A Theory of Monopolistic Competition with Horizontally Heterogeneous Consumers

By Sergey Kokovin, Alina Ozhegova, Shamil Sharapudinov,
Alexander Tarasov, and Philip Ushchev

Online Appendix

In this Appendix, we provide the proofs of some lemmas and propositions as well as some figures.

A Proofs of some Lemmas and Propositions

The Proof of Proposition 2

We proceed in four steps.

Step 1. We start with a series of definitions. First, we define the following function:

$$\pi(\lambda c) \equiv \max_{z\geq 0}[(u'(z) - \lambda c)z].$$

In fact, this is the rescaled profit of a c-type firm under local competitive toughness λ. We define

$$x_{\max} \equiv l^{-1}\left(\frac{\lambda_{\min} f}{\pi(\lambda_{\min} c_{\min})}\right). \tag{A.1}$$

We assume that $x_{\max} < S \iff l(S) < \lambda_{\min} f/\pi(\lambda_{\min} c_{\min})$ (that is, $l(S)$ is sufficiently low). We also define

$$c_{max} \equiv \frac{1}{\lambda_{\min}}\pi^{-1}\left(\frac{\lambda_{\min} f}{l(0)}\right). \tag{A.2}$$

We assume that $c_{\max} > c_{\min} \iff l(0) > \lambda_{\min} f/\pi(\lambda_{\min} c_{\min})$ (that is, $l(0)$ is sufficiently high). Note that, if the latter condition fails to hold, there clearly exists no equilibrium. Indeed, in this case, the most productive firm would not break at $x = 0$, even if the competitive toughness λ is at its minimum possible level: $\lambda = \lambda_{\min} > 0$. Therefore, $l(0) > \lambda_{\min} f/\pi(\lambda_{\min} c_{\min})$ is an absolutely necessary condition for the set of active firms to be non-empty.

Next, we define the *cutoff curve* $C \subset \mathbb{R}^2_+$ as follows:

$$C \equiv \left\{(x,c) \in \mathbb{R}_+^2 \,:\, l(x)\pi(\lambda_{\min} c) = \lambda_{\min} f,\ 0 \le x \le x_{\max},\ c_{\min} \le c \le c_{\max}\right\}.$$

Clearly, C is the set of all a priory feasible solutions $(\overline{x}, \overline{c})$ of the zero-profit condition. Geometrically, C is a downward sloping curve on the (x,c)-plane connecting the points $(0, c_{\max})$ and $(x_{\max}, c_{\min})$, where $x_{\max}$ and $c_{\max}$ are defined, respectively, by (A.1) and (A.2). Note that, from the definition of $c_{\max}$, it follows that $\lambda_{\min} c_{\max} < u'(0)$ (since $\pi(\lambda_{\min} c_{\max}) = \lambda_{\min} f / l(0) > 0$).

Since $x_{\max} < S$, the population decay rate $a(x) \equiv -l'(x)/l(x)$ is a bounded continuous function over $[0, x_{\max}]$.[1] Therefore, using the Weierstrass theorem, we can define:

$$A \equiv \max_{0 \le x \le x_{\max}} a(x) < \infty. \tag{A.3}$$

Step 2. Consider any $\overline{x} \in (0, x_{\max}]$. Because the cutoff curve C is downward sloping, there exists a unique $\overline{c} \in [c_{\min}, c_{\max})$ such that $(\overline{x}, \overline{c}) \in C$. By Picard's theorem (see, e.g., Pontryagin 1962), there exists $\varepsilon > 0$ such that, for any $x \in (\overline{x} - \varepsilon, \overline{x}]$, there exists a unique solution $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ to (3.14) – (3.15) satisfying the boundary conditions: $\lambda_{\overline{x}}(\overline{x}) = \lambda_{\min}$, $c_{\overline{x}}(\overline{x}) = \overline{c}$. Picard's theorem applies here, since the right-hand sides of (3.14) – (3.15) are well-defined and continuously differentiable and, thereby, locally Lipshitz in (λ, c) in the vicinity of $(\lambda_{\min}, \overline{c})$. In particular, the denominator of the right-hand side of (3.15) never equals zero. Indeed, because $(\overline{x}, \overline{c}) \in C$, we have: $\lambda_{\min}\overline{c} < \lambda_{\min} c_{\max} < u'(0)$ (see Step 1).

Next, we show that the above local solution $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ can be extended backwards either on $[x_0, \overline{x}]$, where $x_0 \in [0, \overline{x})$ and $c_{\overline{x}}(x_0) = c_{\min}$, or on $[0, \overline{x}]$. In intuitive geometric terms, it means the following: the solution $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ can be extended backwards either until it hits the plane $\{(x, \lambda, c) \in \mathbb{R}^3 \,:\, x = 0\}$ or up to the plane$\{(x, \lambda, c) \in \mathbb{R}^3 \,:\, c = c_{\min}\}$. Note that the case when $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ hits the intersection line of these two planes, i.e. the straight line $\{(x, \lambda, c) \in \mathbb{R}^3 \,:\, x = 0, c = c_{\min}\}$, is not ruled out.

Assume the opposite: $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ can be **only** extended backwards on $(x_0, \overline{x}]$, where $x_0 \in (0, \overline{x})$ and $\lim_{x \downarrow x_0} c_{\overline{x}}(x) > c_{\min}$. By the continuation theorem for ODE solutions (Pontryagin 1962), this may only hold true in two cases:

Case 1: an "explosion in finite time" occurs, i.e.

$$\limsup_{x \downarrow x_0} \|(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))\| = \infty, \tag{A.4}$$

where $||\cdot||$ stands for the standard Euclidean norm in $\mathbb{R}^2$.

Case 2: the right-hand side of the system (3.14)–(3.15) is not well defined at (x_0, λ, c), where $(\lambda, c) = \lim_{x \downarrow x_0} (\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$.

[1]Observe that $a(x)$ need not be bounded and continuous over the whole range $[0, S]$. To see this, set $S = 1$ and consider a linear symmetric population density: $l(x) = 1 - |x|$ for $x \in (-S, S)$. Then, we have $a(x) = 1/(1 - x)$, which is clearly unbounded over $(0, 1)$.

Let us first explore the possibility of Case 1. One can show that $\lambda_{\overline{x}}(x)$ is bounded on $(x_0, \overline{x}]$. Indeed, we have on $(x_0, \overline{x}]$ (recall that $\mathcal{M}(\lambda c)$ is decreasing in λc, as the price elasticity of demand is increasing)

$$0 > \frac{\mathrm{d}\lambda_{\overline{x}}(x)}{\mathrm{d}x} > -A\mathcal{M}(\lambda_{\min} c_{\min})\lambda_{\overline{x}}(x).$$

This implies that $\mathrm{d}\ln\lambda_{\overline{x}}(x)/\mathrm{d}x$ is uniformly bounded from above in the absolute value, which in turn means that $\lambda_{\overline{x}}(x)$ is bounded from above on $(x_0, \overline{x}]$. Clearly, $c_{\overline{x}}(x)$ is also bounded, as it increases in x and satisfies:

$$0 \le c_{\min} < \lim_{x \downarrow x_0} c_{\overline{x}}(x) \le c_{\overline{x}}(x) \le c_{\overline{x}}(\overline{x}) = \overline{c} < \infty,$$

for all $x \in (x_0, \overline{x}]$. As a result, (A.4) cannot hold, meaning that Case 1 is not possible.

Let us now explore the possibility of Case 2. When $u'(0) = \infty$, this clearly cannot be the case, as the right-hand side of (3.14)–(3.15) is well defined for all $c > c_{\min}$, for all $\lambda > \lambda_{\min}$, and for all $x \ge 0$. Thus, it remains to explore the case when $u'(0) < \infty$. In this case, the ODE system (3.14)–(3.15) is not well defined, when $\lim_{x\downarrow x_0} \lambda_{\overline{x}}(x)c_{\overline{x}}(x) = u'(0)$ (in this case, the denominator of the right-hand side in (3.15) is equal to zero). Assume that this is the case. Then, $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))_{x\in(x_0,\overline{x}]}$ and $\lambda c = u'(0)$ define each a curve in the (λ, c)-plane. Note that $u'(0) > \lambda_{\overline{x}}(x)c_{\overline{x}}(x)$ for any $x \in (x_0, \overline{x}]$, otherwise $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ could not be extended backwards on $(x_0, \overline{x}]$. Hence, the curve $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))_{x\in(x_0,\overline{x}]}$ lies strictly below the curve $\lambda c = u'(0)$ in the (λ, c)-plane and intersects it at $(\lim_{x\downarrow x_0} \lambda_{\overline{x}}(x), \lim_{x\downarrow x_0} c_{\overline{x}}(x))$ (the limits exist, as $\lambda_{\overline{x}}(x)$ and $c_{\overline{x}}(x)$ are monotone and bounded). This in turn implies that

$$\lim_{x\downarrow x_0} \left| \frac{\mathrm{d}c_{\overline{x}}(x)/\mathrm{d}x}{\mathrm{d}\lambda_{\overline{x}}(x)/\mathrm{d}x} \right| \le \frac{u'(0)}{\lim_{x\downarrow x_0} \lambda^2_{\overline{x}}(x)}. \tag{A.5}$$

However, using (3.14)–(3.15), we have:

$$0 > \lim_{x\downarrow x_0} \frac{\mathrm{d}\lambda_{\overline{x}}(x)}{\mathrm{d}x} > -\infty, \qquad \lim_{x\downarrow x_0} \frac{\mathrm{d}c_{\overline{x}}(x)}{\mathrm{d}x} = +\infty,$$

which contradicts the inequality (A.5) when $u'(0) < \infty$. That is, Case 2 is not possible as well. Hence, we observe a contradiction to that $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ can be only extended backwards on $(x_0, \overline{x}]$, where $x_0 \in (0, \overline{x})$ and $\lim_{x\downarrow x_0} c_{\overline{x}}(x) > c_{\min}$.

As a result, the solution $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ can be extended backwards either up to the plane $\{(x, \lambda, c) \in \mathbb{R}^3 : x = 0\}$ or up to the plane$\{(x, \lambda, c) \in \mathbb{R}^3 : c = c_{\min}\}$, or both options hold simultaneously.

Step 3. We now construct an equilibrium without taking into account free entry into the market: i.e., we assume that M_e is given. To do this, we define the following function over $[0, x_{\max}]$:

$$\varphi(\overline{x}) = \begin{cases} c_{\overline{x}}(0) - c_{\min}, & \text{if } (\lambda_{\overline{x}}(x), c_{\overline{x}}(x)) \text{ can be extended up to } \{x = 0\}, \\ -c_{\overline{x}}^{-1}(c_{\min}), & \text{if } (\lambda_{\overline{x}}(x), c_{\overline{x}}(x)) \text{ can be extended up to } \{c = c_{\min}\}. \end{cases} \tag{A.6}$$

By continuity of solutions to ODE w.r.t. initial values (Pontryagin 1962), $\varphi(\overline{x})$ is a continuous function of $\overline{x}$. Furthermore, it is readily verified that the following inequalities hold:

$$\varphi(0) = c_{\max} - c_{\min} > 0, \qquad \varphi(x_{\max}) = -x_{\max} < 0.$$

Hence, by the intermediate value theorem, there exists $\overline{x}^* \in (0, x_{\max})$, such that $\varphi(\overline{x}^*) = 0$. Setting $(\lambda^*(x), c^*(x)) \equiv (\lambda_{\overline{x}^*}(x), c_{\overline{x}^*}(x))$ and $\overline{c}^* \equiv c_{\overline{x}^*}(\overline{x}^*)$, derive a candidate equilibrium:

$$\left\{\overline{x}^*, \overline{c}^*, (\lambda^*(x), c^*(x))_{x\in[0,\overline{x}^*]}\right\}. \tag{A.7}$$

We now verify that the candidate equilibrium (A.7) is indeed an equilibrium when M_e is given. That $(\lambda^*(x), c^*(x))$ is a solution to (3.14) – (3.15) follows by construction. The equality $\varphi(\overline{x}^*) = 0$ means that $(\lambda^*(x), c^*(x))$ can be extended simultaneously up to both planes: $\{x = 0\}$ and $\{c = c_{\min}\}$. This, in turn, is equivalent to $c^*(0) = c_{\min}$, i.e. $(\lambda^*(x), c^*(x))$ satisfies one of the boundary conditions. The other boundary condition, $\lambda^*(\overline{x}^*) = \lambda_{\min}$, is satisfied by construction. Finally, $(\overline{x}^*, \overline{c}^*) \in C$ means that $(\overline{x}^*, \overline{c}^*)$ satisfy the zero-profit condition (3.12).

Step 4. So far, we have been proceeding as if M_e were a constant. However, M_e is endogenous, and is determined by the free entry condition given by:

$$\Pi_e(M_e) \equiv \int_{c_{\min}}^{\overline{c}^*(M_e)} \left[\frac{l(x^*(c, M_e))}{\lambda^*(c, M_e)} \pi\left(\lambda^*(c, M_e)c\right) - f\right] g(c)\mathrm{d}c = f_e, \tag{A.8}$$

where $\lambda^*(c, M_e)$ is a decreasing function parametrically described by the downwards-sloping curve $(\lambda^*(x, M_e),\, c^*(x, M_e))|_{x\in[0,\overline{x}^*]}$, while $x^*(\cdot, M_e)$ is the inverse to $c^*(\cdot, M_e)$. We assume that $l(0)$ is such that

$$f_e < \int_{c_{\min}}^{c_{\max}} \left[\frac{l(0)}{\lambda_{\min}} \pi\left(\lambda_{\min} c\right) - f\right] g(c)\mathrm{d}c. \tag{A.9}$$

Further, we show that this condition is sufficient for equation (A.8) to have a solution $M_e^* > 0$.

First, we show that $\Pi_e(\infty) = 0$. Observe that, when $M_e \to \infty$, equation (3.15) implies that $\mathrm{d}c^*/\mathrm{d}x$ becomes uniformly small. Taking into account that $c^*(0) = c_{\min}$, we have that

$$\lim_{M_e\to\infty} \overline{c}^*(M_e) = c_{\min}, \qquad \lim_{M_e\to\infty} \overline{x}^*(M_e) = x_{\max}.$$

It is straightforward to see that the above implies that $\Pi_e(\infty) = 0$.

Next, we consider $\Pi_e(0)$. Observe that, when $M_e \to 0$, equation (3.15) implies that $\mathrm{d}c^*/\mathrm{d}x$

becomes uniformly large or, equivalently, $\mathrm{d}x^*/\mathrm{d}c$ becomes uniformly small. This implies that

$$\lim_{M_e \to 0} \overline{x}^*(M_e) = 0, \qquad \lim_{M_e \to 0} \overline{c}^*(M_e) = c_{\max}.$$

Hence,

$$\Pi_e(0) = \int_{c_{\min}}^{c_{\max}} \left[\frac{l(0)}{\lambda_{\min}} \pi \left(\lambda_{\min} c \right) - f \right] g(c) \mathrm{d}c.$$

According to our assumption, $\Pi_e(0) > f_e > 0 = \Pi_e(\infty)$. This means that equation (A.8) has a solution $M_e^* > 0$. This completes the proof.

The Proof of Proposition 3

We proceed in four steps. Until Step 4, we ignore the free-entry condition and treat the mass $M_e > 0$ of entrants as exogenous. At Step 4, we take (A.8) into account and show that it uniquely determines M_e.

Step 1. Assume there are at least two equilibrium outcomes corresponding to the same value of M_e:

$$\left\{ \overline{x}^*, \overline{c}^*, \left(\lambda^*(x), c^*(x) \right)_{x \in [0, \overline{x}^*]} \right\} \quad \text{and} \quad \left\{ \overline{x}^{**}, \overline{c}^{**}, \left(\lambda^{**}(x), c^{**}(x) \right)_{x \in [0, \overline{x}^{**}]} \right\}.$$

Note that $\overline{x}^* \neq \overline{x}^{**}$. Indeed, if $\overline{x}^* = \overline{x}^{**}$, then $\overline{c}^* = \overline{c}^{**}$ (since the cutoff curve C is downward-sloping). Hence, $(\lambda^*(x),\, c^*(x))$ and $(\lambda^{**}(x),\, c^{**}(x))$ are solutions to the same system of ODE satisfying the same boundary conditions. By Picard's theorem, this implies that $(\lambda^*(x),\, c^*(x)) = (\lambda^{**}(x),\, c^{**}(x))$ pointwise.

Let us assume without loss of generality that $\overline{x}^* < \overline{x}^{**}$. Because $(\overline{x}^*,\, \overline{c}^*) \in C$ and $(\overline{x}^{**},\, \overline{c}^{**}) \in C$, $\overline{x}^* < \overline{x}^{**}$ implies that $\overline{c}^* > \overline{c}^{**}$. Since $\left\{ \overline{x}^{**}, \overline{c}^{**}, \left(\lambda^{**}(x), c^{**}(x) \right)_{x \in [0, \overline{x}^{**}]} \right\}$ is an equilibrium for given M_e, we have that $c^{**}(0) = c_{\min}$. Furthermore, $\left(c^{**}\right)'_x (x) > 0$. Combining this with $\overline{x}^* < \overline{x}^{**}$, we derive the following inequalities:

$$c^{**}(\overline{x}^{**} - \overline{x}^*) > c^{**}(0) = c_{\min} = c^*(0) = c^*(\overline{x}^* - \overline{x}^*). \tag{A.10}$$

For each $z \in [0, \overline{x}^*]$, define $\Delta(z)$ as follows:

$$\Delta(z) \equiv c^{**}(\overline{x}^{**} - z) - c^*(\overline{x}^* - z). \tag{A.11}$$

As has been shown, $\Delta(\overline{x}^*) > 0$. Taking into account that $\overline{c}^* > \overline{c}^{**}$, $\Delta(0) < 0$. By the intermediate value theorem, there exists $\xi \in (0, \overline{x}^*)$, such that $\Delta\left(\xi\right) = 0$. Let ξ_0 be the smallest of such ξs. Clearly, we have: $c^{**}(\overline{x}^{**} - \xi_0) = c^*(\overline{x}^* - \xi_0)$ and $c^{**}(\overline{x}^{**} - z) < c^*(\overline{x}^* - z)$ for all $z < \xi_0$.

Step 2. Next, we show that

$$\lambda^{**}(\overline{x}^{**} - \xi_0) > \lambda^{*}(\overline{x}^{*} - \xi_0). \tag{A.12}$$

Using (3.14) yields (recall that $\lambda^{**}(\overline{x}^{**}) = \lambda_{\min} = \lambda^{*}(\overline{x}^{*})$)

$$\left(\lambda^{**}(\overline{x}^{**} - z)\right)'_z\Big|_{z=0} = a\left(\overline{x}^{**}\right)\lambda_{\min}\mathcal{M}\left(\lambda_{\min}\overline{c}^{**}\right) > a\left(\overline{x}^{*}\right)\lambda_{\min}\mathcal{M}\left(\lambda_{\min}\overline{c}^{*}\right) = \left(\lambda^{*}(\overline{x}^{*} - z)\right)'_z\Big|_{z=0},$$

which holds true because $a'(x) \geq 0$, $\overline{c}^{*} > \overline{c}^{**}$, and the markup function $\mathcal{M}(\cdot)$ is strictly decreasing. Furthermore, we have:

$$\left(\lambda^{**}(\overline{x}^{**} - z)\right)'_z\Big|_{z=0} > \left(\lambda^{*}(\overline{x}^{*} - z)\right)'_z\Big|_{z=0} > 0.$$

Thus, $\lambda^{**}\left(\overline{x}^{**} - z\right) > \lambda^{*}\left(\overline{x}^{*} - z\right)$ holds true for sufficiently small values of z.

Assume that there is some $\xi_1 \in (0, \xi_0)$, such that $\lambda^{**}(\overline{x}^{**} - \xi_1) = \lambda^{*}(\overline{x}^{*} - \xi_1)$, while $\lambda^{**}(\overline{x}^{**} - z) > \lambda^{*}(\overline{x}^{*} - z)$ for all $z < \xi_1$. Denote $\lambda_1 \equiv \lambda^{*}(\overline{x}^{*} - \xi_1)$. Differentiating the log of the ratio $\lambda^{**}(\overline{x}^{**} - z)/\lambda^{*}(\overline{x}^{*} - z)$ w.r.t. z at $z = \xi_1$ yields (recall that, from the previous step, $c^{**}(\overline{x}^{**} - z) < c^{*}(\overline{x}^{*} - z)$ for all $z < \xi_0$):

$$\left[\ln\left(\frac{\lambda^{**}(\overline{x}^{**} - z)}{\lambda^{*}(\overline{x}^{*} - z)}\right)\right]'_z\Bigg|_{z=\xi_1} = a\left(\overline{x}^{**} - \xi_1\right)\mathcal{M}\left(\lambda_1 c^{**}\left(\overline{x}^{**} - \xi_1\right)\right) - a\left(\overline{x}^{*} - \xi_1\right)\mathcal{M}\left(\lambda_1 c^{*}(\overline{x}^{*} - \xi_1)\right) > 0.$$

By continuity, $\left[\ln\left(\frac{\lambda^{**}(\overline{x}^{**}-z)}{\lambda^{*}(\overline{x}^{*}-z)}\right)\right]'_z > 0$ must hold for any $z \in (\xi_1 - \varepsilon, \xi_1)$, where $\varepsilon > 0$ is sufficiently small. Hence, the ratio $\lambda^{**}(\overline{x}^{**} - z)/\lambda^{*}(\overline{x}^{*} - z)$ increases over $(\xi_1 - \varepsilon, \xi_1)$ and strictly exceeds 1 at $z = \xi_1 - \varepsilon$. Thus, $\lambda^{**}(\overline{x}^{**} - \xi_1)/\lambda^{*}(\overline{x}^{*} - \xi_1)$ also strictly exceeds 1, i.e. $\lambda^{**}(\overline{x}^{**} - \xi_1) > \lambda^{*}(\overline{x}^{*} - \xi_1)$. Based on that, we conclude that ξ_1 does not exist. This proves (A.12).

Step 3. Differentiating the function $\Delta(z)$ defined by (A.11) at $z = \xi_0$, we obtain:

$$\Delta'_z(\xi_0) = -\frac{1}{M_e g\left(c_0^{*}\right)}\left[\frac{\left(V'\right)^{-1}\left(1/\lambda_0^{**}\right)}{u\left(q\left(\lambda_0^{**}c_0^{*}\right)\right)} - \frac{\left(V'\right)^{-1}\left(1/\lambda_0^{*}\right)}{u\left(q\left(\lambda_0^{*}c_0^{*}\right)\right)}\right] < 0. \tag{A.13}$$

where $c_0^{*} \equiv c^{*}(\overline{x}^{*} - \xi_0) = c^{**}(\overline{x}^{**} - \xi_0)$, $\lambda_0^{*} \equiv \lambda^{*}(\overline{x}^{*} - \xi_0)$, and $\lambda_0^{**} \equiv \lambda^{**}(\overline{x}^{**} - \xi_0)$. The inequality (A.13) holds true because, by (A.12), we have $\lambda_0^{**} > \lambda_0^{*}$, while the function $\left(V'\right)^{-1}(1/\lambda)/u\left(q\left(\lambda c\right)\right)$ increases in λ for any given $c > c_{\min}$. However, by definition of ξ_0, $\Delta(z)$ must change sign from negative to positive at $z = \xi_0$. Hence, it must be true that $\Delta'_z(\xi_0) \geq 0$. This contradicts (A.13) and implies that, for any fixed value of M_e, there is a unique equilibrium outcome corresponding to this value of M_e.

Step 4. To finish the proof of uniqueness, it remains to show that $\mathrm{d}\Pi_e(M_e)/\mathrm{d}M_e < 0$ for any

$M_e > 0$. Let us define

$$\mathfrak{N}(c, M_e) \equiv \frac{l(x^*(c, M_e))}{\lambda^*(c, M_e)} \pi\left(\lambda^*(c, M_e) c\right).$$

Then, we have:

$$\frac{\mathrm{d}\Pi_e(M_e)}{\mathrm{d}M_e} = \int\limits_{c_{\min}}^{\overline{c}^*(M_e)} \frac{\partial \mathfrak{N}(c, M_e)}{\partial M_e} g(c) \mathrm{d}c + \left[\mathfrak{N}(\overline{c}^*(M_e), M_e) - f\right] \frac{\mathrm{d}\overline{c}^*(M_e)}{\mathrm{d}M_e},$$

where the last term equals zero due to the cutoff condition. Hence,

$$\frac{\mathrm{d}\Pi_e(M_e)}{\mathrm{d}M_e} = \int\limits_{c_{\min}}^{\overline{c}^*(M_e)} \frac{\partial \mathfrak{N}(c, M_e)}{\partial M_e} \mathrm{d}G(c).$$

Thus, a sufficient condition for $\mathrm{d}\Pi_e(M_e)/\mathrm{d}M_e < 0$ for any $M_e > 0$ is given by

$$\frac{\partial \mathfrak{N}(c, M_e)}{\partial M_e} < 0 \text{ for any } M_e > 0 \text{ and any } c \in \left[c_{\min}, \overline{c}^*(M_e)\right].$$

It is straightforward to see that, due to the envelope theorem, the latter is hold when

$$\frac{\partial \lambda^*(x, M_e)}{\partial M_e} > 0 \text{ for any } M_e > 0 \text{ and any } x \in \left[0, \overline{x}^*(M_e)\right].$$

In fact, it is sufficient to show that

$$\frac{\partial \lambda^*(x, M_e)}{\partial M_e} \geq 0 \text{ for any } M_e > 0 \text{ and any } x \in \left[0, \overline{x}^*(M_e)\right]$$

and $\partial \lambda^*(x, M_e)/\partial M_e > 0$ on some non-zero measure subset of $[0, \overline{x}^*(M_e)]$. The rest of the proof amounts to establishing the latter statement.

Assume that, on the contrary, for some $M_e > 0$, there exists a compact interval $[x_1, x_2] \subseteq [0, \overline{x}^*(M_e)]$, such that $\partial \lambda^*(x, M_e)/\partial M_e \leq 0$ for all $x \in [x_1, x_2]$. Without loss of generality, let us also assume that $[x_1, x_2]$ cannot be extended further without violating the condition $\partial \lambda^*(x, M_e)/\partial M_e \leq 0$ (otherwise, we can replace it with a larger one). We will therefore refer to $[x_1, x_2]$ as a *non-extendable* interval. We consider several possible cases.

Case 1: Assume that $x_1 = 0$. In this case, we have: $c^*(x_1, M_e) = c_{\min}$, hence $\partial c^*(x_1, M_e)/\partial M_e = 0$. Recall that

$$\frac{\mathrm{d}c}{\mathrm{d}x} = \frac{1}{M_e} \frac{\left(V'\right)^{-1}(1/\lambda)}{g(c)\, u(q_x)}.$$

Since $\partial \lambda^*(x_1, M_e)/\partial M_e \leq 0$, $\partial c^*(x_1, M_e)/\partial M_e = 0$, and M_e rises, $\partial \left(c^*\right)'_x (x_1, M_e)/\partial M_e < 0$ (the right-hand side of the above equation decreases at $x_1 = 0$ with a rise in M_e). Note that $\partial c^*(x_1, M_e)/\partial M_e = 0$ and $\partial \left(c^*\right)'_x (x_1, M_e)/\partial M_e < 0$ imply that $\partial c^*(x, M_e)/\partial M_e < 0$ in some right neighborhood of $x_1 = 0$.

Case 2: Assume that $x_2 = \overline{x}^*(M_e)$. We have $\lambda^*(\overline{x}^*(M_e), M_e) = \lambda_{\min}$. This implies that

$$\frac{\partial\lambda^*(\overline{x}^*(M_e), M_e)}{\partial x}\frac{d\overline{x}^*(M_e)}{dM_e} + \frac{\partial\lambda^*(\overline{x}^*(M_e), M_e)}{\partial M_e} = 0.$$

The second term in the left-hand side of the above equation is non-positive (as assumed). Recall that $\lambda^*(x, M_e)$ is strictly decreasing in x. As a result, $d\overline{x}^*(M_e)/dM_e \le 0$. Combining this with the fact $(\overline{x}^*(M_e), \overline{c}^*(M_e)) \in C$, where C is the downward sloping cutoff curve, we get: $\mathrm{d}\overline{c}^*(M_e)/\mathrm{d}M_e \ge 0$. That is,

$$\frac{\partial c^*\left(\overline{x}^*(M_e), M_e\right)}{\partial x}\frac{\mathrm{d}\overline{x}^*(M_e)}{\mathrm{d}M_e} + \frac{\partial c^*\left(\overline{x}^*(M_e), M_e\right)}{\partial M_e} \ge 0,$$

where the first term is non-positive because, as shown above, $\mathrm{d}\overline{x}^*(M_e)/\mathrm{d}M_e \le 0$, while

$\partial c^*\left(\overline{x}^*(M_e), M_e\right)/\partial x > 0$. Hence, the second term, $\partial c^*\left(\overline{x}^*(M_e), M_e\right)/\partial M_e$, must be non-negative. If $\partial c^*\left(\overline{x}^*(M_e), M_e\right)/\partial M_e = 0$, then one can show that $\partial\left(c^*\right)'_x\left(\overline{x}^*(M_e), M_e\right)/\partial M_e < 0$. Here, we use again the fact that

$$\frac{\mathrm{d}c}{\mathrm{d}x} = \frac{1}{M_e}\frac{\left(V'\right)^{-1}(1/\lambda)}{g(c)\,u(q_x)}.$$

This in turn implies that $\partial c^*\left(\overline{x}^*(M_e), M_e\right)/\partial M_e > 0$ in some left neighborhood of $x_2 = \overline{x}^*(M_e)$.

Case 3: Assume that $0 < x_1 < x_2 < \overline{x}^*(M_e)$. Because $[x_1, x_2]$ is non-extendable, there exists a small open left half-neighborhood $\mathcal{N}_1$ of x_1, and a small right half-neighborhood $\mathcal{N}_2$ of x_2, such that $\partial\lambda^*\left(x, M_e\right)/\partial M_e > 0$ for all $x \in \mathcal{N} \equiv \mathcal{N}_1 \cup \mathcal{N}_2$. Hence, for a c-type firm where $c = c^*\left(x, M_e\right)$ with $x \in [x_1, x_2]$, relocating marginally beyond $[x_1, x_2]$ in response to a marginal increase in M_e is not profit-maximizing behavior. Indeed, that $\partial\lambda^*\left(x, M_e\right)/\partial M_e \le 0$ over $[x_1, x_2]$ means that the profit function increases uniformly over $[x_1, x_2]$, while $\partial\lambda^*\left(x, M_e\right)/\partial M_e > 0$ for all $x \in \mathcal{N}$ means that relocating from $[x_1, x_2]$ into $\mathcal{N}$ would lead to a reduction of maximum feasible profit.[2] This immediately imply that

$$\frac{\partial c^*(x_1, M_e)}{\partial M_e} \le 0, \qquad \frac{\partial c^*(x_2, M_e)}{\partial M_e} \ge 0.$$

Moreover, for $j = 1, 2$ we have (the proof is the same as in the previous cases)

$$\frac{\partial c^*(x_j, M_e)}{\partial M_e} = 0 \Rightarrow \frac{\partial\left(c^*\right)'_x(x_j, M_e)}{\partial M_e} < 0.$$

The findings in the above cases allow us to formulate the following important result. *There exists a location x_4 in an arbitrary small right half-neighborhood of x_1, such that $\partial c^*(x_4, M_e)/\partial M_e < 0$. Similarly, there exists a location x_5 in an arbitrary small left half-neighborhood of x_2, such that*

[2]One may wonder why no firm would relocate from $[x_1, x_2]$ to somewhere beyond $\mathcal{N}$ in response to a marginal increase of M_e. This would mean, for at least some firm type c, that the firm's profit-maximizing location choice $x^*(c, M_e)$ has a discontinuity in M_e. However, by the maximum theorem (Sundaram 1996), $x^*(c, M_e)$ must be upper-hemicontinuous in M_e. Furthermore, by strict quasi-concavity of the profit function, $x^*(c, M_e)$ is single-valued. For single-valued mappings, upper-hemicontinuity implies continuity. Hence, $x^*(c, M_e)$ cannot exhibit discontinuities.

$\partial c^*(x_5, M_e)/\partial M_e > 0$.

By the intermediate value theorem, there must exist a location $x_3 \in (x_4, x_5) \subset [x_1, x_2]$ such that

$$\frac{\partial c^*(x_3, M_e)}{\partial M_e} = 0, \quad \frac{\partial \left(c^*\right)'_x (x_3, M_e)}{\partial M_e} \geq 0.$$

The non-negative sign of the derivative follows from the fact that $c^*(x, M_e)$ is increasing in x. This in turn implies that the derivative of

$$\frac{1}{M_e} \frac{\left(V'\right)^{-1} \left(1/\lambda^*(x_3, M_e)\right)}{g(c^*(x_3, M_e))\, u(q(\lambda^*(x_3, M_e)c^*(x_3, M_e)))}$$

with respect to M_e is non-negative. That is, the derivative of

$$\frac{\left(V'\right)^{-1} \left(1/\lambda^*(x_3, M_e)\right)}{g(c^*(x_3, M_e))u(q(\lambda^*(x_3, M_e)c^*(x_3, M_e)))}$$

with respect to M_e is strictly positive. This means that $\partial\lambda^* \left(x_3, M_e\right)/\partial M_e > 0$ (recall that $\partial c^*(x_3, M_e)/\partial M_e = 0$). However, since $x_3 \in [x_1, x_2]$, it must be that $\partial\lambda^*(x_3, M_e)/\partial M_e \leq 0$, which is a contradiction. This completes the proof of uniqueness of the equilibrium.

The proof of Proposition 4

To prove the proposition, we use the equilibrium conditions for $\lambda'(x)$ and $c'(x)$. Specifically, from (3.11) and (3.9),

$$\lambda'(x) = \frac{l'(x)\lambda(x)}{l(x)} \frac{p(x, c(x)) - c(x)}{p(x, c(x))},$$

$$M_e g\left(c(x)\right) c'(x) u\left(q(x, c(x))\right) = \left(V'\right)^{-1} \left(1/\lambda(x)\right) \iff c'(x) = \frac{\left(V'\right)^{-1} \left(1/\lambda(x)\right)}{M_e g\left(c(x)\right) u\left(q(x, c(x))\right)}.$$

Hence,

$$\left(\lambda(x)c(x)\right)'_x = c(x)\lambda'(x) + \lambda(x)c'(x)$$

$$= \frac{\lambda(x)}{g\left(c(x)\right)} \left[c(x)g\left(c(x)\right) \frac{l'(x)}{l(x)} \frac{p(x, c(x)) - c(x)}{p(x, c(x))} + \frac{\left(V'\right)^{-1} \left(1/\lambda(x)\right)}{M_e u\left(q(x, c(x))\right)}\right].$$

Consider,

$$\left(\lambda(x)c(x)\right)'_{x=0} = \frac{\lambda(0)}{g\left(c_{\min}\right)} \left(c_{\min}\, g\left(c_{\min}\right) \frac{l'(0)}{l(0)} \frac{p(0, c_{\min}) - c_{\min}}{p(0, c_{\min})} + \frac{\left(V'\right)^{-1} \left(1/\lambda(0)\right)}{M_e u\left(q(0, c_{\min})\right)}\right).$$

Since $g\left(c\right)$ is a density function, $lim_{c_{\min}\to 0} c_{\min}\, g\left(c_{\min}\right) = 0$. Hence, if $\mid l'(0) \mid < \infty$, then for sufficiently low $c_{\min}$,

$$c_{\min} g\left(c_{\min}\right) \frac{l'(0)}{l(0)} \frac{p(0, c_{\min}) - c_{\min}}{p(0, c_{\min})} + \frac{\left(V'\right)^{-1}\left(1/\lambda(0)\right)}{M_e u\left(q(0, c_{\min})\right)} > 0.$$

Similarly,

$$\left(\lambda(x)c(x)\right)'_{x=\bar{x}} = \frac{\lambda(\bar{x})}{g\left(\bar{c}\right)} \left(\bar{c}\, g\left(\bar{c}\right) \frac{l'(\bar{x})}{l(\bar{x})} \frac{p(\bar{x}, \bar{c}) - \bar{c}}{p(\bar{x}, \bar{c})} + \frac{\left(V'\right)^{-1}\left(1/\lambda(\bar{x})\right)}{M_e u\left(q(\bar{x}, \bar{c})\right)} \right).$$

Note that, as there is the fixed cost of production f, $p(\bar{x}, \bar{c}) > \bar{c}$. Moreover, $\lambda(\bar{x}) = 1/V'(0)$ in the equilibrium, implying that $\left(V'\right)^{-1}\left(1/\lambda(\bar{x})\right) = 0$ (this also means that $c'(\bar{x}) = 0$). As a result, since $l'(\bar{x}) < 0$,

$$\bar{c}\, g\left(\bar{c}\right) \frac{l'(\bar{x})}{l(\bar{x})} \frac{p(\bar{x}, \bar{c}) - \bar{c}}{p(\bar{x}, \bar{c})} + \frac{\left(V'\right)^{-1}\left(1/\lambda(\bar{x})\right)}{M_e u\left(q(\bar{x}, \bar{c})\right)} < 0.$$

To prove the third statement of the proposition, we rewrite $\left(\lambda(x)c(x)\right)'_x$ in the following way:

$$\left(\lambda(x)c(x)\right)'_x = \frac{\lambda(x)}{g\left(c\left(x\right)\right)} \left(\frac{l'(x)}{l(x)} c\left(x\right) g\left(c\left(x\right)\right) \mathcal{M}(\lambda(x)c(x)) + \frac{\left(V'\right)^{-1}\left(1/\lambda(x)\right)}{M_e u\left(q(\lambda(x)c(x))\right)} \right),$$

where $\mathcal{M}(.)$ is the markup function. Let us denote $\tilde{x} \in (0, \bar{x})$ as an interior extremum of $\lambda(x)c(x)$: $\left(\lambda(\tilde{x})c(\tilde{x})\right)'_x = 0$. We know that $\left(\lambda(x)c(x)\right)'_{x=0} > 0$ and $\left(\lambda(x)c(x)\right)'_{x=\bar{x}} < 0$. Hence, $\lambda(x)c(x)$ has at least one interior local maximizer.

Next, we show that, for any $\tilde{x}$, $\left(\lambda(\tilde{x})c(\tilde{x})\right)''_{xx} < 0$. We have

$$\left(\lambda(\tilde{x})c(\tilde{x})\right)''_{xx} = \left(\tfrac{\lambda(\tilde{x})}{g(c(\tilde{x}))}\right)' \left(\tfrac{l'(\tilde{x})}{l(\tilde{x})} c\left(\tilde{x}\right) g\left(c\left(\tilde{x}\right)\right) \mathcal{M}(\lambda(\tilde{x})c(\tilde{x})) + \tfrac{\left(V'\right)^{-1}\left(1/\lambda(\tilde{x})\right)}{M_e u(q(\lambda(\tilde{x})c(\tilde{x})))} \right)$$

$$+ \tfrac{\lambda(\tilde{x})}{g(c(\tilde{x}))} \left(\tfrac{l'(\tilde{x})}{l(\tilde{x})} c\left(\tilde{x}\right) g\left(c\left(\tilde{x}\right)\right) \mathcal{M}(\lambda(\tilde{x})c(\tilde{x})) + \tfrac{\left(V'\right)^{-1}\left(1/\lambda(\tilde{x})\right)}{M_e u(q(\lambda(\tilde{x})c(\tilde{x})))} \right)'_x.$$

Note that the first term in the right hand side of the above formula is equal to zero. Thus, we have (recall that $\left(\lambda(\tilde{x})c(\tilde{x})\right)'_x = 0$)

$$\left(\lambda(\tilde{x})c(\tilde{x})\right)''_{xx} = \tfrac{\lambda(\tilde{x})}{g(c(\tilde{x}))} \left(\tfrac{l'(\tilde{x})}{l(\tilde{x})} c\left(\tilde{x}\right) g\left(c\left(\tilde{x}\right)\right) \mathcal{M}(\lambda(\tilde{x})c(\tilde{x})) + \tfrac{\left(V'\right)^{-1}\left(1/\lambda(\tilde{x})\right)}{M_e u(q(\lambda(\tilde{x})c(\tilde{x})))} \right)'_x$$

$$= \tfrac{\lambda(\tilde{x})}{g(c(\tilde{x}))} \left(\left(\tfrac{l'(\tilde{x})}{l(\tilde{x})} c\left(\tilde{x}\right) g\left(c\left(\tilde{x}\right)\right) \right)'_x \mathcal{M}(\lambda(\tilde{x})c(\tilde{x})) + \tfrac{\left(\left(V'\right)^{-1}\left(1/\lambda(\tilde{x})\right)\right)'_x}{M_e u(q(\lambda(\tilde{x})c(\tilde{x})))} \right).$$

We have

$$\left(\frac{l'(x)}{l(x)} c(x) g\left(c(x)\right) \right)'_x = \frac{l'(x)}{l(x)} \left(c(x) g\left(c(x)\right)\right)'_x + c(x) g\left(c(x)\right) \left(\frac{l'(x)}{l(x)} \right)'_x < 0,$$

since $c'(x) > 0$, $g'(c) \geq 0$, and $(l'(x)/l(x))'_x \leq 0$. At the same time, $(V')^{-1}(1/\lambda(x))$ is decreasing in x as $V''(\cdot) < 0$ and $\lambda'(x) < 0$. Hence, $(\lambda(\tilde{x})c(\tilde{x}))''_{xx} < 0$.

We now finish the proof of part *(iii)* of Proposition 3. As derived above, $\lambda(x)c(x)$ has no interior local minimum over $(0, \overline{x})$ and at least one interior local maximizer. Assume that $\lambda(x)c(x)$ has at least two distinct local maximizers. Then, there must be a local minimizer in between, which contradicts our above finding. We conclude that $\lambda(x)c(x)$ is bell-shaped in x, while the markup function $\mathcal{M}(\lambda(x)c(x))$ is U-shaped in x. This completes the proof.

The proof of Lemma 2

Note that in this proof it is important that $\partial\lambda(x, M_e, \delta)/\partial\delta$ and $\partial c(x, M_e, \delta)/\partial\delta$ are analytic in x over $(0, \overline{x})$, meaning that they can be represented by convergent power series (this is the case, when, for instance, the primitives in the model are analytic):

$$\frac{\partial\lambda(x, M_e, \delta)}{\partial\delta} = \sum_{k=0}^{\infty} a_k(M_e, \delta)x^k, \qquad \frac{\partial c(x, M_e, \delta)}{\partial\delta} = \sum_{k=0}^{\infty} b_k(M_e, \delta)x^k.$$

This makes the case when $\partial\lambda(x, M_e, \delta)/\partial\delta = 0$ and $\partial(\lambda)'_x(x, M_e, \delta)/\partial\delta = 0$ at some x impossible. Why? If this is the case, then $\partial c(x, M_e, \delta)/\partial\delta = 0$ and $\partial(c)'_x(x, M_e, \delta)/\partial\delta = 0$ as well implying that the derivatives of all orders of $\partial\lambda(x, M_e, \delta)/\partial\delta$ w.r.t. x at this point equal to zero. An analytic function with this property must be identically zero (Courant and John 2012, p. 545). This in turn means that $\lambda(x)$ does not change on the whole interval $[0, \overline{x}]$ when δ changes, which is impossible. For the same reason, it is not possible that $\partial c(x, M_e, \delta)/\partial\delta = 0$ and $\partial(c)'_x(x, M_e, \delta)/\partial\delta = 0$ at some x.

To simplify the exposition of the proof, we divide it into several parts.

Part 1

In this part, we prove that $\partial\overline{x}(M_e, \delta)/\partial\delta > 0$. Assume, on the contrary, that $\partial\overline{x}(M_e, \delta)/\partial\delta \leq 0$. Then, because an increase in δ leads to an upward shift of the cutoff curve C, it must be that $\partial\overline{c}(M_e, \delta)/\partial\delta > 0$. Note also that if $\partial\overline{x}(M_e, \delta)/\partial\delta < 0$, then (by continuity) $\lambda(x, M_e, \delta)$ must decrease w.r.t. δ in some neighborhood of $\overline{x}$ (as $\lambda(x, M_e, \delta)$ is decreasing in x). If $\overline{x}$ does not change with the change in δ, one can derive from (3.14) that $\partial\left(-(\lambda)'_x(\overline{x}, M_e, \delta)\right)\partial\delta < 0$. This is because $\partial\overline{c}(M_e, \delta)/\partial\delta > 0$ and $\lambda(\overline{x}, M_e, \delta) = \lambda_{\min}$. This in turn also means that $\partial\lambda(x, M_e, \delta)/\partial\delta < 0$ in some neighborhood of $\overline{x}$. That is, if $\partial\overline{x}(M_e, \delta)/\partial\delta \leq 0$, $\lambda(x, M_e, \delta)$ must decrease w.r.t. δ over some interval $(x_1, \overline{x})$. Two cases may arise.

Case 1: $x_1 = 0$. In this case, $\partial\lambda(0, M_e, \delta)/\partial\delta < 0$. Then, taking into account the boundary condition $c(0, M_e, \delta) = c_{\min}$, it is straightforward to see from the equilibrium condition in (3.15) that $\partial(c)'_x(0, M_e, \delta)/\partial\delta < 0$. This in turn implies that $\partial c(x, M_e, \delta)/\partial\delta < 0$ in the vicinity of $x = 0$ (since $c(0, M_e, \delta) = c_{\min}$ is not affected by δ). As a result, we have the following situation:

given the rise in δ, $c(x)$ falls in the neighborhood of zero and rises in the neighborhood of $\overline{x}$ as $\partial\overline{c}(M_e,\delta)/\partial\delta > 0$. This implies that there exists $x_2 \in (0,\overline{x})$ such that $\partial c(x_2, M_e, \delta)/\partial\delta = 0$ - the value of $c(x)$ at x_2 is not affected by the rise in δ. Moreover, $\partial(c)'_x(x_2, M_e, \delta)/\partial\delta > 0$ (as $c(x)$ falls around zero). This in turn means (here we use the equilibrium condition in (3.15)) that $\partial\lambda(x_2, M_e, \delta)/\partial\delta > 0$ which contradicts the assumption that $\partial\lambda(x, M_e, \delta)/\partial\delta < 0$ for all $x > 0$. *Note that we will use this particular way of deriving the contradiction throughout the whole proof of the lemma.*

Case 2 $x_1 > 0$. In this case, it must be true that $\partial\lambda(x_1, M_e, \delta)/\partial\delta = 0$. Moreover, the absolute value of the slope of $\lambda(x)$ at this point increases: $\partial\left(-(\lambda)'_x(x_1, M_e, \delta)\right)/\partial\delta > 0$, as $\partial\lambda(x, M_e, \delta)/\partial\delta < 0$ on $(x_1, \overline{x})$. In this case, from the equilibrium condition in (3.14) we derive that $\partial c(x_1, M_e, \delta)/\partial\delta < 0$. Now, we use the same argument as in the previous case. There exists $x_3 \in (x_1, \overline{x})$ such that $\partial c(x_3, M_e, \delta)/\partial\delta = 0$ and $\partial(c)'_x(x_3, M_e, \delta)/\partial\delta > 0$. This in turn implies that $\partial\lambda(x_3, M_e, \delta)/\partial\delta > 0$ which contradicts the assumption that $\partial\lambda(x, M_e, \delta)/\partial\delta < 0$ for all $x > x_1$.

Thus, we show that $\partial\overline{x}(M_e, \delta)/\partial\delta > 0$.

Part 2

Next, we show that $\partial\lambda(x, M_e, \delta)/\partial\delta > 0$ for all x. Assume that, on the contrary, there exists a non-extendable interval $(x_4, x_5) \subset [0, \overline{x}]$ such that $\partial\lambda(x, M_e, \delta)/\partial\delta \leq 0$ on this interval. Note that since $\overline{x}$ rises (implying that $\partial\lambda(x, M_e, \delta)/\partial\delta > 0$ in some neighborhood of $\overline{x}$), $x_5 < \overline{x}$. Consider again two cases.

Case 1: $x_4 > 0$. In this case, because (x_4, x_5) is a non-extendable interval where $\partial\lambda(x, M_e, \delta)/\partial\delta < 0$, it must be that:

$$\frac{\partial\lambda(x_4, M_e, \delta)}{\partial\delta} = 0 = \frac{\partial\lambda(x_5, M_e, \delta)}{\partial\delta}.$$

Moreover,

$$\frac{\partial\left(-(\lambda)'_x(x_4, M_e, \delta)\right)}{\partial\delta} > 0 > \frac{\partial\left(-(\lambda)'_x(x_5, M_e, \delta)\right)}{\partial\delta}.$$

In this case, (3.14) implies that

$$\frac{\partial c(x_4, M_e, \delta)}{\partial\delta} < 0 < \frac{\partial c(x_5, M_e, \delta)}{\partial\delta}.$$

Hence, there exists $x_6 \in (x_4, x_5)$, such that

$$\frac{\partial c(x_6, M_e, \delta)}{\partial\delta} = 0, \qquad \frac{\partial\left(c\right)'_x(x_6, M_e, \delta)}{\partial\delta} > 0.$$

This means that $\partial\lambda(x_6, M_e, \delta)/\partial\delta > 0$, which contradicts the assumption that $\partial\lambda(x, M_e, \delta)/\partial\delta \leq 0$ for all $x \in (x_4, x_5)$.

Case 2: $x_4 = 0$. In this case, it can potentially be that $\partial\lambda(0, M_e, \delta)/\partial\delta = 0$ or $\partial\lambda(0, M_e, \delta)/\partial\delta < 0$. Note that if $\partial\lambda(0, M_e, \delta)/\partial\delta = 0$, then $\partial(\lambda)'_x(x, M_e, \delta)/\partial\delta = 0$ (as $\partial c(0, M_e, \delta)/\partial\delta = 0$). As discussed at the beginning of the proof, this case is impossible. If $\partial\lambda(0, M_e, \delta)/\partial\delta < 0$, then from (3.15), $\partial(c)'_x(0, M_e, \delta)/\partial\delta < 0$, meaning that in some neighborhood of zero $c(x)$ falls with the rise in δ. Then, we use again the logic from the previous case and, thereby, derive the contradiction.

Part 3

The next step is to show that $\partial c(x, M_e, \delta)/\partial\delta > 0$ for all $x \in (0, \overline{x}]$. Assume that, on the contrary, that there exists a non-extendable interval $(x_7, x_8) \subset [0, \overline{x}]$, such that $\partial c(x, M_e, \delta)/\partial\delta \leq 0$ on this interval. If $x_7 = 0$, then $\partial(c)'_x(0, M_e, \delta)/\partial\delta \leq 0$ and $\partial c(0, M_e, \delta)/\partial\delta = 0$. In this case, $\partial\lambda(0, M_e, \delta)/\partial\delta \leq 0$ which contradicts our previous results. If $x_7 > 0$, then again $\partial c(x_7, M_e, \delta)/\partial\delta = 0$ and $\partial\,(c)'_x\,(x_7, M_e, \delta)/\partial\delta < 0$ (recall that $\partial\,(c)'_x\,(x_7, M_e, \delta)/\partial\delta$ cannot be equal to zero). That is, we derive the contradiction: $\partial\lambda(x_7, M_e, \delta)/\partial\delta < 0$.

Finally, since $\partial c(x, M_e, \delta)/\partial\delta > 0$, $\partial\overline{x}(M_e, \delta)/\partial\delta > 0$, and $(c)'_x > 0$, $\partial\overline{c}(M_e, \delta)/\partial\delta > 0$.

The proof of Proposition 5

(i) Totally differentiating both sides of the FOCs, $\Pi_p = 0$ and $\Pi_x = 0$, w.r.t. c yields

$$\begin{pmatrix} dp(c)/dc \\ dx(c)/dc \end{pmatrix} = -\begin{pmatrix} \Pi_{pp} & \Pi_{px} \\ \Pi_{px} & \Pi_{xx} \end{pmatrix}^{-1} \begin{pmatrix} \Pi_{cp} \\ \Pi_{cx} \end{pmatrix}, \tag{A.14}$$

where the right-hand side is evaluated at $(p, x) = (p(c),\, x(c))$. As implied by the FOCs and the definition of the profit function, we have: $\Pi_{cp} = -Q_p > 0$, $\Pi_{cx} = -Q_x = \frac{\Pi_x}{p-c} = 0$. Plugging these expressions for Π_{cp} and Π_{cx} back to (A.14) yields

$$\begin{pmatrix} dp(c)/dc \\ dx(c)/dc \end{pmatrix} = \frac{1}{\Pi_{pp}\Pi_{xx} - \Pi_{px}^2} \begin{pmatrix} \Pi_{xx}Q_p \\ -\Pi_{px}Q_p \end{pmatrix}. \tag{A.15}$$

Using (A.15) and the chain rule, and taking into account that $Q_x = 0$, we obtain:

$$\frac{dp(c)}{dc} = \frac{\Pi_{xx}}{\Pi_{pp}\Pi_{xx} - \Pi_{px}^2} Q_p > 0,$$

$$\frac{d}{dc}Q(p(c), x(c)) = \frac{\Pi_{xx}}{\Pi_{pp}\Pi_{xx} - \Pi_{px}^2} Q_p^2 < 0,$$

where both inequalities hold due to the SOC. This proves the inequalities in (30).

(ii) The equivalence of the inequality in (31) to $\mathrm{d}x(c)/\mathrm{d}c > 0$ follows immediately from (A.15) and the SOC.

B Some Figures

Figure 1: Basic Units in the City of Bergen

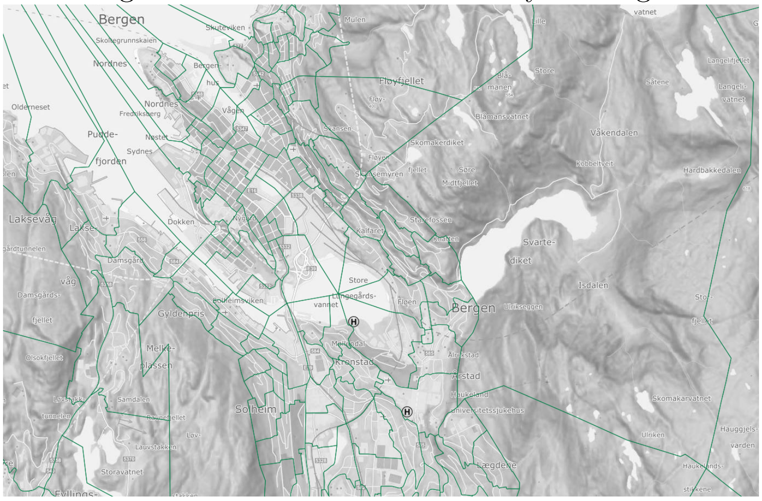
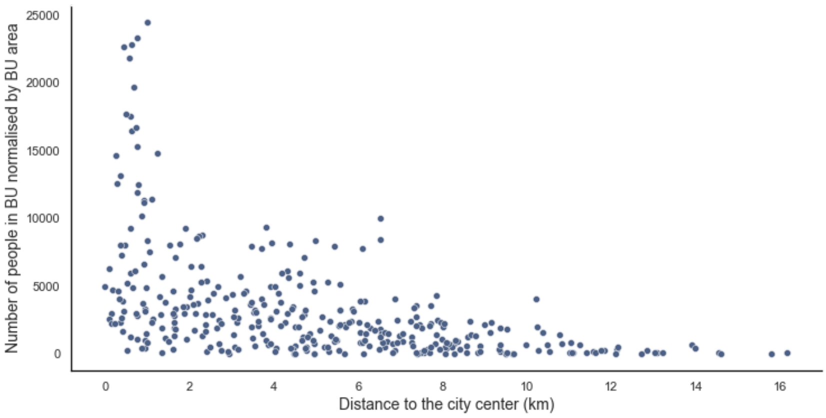


Figure 2: Distribution of population in Bergen



Note: Each dot in the figure represents the number of people living in a certain basic unit of Bergen divided by the basic unit area.

References

Courant, R. and F. John (2012). *Introduction to Calculus and Analysis I.* Springer Science & Business Media.

Pontryagin, L. S. (1962). *Ordinary Differential Equations.* Elsevier, Amsterdam.

Sundaram, R. K. (1996). *A First Course in Optimization Theory.* Cambridge University Press.