for both agents, it now follows that agent i wins with a higher probability in equilibrium if and only if $\tau_i > \tau_j$.

If rationing is ruled out, then performance with negative likelihood-ratio come into play. Given τ_i, agent i's score is in equilibrium distributed according to

$$K_i(s_i|\tau_i) = e^{\frac{s_i}{\tau_i}-1},\ s_i \in (-\infty, \tau_i]$$

with density

$$k_i(s_i|\tau_i) = \frac{1}{\tau_i} e^{\frac{s_i}{\tau_i}-1},\ s_i \in (-\infty, \tau_i].$$

Without loss of generality, arrange agents in ascending order based on their τ_i, with $\tau_1 \leq \tau_2 \leq ...\tau_N$. Let $\tau_0 = -\infty$. A score above τ_j automatically beats agent

j. Hence, agent i's equilibrium winning probability can then be written as

$$
\begin{aligned}
P_i^*(\tau_1, \tau_2, ..., \tau_N) &= \int_{\tau_0}^{\tau_1} \left(\prod\nolimits_{j \geq 1, j \neq i} K_j(s|\tau_j) \right) k_i(s|\tau_i) ds \\
&\quad + \int_{\tau_1}^{\tau_2} \left(\prod\nolimits_{j \geq 2, j \neq i} K_j(s|\tau_j) \right) k_i(s|\tau_i) ds \\
&\quad + ... + \int_{\tau_{i-1}}^{\tau_i} \left(\prod\nolimits_{j \geq i, j \neq i} K_j(s|\tau_j) \right) k_i(s|\tau_i) ds \\
&= \frac{1}{\tau_i} \sum_{m=1}^{i} \alpha_m,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_m &= \int_{\tau_{m-1}}^{\tau_m} e^{\sum_{j \geq m} \left(\frac{s}{\tau_j} - 1 \right)} ds \\
&= \frac{1}{\sum_{j \geq m} \frac{1}{\tau_j}} \left(e^{\sum_{j \geq m} \left(\frac{\tau_m}{\tau_j} - 1 \right)} - e^{\sum_{j \geq m} \left(\frac{\tau_{m-1}}{\tau_j} - 1 \right)} \right).
\end{aligned}
$$

Going forward, for $i = 2, ..., n$, it is useful to compare

$$
\alpha_i = \frac{1}{\sum_{j \geq i} \frac{1}{\tau_j}} \left(e^{\sum_{j \geq i} \left(\frac{\tau_i}{\tau_j} - 1 \right)} - e^{\sum_{j \geq i} \left(\frac{\tau_{i-1}}{\tau_j} - 1 \right)} \right)
$$

and

$$
\begin{aligned}
\sum_{m=1}^{i-1} \alpha_m &\leq \int_{\tau_0}^{\tau_{i-1}} e^{\sum_{j \geq i-1} \left(\frac{s}{\tau_j} - 1 \right)} ds. \\
&= \frac{1}{\sum_{j \geq i-1} \frac{1}{\tau_j}} e^{\sum_{j \geq i-1} \left(\frac{\tau_{i-1}}{\tau_j} - 1 \right)} \\
&= \frac{1}{\sum_{j \geq i-1} \frac{1}{\tau_j}} e^{\sum_{j \geq i} \left(\frac{\tau_{i-1}}{\tau_j} - 1 \right)}.
\end{aligned}
$$

Then, note that for $i = 2, ..., n$,

$$\begin{aligned}
P_i^* - P_{i-1}^* &= \frac{1}{\tau_i}\sum_{m=1}^{i}\alpha_m - \frac{1}{\tau_{i-1}}\sum_{m=1}^{i-1}\alpha_m \\
&= \frac{1}{\tau_i}\alpha_i - \left(\frac{1}{\tau_{i-1}} - \frac{1}{\tau_i}\right)\sum_{m=1}^{i-1}\alpha_m \\
&\geq \frac{1}{\tau_i}\frac{1}{\sum_{j\geq i}\frac{1}{\tau_j}}\left(e^{\sum_{j\geq i}\left(\frac{\tau_i}{\tau_j}-1\right)} - e^{\sum_{j\geq i}\left(\frac{\tau_{i-1}}{\tau_j}-1\right)}\right) \\
&\quad - \left(\frac{1}{\tau_{i-1}} - \frac{1}{\tau_i}\right)\frac{1}{\sum_{j\geq i-1}\frac{1}{\tau_j}}e^{\sum_{j\geq i}\left(\frac{\tau_{i-1}}{\tau_j}-1\right)},
\end{aligned}$$

and where, defining $x_i = \sum_{j\geq i}\frac{1}{\tau_j}$, the latter is proportional to

$$\begin{aligned}
\Delta_i &= (1+\tau_{i-1}x_i)\left(e^{\tau_i x_i} - e^{\tau_{i-1}x_i}\right) - (\tau_i - \tau_{i-1})\,x_i e^{\tau_{i-1}x_i} \\
&= (1+\tau_{i-1}x_i)\,e^{\tau_i x_i} - (1+\tau_i x_i)\,e^{\tau_{i-1}x_i} > 0
\end{aligned}$$

when $\tau_i > \tau_{i-1}$. Hence, it now follows that winning probabilities are arranged in the same order as the τ_i's. ■

Given a vector $\boldsymbol{\tau}$ that lists all τ_i's, it is in principle possible to derive the CSF – the probability that agent i wins for any given action profile $\mathbf{a}$ – by integrating out the uncertainty over performance, i.e. by calculating

$$\int\left(\int P_i(q_i, \mathbf{q}_{-i})g_i(q_i|a_i)dq_i\right)\prod_{j\neq i}g_j(q_j|a_j)d\mathbf{q}_{-i}.$$

In the best-shot model, however, a more direct argument is also possible. This is illustrated in the proof of the next proposition, under the assumption that rationing is ruled out and that all agents are active. In this case, negative scores have a chance of winning.

Proposition 10 *Under the assumptions in Proposition 9, if* $\mathbf{a}^*$ *is the equilibrium* *action profile and $\tau_i \leq \tau_j$ for all $j \in N$, then agent i wins with probability*

$$\widehat{p}_i(\mathbf{a}|\boldsymbol{\tau}) = \left(\prod\nolimits_{j\in N\setminus\{i\}} e^{\frac{(\tau_i-\tau_j)f_j(a_j)}{\tau_j f_j(a_j^*)}}\right) \frac{\frac{f_i(a_i)}{\tau_i f_i(a_i^*)}}{\sum_{j\in N}\frac{f_j(a_j)}{\tau_j f_j(a_j^*)}}, \tag{22}$$

when rationing is ruled out, for any action profile $\mathbf{a}$ with $a_i > 0$.

Proof. To start, note that the distribution of agent i's score is

$$S_i(s|a_i) = \left(e^{s-\tau_i}\right)^{\frac{f_i(a_i)}{\tau_i f_i(a_i^*)}}, \quad s \in (-\infty, \tau_i]$$

when he takes action a_i rather than a_i^*. It is as if he draws $\frac{f_i(a_i)}{\tau_i f_i(a_i^*)}$ scores from the distribution $e^{s-\tau_i}$, but only the best score is counted. The range of scores depends on the identity of the agent, with τ_i describing the highest possible score that agent i can achieve. Assume agent i is the agent with the lowest τ, or $\tau_i \leq \tau_j$. Then, in order for agent i to win it is necessary that all other agents score below τ_i, the probability of which is

$$\left(\prod\nolimits_{j\in N\setminus\{i\}} e^{\frac{(\tau_i-\tau_j)f_j(a_j)}{\tau_j f_j(a_j^*)}}\right). \tag{23}$$

Given this event, however, the conditional distribution of agent j's score is

$$\frac{S_j(s|a_j)}{S_j(\tau_i|a_j)} = \left(e^{s-\tau_i}\right)^{\frac{f_j(a_j)}{\tau_j f_j(a_j^*)}}, \quad s \in (-\infty, \tau_i].$$

Hence, it is as if all agents draw scores from the *same* distribution, $e^{s-\tau_i}$. Since each draw therefore has an equal chance of winning, the conditional probability that agent i wins is

$$\frac{\frac{f_i(a_i)}{\tau_i f_i(a_i^*)}}{\sum_{j\in N}\frac{f_j(a_j)}{\tau_j f_j(a_j^*)}}. \tag{24}$$

Combining (23) and (23) yields the CSF in the proposition. ■

As a consistency check, note that if $\tau_i = \tau_j$ for all $j \in N$ then $\widehat{p}_i(\mathbf{a}^*|\boldsymbol{\tau}) = \frac{1}{n}$ and all agents win with equal probability in equilibrium. Note that the first term in (22) depends on the action profile, for reasons that are carefully explained in the proof of the proposition. Due to this distortion, (22) is *not* a lottery CSF (except in the special case where $\tau_i = \tau_j$ for all $j \in N$).

B. The feasible set with and without rationing

Next, the feasible set of implementable actions is characterized. As explained in the main text, it is assumed that

$$\lim_{a_i \to 0} c_i'(a_i) \frac{f_i(a_i)}{f_i'(a_i)} = 0,$$

for all i. Now, the highest possible implementable action of agent i, $\overline{a}_i$, can be characterized succinctly in the best-shot model. This follows from the proof of Proposition 1.

Corollary 4 *In the best-shot model, any action no greater than the unique solution $\overline{a}_i$ to*

$$c_i'(\overline{a}_i) \frac{f_i(\overline{a}_i)}{f_i'(\overline{a}_i)} = \frac{v_i}{e} \tag{25}$$

can be implemented by appropriately designing the assignment rule.

Proof. In the best-shot model, where $\widehat{q}_i(a_i^t) = H^{-1}\left(e^{-\frac{1}{f_i(a_i^t)}}\right)$ or $H(\widehat{q}_i(a_i)) = e^{-\frac{1}{f_i(a_i^t)}}$, (17) is

$$\overline{U}_i(a_i) = v_i\left(1 - e^{-\frac{f_i(a_i)}{f_i(a_i^t)}}\right) - c_i(a_i)$$

and (18) simplifies to

$$\frac{1}{c_i'(a_i^t)} \frac{f_i'(a_i^t)}{f_i(a_i^t)} \geq \frac{e}{v_i}.$$

By concavity of f_i and convexity of c_i, the left hand side is decreasing. Hence, the condition is satisfied if and only if a_i^t is no greater than the solution to (25). By Proposition 1, it is then possible to implement the action. ■

Using similar logic, it is possible to characterize the corners of the frontier of the feasible set when there are two groups of agents and the rules are group-symmetric. As in Section II, let $\overline{a}_i^s$ denote the highest possible action is group i when rules are group-symmetric. When $n_i = 1$, $\overline{a}_i^s = \overline{a}_i$ but otherwise $\overline{a}_i^s < \overline{a}_i$.

Similarly, let $\underline{a}_i^s$ denote the smallest possible action along the frontier for an agent in group i when rules are group-symmetric. This is the action that is implemented when $a_j = \overline{a}_j^s$ in the other group, $j \neq i$. This means that an agent in group i has a chance of winning only if all agents in group j have negative likelihood-ratios. If $n_i \geq 2$, then competition within group i still ensures that $\underline{a}_i^s > 0$. However, if $n_i = 1$, then agent i is simply the "residual claimant" of the prize and has no incentive to exert effort.

Corollary 5 *In the best-shot model with two groups, group-symmetric rules, and no rationing, the frontier of the feasible set contains the corners* $(\overline{a}_1^s, \underline{a}_2^s)$ *and* $(\underline{a}_1^s, \overline{a}_2^s)$*, where* $\overline{a}_i^s$ *and* $\underline{a}_i^s$ *solve*

$$c_i'(\overline{a}_i^s)\frac{f_i(\overline{a}_i^s)}{f_i'(\overline{a}_i^s)} = v_i\frac{n_i - 1 + e^{-n_i}}{n_i^2} \text{ and } c_i'(\underline{a}_i^s)\frac{f_i(\underline{a}_i^s)}{f_i'(\underline{a}_i^s)} = v_i e^{-n_j}\frac{n_i - 1}{n_i^2}, \quad i,j = 1,2 \text{ and } j \neq i.$$

Here, $\overline{a}_i^s$ *is strictly decreasing in* n_i *and independent of* n_j*. Similarly,* $\underline{a}_i^s$ *is strictly positive if and only if* $n_i \geq 2$ *and it is then strictly decreasing in both* n_i *and* n_j*.*

Proof. To implement $\overline{a}_i^s$, the contest rules must imply that an agent in group i wins if and only if his likelihood-ratio is positive and (by group-symmetry and the MLRP) if his performance is higher than the performance of all other agents in his group. Given all other agents in group i takes action $\overline{a}_i^s$, the relevant first-order condition is

$$\begin{aligned} c_i'(\overline{a}_i^s) &= v_i \int_{\widehat{q}_i(\overline{a}_i^s)}^{\overline{q}_i} \left(G_i(q_i|\overline{a}_i^s)\right)^{n_i-1} L_i(q_i|\overline{a}_i^s) g_i(q_i|\overline{a}_i^s) dq_i \\ &= v_i \frac{f_i'(\overline{a}_i^s)}{f_i(\overline{a}_i^s)} \int_{\widehat{q}_i(\overline{a}_i^s)}^{\overline{q}_i} \left(G_i(q_i|\overline{a}_i^s)\right)^{n_i-1} \left(1 + \ln G_i(q_i|\overline{a}_i^s)\right) g_i(q_i|\overline{a}_i^s) dq_i \end{aligned}$$

Substituting the equilibrium quantiles of agent i's performance, $z = G_i(q_i|\overline{a}_i^s)$ and $dz = g_i(q_i|\overline{a}_i^s)dq_i$, yields

$$\begin{aligned} c_i'(\overline{a}_i^s)\frac{f_i(\overline{a}_i^s)}{f_i'(\overline{a}_i^s)} &= v_i \int_{e^{-1}}^{1} z^{n_i-1}\left(1 + \ln z\right) dz \\ &= v_i \frac{n_i - 1 + e^{-n_i}}{n_i^2}. \end{aligned}$$

Simple differentiation shows that the right hand side is decreasing in n_i.

Turning to $\underline{a}_i^s$, this is the action that is implemented when the other group is induced to take action $a_j = \overline{a}_j^s$, $j \neq i$. Thus, an agent in group i has a chance of winning only if all agents in group j have negative likelihood-ratios. This occurs with probability e^{-n_j}. Conditional on this event, the agent must moreover (by group-symmetry and the MLRP) outperform all other agents in his group. Since rationing is ruled out, such an agent may win even if his own likelihood-ratio is negative. Thus, the first-order condition is

$$c_i'(\underline{a}_i^s) = v_i e^{-n_j} \int_{\underline{q}_i}^{\overline{q}_i} \left(G_i(q_i|\underline{a}_i^s)\right)^{n_i-1} L_i(q_i|\underline{a}_i^s) g_i(q_i|\underline{a}_i^s) dq_i,$$

which reduces to the statement in the corollary. The rest follows by simple differentiation. ■

Next, the interior part of the frontier is characterized.

Proposition 11 *In the best-shot model with two groups, group-symmetric rules, and no rationing, the frontier of the feasible set contains the corners* $(\overline{a}_1^s, \underline{a}_2^s)$ *and* $(\underline{a}_1^s, \overline{a}_2^s)$. *The remaining action profiles on the frontier can be traced out by varying* $\tau_1 > 0$ *and* $\tau_2 > 0$, *where the equilibrium action of an agent in group* i *is determined by*

$$c_i'(a_i^*) \frac{f_i(a_i^*)}{f_i'(a_i^*)} = v_i F\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right),$$

with

$$F\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right) = \begin{cases} e^{n_j\left(\frac{\tau_i}{\tau_j}-1\right)} \frac{n_j \frac{\tau_i}{\tau_j}+n_i-1}{\left(n_j\frac{\tau_i}{\tau_j}+n_i\right)^2} & \text{if } \frac{\tau_i}{\tau_j} \in (0,1) \\ \frac{n_i-1}{n_i^2} + e^{n_i\left(\frac{1}{\tau_i/\tau_j}-1\right)} \frac{n_j}{n_i^2} \frac{n_j\left(\frac{\tau_i}{\tau_j}\right)^2 + \frac{\tau_i}{\tau_j} n_i(2-n_j) - n_i^2}{\left(n_j\frac{\tau_i}{\tau_j}+n_i\right)^2} & \text{if } \frac{\tau_i}{\tau_j} \geq 1 \end{cases}.$$

Here, $F\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right)$ *is strictly increasing in* $\frac{\tau_i}{\tau_j}$ *and satisfies* $F(1|n_i, n_j) = \frac{n-1}{n^2}$. *Hence,* a_i^* *is strictly increasing in* $\frac{\tau_i}{\tau_j}$.

Proof. The corners are described in Corollary 5. To describe actions away from the corners of the frontier, note that such actions must be interior and the two first-order conditions must therefore be solved simultaneously. Assume first that $0 < \tau_i \leq \tau_j$. Then, regardless of his performance, an agent in group i wins with a probability strictly less than one when $\tau_i < \tau_j$. If his performance is q_i, then he beats an agent in group j if and only if $s_i(q_i) \geq s_j(q_j)$, which occurs if and only if q_i and q_j are such that

$$e^{\frac{\tau_i-\tau_j}{\tau_j}} G_i(q_i|a_i^*)^{\frac{\tau_i}{\tau_j}} \geq G_j(q_j|a_j^*),$$

where the term on the right hand side is the equilibrium distribution of the performance of a member of group j. Hence, the interim probability that agent i with performance q_i beats such an agent is $e^{\frac{\tau_i-\tau_j}{\tau_j}} G_i(q_i|a_i^*)^{\frac{\tau_i}{\tau_j}}$. To win, the agent has to beat all agents in group j as well as all the other agents in group i. With this in mind, agent i's first-order condition in equilibrium is

$$v_i \int_{\underline{q}_i}^{\overline{q}_i} L_i(q_i|a_i^*) \left(e^{\frac{\tau_i-\tau_j}{\tau_j}} G_i(q_i|a_i^*)^{\frac{\tau_i}{\tau_j}} \right)^{n_j} G_i(q_i|a_i^*)^{n_i-1} g_i(q_i|a_i^*) dq_i - c_i'(a_i^*) = 0$$

or

$$v_i \frac{f_i'(a_i^*)}{f_i(a_i^*)} e^{n_j \frac{\tau_i-\tau_j}{\tau_j}} \int_{\underline{q}_i}^{\overline{q}_i} \left(1 + \ln G_i(q_i|a_i^*)\right) G_i(q_i|a_i^*)^{n_j \frac{\tau_i}{\tau_j} + n_i - 1} g_i(q_i|a_i^*) dq_i - c_i'(a_i^*) = 0.$$

As in Corollary 5, substituting the equilibrium quantiles of agent i's performance, $z = G_i(q_i|a_i^*)$ and $dz = g_i(q_i|a_i^*)dq_i$. This gives

$$\begin{aligned} c_i'(a_i^*) \frac{f_i(a_i^*)}{f_i'(a_i^*)} &= v_i e^{n_j \frac{\tau_i-\tau_j}{\tau_j}} \int_0^1 \left(1 + \ln z\right) z^{n_j \frac{\tau_i}{\tau_j} + n_i - 1} dz \\ &= v_i e^{n_j \left(\frac{\tau_i}{\tau_j} - 1\right)} \frac{n_j \frac{\tau_i}{\tau_j} + n_i - 1}{\left(n_j \frac{\tau_i}{\tau_j} + n_i\right)^2}, \end{aligned}$$

which nails down a_i^* since the left hand side is strictly increasing in a_i^*.

Assume now that $\tau_i > \tau_j > 0$. In this case, agent i beats any agent in group j with probability one if his performance is high enough, or specifically if $q_i \geq \widetilde{q}_i$ where

$$e^{\frac{\tau_i-\tau_j}{\tau_j}} G_i(\widetilde{q}_i|a_i^*)^{\frac{\tau_i}{\tau_j}} = 1,$$

which implies that

$$G_i(\widetilde{q}_i|a_i^*) = e^{\frac{\tau_j-\tau_i}{\tau_i}}$$

and

$$1+\ln G_i(\widetilde{q}_i|a_i^*) = \frac{\tau_j}{\tau_i}.$$

Agent i's first order condition is now

$$v_i \int_{\underline{q}_i}^{\widetilde{q}_i} L_i(q_i|a_i^*)\left(e^{\frac{\tau_i-\tau_j}{\tau_j}} G_i(q_i|a_i^*)^{\frac{\tau_i}{\tau_j}}\right)^{n_j} G_i(q_i|a_i^*)^{n_i-1} g_i(q_i|a_i^*)dq_i$$
$$+ v_i \int_{\widetilde{q}_i}^{\overline{q}_i} L_i(q_i|a_i^*) G_i(q_i|a_i^*)^{n_i-1} g_i(q_i|a_i^*)dq_i - c_i'(a_i^*) = 0.$$

The same substitution as before yields

$$c_i'(a_i^*)\frac{f_i(a_i^*)}{f_i'(a_i^*)} = v_i e^{n_j\frac{\tau_i-\tau_j}{\tau_j}} \int_0^{e^{\frac{\tau_j-\tau_i}{\tau_i}}} (1+\ln z)\, z^{n_j\frac{\tau_i}{\tau_j}+n_i-1} dz + v_i \int_{e^{\frac{\tau_j-\tau_i}{\tau_i}}}^1 (1+\ln z)\, z^{n_i-1} dz$$

or

$$c_i'(a_i^*)\frac{f_i(a_i^*)}{f_i'(a_i^*)} = v_i \left(\frac{n_i-1}{n_i^2} + e^{n_i\left(\frac{1}{\tau_i/\tau_j}-1\right)} \frac{n_j}{n_i^2} \frac{n_j\left(\frac{\tau_i}{\tau_j}\right)^2 + \frac{\tau_i}{\tau_j} n_i\,(2-n_j) - n_i^2}{\left(n_j\frac{\tau_i}{\tau_j}+n_i\right)^2} \right).$$

As before, a_i^* is nailed down because the left hand side is strictly increasing. The characterization result in the proposition now follows. The fact that $F\left(\frac{\tau_i}{\tau_j}|n_i,n_j\right)$ is strictly increasing in $\frac{\tau_i}{\tau_j}$ is verified by differentiation, and it is straightforward to verify that

$$F(1|n_i,n_j) = \frac{n-1}{n^2}.$$

■

It can be verified that as $\frac{\tau_i}{\tau_j}$ converges to infinity or zero, a_i converges to $\overline{a}_i^s$ and $\underline{a}_i^s$ as described in Corollary 5, respectively.

The frontier of the feasible set is described in a similar fashion when rationing is allowed.

Proposition 12 *In the best-shot model with two groups, group-symmetric rules,* *and with rationing allowed, the action profiles on the frontier of the feasible set, away from the corners, can be traced out by varying $\tau_1 > 0$ and $\tau_2 > 0$. The equilibrium action of an agent in group i is determined by*

$$c_i'(a_i^*)\frac{f_i(a_i^*)}{f_i'(a_i^*)} = v_i F_R\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right),$$

where

$$F_R\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right) = F\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right) + \frac{e^{-(n_i+n_j)}}{\left(n_j\frac{\tau_i}{\tau_j} + n_i\right)^2}.$$

Here, $F_R\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right)$ is strictly increasing in $\frac{\tau_i}{\tau_j}$ and satisfies $F_R\left(1|n_i, n_j\right) = \frac{n-1}{n^2} + \frac{e^{-n}}{n^2}$. Hence, a_i^* is strictly increasing in $\frac{\tau_i}{\tau_j}$.

Proof. The proof follows the same steps as in the proof of Proposition 11. The only difference is that an agent in group i now has zero probability of winning if $q_i < \widehat{q}_i(a_i^*)$ or, using the same substitution as in Proposition 11, if $z \leq G_i(\widehat{q}_i(a_i^*)|a_i^*) = e^{-1}$. Hence, the lower bounds on the integrals that are evaluated in the proof of Proposition 11 change. This produces $F_R(\frac{\tau_i}{\tau_j})$ as stated in the proposition. Monotonicity can be verified by differentiation. ■

Note that $F_R(\frac{\tau_i}{\tau_j}) > F(\frac{\tau_i}{\tau_j})$. Since t_i is increasing, it follows, as expected, that the action profile for any given $\frac{\tau_i}{\tau_j}$ is higher when rationing is allowed than when it is not. It can be verified that Lemma 1 is unchanged when rationing is allowed.

Winning probabilities can be computed using the method described in the proof of Proposition 6, except the integration is performed only over non-negative scores. With $\kappa_i = \frac{\tau_i}{\tau_j}$, this yields equilibrium winning probabilities for an agent in group i of

$$W_R\left(\kappa_i|n_i, n_j\right) = \begin{cases} e^{n_j(\kappa_i-1)}\frac{1}{n_j\kappa_i+n_i} - \frac{e^{-(n_i+n_j)}}{n_j\kappa_i+n_i} & \text{if } \kappa_i \in (0,1) \\ \frac{1}{n_i}\left(1 - n_j e^{n_i\left(\frac{1}{\kappa_i}-1\right)}\frac{\kappa_i}{n_j\kappa_i+n_i}\right) - \frac{e^{-(n_i+n_j)}}{n_j\kappa_i+n_i} & \text{if } \kappa_i \geq 1 \end{cases}$$

or simply

$$W_R\left(\kappa_i|n_i, n_j\right) = W\left(\kappa_i|n_i, n_j\right) - \frac{e^{-(n_i+n_j)}}{n_j\kappa_i + n_i},\ \kappa_i \in (0, \infty).$$

With the proof of Corollary 2 in mind, it can be verified that

$$W_R(\kappa_i|n_i, n_j) - F_R(\kappa_i|n_i, n_j)$$

is increasing in κ_i whenever $W(\kappa_i|n_i, n_j) - F(\kappa_i|n_i, n_j)$ is. The only case in which $W(\kappa_i|n_i, n_j) - F(\kappa_i|n_i, n_j)$ is not increasing is when $n_i = 1$ and $\kappa_i < 1$. Checking this case, however, it turns out that $W_R(\kappa_i|n_i, n_j) - F_R(\kappa_i|n_i, n_j)$ is in fact increasing. In other words, $W_R(\kappa_i|n_i, n_j) - F_R(\kappa_i|n_i, n_j)$ is globally increasing in κ_i for all (n_i, n_j). By the argument in Corollary 2, it follows that expected utility to agents in group i is strictly increasing in κ_i.

C. Microfoundations for the biased lottery CSF

It is possible to use the best-shot model to provide microfoundations for (15).

Proposition 13 *Consider the best-shot model with $H_i(q_i) = H(q_i)$, $i \in N$. Assign agent i with performance q_i a base score of $s_i^B(q_i) = H(q_i)^{1/b_i} \in [0,1]$, $b_i > 0$. Draw an auxiliary score s_i^{AUX} for agent i from the distribution $\left(s_i^{AUX}\right)^{\delta_i}$, $s_i^{AUX} \in [0,1]$, $\delta_i \geq 0$. Let agent i's final score be $s_i^{FM}(q_i) = \max\{s_i^B(q_i), s_i^{AUX}\}$. Finally, draw a score s^D for the designer from the distribution $\left(s^D\right)^z$, $s^D \in [0,1]$, $z \geq 0$. Let the individual (agent or designer) with the highest score win. Then, the CSF is given by (15).*

Proof. Agent i's final score is below s_i if and only if both s_i^B and s_i^{AUX} are below s_i. First, $s_i^B \leq s_i$ when $q_i \leq H^{-1}\left(s_i^{b_i}\right)$, the probability of which is $H(q_i)^{f_i(a_i)} = s_i^{b_i f_i(a_i)}$. Second, the probability that $s_i^{AUX} \leq s_i$ is $s_i^{\delta_i}$. Hence, the probability that the final score is below s_i is $s_i^{b_i f_i(a_i)} s_i^{\delta_i} = s_i^{b_i f_i(a_i) + \delta_i}$. It is as if agent i draws $b_i f_i(a_i) + \delta_i$ "ideas" from a uniform distribution. Similarly, the designer draws z "ideas" from a uniform distribution. Since each "idea" is equally likely to be the best, the ex ante probability that agent i wins is (15). ∎

The transformation of q_i into a base score maps the idea from the support $[\underline{q}, \overline{q}]$ into a quality index on $[0,1]$, where the index is identity dependent via b_i. Given action a_i, agent i then draws $b_i f_i(a_i)$ ideas from a uniform distribution on this index. He is then given δ_i fake ideas by the designer, again drawn from a uniform distribution. The agent now has a total of $b_i f_i(a_i) + \delta_i$ real and fake ideas. The designer also draws z fake ideas from a uniform distribution. Each idea, real or fake, has an equal chance of winning, yielding (15).

The stochastic nature of the fake ideas may or may not be palatable. Thus, Proposition 13 should not be taken as a defense of (15) but rather as a clarification of the lengths one must go to in order to justify it. The transformation of the performance into a quality index seems more appealing. However, this particular transformation is still ad hoc.[2] In fact, Proposition 13 merely shows that (15) can be implemented in the Fullerton and McAfee (1999) model. Hence, it follows that the set of implementable actions must in reality be strictly larger than the set of actions that can be implemented by using (15).[3]

One drawback of using (15) for contest design is that it says little about how to implement the optimal design in practice. For instance, how exactly is the playing field supposed to be made level if the designer does not observe actions? Proposition 13 tells us how this can be achieved by linking design to the observable signals. In other words, the kind of story embodied in Proposition 13 is important if the desire is to apply lessons from (15) in practice. The issue is that (15) pushes the performance profile to the back, which is unfortunate since this is the observable variable. The stochastic performance approach in the current paper has the distinct advantage that it starts directly from the observables.

[2]Similarly, giving agents a multiplicative bonus in Hirschleifer and Riley's (1992) model yields $p_i(\mathbf{a}|\mathbf{0}, \mathbf{b}, 0)$. This can also be obtained by variying the β_i parameter in Clark and Riis' (1996) random utility framework. Again, these are ad hoc ways to manipulate the contest.

[3]Fu and Wu (2020) and most of the prior literature restrict head starts to be non-negative. Drugov and Ryvkin (2017) show that negative head starts may be better. However, negative head starts cannot be justified by Proposition 13.