Online Appendix for
"Similarity Nash Equilibria in Statistical Games"
by Rossella Argenziano and Itzhak Gilboa

a) Calculation of $b(L, W, l, 1.5) > 0$.

We evaluate the function $b(L, W, l, \omega)$ at $\omega = 1.5$ and find the following expression:

$$\frac{L * g(L, W, l)}{(-1+L+W)^2(L+2lL+2lW)^2(L+2lL+2L^2+2lW+2LW)^2}$$

where $g(L, W, l)$ can be expressed as a polynomial in W :

$$\begin{aligned}
& g(L, W, l) \\
= \;& (32l^4 + 64l^3L + 32l^2L^2)W^5 \\
& +16l\left[2L^3 + l^3(8L-1) + 2l^2L(1+8L) + lL^2(5+8L)\right] W^4 \\
& +4lL\left[12l^3(4L-1) + 3lL^2(21+16L) + L^2(4+27L) + 8l^2(-1+3L+12L^2)\right] W^3 \\
& +2L^2 \left[\begin{array}{c} -3(L-2)L^2 + 8l^4(8L-3) + 16l^3(-2+3L+8L^2)+ \\ 2l^2(-6-6L+69L^2+32L^3) + 2lL(6-L+33L^2) \end{array} \right] W^2 \\
& + \left[\begin{array}{c} 16l^4(2L-1) + 3L(2+5L-4L^2) + 32l^3(-1+L+2L^2) \\ +4l^2(-3-12L+29L^2+8L^3) + 4l(-6+15L-14L^2+17L^3) \end{array} \right] L^3W \\
& -3L^4(L-1)^2(3+2L) + 12l^2L^4(L-1)^2 + 12lL^4(L-1)^3
\end{aligned}$$

Notice that for $W, L > 2$ and $l > 0$ the terms multiplying W^5, W^4, and W^3 are positive. The terms multiplying W^2 and L^3W and the constant are polynomials in l. For $l > 0$, all three are increasing in l, as the coefficients of the positive powers of l are positive. Moreover, all three are positive when evaluated are $l = 1$, hence for all $l > 1$ as well. . In particular, the coefficient of W^2 evaluated at $l = 1$ is equal to $-68 + 112L + 270L^2 + 127L^3 > 0$. The coefficient of L^3W evaluated at $l = 1$ is equal to $-84+82L+139L^2+88L^3 > 0$. Finally, the constant evaluated at $l = 1$ is equal to $3L^4(2L-3)(L-1)^2 > 0$.

We have proved that $g(L, W, l) > 0$. Since $\frac{L}{(-1+L+W)^2(L+2lL+2lW)^2(L+2lL+2L^2+2lW+2LW)^2} > 0$, this concludes the proof.

b) Calculation of $b(L, W, l, -0.5) > 0$ **for** $l > 1$ **and** $\left[\frac{lW}{L}\right] \geq 1$.

We evaluate the function $b(L, W, l, \omega)$ at $\omega = -0.5$ and find the following expression:

$$\frac{-L * h\,(L, W, l)}{(-1+L+W)^2(-3L+2lL+2lW)^2(-3L+2lL+2L^2+2lW+2LW)^2}$$

where $h\,(L, W, l)$ can be expressed as a polynomial in L :

$$\begin{aligned}
& h\,(L, W, l) \\
= \;& (20l-18)\,L^7 + \left[-4l^2(8W-5)+l(-76+92W)+45-36W\right] L^6 \\
& + \left[-36+4(23-10l)l+(81-4l(90+l(-75+16l)))W-2(9-78l+64l^2)W^2\right] L^5 \\
& + \left[\begin{array}{c} 9-36l+20l^2+(-54+388l-528l^2+224l^3-32l^4)W \\ +(36-492l+780l^2-256l^3)W^2+(116l-192l^2)W^3 \end{array}\right] L^4 \\
& + \left[\begin{array}{c} -4l(18-43l+24l^2-4l^3)-4l(-74+234l-168l^2+32l^3)W \\ -4l(52-185l+96l^2)W^2-4l(32l-8)W^3 \end{array}\right] WL^3 \\
& -8l^2W^2\left[-19+56W-30W^2+4W^3+l^2(-6+24W)+l(24-84W+32W^2)\right] L^2 \\
& -16l^3W^3\left[6-3l+(8l-14)W+4W^2\right] L-16l^4W^4(2W-1)
\end{aligned}$$

In what follows, we prove that $h\,(L, W, l) < 0$ for all $l > 0$ and $L, W > 2$. The constant term is negative. The coefficient of L is negative because it is the product of a negative term and a quadratic expression in W with a positive coefficient on the square which is positive and increasing at $W = 2$, hence for any larger W too. Similarly, the coefficient of L^2 is negative because it is the product of a negative term and a quadratic expression in l with a positive coefficient on the square which is positive and increasing at $l = 2$, hence for any larger l too.

The coefficient of L^3 is the product of W, which is positive, and a third degree polynomial in W which can be shown to be negative in the relevant range. In particular, the polynomial has a negative coefficient on the third and second power. At $W = 2$, this polynomial is equal to $-56l + 236l^2 - 288l^3 - 240l^4$ which is negative for all $l > 1$. Moreover, its derivative at $W = 2$ is equal to $-152l + 488l^2 - 864l^3 - 128l^4$ which is also negative for all $l > 1$. Finally, the fact that this derivative is negative $W = 2$ implies that it is also negative for all values of $W > 2$, because the negative coefficients on the third and second powers of W guarantee that the function is concave in W for positive W.

The coefficient of L^4 is a third degree polynomial in W which can be shown to be negative in the relevant range ($l > 1$, $W > 2$). The polynomial has a negative coefficient on the third power. Evaluated at $W = 2$, it takes value $45 - 300l + 548l^2 - 576l^3 - 64l^4 < 0$ for all $l > 1$. Moreover, its derivative w.r.t. W evaluated at $W = 2$ is equal to $90 - 188l + 288l^2 - 800l^3 - 32l^4$ which is also negative for all $l > 1$. Finally, its second derivative w.r.t. W is equal to $-8\left(-9 + 123l - 195l^2 + 64l^3 + (144l - 87)lW\right)$ which is negative at $W = 2$ and decreasing in W for all positive values of W.

The coefficient of L^5 is a quadratic function of W with a negative coefficient on the square, which is negative and decreasing at $W = 3$, hence negative for all larger values of W too. The coefficient of L^6 is a quadratic function of l with a negative coefficient on the square, which is positive for $l = 2$ and negative for all larger values of l. The coefficient of L^7 is positive.

Since the coefficient L^7 is positive, and we want to prove that the whole polynomial in L is negative, we prove that the sum of the terms in L^7 and L^5 is negative.

First, notice that the condition $\frac{lW}{L} \geq \frac{1}{2}$ implies that $L \leq 2lW$, which in turn implies:

$$(20l - 18)\, L^7 < 4\,(20l - 18)\, L^5 l^2 W^2$$

which in turn implies that

$$\begin{aligned}
& (20l - 18)\, L^7 + \left[\begin{array}{c} -36 + 4(23 - 10l)l \\ +(81 - 4l(90 + l(-75 + 16l)))W - 2(9 - 78l + 64l^2)W^2 \end{array}\right] L^5 \\
< \; & 4\,(20l - 18)\, L^5 l^2 W^2 + \left[\begin{array}{c} -36 + 4(23 - 10l)l \\ +(81 - 4l(90 + l(-75 + 16l)))W - 2(9 - 78l + 64l^2)W^2 \end{array}\right] L^5 \\
= \; & \left[\begin{array}{c} (80l - 72)\, l^2 W^2 - 36 + 4(23 - 10l)l \\ +(81 - 4l(90 + l(-75 + 16l)))W - 2(9 - 78l + 64l^2)W^2 \end{array}\right] L^5 \\
= \; & \left[\left(92l - 40l^2 - 36\right) + \left(300l^2 - 64l^3 - 360l + 81\right) W + (-128l^2 + 236l - 90)W^2\right] L^5
\end{aligned}$$

The last expression is a quadratic in W which is negative for all $W > 2$. In particular, it has a negative coefficient on the square, hence it is concave. Evaluated at $W = 2$ it is equal to $-128l^3 + 48l^2 + 316l - 234 < 0$ for all $l > 1$. Moreover, its derivative evaluated at $W = 2$ is equal to $-64l^3 - 212l^2 + 584l - 279 < 0$ for all $l > 1$.

To conclude the proof that the whole polynomial in L is negative, we still need to address the fact that the coefficient of L^6 is positive at $l = 2$.

In particular, we do so by proving that the sum of the terms in L^6 and L^4 is negative at $l = 2$. First, notice that the condition $\frac{lW}{L} \geq \frac{1}{2}$ implies that $L \leq 2lW$, which in turn implies:

$$\begin{aligned} & \left[-4l^2(8W-5)+l(-76+92W)+45-36W\right]/_{l=2}L^6 \\ < \;\; & 4\left[-4l^2(8W-5)+l(-76+92W)+45-36W\right]/_{l=2}L^4l^2W^2 \end{aligned}$$

which in turn implies that

$$\begin{aligned} = \;\; & \left[-4l^2(8W-5)+l(-76+92W)+45-36W\right]/_{l=2}L^6 \\ & +L^4\left[\begin{array}{c} 9-36l+20l^2+(-54+388l-528l^2+224l^3-32l^4)W \\ +(36-492l+780l^2-256l^3)W^2+(116l-192l^2)W^3 \end{array}\right]/_{l=2} \\ < \;\; & 4\left[-4l^2(8W-5)+l(-76+92W)+45-36W\right]/_{l=2}L^4l^2W^2 \\ & +L^4\left[\begin{array}{c} 9-36l+20l^2+(-54+388l-528l^2+224l^3-32l^4)W \\ +(36-492l+780l^2-256l^3)W^2+(116l-192l^2)W^3 \end{array}\right]/_{l=2} \\ = \;\; & \left(-216W^3-308W^2-110W+17\right)L^4 < 0 \text{ for all } W > 2. \end{aligned}$$

This concludes the proof that $b\,(L, W, l, -0.5) > 0$ for $l > 1$.

Calculation of $b\,(L, W, l, 0.5) > 0$ **for** $l > 1$ **and** $\left[\frac{lW}{L}\right] = 0$.

We evaluate the function $b\,(L, W, l, \omega)$ at $\omega = 0.5$ and find the following expression:

$$\frac{L * \eta\,(L, W, l)}{(L+W-1)^2\,(-L+2LW+2lW+2Ll+2L^2)\,(-L+2lW+2Ll)}$$

where $\eta\,(L, W, l)$ can be expressed as a polynomial in L : *in* which all the coefficients, as well as the constant, are positive:

$$
\begin{aligned}
&\eta(L,W,l) \\
=\;& (12l-2)L^7 + \left[W(4l-4) + l(32W-28) + 32l^2W + 12l^2 + 3\right]L^6 \\
&+ \left[\begin{array}{c} 64l^3W + l^2W(100W-44) + l^2(28W^2-24) \\ +W^2(6l-2) + lW(30W-72) + 20l + 7W \end{array}\right]L^5 \\
&+ \left[\begin{array}{c} 6l^4W + l^3W(156W-96) + l^2W^2(192W-204) + 16l^2W(l^2-1) + 12lW^3 \\ +lW^2(100l^2-60) + (44lW-1) + l(12l-4) + 2W(2W-1) \end{array}\right]L^4 \\
&+ \left[\begin{array}{c} l^4W(128W-16) + l^3W^2(300W-288) + 32l^3W + l^2W^3(128W-228) \\ +40l^2W^2 + 20l^2W + lW^3(84l^2-16) + lW(24W-8) \end{array}\right]L^3 \\
&+ \left[\begin{array}{c} l^4W^2(192W-48) + l^2W^4(156l-80) + l^3W^3(100W-288) + 64l^3W^2 \\ +32l^2W^5 + 32l^2W^3 + 8l^2W^2 \end{array}\right]L^2 \\
&+ \left[l^4W^3(128W-48) + l^3W^4(64W-96) + 32l^3W^3\right]L + 16l^4W^4(2W-1)
\end{aligned}
$$

c) Calculation of $\frac{\partial b(L,W,l,\omega)}{\partial\omega} < 0$ for all $\omega \geq -\frac{W}{L}$ for the case $l = 1$

For $l = 1$, the $b(L,W,l,\omega)$ function and its derivative with respect to ω are

$$
\begin{aligned}
b(L,W,1,\omega) &= \frac{LW(L+W)}{(L+W-1)^2} + \frac{L+W+L\omega}{W+L\omega} \\
&- \frac{(1+L)(W+LW+L\omega)(L+L^2+W+LW+L\omega)}{(L^2+W+LW+L\omega)^2}
\end{aligned}
$$

$$
\frac{\partial b(L,W,1,\omega)}{\partial\omega} = \frac{-L^3\phi(L,W,\omega)}{(W+L\omega)^2(L^2+W+LW+L\omega)^3}
$$

where $\phi(L,W,\omega)$ is the following cubic expression in ω in which all the coefficients, including the constant, are positive.

$$
\begin{aligned}
&\phi(L,W,\omega) \\
=\;& L^5 + 3L^3W + 3L^4W + 4LW^2 + 8L^2W^2 + 4L^3W^2 + 2W^3 + 4LW^3 + 2L^2W^3 \\
&+\omega\left(3L^4 + 8L^2W + 10L^3W + 2L^4W + 4LW^2 + 6L^2W^2 + 2L^3W^2\right) \\
&+\omega^2\left(4L^3 + 2L^4 + L^5 + 2L^2W + 3L^3W + L^4W\right) + \omega^3L^4
\end{aligned}
$$

The sign of the coefficients guarantees that the expression is positive, for all $\omega \geq 0$. To examine the sign of $\phi(L,W,\omega)$ for $w \in [-\frac{W}{L}, 0)$, notice that:

a) $\phi\left(L,W,-\frac{W}{L}\right)=L^2(L+W)^3>0$

b) $\phi\left(L,W,0\right)=L^5+3L^3W+3L^4W+4LW^2+8L^2W^2+4L^3W^2+2W^3+4LW^3+2L^2W^3>0$

c)

$$\begin{aligned}
\frac{\partial\phi\left(L,W,\omega\right)}{\partial\omega} &= \left(3L^4+8L^2W+10L^3W+2L^4W+4LW^2+6L^2W^2+2L^3W^2\right) \\
&\quad +2\omega\left(4L^3+2L^4+L^5+2L^2W+3L^3W+L^4W\right)+3L^4\omega^2 \\
&\geq \left(3L^4+8L^2W+10L^3W+2L^4W+4LW^2+6L^2W^2+2L^3W^2\right) \\
&\quad +2\omega\left(4L^3+2L^4+L^5+2L^2W+3L^3W+L^4W\right) \\
&> \left(3L^4+8L^2W+10L^3W+2L^4W+4LW^2+6L^2W^2+2L^3W^2\right) \\
&\quad -2\frac{W}{L}\left(4L^3+2L^4+L^5+2L^2W+3L^3W+L^4W\right) \\
&= 3L^3\left(L+2W\right)>0
\end{aligned}$$

where the first inequality follows from the fact that $3L^4\omega^2\geq 0$ and the second from the fact that $\omega>-\frac{W}{L}$.

Hence we can conclude that $\phi\left(L,W,\omega\right)$ is positive and increasing in the whole interval $\left(-\frac{W}{L},0\right)$, hence the function $b\left(L,W,1,\omega\right)$ is decreasing for all $\omega>-\frac{W}{L}$.